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Extension of a positivity trick and estimates involving L-functions at the edge of the critical strip

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A positivity trick and $L$-functions at the edge of the critical strip

Xiannan Li

Kansas State University

May 2019
A classic calculation
Recall that
\[ \zeta(s) = \sum_{n \geq 1} \frac{1}{n^s} = \prod_p \left( 1 - \frac{1}{p^s} \right)^{-1}, \]
valid for \( \Re(s) > 1. \)

\[ -\frac{\zeta'}{\zeta}(s) = \sum_n \frac{\Lambda(n)}{n^s}, \]
where \( \Lambda \) is the usual von Mangoldt function supported on prime powers.
We will refer to a sum like \( \sum_n \frac{\Lambda(n)}{n^s} \) as a sum over primes.
Recall that
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\textbf{CBMS}\textbf{\hspace{1cm}L-functions at the edge of the critical strip}
Recall that

$$\xi(s) = \frac{1}{2} s(s - 1) \pi^{-\frac{1}{2} s} \Gamma \left( \frac{s}{2} \right) \zeta(s)$$

is an entire function of order 1. Entire functions are like polynomials in some ways. The way that is useful for us is to write

$$\xi(s) = e^{A+Bs} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{\frac{s}{\rho}},$$

where $\rho$ always denotes zeros of $\xi(s)$. 
Recall that

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\xi(s) = e^{A+B s} \prod_{\rho} \left( 1 - \frac{s}{\rho} \right) e^{\frac{s}{\rho}},
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where $\rho$ always denotes zeros of $\xi(s)$. 
Logarithmically differentiating gives
\[- \frac{\zeta'}{\zeta}(s) = - \sum_{\rho} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right) + \frac{1}{s - 1} + O(\log(|t| + 2)), \]

writing $s = \sigma + it$.

Suppose that we are interested in an upper bound for $- \text{Re} \frac{\zeta'}{\zeta}(s)$ away from 1 and write
\[- \text{Re} \frac{\zeta'}{\zeta}(s) = - F(s) + O(\log(|t| + 2)), \]

where
\[ F(s) = \sum_{\rho} \text{Re} \left( \frac{1}{s - \rho} + \frac{1}{\rho} \right). \]
Recall
\[-\text{Re} \frac{\zeta'}{\zeta}(s) = -F(s) + O(\log(|t| + 2)),\]

Writing \(\rho = \beta + i\gamma\),
\[F(s) = \sum_{\rho} \frac{\sigma - \beta}{|s - \rho|^2} + \frac{\beta}{|\rho|^2} \geq 0,\]

for \(\sigma \geq 1 \geq \beta \geq 0\). Thus
\[-\text{Re} \frac{\zeta'}{\zeta}(s) \leq A \log(|t| + 2),\]

for \(s\) away from 1 and some constant \(A\).
Recall

\[-\text{Re} \frac{\zeta'}{\zeta}(s) = -F(s) + O(\log(|t| + 2)),\]

Writing \(\rho = \beta + i\gamma,\)

\[F(s) = \sum_{\rho} \frac{\sigma - \beta}{|s - \rho|^2} + \frac{\beta}{|\rho|^2} \geq 0,\]

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\[-\text{Re} \frac{\zeta'}{\zeta}(s) \leq A \log(|t| + 2),\]

for \(s\) away from 1 and some constant \(A.\)
When studying $\zeta(s)$ near the Res $= 1$ line, one intuition is that if

$$\zeta(1 + it) = 0,$$

$$-\frac{\zeta'}{\zeta}(1+it) = -\infty \text{ from the right},$$

then $p^{it} \approx -1$ for many small $p$, and so $p^{2it} \approx 1$ for many small $p$. Then we expect there to be a pole at $\zeta(1 + 2it)$ or

$$-\frac{\zeta'}{\zeta}(1 + 2it) = \infty$$

from the right.

We see that in order to see that there is no zero of $\zeta$ near $1 + it$, one needs an upper bound on $-\operatorname{Re}\frac{\zeta'}{\zeta}(s)$, and this is done on the last slide.
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We see that in order to see that there is no zero of $\zeta$ near $1 + it$, one needs an upper bound on $-\text{Re}\frac{\zeta'}{\zeta}(s)$, and this is done on the last slide.
Upper bounds on general $L$-functions
More general $L$-functions: Rankin-Selberg $L$-functions

Cuspidal automorphic representations for $GL(n)$ are central objects in Langland’s program. Part of their interest can be conveyed by the belief that ”all $L$-functions arise from some automorphic representation.”

1. It’s interesting that $L$-functions attached to elliptic curves are automorphic (indeed, modular - Breuil, Conrad, Diamond, Taylor).

2. It’s conjectured that Artin $L$-functions are automorphic.

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3. It’s conjectured that Rankin-Selberg \( L \)-functions are automorphic.
Let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations for $GL(n)$ and $GL(m)$ respectively. Let

$$R(\pi_1, \pi_2) = \begin{cases} 
\text{Res}_{s=1} L(s, \pi_1 \times \pi_2) & \text{if } L(s, \pi_1 \times \pi_2) \text{ has a pole}, \\
L(1, \pi_1 \times \pi_2) & \text{otherwise}.
\end{cases}$$

Let $C$ denote the conductor of $L(s, \pi_1 \times \pi_2)$. Roughly speaking, one can think of $C$ as a measure of the complexity of $L(s, \pi_1 \times \pi_2)$.

- The standard convexity bound of $R \ll \epsilon C^\epsilon$ is known unconditionally for $m, n \leq 2$ by the work of Molteni (follows work of Iwaniec).
- Brumley extended this to $m, n \leq 4$ (required functoriality results).

**Theorem**

$$R(\pi_1, \pi_2) \ll \exp \left( C \frac{\log C}{\log \log C} \right) \ll C^\epsilon,$$
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Let $\mathfrak{C}$ denote the conductor of $L(s, \pi_1 \times \pi_2)$. Roughly speaking, one can think of $\mathfrak{C}$ as a measure of the complexity of $L(s, \pi_1 \times \pi_2)$.

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**Theorem**

$$R(\pi_1, \pi_2) \ll \exp \left( C \frac{\log \mathfrak{C}}{\log \log \mathfrak{C}} \right) \ll \mathfrak{C}^\varepsilon,$$
Upper bounds for Rankin-Selberg $L$-functions

Let $\pi_1$ and $\pi_2$ be cuspidal automorphic representations for $GL(n)$ and $GL(m)$ respectively. Let

$$R(\pi_1, \pi_2) = \begin{cases} \text{Res}_{s=1}L(s, \pi_1 \times \pi_2) & \text{if } L(s, \pi_1 \times \pi_2) \text{ has a pole}, \\ L(1, \pi_1 \times \pi_2) & \text{otherwise}. \end{cases}$$

Let $\mathcal{C}$ denote the conductor of $L(s, \pi_1 \times \pi_2)$. Roughly speaking, one can think of $\mathcal{C}$ as a measure of the complexity of $L(s, \pi_1 \times \pi_2)$.

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**Theorem**

$$R(\pi_1, \pi_2) \ll \exp \left( C \frac{\log \mathcal{C}}{\log \log \mathcal{C}} \right) \ll \mathcal{C}^\epsilon,$$
The method of proof

On \( L \)-functions

The following method applies to all Dirichlet series with Euler product which is absolutely convergent for \( \Re s > 1 \), and has analytic continuation and functional equation. We write

\[
L(s) = L(s, A) = \sum_n \frac{a(n)}{n^s},
\]

and

\[
- \frac{L'(s)}{L(s)} = \sum_n \frac{\Lambda_A(n)}{n^s}
\]

for \( \Re s > 1 \).
A first attempt

We can try to get upper bounds on $L(1, \mathcal{A})$ by summing up the Dirichlet series when $s = 1 + \epsilon$ say.

Main obstacle

The coefficients of the $L$-function might be too large. The Ramanujan-Peterson conjecture states that $a(n) \ll n^\epsilon$ but the bounds available are almost as bad as

$$a(n) \ll n.$$
Zeros alone

Similar to the $\zeta$ function,

$$-\text{Re} \frac{L'}{L}(s) = \frac{\log \mathcal{C}}{2} - F(s) + \text{uninteresting terms}$$

where

$$F(s) = \text{Re} \left( \sum_\rho \frac{1}{s - \rho} + \frac{1}{\rho} \right) \geq 0,$$

for $\sigma \geq 1$. Integrating from 1 to $1 + \epsilon$ gives

$$L(1) \ll |L(1 + \epsilon)| \epsilon^{\epsilon/2}.$$

Do we understand $L(1 + \epsilon)$?
Similar to the $\zeta$ function,

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Do we understand $L(1 + \epsilon)$?
Primes and zeros

Lemma of Selberg:

$$-\frac{L'}{L}(s) = \text{Sum over primes} + \text{Sum over zeros}.$$ 

Actual formula (variation due to Sound):

$$-\frac{L'}{L}(s) = \sum_{n \leq x} \Lambda_A(n) \frac{\log(x/n)}{n^s \log x} + \frac{1}{\log x} \left( \frac{L'}{L}(s) \right)' + \frac{1}{\log x} \sum_{\rho} \frac{x^{\rho-s}}{\rho - s} + \text{uninteresting terms}$$

Note $x$ controls the relative weight of the sum over primes and sum over zeros.
Lemma of Selberg:

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Actual formula (variation due to Sound):

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Note $x$ controls the relative weight of the sum over primes and sum over zeros.
Immediate problem
How do we deal with the sum over zeros? These terms blow up if we have zeros near 1.

Intuition from the classical trick
If there are zeros very close to 1, then \( L(s) \) should be small there. Can we arrange for the contribution of the zeros to be negative?

Two components:
1. The contribution of \( \sum_{\rho} \frac{x^{\rho-s}}{(\rho-s)^2} \) is relatively small due to the \( x^{\rho-s} \) weight.
2. After integrating, \( \left( \frac{L'}{L}(s) \right)' \) gives a \( \frac{\log x}{2} - F(s) \) just as before.
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Following this, integrate the above from $\sigma_0$ to $\infty$ and get

$$ \log |L(\sigma_0)| \leq \text{Re} \sum_{n \leq x} \frac{\Lambda_A(n) \log(x/n)}{n^{\sigma_0} \log n \log x} + \frac{1}{\log x} \left( \frac{\log \mathcal{C}}{2} \right) + O(1), $$

Then choosing $x = \log \mathcal{C}$ suffices.

**Summary**

When we don’t have good information about the coefficients, we put most of the burden onto the sum over zeros. These are understood via an extension of the positivity trick.
This technique will absorb and use advances in functoriality. For instance, we have for $\pi$ cuspidal automorphic representation of $GL(n)$ that

$$L(1, \pi) \ll \exp(C\sqrt{\log \mathcal{C}}).$$

This results purely from the Rankin-Selberg method. When $n = 2$, can show that

$$L(1, \pi) \ll \epsilon \exp(C(\log \mathcal{C})^{1/8} + \epsilon).$$

This comes from the functoriality of the symmetric fourth due to Kim.
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An application about primes
Recall that for \((n, q) = 1\), \(n\) is a quadratic residue modulo \(q\) if there exists \(a\) such that

\[ n \equiv a^2 \pmod{q}. \]

We call \(n\) a quadratic non-residue otherwise.

Let \(N\) denote the least quadratic non-residue modulo \(p\), with \(p\) prime. A difficult classical problem is to find good upper bounds for \(N\).

Vinogradov showed that \(N \ll p^{\frac{1}{2\sqrt{e}} + o(1)}\), and later work of Burgess improved this bound to \(N \ll p^{\frac{1}{4\sqrt{e}} + o(1)}\).
A classical problem about quadratic non-residues

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Vinogradov showed that \(\mathcal{N} \ll p^{\frac{1}{2\sqrt{e}} + o(1)}\), and later work of Burgess improved this bound for \(\mathcal{N} \ll p^{\frac{1}{4\sqrt{e}} + o(1)}\).
The best result known arises from considerations of cancellation in character sums. To be more specific, let $\chi$ be the quadratic character with modulus $p$. Then we say that $\chi$ exhibits cancellation at $x = x(p)$ if

$$\sum_{n \leq x} \chi(n) = o(x).$$

Recall that $\chi(n) = 1$ if $n$ is a quadratic residue and $\chi(n) = -1$ if $n$ is a quadratic non-residue.

Polya-Vinogradov tells us that $\sum_{n \leq x} \chi(n) \ll \sqrt{p} \log p$, and this tells us that cancellation occurs for $x = p^{1/2 + o(1)}$.

Burgess showed that cancellation occurs at $x = p^{1/4 + o(1)}$. 
Burgess’s result implies that at least $1/2 - o(1)$ of the numbers less than $x = p^{1/4 + o(1)}$ are quadratic non-residues so certainly $\mathcal{N} \ll x$.

Burgess’s bound combined with Vinogradov’s method implies that

$$\mathcal{N} \ll p^{\frac{1}{4\sqrt{e}} + o(1)}.$$  \hspace{1cm} (1)

How Vinogradov’s method works

- Recall that $n$ is called $y$ smooth if all prime factors of $n$ are less than or equal to $y$.
- Observe that if all $n \leq y$ are quadratic residues, then all $y$ smooth numbers are also quadratic residues.
- The number of $y$ smooth numbers less than $x$ is $\leq x/2$ precisely when $y \leq x^{\frac{1}{\sqrt{e}}}$.
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Burgess’s result implies that at least $1/2 - o(1)$ of the numbers less than $x = p^{1/4 + o(1)}$ are quadratic non-residues so certainly $N \ll x$.

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- Recall that $n$ is called $y$ smooth if all prime factors of $n$ are less than or equal to $y$.
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- The number of $y$ smooth numbers less than $x$ is $\leq x/2$ precisely when $y \leq x^{\frac{1}{\sqrt{e}}}$.
Let $K$ be a number field of degree $l$ with discriminant $d_K$. Let $p$ be a prime. We say that $p$ splits completely in $K$ if $p$ factors as

$$p\mathfrak{O}_K = p_1 p_2 \ldots p_l,$$

for $p_i$ prime ideals in $\mathfrak{O}_K$ (here, $\mathfrak{O}_K$ is the ring of integers of $K$).

Let $N$ the least prime which does not split completely.

In the case that $K$ is a quadratic field, $N$ is the least quadratic non-residue.

A very familiar example (with classical arithmetic significance): primes which are $1 \text{ mod } 4$ factor in $\mathbb{Z}[i]$ as $(a + bi)(a - bi)$, whereas primes which are $3 \text{ mod } 4$ are inert.
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A very familiar example (with classical arithmetic significance): primes which are 1 mod 4 factor in $\mathbb{Z}[i]$ as $(a + bi)(a - bi)$, whereas primes which are 3 mod 4 are inert.
K. Murty proved a result roughly of the form

\[ \mathcal{N} \ll d_K^{2(l-1)}, \]  

(2)

This type of result was explicitly proved later using essentially elementary methods by Vaaler and Voloch. Their result is that

\[ \mathcal{N} \leq 26l^2 d_K^{2(l-1)}. \]  

(3)

provided that

\[ d_K \geq \frac{1}{8} e^{2(l-1) \max(105, 25 \log^2 l)}. \]
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Can this result be improved by some generalization of Vinogradov’s method? Interestingly enough, this is not the case. In fact, the best result from Vinogradov’s approach is also a bound of the form $\mathcal{N} \ll d_K^{1\over 2(l-1)}$, suggesting that this is a natural barrier.

We will derive the following bound for large $l$ using analytic methods:

$$\mathcal{N} \ll d_K^{4(l-1)}.$$
Can this result be improved by some generalization of Vinogradov’s method? Interestingly enough, this is not the case. In fact, the best result from Vinogradov’s approach is also a bound of the form $N \ll d_K^{\frac{1}{2(l-1)}}$, suggesting that this is a natural barrier. We will derive the following bound for large $l$ using analytic methods:

$$N \ll d_K^{\frac{1}{4(l-1)}}.$$
The Dedekind ζ function for \( K \) is defined to be

\[
\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N(\mathfrak{a})^s} = \prod_{\mathfrak{p}} \left( 1 - \frac{1}{N(\mathfrak{p})^s} \right)^{-1}
\]

for \( \Re s > 1 \), where the sum is over ideals of \( \mathfrak{o}_K \), and the product is over prime ideals \( \mathfrak{p} \). Here, \( N(\mathfrak{p}) \) denotes the norm of \( \mathfrak{p} \) and is a power of an integer prime. The completed Dedekind zeta function defined by

\[
\Lambda(s) = s(s-1) \left( \frac{d_K}{2^{2r_2} \pi^{r_1}} \right)^{s/2} \Gamma \left( \frac{s}{2} \right)^{r_1+r_2} \Gamma \left( \frac{s+1}{2} \right)^{r_2} \zeta_K(s),
\]

satisfies \( \Lambda(s) = \Lambda(1-s) \) and is entire of order 1.
The Dedekind $\zeta$ function

The Dedekind zeta function for $K$ is defined to be

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for $\Re s > 1$, where the sum is over ideals of $\mathfrak{o}_K$, and the product is over prime ideals $\mathfrak{p}$. Here, $N(\mathfrak{p})$ denotes the norm of $\mathfrak{p}$ and is a power of an integer prime. The completed Dedekind zeta function defined by

$$\Lambda(s) = s(s-1) \left( \frac{d_K}{2^{2r_2} \pi^{r_1}} \right)^{s/2} \Gamma \left( \frac{s}{2} \right)^{r_1+r_2} \Gamma \left( \frac{s+1}{2} \right)^{r_2} \zeta_K(s),$$

satisfies $\Lambda(s) = \Lambda(1-s)$ and is entire of order 1.
The analytic starting point

If all integer primes split over $K$, then

$$\zeta_K(s) = \prod_{p} \prod_{p|p} \left(1 - \frac{1}{Np^s}\right)^{-1} = \prod_{p} \prod_{p|p} \left(1 - \frac{1}{p^s}\right)^{-1}$$

is $\zeta(s)^l$, where as usual $\zeta(s)$ denotes the Riemann zeta function.

1. $\zeta(s)^l$ has a pole of order $l$ at $s = 1$ and $\zeta_K(s)$ has only a simple pole at $s = 1$, we see that not all primes split.

2. This also leads to quantifications of the statement that the least prime which does not split cannot be too large, as in K. Murty’s work.

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If all integer primes split over $K$, then

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The approach

Again, start by logarithmically differentiating the completed Dedekind zeta function to get

\[- \operatorname{Re} \frac{\zeta'_K(s)}{\zeta_K(s)} = \operatorname{Re} \left( \frac{1}{2} \log d_K - F(s) + \frac{1}{s-1} \right) + \text{insignificant terms},\]

where

\[F(s) = \operatorname{Re} \sum_{\rho} \frac{1}{s-\rho} = \sum_{\rho} \frac{\sigma - \beta}{(\sigma - \beta)^2 + (t - \gamma)^2}.\]

As usual, write \(s = \sigma + it\) and \(\rho = \beta + i\gamma\).

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Then we have that

\[- \frac{\zeta_K'(\sigma)}{\zeta_K(\sigma)} = \sum_{n \geq 1} \frac{\Lambda_K(n)}{n^\sigma} \leq \frac{1}{2} \log d_K + \frac{1}{\sigma - 1}.\] (4)

Note \(\Lambda_K(n) = 0\) if \(n\) is not a power of a prime, and \(0 \leq \Lambda_K(p^r) \leq l \log p\) and that if \(p\) splits that \(\Lambda_K(p) = l \log p\).

The starting point

If all the primes up to \(x\) split completely, then \(\Lambda_K(p)\) is too large up to \(x\), and thus makes \(- \frac{\zeta_K'(\sigma)}{\zeta_K(\sigma)}\) larger than it should be.
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If all the primes up to $x$ split completely, then $\Lambda_K(p)$ is too large up to $x$, and thus makes $-\frac{\zeta_K'(\sigma)}{\zeta_K(\sigma)}$ larger than it should be.
Set $\sigma = 1 + \frac{2l}{\log d_K}$ so that morally,

$$\sum_{n \leq x} \frac{\Lambda_K(n)}{n} \leq \sum_{n \geq 1} \frac{\Lambda_K(n)}{n^\sigma} = -\frac{\zeta'_K(\sigma)}{\zeta_K(\sigma)} \leq \frac{1}{2} \log d_K.$$ 

If we assume that all primes up to $x$ split, then

$$l \log x \leq \frac{\log d_K}{2},$$

whence $x \leq d_K^{\frac{1}{2l}}$. Morally, we see that if we can prove a bound of the form

$$-\frac{\zeta'_K(\sigma)}{\zeta_K(\sigma)} \leq c \log d_K,$$

then we get a bound

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How do we improve the bound on \(- \frac{\zeta'_K(\sigma)}{\zeta_K(\sigma)}\)?

Remember that we had neglected the contribution of

\[ F(\sigma) = \sum_{\rho} \frac{\sigma - \beta}{(\sigma - \beta)^2 + \gamma^2}. \]

How do we account for the contribution \( F(\sigma) \) in

\[- \frac{\zeta'_K(\sigma)}{\zeta_K(\sigma)} = \sum_{n \geq 1} \frac{\Lambda_K(n)}{n^{\sigma}} \leq \frac{1}{2} \log d_K + \frac{1}{\sigma - 1} - F(\sigma) ?\]

What kind of lower bound can we prove for \( F(\sigma) ? \)

The individual terms in \( F(\sigma) \) are smallest when \( \beta \) is close to 1. However, in that case, \( 1 - \beta \) is also the real part of a zeta of \( \zeta_K(s) \) and that term will be large. One expects by this reasoning alone to be able to show that \( F(\sigma) \geq c \log d_K \) for some \( c \). In fact, this was done by Stechkin with \( c = \frac{1}{2\sqrt{5}} \).
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CBMS L-functions at the edge of the critical strip
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By adapting a method of Heath-Brown, we can in fact do a bit better and show something roughly of the form

\[ \frac{\Lambda_K(n)}{n^\sigma} \leq \frac{1}{4} \log d_K + \frac{1}{\sigma - 1}. \]

This leads us to a bound roughly of the form

\[ \mathcal{N} \ll \epsilon \, d_K^{\frac{1+\epsilon}{4(l-1)}}. \]

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Heath-Brown’s formula

Let $C_R$ denote the circle of radius $R$ with center $\sigma_0$ with no zeros of $f(s) = (s - 1)\zeta_K(s)$ on $C_R$. The technique depends on an identity of the form

$$-\Re \frac{f'}{f}(\sigma_0) = -\sum' \left( \frac{1}{\sigma_0 - \rho} - \frac{\sigma_0 - \rho}{R^2} \right)$$

$$- \frac{1}{\pi R} \int_0^{2\pi} \cos \theta \log |f(\sigma_0 + Re^{i\theta})| d\theta$$

where $\sum'$ denotes a sum over all zeros of $f$ within $C_R$.

Origins

This comes from evaluating the integral

$$I = \frac{1}{2\pi i} \oint_{|z|=R} \left( \frac{1}{z} - \frac{z}{R^2} \right) \frac{f'}{f}(z + \sigma_0)dz$$
The bound

\[-\Re\frac{f'}{f}(\sigma_0) = -\sum' \left( \frac{1}{\sigma_0 - \rho} - \frac{\sigma_0 - \rho}{R^2} \right) \]

\[\text{neglect by positivity trick}\]

\[-\frac{1}{\pi R} \int_0^{2\pi} \cos \theta \log |f(\sigma_0 + Re^{i\theta})| d\theta \]

\[\text{bound using convexity bound}\]
Cubic and Biquadratic case

In the case when $K$ is a cubic or biquadratic field, more can be proven using how multiplicative functions interact. In this case, the best results are reached using methods involving multiplicative functions.

Upper bound on $\kappa$

Let $\kappa = \operatorname{Res}_{s=1} \zeta_K(s)$. The discussion above is related to deriving good upper bounds on $\kappa$ of the form

$$\kappa \ll \left( \frac{\left(\frac{1}{4} + o(1)\right) e^{\gamma + o(1)} \log d_K}{l} \right)^{l-1}.$$
The End