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High moments of L-functions

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High moments of $L$-functions

Vorrapan (Fai) Chandee
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joint work with
Xiannan Li
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NSF-CBMS Conference: L-functions and Multiplicative Number Theory
May 21, 2019
Outline of the talk

• Moment results for various families of L-functions
  • Riemann zeta function
  • Dirichlet L-functions
  • $\Gamma_1(q)$ L-functions
  • $GL(k) \times GL(2)$ Rankin Selberg L-functions
• Few words about the proof of high moments.
Moments of the Riemann zeta function
The Riemann zeta function

**The Riemann hypothesis (RH):** all non-trivial zeros of the Riemann zeta function lie on the critical line $\text{Re}(s) = 1/2$.

One consequence of RH is the **Lindelöf hypothesis (LH)**, which states that

$$\zeta \left( \frac{1}{2} + it \right) = O(t^\epsilon).$$
The Riemann zeta function

**The Riemann hypothesis (RH):** all non-trivial zeros of the Riemann zeta function lie on the critical line \( \Re(s) = 1/2 \).

One consequence of RH is the **Lindelöf hypothesis** (LH), which states that

\[
\zeta\left(\frac{1}{2} + it\right) = O(t^\epsilon).
\]

An equivalent statement of LH is connected to the moments of the Riemann zeta function. In particular,

**Theorem**

*LH is true if and only if for all \( k \in \mathbb{N} \),

\[
\frac{1}{T} \int_0^T \left| \zeta\left(\frac{1}{2} + it\right) \right|^{2k} \, dt = O(T^\epsilon).
\]

Moreover, understanding higher moments of \( \zeta(s) \) gives us progressively better bounds for \( \zeta(1/2 + it) \).
• Second moment: \( k = 1 \rightarrow \zeta(1/2 + it) \ll T^{\frac{1}{2} + \epsilon} \)

• Fourth moment: \( k = 2 \rightarrow \zeta(1/2 + it) \ll T^{\frac{1}{4} + \epsilon} \) (convexity bound)

• Sixth moment and higher: \( k \geq 3 \rightarrow \zeta(1/2 + it) \ll T^{\frac{1}{2k} + \epsilon} \) (subconvexity bound)
History of the moments of the Riemann zeta function

Let \( I_k(T) = \int_0^T |\zeta \left( \frac{1}{2} + it \right) |^{2k} \, dt \).

\( k = 1 \)

Hardy and Littlewood (1918) showed that

\[ I_1(T) \sim T(\log T). \]
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\( k = 2 \)
Ingham (1926) showed that
\[
I_2(T) \sim 2a_2 T \frac{(\log T)^4}{4!}.
\]

It remains unsolved! However, we have a good conjecture, lower bound and upper bound for it.
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It remains unsolved! However, we have a good conjecture, lower bound and upper bound for it.
Conjecture for the moments of $\zeta(s)$

Keating and Snaith (2000): Through random matrix theory and relating distribution of the zeros $\zeta(s)$ with distribution of eigenvalues of matrices in circular unitary ensembles, they conjectured that for any positive integers $k$,

$$I_k(T) \sim \frac{g_k a_k}{k^2!} T (\log T)^{k^2},$$

where $a_k$ is the coefficient of the leading term of $\sum_{n \leq T} \frac{d_k^2(n)}{n}$, and

$$g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}.$$

Note: $g_1 = 1$, $g_2 = 2$, $g_3 = 42$, and $g_4 = 24024$. 
Conjecture (cont’d)

Conrey, Farmer, Keating, Rubinstein, Snaith (2005) gave a more precise conjecture of $I_k(T)$ including an asymptotic expansion for lower order terms through the shifted moments. Their recipe also applies to moments of other families of $L$-functions.

Diaconu, Goldfeld and Hoffstein (2003) gave an alternative approach, based on multiple Dirichlet series, to give the same conjectures.

Conrey and Keating (2016-2017) conjectured the $2k^{th}$ moments and shifted moments of the Riemann zeta function using long Dirichlet polynomial and divisor correlations.

There are good upper and lower bound results (Sound will speak about it tomorrow).
Why is it difficult to compute high moments \((k > 2)\) ?

Approximate functional equation

\[
\zeta \left( \frac{1}{2} + it \right) = \sum_{n \ll \sqrt{T}} \frac{1}{n^{1/2+it}} + \text{gamma factor} \sum_{m \ll \sqrt{T}} \frac{1}{m^{1/2-it}}
\]

\[
\int_T^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} \, dt \sim Tc_k \sum_{n \leq T^{k/2}} \frac{d_k^2(n)}{n} + \text{off-diagonal terms } m \neq n .
\]

- When \(k = 1, 2\) the main term solely comes from the diagonal terms.

\[
\int_T^{2T} \left| \zeta \left( \frac{1}{2} + it \right) \right|^{2k} \, dt \sim Tc_k \sum_{n \leq T^{k/2}} \frac{d_k^2(n)}{n} + \text{off-diagonal terms } m \neq n .
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$$

$$
\int_{T}^{2T} |\zeta \left( \frac{1}{2} + it \right)|^{2k} \, dt \sim Tc_k \sum_{n \leq T^{k/2}} \frac{d_k^2(n)}{n} + \text{off-diagonal terms } m \neq n.
$$

- When $k = 1, 2$ the main term solely comes from the diagonal terms.
- When $k \geq 3$, the Dirichlet polynomial is too long (the length goes up to $T^{k/2}$). The off-diagonal terms also give main contribution, but we do not know how to deal with the off-diagonal terms. It is related to shifted convolution sums of the form $\sum_{n \leq x} d_k(n)d_k(n + f)$.
- The off-diagonal terms are also a main difficulty for high moments of other families of $L$-functions.
Moments of Dirichlet $L$-functions
Dirichlet $L$-functions

$$L(s, \chi) = \sum_{n \geq 1} \frac{\chi(n)}{n^s} = \prod_p \left(1 - \frac{\chi(p)}{p^s}\right)^{-1},$$

where $\chi$ is a primitive Dirichlet character modulo $q$.

- The family of all primitive Dirichlet $L$-functions of modulus $q$ is analogous in some ways to the Riemann zeta function in $t$-aspects.
- In fact, this family is associated to unitary ensemble.
- The moments of this family should behave similarly to the moments of the $\zeta(1/2 + it)$. 

Notation: $\sum^*$ is the sum over all primitive characters mod $q$.

$\phi^*(q)$ is number of primitive characters mod $q$. 

V. Chandee and X. Li  High moments of $L$-functions
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V. Chandee and X. Li
High moments of $L$-functions
Moments of $L(s, \chi)$

Let $M_k(q) = \sum_{\chi \mod q} \* \chi(\mod q) \ |L(1/2, \chi)|^{2k}$.

$k = 1$: $M_1(q) \sim \phi^*(q) \log q$.

$k = 2$: Heath-Brown (1981), Soundararjan (2007), Young (2010) showed that

$$\sum_{\chi \mod q} \* \chi(\mod q) \ |L(1/2, \chi)|^{4} \sim 2b_2 \phi^*(q) \frac{(\log q)^4}{4!}$$

$k \geq 3$: Unknown. It is conjectured that

$$\sum_{\chi \mod q} \* \chi(\mod q) \ |L(1/2, \chi)|^{2k} \sim g_k b_k \phi^*(q) \frac{(\log q)^{k^2}}{k^2!},$$

where $g_k = k^2! \prod_{j=0}^{k-1} \frac{j!}{(k+j)!}$ (same as the constant in the asymptotic formula of Riemann zeta function case).
From the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith, it is conjectured that

\[
\frac{1}{\phi^*(q)} \sum_{\chi \mod q}^* |L(1/2, \chi)|^6 \sim 42a_3 \prod_{p|q} \frac{(1 - \frac{1}{p})^5}{\left(1 + \frac{4}{p} + \frac{1}{p^2}\right)} \frac{(\log q)^9}{9!}
\]
The sixth moment of Dirichlet $L$-functions

From the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith, it is conjectured that

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* |L(1/2, \chi)|^6 \sim 42a_3 \prod_{p|q} \left(1 - \frac{1}{p}\right)^5 \left(\frac{1 - \frac{1}{p}}{1 + \frac{4}{p} + \frac{1}{p^2}}\right)^{9!} (\log q)^9$$

By using large sieve inequality, Huxley (1970) showed that

$$\sum_{q \sim Q} \sum_{\chi \pmod{q}}^* |L(1/2, \chi)|^6 \ll Q^2 (\log Q)^9.$$
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The average over $q$ significantly increases the size of our family of $L$-functions from a family of about $q$ $L$-functions of conductor $q$ to a family of about $Q^2$ $L$-functions with conductor around $Q$. 

V. Chandee and X. Li
High moments of $L$-functions
Question: Is there an asymptotic formula of the sixth moment for this larger family of Dirichlet $L$-functions?

Almost!

Conrey, Iwaniec and Soundararajan (2012) can prove the following

$$\sum_{q \leq Q} \sum_{\star \chi \mod q} \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, \chi\right) \right|^6 \frac{dt}{\Gamma\left(\frac{1}{2} + it\right)^6} \sim 42 a^3 \sum_{q \leq Q} \prod_p \left| \frac{q(1 - 1/p)^5}{1 + 4p + p^2} \phi^\star (q) \log q \right|^{9/9!} \int_{-\infty}^{\infty} \left| \frac{1}{\Gamma\left(\frac{1}{2} + it\right)^6} \right| dt \sim 42 \tilde{a}^3 Q^2 \log^9 Q \frac{1}{9!} \int_{-\infty}^{\infty} \left| \frac{1}{\Gamma\left(\frac{1}{2} + it\right)^6} \right| dt.$$
The sixth moment (cont’d)

**Question:** Is there an asymptotic formula of the sixth moment for this larger family of Dirichlet $L$-functions?

**Answer:** Almost!

Conrey, Iwaniec and Soundararajajan (2012) can prove the following

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \int_{-\infty}^{\infty} \left| L \left( \frac{1}{2} + it, \chi \right) \right|^6 \left| \Gamma \left( \frac{1/2 + it}{2} \right) \right|^6 dt$$

$$\sim 42 a_3 \sum_{q \leq Q} \prod_{p|q} \frac{\left( 1 - \frac{1}{p} \right)^5}{\left( 1 + \frac{4}{p} + \frac{1}{p^2} \right)} \phi^*(q) \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma \left( \frac{1/2 + it}{2} \right) \right|^6 dt$$

$$\sim 42 \tilde{a}_3 Q^2 \frac{(\log Q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma \left( \frac{1/2 + it}{2} \right) \right|^6 dt.$$
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\sim 42a_3 \sum_{q \leq Q} \prod_{p|q} \left(1 - \frac{1}{p}\right)^5 \frac{\phi^*(q)(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma \left(\frac{1}{2} + \frac{it}{2}\right) \right|^6 dt
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The sixth moments (cont’d)

- Off-diagonal terms also contribute to the main term.
- They also state a more precise technical result which gives the asymptotic for the sixth moment including shifts with a power saving error term of size $Q^{2-1/10+\epsilon}$.
- Conrey, Iwaniec and Soundararajan’s proof is related to their work on the ”asymptotic large sieve”.
- The limitation of the method should be the eighth moment.
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- The limitation of the method should be the eighth moment.
- The integration over $t$ is fairly short due to the rapid decay of the $\Gamma$ function along vertical lines.
- Deriving an analogous result without the average over $t$ remains open.
- C., Li, Matomäki, Radziwiłł (on-going): Obtain an asymptotic formula without the average over $t$. 
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The eighth moment of Dirichlet $L$-functions

From the recipe of Conrey, Farmer, Keating, Rubinstein and Snaith, it is conjectured that

$$\frac{1}{\phi^*(q)} \sum_{\chi \pmod{q}}^* |L(1/2, \chi)|^8 \sim 24024 a_4 \prod_{p | q} \frac{(1 - \frac{1}{p})^7}{\left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right)} \frac{(\log q)^{16}}{16!}.$$ 

Note that the constant 24024 appears in the leading term of the eighth moment of $\zeta(s)$. 

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Note that the constant 24024 appears in the leading term of the eighth moment of $\zeta(s)$. By using large sieve inequality, Huxley (1970) showed that

$$\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* |L(1/2, \chi)|^8 \ll Q^2(\log Q)^{16}.$$
Theorem (C., Li (2014))

On GRH, we have

\[
\sum_{q \leq Q} \sum_{\chi \pmod{q}}^* \int_{-\infty}^{\infty} \left| L\left(\frac{1}{2} + it, \chi\right) \right|^8 \left| \Gamma \left(\frac{1/2 + it}{2}\right) \right|^8 dt 
\]

\[
\sim 24024 a_4 \sum_{q \leq Q} \prod_{p | q} \frac{(1 - \frac{1}{p})^7}{\left(1 + \frac{9}{p} + \frac{9}{p^2} + \frac{1}{p^3}\right) \phi^*(q) \frac{(\log q)^{16}}{16!}} 
\]

\[
\times \int_{-\infty}^{\infty} \left| \Gamma \left(\frac{1/2 + it}{2}\right) \right|^8 dt 
\]

\[
\sim 24024 \tilde{a}_4 Q^2 \frac{(\log Q)^{16}}{16!} \int_{-\infty}^{\infty} \left| \Gamma \left(\frac{1/2 + it}{2}\right) \right|^8 dt. 
\]

**Note:** We cannot get power saving error terms. Our error term is of size \( Q^2(\log Q)^{15+\epsilon} \).
Moments of $\Gamma_1(q)$ automorphic $L$-functions
(GL(2) $L$-functions)
Holomorphic $L$-functions

Let $q$ be a prime number. Let $S_k(\Gamma_0(q), \chi)$ be the space of cuspidal holomorphic forms of weight $k$ with respect to the congruence subgroup $\Gamma_0(q)$ and the character $\chi \mod q$.

$$\Gamma_0(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \bigg| \ c \equiv 0 \mod q \right\},$$

and we define $S_k(\Gamma_1(q))$ be the space of cuspidal holomorphic forms of weight $k$ with respect to the congruence subgroup $\Gamma_1(q)$, where

$$\Gamma_1(q) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \ \bigg| \ c \equiv 0 \mod q, \ a \equiv d \equiv 1 \mod q \right\}$$
Let $\mathcal{H}_k(q, \chi) \subset S_k(\Gamma_0(q), \chi)$ be the set of orthogonal basis of $S_k(\Gamma_0(q), \chi)$. Let $f$ be a normalized cusp form in $\mathcal{H}_k(q, \chi)$.

An $L$-function $L(f, s)$ associated to the normalized cusp form $f$ is defined for $\text{Re}(s) > 1$ as

$$L(f, s) = \sum_{n \geq 1} \frac{\lambda_f(n)}{n^s} = \prod_p \left( 1 - \frac{\lambda_f(p)}{p^s} + \frac{\chi(p)}{p^{2s}} \right)^{-1},$$

where $\lambda_f(n)$ is the coefficient from the Fourier expansion of $f$. 
Some results on moments of automorphic $L$-functions

The second moment: $\sum_{f \in \mathcal{H}_2(q, \chi_0)} L(f, 1/2)^2 \sim \log q$ is obtained by Iwaniec and Sarnak

The fourth moment: Kowalski, Michel and Vanderkam (2000) obtained the result for $\sum_{f \in \mathcal{H}_2(q, \chi_0)} L(f, 1/2)^4 \sim c_k(\log q)^6$
Some results on moments of automorphic $L$-functions

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Higher moment: Unknown. Lindelöf type bounds (without GRH assumption) for higher moments are also challenging.

$$\sum_{f \in \mathcal{H}_k(q, \chi)} |L(f, 1/2)|^{2k} \ll q^\epsilon.$$
• To obtain good upper bounds/asymptotic formulae for the sixth and the eighth moment of Dirichlet $L$-functions, we need to enlarge the size of the family we average on.

• In this case, we will also increase the size of family of $L$-functions to get an asymptotic formula and Lindelöf upper bounds for the sixth and the eighth moment.
The spaces $S_k(\Gamma_0(q))$ vs $S_k(\Gamma_1(q))$

Dimension of the spaces

$$\dim S_k(\Gamma_0(q)) \sim \frac{k - 1}{12} q \prod_{p|q} (1 + p^{-1}),$$

and

$$\dim S_k(\Gamma_1(q)) \sim \frac{k - 1}{24} q^2 \prod_{p|q} (1 - p^{-2}).$$

They are connected by

$$S_k(\Gamma_1(q)) = \bigoplus_{\chi \mod q, \chi(-1)=(-1)^k} S_k(\Gamma_0(q), \chi).$$
The spaces \( S_k(\Gamma_0(q)) \) vs \( S_k(\Gamma_1(q)) \)

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<table>
<thead>
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<th>Dirichlet</th>
<th>Holomorphic</th>
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<td>Dirichlet characters mod ( q ) (size ( q ))</td>
<td>( \Gamma_0(q) ) modular forms</td>
</tr>
<tr>
<td>All Dirichlet characters mod ( q \sim Q ) (size ( Q^2 ))</td>
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Djankovic’s work

For fixed weight $k$ and prime $q$, Djankovic (2011) considered the sixth moment of family of $L$-functions associated with modular forms in $S_k(\Gamma_1(q))$. In particular, he considered

$$M_6(q) = \frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{\chi(-1)=(-1)^k}^{\chi \in \mathcal{H}_k(q,\chi)} |L(f,1/2)|^6.$$ 

This family also admits the unitary symmetry, so it has similar conjectures to the moment of $\zeta(s)$.
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This family also admits the unitary symmetry, so it has similar conjectures to the moment of $\zeta(s)$.

**Theorem (Djankovic)**

*For* $k \geq 3$,

$$M_6(q) \ll q^\epsilon.$$
• This bound is consistent with the Lindelöf hypothesis on average.

• A main tool of the proof is an asymptotic large sieve for the family of \( \Gamma_1(q) \) developed by Iwaniec and Xiaoqing Li (2007):

\[
\frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{\substack{h \\ \chi(-1)=(-1)^k}} \left| \sum_{n \leq N} a_n \lambda_f(n) \right|^2.
\]
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\]

We cannot find an asymptotic formula for \( M_6(q) \). Instead, we can obtain an asymptotic formula if we include a short integration over \( t \).

\[
\mathcal{I}_6(q) := \frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{f \in \mathcal{H}_k(q,\chi)} \int_{-\infty}^{\infty} \left| L \left( f, \frac{1}{2} + it \right) \right|^6 \left| \Gamma \left( \frac{k}{2} + it \right) \right|^6 dt.
\]
Theorem (C. and Li, 2016)

For \( k \geq 4 \), we have

\[
I_6(q) = \frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{f \in \mathcal{H}_k(q, \chi)}^{h} \int_{-\infty}^{\infty} \left| L \left( f, \frac{1}{2} + it \right) \right|^6 \left| \Gamma \left( \frac{k}{2} + it \right) \right|^6 dt
\]

\[
\sim 42b_3 \frac{(\log q)^9}{9!} \int_{-\infty}^{\infty} \left| \Gamma \left( \frac{k}{2} + it \right) \right|^6 dt
\]

In fact, we prove a more precise asymptotic formula including shifts with a power saving error term of size \( q^{-1/4 + \epsilon} \).
The eighth moment of $\Gamma_1(q)$ automorphic $L$-functions

**Question:** Could we get the Lindelöf type of upper bounds for the eighth moment of this family by the same method as the sixth moment?

\[ M_8(q) = \frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{\substack{h \in \mathcal{H}_k(q, \chi) \atop \chi(-1)=(-1)^k}} |L(f, 1/2)|^8 \ll q^{\epsilon}. \]
The eighth moment of $\Gamma_1(q)$ automorphic $L$-functions

**Question:** Could we get the Lindelöf type of upper bounds for the eighth moment of this family by the same method as the sixth moment?

$$M_8(q) = \frac{2}{\phi(q)} \sum_{\chi \mod q} \sum_{f \in \mathcal{H}_k(q, \chi)^h} |L(f, 1/2)|^8 \ll q^\epsilon.$$ 

**Answer:** No!

Part of the difficulty in this family is from the fact that the asymptotic large sieve of Iwaniec and Li is not perfectly orthogonal.
"Perfectly orthogonal" large sieve: Let $\mathcal{X}$ be a finite set of "nice" sequences.

$$\sum_{x \in \mathcal{X}} \left| \sum_{n \leq N} a_n x(n) \right|^2 \ll (|\mathcal{X}| + N) \sum_{n \leq N} |a_n|^2.$$ 

For example, the large sieve for primitive Dirichlet $L$-functions:

$$\sum_{q \sim Q} \sum_{\chi \pmod{q}}^{*} \left| \sum_{n \leq N} a_n \chi(n) \right|^2 \ll (Q^2 + N) \sum_{n \leq N} |a_n|^2.$$
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\]

However Iwaniec and Li’s large sieve is of the form

\[
\frac{2}{\phi(q)} \sum_{\chi \mod q}^* \sum_{f \in \mathcal{H}_k(q, \chi)} \left| \sum_{n \leq N} a_n \lambda_f(n) \right|^2 \ll \text{Main term} + O \left( q^{-2} (N + q^2) \sum |a_n|^2 \right).
\]
Main term composes of Bessel function and Kloosterman sum. It can be larger than expected if the coefficients are chosen to look like certain Bessel functions twisted by Kloosterman sums.
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We surmount this difficulty, in the process; we need to verify that the coefficients $d_4(n)$ from the eighth moment are not correlated to Bessel function twisted with Kloosterman sums in certain ranges.

**Theorem (C. and X. Li, 2018)**

For $k \geq 4$, we have

$$M_8(q) = \frac{2}{\phi(q)} \sum_{\chi \bmod q} \sum_{\substack{f \in \mathcal{H}_k(q,\chi) \atop \chi(-1)=-1}} \left| L(f, \frac{1}{2}) \right|^8 \ll q^\epsilon.$$
Moments of families of $GL(k) \times GL(2)$ Rankin-Selberg $L$-functions at special points
Let $\phi^{(k)}$ be a fixed Hecke Maass form for $SL(k, \mathbb{Z})$ when $k = 3$ or $4$ and $u_j$ be an orthonormal basis of Hecke-Maass forms on $SL(2, \mathbb{Z})$ with Laplace eigenvalues $\frac{1}{4} + t_j^2$.

**The special points are** $s = \frac{1}{2} + it_j$

The $GL(k) \times GL(2)$ Rankin-Selberg $L$-function has Dirichlet series

$$L(u_j \times \phi^{(k)}, s) = \sum_{m,n \geq 1} \lambda_j(n) B_k(m, n) \frac{m^{2s} n^s}{m^{2s} n^s},$$

where $\lambda_j(n)$ is the Fourier coefficient from $u_j$, and $B_k(m, n)$ is the coefficient from $\phi^{(k)}$. 
At special points

The special points $1/2 + it_j$ lead to conductor dropping phenomena. Let $t_j \sim T$.

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Gamma factor in functional equation:

$$\prod_{i=1}^{k} \Gamma \left( \frac{s - it_j - \alpha_i}{2} \right) \Gamma \left( \frac{s + it_j - \alpha_j}{2} \right).$$

The conductor dropping is somewhat analogous to enlarging the size of Dirichlet and holomorphic GL(2) $L$-functions families.
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The conductor dropping is somewhat analogous to enlarging the size of Dirichlet and holomorphic $GL(2)$ $L$-functions families.

- This makes it possible to study the second moment for both families. $\sum_{t_j \leq T} \left| L \left( u_j \times \phi^{(k)}, \frac{1}{2} + it_j \right) \right|^2$
- Note that $\sum_{t_j \leq T} \left| L \left( u_j \times \phi^{(k)}, \frac{1}{2} \right) \right|^2$ is very difficult to evaluate.

V. Chandee and X. Li

High moments of $L$-functions
Theorem (Young, 2011)

Let $L \left( u_j \times \phi^{(3)}, s \right)$ be $GL(3) \times GL(2)$ Rankin- Selberg $L$-functions. Then

$$\sum_{t_j \leq T} \left| L \left( u_j \times \phi^{(3)}, \frac{1}{2} + it_j \right) \right|^2 \ll T^{2+\epsilon}.$$
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\sum_{t_j \leq T} \left| L \left( u_j \times \phi^{(3)}, \frac{1}{2} + it_j \right) \right|^2 \ll T^{2+\epsilon}.
$$

Theorem (C. and X. Li, 2018-19)

Let $L \left( u_j \times \phi^{(4)}, s \right)$ be $GL(4) \times GL(2)$ Rankin- Selberg $L$-functions. Then

$$
\sum_{t_j \leq T} \left| L \left( u_j \times \phi^{(4)}, \frac{1}{2} + it_j \right) \right|^2 \ll T^{2+\epsilon}.
$$

Note: The main technical evaluation of these moments requires careful understanding of certain exponential sums restricted to narrow regions.
• These bounds represent Lindelöf hypothesis on average.
• The limits of the current method is the second moment for the $GL(4) \times GL(2)$ family where our moment bound immediately implies the convexity bound for $L$-functions in this family, which is $T^{1+\epsilon}$.

$$\sum t_j \leq |\mathcal{L}(u_j, 1/2 + it_j)|^{2k} \ll T^{2+\epsilon},$$

$$\sum t_j \sim |\mathcal{L}(u_j, 1/2 + it_j)|^6 \ll T^{9/4+\epsilon},$$

$$\sum t_j \sim |\mathcal{L}(u_j, 1/2 + it_j)|^8 \ll T^{5/2+\epsilon}.$$
• These bounds represent Lindelöf hypothesis on average.
• The limits of the current method is the second moment for the $GL(4) \times GL(2)$ family where our moment bound immediately implies the convexity bound for $L$-functions in this family, which is $T^{1+\epsilon}$.
• These methods lead to essentially optimal upper bounds for the sixth and eighth moment of $GL(2) L$-functions associated with Hecke-Mass cusp forms.

$$\sum_{t_j \leq T} \left| L \left( u_j, \frac{1}{2} + it_j \right) \right|^{2k} \ll T^{2+\epsilon},$$

where $k = 3, 4$.

• Previously Luo (1995) applied a large sieve type inequality he derived for this family and obtained that

$$\sum_{t_j \sim T} \left| L \left( u_j, \frac{1}{2} + it_j \right) \right|^6 \ll T^{9/4+\epsilon}, \quad \sum_{t_j \sim T} \left| L \left( u_j, \frac{1}{2} + it_j \right) \right|^8 \ll T^{5/2+\epsilon}.$$
**Few words about the proof**

High moments for each families require different techniques as their structures are different. However, there are some similar phenomenon appearing in the proof.

- The drop in conductor.
  (This is helpful especially for the sixth moment)

- Understanding the sums in narrow regions.
  (This is especially apparent in the eighth moment.)
Moments of Dirichlet $L$-functions

We consider

$$\sum_q \psi \left( \frac{q}{Q} \right) \sum_{\chi \pmod q}^* \int_{-\infty}^{\infty} \left| L\left( \frac{1}{2} + it, \chi \right) \right|^{2k} \left| \Gamma \left( \frac{1/2 + it}{2} \right) \right|^{2k} dt$$

where $k = 3$ (the sixth moment) or 4 (the eighth moment), and $\psi$ is a smooth function compactly supported in $[1, 2]$. 
After approximate functional equation and orthogonality relation of Dirichlet characters, we roughly need to understand the sum of the form

\[ Q \sum_q \psi \left( \frac{q}{Q} \right) \sum_{m,n \leq Q^{k/2+\epsilon}} \frac{d_k(n)d_k(m)}{\sqrt{mn}} \]

- If we do not have the integration over \( t \), the main contribution will come from \( mn \ll Q^{k+\epsilon} \) instead of \( m, n \leq Q^{k/2+\epsilon} \).
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\[ m \equiv n \mod q \]

- If we do not have the integration over \( t \), the main contribution will comes from \( mn \ll Q^{k+\epsilon} \) instead of \( m, n \leq Q^{k/2+\epsilon} \).
- The diagonal term \( m = n \) is easy to understand.
- For the off-diagonal term, write \( m - n = hq \), where \( h \neq 0 \). So \( h \ll \frac{|m-n|}{Q} \).
- Conrey, Iwaniec and Soundararajan use the complementary divisor trick, which is replacing the congruence condition modulo \( q \) with a congruence condition modulo \( h \) (reduce the size of the conductor).
The original size of conductor is $q \asymp Q$, and $h \asymp \frac{|m-n|}{Q}$.

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For the eighth moment, before switching the character, we truncate the sum over $m, n$ to $Q^2/(\log Q)^{\alpha}$ by using large sieve inequality. Now

$$h \asymp \frac{Q}{(\log Q)^{\alpha}}.$$
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\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|}
\hline
 & the sixth moment & the eighth moment \\
\hline
\hline
critical range & $m \asymp n \asymp Q^{3/2+\epsilon}$ & $m \asymp n \asymp Q^{2+\epsilon}$ \\
\hline
size of $h$ & $\ll Q^{1/2+\epsilon}$ & $\ll Q^{1+\epsilon}$ \\
\hline
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For the eighth moment, before switching the character, we truncate the sum over $m, n$ to $Q^{2}/(\log Q)^{\alpha}$ by using large sieve inequality. Now

$$h \asymp \frac{Q}{(\log Q)^{\alpha}}.$$ 

Note: if we do not have integration over $t$, the size of $h$ can be of size $Q^{2}$ for the sixth moment and of size $Q^{3}$ for the eighth moment. (too big!)
For the eighth moment, we express the congruence condition modulo $h$ using orthogonality relation of characters mod $h$. Roughly, we study

$$Q \sum_{h} \frac{1}{\phi(h)} \sum_{\chi \mod h} \sum_{m \leq \frac{Q^2}{(\log Q)^\alpha}} \sum_{n \leq \frac{Q^2}{(\log Q)^\alpha}} \frac{d_4(m)d_4(n)\chi(m)\overline{\chi(n)}}{\sqrt{mn}} \times \psi\left(\frac{|m-n|}{hQ}\right) U(m, n, h).$$

- The principal character mod $h$ contributes to the main term, and the rest gives the error term.
- Consider the smooth factor $\psi\left(\frac{|m-n|}{hQ}\right) U(m, n, h)$. When $h$ is small, $|m-n| \ll hQ$ is also small. Hence, for fixed $n$, the sums over $m$ is restricted to an interval much shorter than $Q^2$. Technically, GRH is then used to find cancellation in the sums over $m, n$ restricted to a narrow region.
The sixth moments of $\Gamma_1(q)$ $L$-functions

Roughly speaking, after the approximate functional equation, we consider $M(q)$.

$$\frac{2}{\phi(q)} \sum_{\chi \pmod{q} \chi(-1)=-1}^{h} \sum_{f \in \mathcal{H}_k(q,\chi)} \sum_{m,n \ll q^{3/2+\epsilon}} \frac{\lambda_f(n)d_3(n)}{n^{1/2}} \frac{\lambda_f(m)d_3(m)}{m^{1/2}} V(m,n).$$

Remark

- **The most important region is when** $m, n \asymp q^{3/2+\epsilon}$. **In a simple model, we will do analysis there.**
- **The integration over $t$ is absorbed in** $V(m,n)$. **If we do not have the integration over $t$, then the main contribution comes from when** $mn \leq q^{3+\epsilon}$. 

We then apply Petersson’s formula for $\sum_{f \in \mathcal{H}_k(q,\chi)}^h$. We can write

$$M(q) = D(q) + G(q),$$

where

- $D(q)$ is the contribution from diagonal terms $m = n$. (This contributes to the main term.)
- $G(q)$ is the contribution from off-diagonal terms, also composing of Kloosterman sums and Bessel functions.
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The Bessel function factor of \( G(q) \) is \( J_{k-1} \left( \frac{4\pi}{cq} \sqrt{mn} \right) \). The most important region of \( J_{k-1}(x) \) to consider is the transition region when \( x \approx 1 \). In other words, when \( c \approx \frac{\sqrt{mn}}{q} \approx q^{1/2+\epsilon} \). (Recall that here we consider only the case when \( m, n \approx q^{3/2+\epsilon} \).)
After applying the orthogonality relation for characters, essentially, we need to understand the sum of the form

\[ \sum_{n,m \approx q^{3/2+\epsilon}} \frac{d_3(n)}{n^{1/2}} \frac{d_3(m)}{m^{1/2}} V(m, n) \sum_{c \ll q^{1/2+\epsilon}} \frac{1}{cq} J_{k-1} \left( \frac{4\pi}{cq} \sqrt{mn} \right) \]

\[ \times \sum_{a \mod cq}^* e \left( \frac{am + \bar{a}n}{cq} \right). \]
After applying the orthogonality relation for characters, essentially, we need to understand the sum of the form

\[
\sum_{n,m \approx q^{3/2+\epsilon}} \frac{d_3(n)}{n^{1/2}} \frac{d_3(m)}{m^{1/2}} V(m, n) \sum_{c \ll q^{3/2+\epsilon}} \frac{1}{cq} J_{k-1} \left( \frac{4\pi}{cq} \sqrt{mn} \right) \times \sum^* \quad e \left( \frac{am + \bar{a}n}{cq} \right) \quad \text{mod} \quad cq \mod q
\]

Our next step is to apply Voronoi summation to the sum over \( m \) and \( n \). Now the conductor is of size \( cq \ll q^{3/2+\epsilon} \). If we applied Voronoi summation, then

<table>
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<tr>
<th>Original sum</th>
<th>the dual sum</th>
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<tr>
<td>( q^{3/2+\epsilon} )</td>
<td>( \frac{(cq)^3}{q^{3/2+\epsilon}} \ll q^{3+\epsilon} )</td>
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The dual sum is longer than the original sum, and more difficult to handle! We try to reduce the conductor in the exponential sum.
Treatment of the exponential sum

By Chinese Remainder Theorem and reciprocity, we may factor our exponential sum and reduce the conductor to the size $c \asymp q^{1/2+\epsilon}$. Then the length of dual sum is

$$\frac{c^3}{q^{3/2+\epsilon}} \asymp q^\epsilon.$$
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**Note:** Recall that this phenomenon of reducing the size of conductor also appeared in the case of the moments of Dirichlet $L$-functions.

Applying Voronoi summation for the sum over $m$ (fixed $n$), we roughly have

$$\text{main terms} + \sum_{m' \ll q^\epsilon} a(m', n),$$

We can bound the error term trivially since the sum is short.
The important region for the eighth moment is when \( m, n \approx q^{2+\epsilon} \).

This yields that the conductor \( c \) is around the size \( q^{1+\epsilon} \). After Voronoi summation, the length of dual sum is around the size

\[
\frac{c^4}{q^{2+\epsilon}} \lesssim q^{2+\epsilon}.
\]

The length is not short like the sixth moment case, so the error term is a lot harder to deal with.
Upper bound for the eighth moment

Let $T, H \leq q$. After applying the asymptotic large sieve from Iwaniecie and Li’s work, roughly we consider the sum of the form

$$\sum_{t \sim T} \sum_{h \sim H} \left| \sum_{n \sim q^2} \frac{d_4(n)}{\sqrt{n}} S(hq, n; t) J_{k-1} \left( \frac{4\pi}{t} \sqrt{\frac{hn}{q}} \right) \right|^2.$$
Let $T, H \leq q$. After applying the asymptotic large sieve from Iwaniec and Li’s work, roughly we consider the sum of the form

$$\sum_{t \sim T} \sum_{h \sim H} \left| \sum_{n \sim q^2} d_4(n) \frac{1}{\sqrt{n}} S(h\bar{q}, n; t) J_{k-1} \left( \frac{4\pi}{t} \sqrt{\frac{hn}{q}} \right) \right|^2.$$

- We can apply Voronoi summation to $d_4(n)$, but it will give hyper-Kloosterman sum which is quite complicated to deal with.

- Instead, we write $d_4(n) = \sum_{n_1 n_2 = n} d_2(n_1) d_2(n_2)$. Then apply Voronoi summation to only one of the variable $n_i$. The important region is when $n_1 \sim n_2 \sim q$.

$$\sum_{t \sim T} \sum_{h \sim H} \left| \sum_{n_1, n_2 \sim q} \frac{d_2(n_1) d_2(n_2)}{\sqrt{n_1 n_2}} S(h\bar{q}, n_1 n_2; t) J_{k-1} \left( \frac{4\pi}{t} \sqrt{\frac{hn_1 n_2}{q}} \right) \right|^2.$$
• Without loss of generality, we do Voronoi summation in $n_2$.
• Transition region for the Bessel function is $\frac{1}{T} \sqrt{Hq} \approx 1$.

**Case 1:** $\frac{1}{T} \sqrt{Hq} \ll q^{\epsilon}$. The arguments inside the Bessel function are small, and the dual sum after Voronoi has length $\frac{T^2}{q} \leq q$. This turns out to be an easy case.
• Without loss of generality, we do Voronoi summation in $n_2$.
• Transition region for the Bessel function is $\frac{1}{T}\sqrt{Hq} \asymp 1$.

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Case 2: $\frac{1}{T}\sqrt{Hq} \gg q^\epsilon$. The arguments inside the Bessel function are large. This affects the length of dual sum after Voronoi. In particular, the dual sum now is of length $H \gg q^\epsilon \frac{T^2}{q}$.

After opening up the square, Poisson summation in $h$ and some analysis, we need to understand

$$
\sum \sum \sum \sum \sum d_2(n_1)d_2(n_1')d_2(\ell)d_2(\ell')\Psi \left( \frac{n_1\ell' - n_1'\ell}{qHR} \right)
\times V(n_1, n_1', \ell, \ell', h, t)
$$

where $R \asymp \frac{T}{\sqrt{Hq}} \ll \frac{1}{q^\epsilon}$. Again we have to understand these sums in a narrow region.
THANK YOU VERY MUCH!