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An effective Chebotarev density theorem for families of fields, with an application to class groups

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A Chebotarev density theorem for families of fields, with an application to class groups

Caroline Turnage-Butterbaugh
Carleton College

(Joint work with Lillian Pierce and Melanie Matchett Wood)

NSF-CBMS Conference
L-functions and Multiplicative Number Theory
University of Mississippi
May 20, 2019
(a, b, c) := ax^2 + bxy + cy^2, \quad a, b, c \text{ integers.}
Binary Quadratic Forms

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- classified the binary quadratic forms with a given discriminant \(D := b^2 - 4ac;\)
**Binary Quadratic Forms**

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- formed the *class group*, the group of equivalence classes of binary quadratic forms of a given \(D\) with group action Gauss composition;
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Gauss

• classified the binary quadratic forms with a given discriminant \( D := b^2 - 4ac \);

• formed the class group, the group of equivalence classes of binary quadratic forms of a given \( D \) with group action Gauss composition;

• showed that, for any given discriminant \( D \), there exist only finitely many equivalence classes of binary quadratic forms.
Let $K = \mathbb{Q}(\sqrt{D})$ be a quadratic number field. To each form

$$(a, b, c) := ax^2 + bxy + cy^2$$

with discriminant $D = b^2 - 4ac$, we may associate an ideal $I$ of $\mathcal{O}_K$, where

$$I = \left\langle a, \frac{-b + \sqrt{D}}{2} \right\rangle.$$
Binary quadratic forms $\leftrightarrow$ Nonzero ideals of $\mathcal{O}_{\mathbb{Q}[\sqrt{D}]}$

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equivalent forms $\leftrightarrow$ equivalent ideals

composition of equivalence classes of forms $\leftrightarrow$ multiplication of equivalence classes of ideals
<table>
<thead>
<tr>
<th>Binary quadratic forms</th>
<th>←→</th>
<th>Nonzero ideals of $O_{\mathbb{Q}[\sqrt{D}]}$</th>
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| equivalent binary quadratic forms          | ←→ | equivalent ideals                          |
| composition of equivalence classes of forms| ←→ | multiplication of equivalence classes of ideals |

Cl$_K := \text{the ideal class group of } K = \mathbb{Q}(\sqrt{D})$

$h(K) = |\text{Cl}_K| := \text{the class number of } K = \mathbb{Q}(\sqrt{D})$

**Note:** $h(K)$ is finite via the correspondence.
**Class group of $K$, $[K : \mathbb{Q}] \geq 2$**

The ideal class group of $K$ is defined by

$$\text{Cl}_K := J_K/P_K$$

- $J_K :=$ the group of fractional ideals of $K$
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$h(K) = 1 \iff \text{Cl}_K = \{\text{id}\} \iff \mathcal{O}_K$ is a PID \iff $\mathcal{O}_K$ is a UFD

**Question:** How big is $|\text{Cl}_K|$ in general?
Landau observed that if \([K : \mathbb{Q}] = n\), then

\[
|\text{Cl}_K| \ll_n D_K^{1/2+\varepsilon}
\]

We may conclude that \(\text{Cl}_K\) is a finite abelian group.
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For any integer \(\ell > 1\), the \(\ell\)-torsion subgroup of \(\text{Cl}_K\) is given by

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**Natural Question:** What is the size of \(\text{Cl}_K[\ell]\) as \(K\) varies within a family of fields of fixed degree?
**HOW BIG IS** \(|\text{Cl}_K[\ell]|\)?

**Trivial Bound** – For \([K : \mathbb{Q}] = n\), any integer \(\ell \geq 1\), and \(\varepsilon > 0\)

\[|\text{Cl}_K[\ell]| \leq |\text{Cl}_K| \ll_{n, \varepsilon} D_K^{1/2 + \varepsilon}\]
**How Big is $|\text{Cl}_K[\ell]|$?**

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**Conjecture** – For $[K : \mathbb{Q}] = n$, any integer $\ell \geq 1$, and $\varepsilon > 0$

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Recorded by

- Brumer-Silverman, ’96
- Zhang, ’05
- Ellenberg-Venkatesh, ’07
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Implied by

- Cohen-Lenstra-Martinet heuristics on the distribution of class groups and $\ell$-torsion subgroups within families
What do we know is true?

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**Theorem (Gauss)**

For all quadratic fields $K$, we have $|\text{Cl}_K[2]| \ll_\varepsilon D_K^\varepsilon.$
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*For all quadratic fields $K$, we have $|\text{Cl}_K[2]| \ll \varepsilon D_K^\varepsilon.$*

- This is the only case (for $\ell$ prime) in which the conjecture has been proved.
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- **Question**: Are there cases for which nontrivial bounds known?
**Nontrivial bounds on $|\text{Cl}_K[\ell]|$**

**Theorem (Ellenberg & Venkatesh, 2007)**

Let $K/\mathbb{Q}$ be a number field of degree 2 or 3. We have

$$|\text{Cl}_K[3]| \ll n, \varepsilon D_K^{\frac{1}{3}+\varepsilon}.$$

**Theorem (Bhargava, Shankar, Taniguchi, Thorne, Tsimerman & Zhao, 2017)**

Let $K/\mathbb{Q}$ be a number field of degree $n > 2$. For some $\delta_n > 0$ we have

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Let $K/\mathbb{Q}$ be a non-$D_4$ number field of degree 4. We have

$$|\text{Cl}_K[3]| \ll \varepsilon \ D_K^{\frac{1}{2} - \frac{1}{168} + \varepsilon}.$$
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Theorem ( Ellenberg & Venkatesh, 2007)

Let $K/\mathbb{Q}$ be a number field of degree $n$ and $\ell$ a positive integer. Assuming GRH, we have

$$|\text{Cl}_K[\ell]| \ll_{n, \ell, \varepsilon} D_K^{\frac{1}{2}} - \frac{1}{2\ell(n-1)} + \varepsilon.$$
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• **Question:** What can we say unconditionally for all but a possible exceptional set of fields $K$ within a family?
Theorem (Soundararajan, 2000)

Let \( \ell \) be prime. For all but a possible zero-density exceptional family of imaginary quadratic fields \( K/\mathbb{Q} \), we have

\[
|\text{Cl}_K[\ell]| \ll_{\ell, \varepsilon} D_K^{\frac{1}{2} - \frac{1}{2\ell} + \varepsilon}.
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**Nontrivial bounds on $|\text{Cl}_K[\ell]|$ ... in families**

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**Theorem (Heath-Brown & Pierce, 2014)**

Let $\ell \geq 5$ be prime. For all but a possible zero-density exceptional family of imaginary quadratic fields $K/\mathbb{Q}$, we have

$$|\text{Cl}_K[\ell]| \ll \ell, \varepsilon D_K^{\frac{1}{2}} \frac{3}{2\ell + 2} + \varepsilon.$$
**Theorem (Ellenberg, Pierce, & Wood, 2016)**

Let $\ell \geq 1$, and let $[K : \mathbb{Q}] = 2, 3$ or $5$. For all but a possible zero-density exceptional family of fields $K/\mathbb{Q}$, we have

$$|\text{Cl}_K[\ell]| \ll n, \ell, \varepsilon D_K^{\frac{1}{2} - \frac{1}{2\ell(n-1)}} + \varepsilon.$$  

If $[K : \mathbb{Q}] = 4$, then the same bound applies for $K$ non-$D_4$. 

**Note that the bound is as strong as on GRH.**

Pierce, T., and Wood, (2017 preprint) Under certain conditions (but never under GRH), we extend this result to different families in which $[K : \mathbb{Q}] \geq 2$. 


Nontrivial bounds on $|\text{Cl}_K[\ell]| \ldots$ in families

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Suppose that there are $M$ rational primes $p_1, p_2, \ldots, p_M$ that split completely in $K$, where $p_j \leq D_K^\delta$ and $\delta < \frac{1}{2\ell(n-1)}$. Then for any $\varepsilon > 0$,

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**Question:** How might one go about finding small primes that split completely in $K$?
Starting Point

Theorem (Ellenberg & Venkatesh, 2007)

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Question: How might one go about finding small primes that split completely in $K$?

Answer: via a Chebotarev Density Theorem
**An Effective Chebotarev Density Theorem**

Theorem (Lagarias-Odlyzko*, 1975)

If GRH holds for $\zeta_K(s)$, then

$$\left| \# \{ p \leq x \text{ that split completely in } K \} - \frac{\text{Li}(x)}{|G|} \right| \leq \frac{C_0}{|G|} x^{1/2} \log(D_Kx^{n_K})$$

for every $x \geq 2$ and $C_0$ is effectively computable.

\[ \text{Gal}(K/Q) \cong G \]
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*This is a special case of their theorem.*

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- We may take $x = D_{K}^{\delta - \epsilon_0}$, with $\delta = \frac{1}{2\ell(n - 1)}$. 

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for every $x \geq 2$ and $C_0$ is effectively computable.

*This is a special case of their theorem.

• We may take $x = D_K^{\delta-\epsilon_0}$, with $\delta = \frac{1}{2\ell(n-1)}$.

• Obtain at least $M \gg D_K^{1/(2\ell(n-1))-\epsilon_0}$ sufficiently small primes that split completely in $K$. 
Bounding \( \ell \)-torsion assuming GRH

Ellenberg-Venkatesh (2007)

\[ |\text{Cl}_K[\ell]| \ll \ell, n, \varepsilon \quad D_K^{\frac{1}{2}+\varepsilon} M^{-1} \]
Bounding $\ell$-torsion assuming GRH

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$$|\text{Cl}_K[\ell]| \ll \ell, n, \varepsilon \quad D_K^{\frac{1}{2} + \varepsilon} M^{-1}$$

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Assuming GRH, we have $|Cl_K[\ell]| \ll \ell, n, \varepsilon \quad D_K^{\frac{1}{2} - \frac{1}{2\ell(n-1)} + \varepsilon}$

Goal: Remove GRH and obtain the same $\ell$-torsion bound.
Bounding $\ell$-torsion assuming GRH

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$|\text{Cl}_K[\ell]| \ll_{\ell,n,\varepsilon} D_K^{1/2+\varepsilon} M^{-1}$

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Assuming GRH, we have $|\text{Cl}_K[\ell]| \ll_{\ell,n,\varepsilon} D_K^{1/2 - 1/2\ell(n-1) + \varepsilon}$

Goal: Remove GRH and obtain the same $\ell$-torsion bound.
– We can do this at the cost of proving the result for all but a possible zero-density family of fields.
SAME STARTING POINT AS BEFORE

Theorem (Ellenberg & Venkatesh, 2007)

Suppose that there are $M$ rational primes $p_1, p_2, \ldots, p_M$ that split completely in $K$, where $p_j \leq D^K_\delta$ and $\delta < \frac{1}{2\ell(n-1)}$. Then for any $\varepsilon > 0$,

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We need an effective Chebotarev density theorem for a family of fields $K$

- that does not assume GRH, and
SAME STARTING POINT AS BEFORE

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*that split completely in* \( K \), *where* \( p_j \leq D_K^\delta \) *and* \( \delta < \frac{1}{2\ell(n-1)} \). *Then for any* \( \varepsilon > 0 \),

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We need an effective Chebotarev density theorem for a family of fields \( K \)

- that does not assume GRH, and
- has a low threshold on \( x \).
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We need an effective Chebotarev density theorem for a family of fields $K$

- that does not assume GRH, and
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Let us first recall how to count primes.
Motivating Question

Given a large number \( x \), how many primes are there less than or equal to \( x \)?
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That is, if we let

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Given a large number \( x \), how many primes are there less than or equal to \( x \)?

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\]

how does \( \pi(x) \) behave as \( x \to \infty \)?
Prime Number Theorem (Hadamard, de la Vallée Poussin 1896)

$$\pi(x) \sim \text{Li}(x), \quad x \to \infty$$
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\log p, & \text{if } n = p^k, k \geq 1, \\
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Heuristic:

\[ \psi(x) = \sum_{n \leq x} \Lambda(n) \approx \sum_{p \leq x} \log p \approx \pi(x) \log x \]

\[ \psi(x) \sim x \iff \pi(x) \sim \frac{x}{\log x} \]
**Proving** $\psi(x) \sim x$
Proving $\psi(x) \sim x$

Explicit Formula (truncated version)

We have

$$\psi(x) = x - \sum_{|\gamma| \leq x} \frac{x^\rho}{\rho} + O(\log^2 x)$$

where the sum is over the nontrivial zeros of $\zeta(s)$. 

• Since $|x^\rho| = x^\beta$, if $\beta < 1$, then the contribution from the nontrivial zeros is not too big.

• Key to proof of the Prime Number Theorem: $\zeta(s) \neq 0$ for $\Re(s) = 1$. 

---

$\rho = \beta + i\gamma$ is a nontrivial zero of $\zeta(s)$. 

$\zeta(s) = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \left(1 - \frac{1}{p^s}\right)^{-1}$ for $\Re(s) > 1$. 

---

$\Phi(x)$ is the Euler totient function counting the number of positive integers $\leq x$ that are relatively prime to $x$. 

$\psi(x)$ counts the number of positive integers $\leq x$ that have an odd number of prime factors. 

---

$\zeta(s)$ is the Riemann zeta function.
**Proving $\psi(x) \sim x$**

Explicit Formula (truncated version)

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**Proving** \( \psi(x) \sim x \)

Explicit Formula (truncated version)

We have

\[
\psi(x) = x - \sum_{|\gamma| \leq x} \frac{x^\rho}{\rho} + O \left( \log^2 x \right)
\]

where the sum is over the nontrivial zeros of \( \zeta(s) \).

\( \rho = \beta + i\gamma \) is a nontrivial zero of \( \zeta(s) \):

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Siegel-Walfisz Theorem (1935)

If $n \geq 2$ and $a$ is coprime to $q$ then as $x \to \infty$,

$$\pi(x; a, q) := \sum_{\substack{p \leq x \\ p \equiv a \ (\text{mod} \ q)}} 1 = \frac{1}{\varphi(q)} \text{Li}(x) + \text{"error term"}.$$
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$$\pi(x; a, q) := \sum_{\substack{p \leq x \atop p \equiv a \pmod{q}}} 1 = \frac{1}{\varphi(q)} \text{Li}(x) + "\text{error term}".$$ 

• The error term depends on the zero-free region of the Dirichlet $L$-function:

$$L(s, \chi_q) := \sum_{n=1}^{\infty} \frac{\chi_q(n)}{n^s}, \quad \Re(s) > 1$$
Prime Ideal Theorem (Landau 1918)

As $x \to \infty$,

$$\pi(x; k) := \sum_{\text{p} \subset \mathcal{O}_k \atop \text{Nm}_{k/Q} \text{p} \leq x} 1 = \text{Li}(x) + \text{"error term"}.$$
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The error term depends on the zero-free region of the Dedekind zeta-function of \( k \):
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The error term depends on the zero-free region of the Dedekind zeta-function of $k$:
\[
\zeta_k(s) := \sum_{I \subset \mathcal{O}_k} \frac{1}{(\text{Nm}_{k/\mathbb{Q}} I)^s} = \prod_{p \subset \mathcal{O}_k} \left(1 - \frac{1}{(\text{Nm}_{k/\mathbb{Q}} p)^s}\right)^{-1}, \quad \Re(s) > 1
\]
**Counting prime ideals in number fields**

\[ \mathcal{O}_k \quad k \quad \mathbb{Z} \quad \mathbb{Q} \]

**Prime Ideal Theorem (Landau 1918)**

As \( x \to \infty \),

\[ \pi(x; k) := \sum_{p \subset \mathcal{O}_k \text{Nm}_{k/\mathbb{Q}} p \leq x} 1 = \text{Li}(x) + "error \ term". \]

The error term depends on the zero-free region of the Dedekind zeta-function of \( k \):

\[ \zeta_k(s) := \sum_{I \subset \mathcal{O}_k} \frac{1}{(\text{Nm}_{k/\mathbb{Q}} I)^s} = \prod_{p \subset \mathcal{O}_k} \left( 1 - \frac{1}{(\text{Nm}_{k/\mathbb{Q}} p)^s} \right)^{-1}, \quad \Re(s) > 1 \]

**Example 1:** When \( k = \mathbb{Q} \), we have \( \zeta_k(s) = \zeta(s) \).
Prime Ideal Theorem (Landau 1918)
As $x \to \infty$,
\[
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\]

**Example 1:** When $k = \mathbb{Q}$, we have $\zeta_k(s) = \zeta(s)$.

**Example 2:** When $k = \mathbb{Q}(\sqrt{q})$, one can show $\zeta_k(s) = \zeta(s)L(s, \chi_q)$. 
Prime Ideal Theorem (Landau 1918)

As \( x \to \infty \),

\[
\pi(x; k) := \sum_{p \in \mathcal{O}_k} 1 = \text{Li}(x) + \text{"error term"}.
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The error term depends on the zero-free region of the Dedekind zeta-function of \( k \):

\[
\zeta_k(s) := \sum_{I \subset \mathcal{O}_k} \frac{1}{(Nm_{k/Q} I)^s} = \prod_{p \in \mathcal{O}_k} \left( 1 - \frac{1}{(Nm_{k/Q} p)^s} \right)^{-1}, \quad \Re(s) > 1
\]

**Generalized Riemann Hypothesis:** Nontrivial zeros of \( \zeta_K(s) \) have real part equal to 1/2.
Let $L/k$ be a normal extension with Galois group $G = \text{Gal}(L/k)$. 

π\(_C \left( x, \frac{L}{k} \right) := \# \{ p \subset O_k : p \text{ unramified in } \frac{L}{k}, \left[ \frac{L}{k} \right] p = C, \text{Nm}_{k/Q} p \leq x \}$
Let $L/k$ be a normal extension with Galois group $G = \text{Gal}(L/k)$.

\[
\begin{array}{c}
L \\
\downarrow \\
\text{Gal}(L/k) \cong G \\
\downarrow \\
k \\
\downarrow \\
\mathbb{Q}
\end{array}
\]
Let $L/k$ be a normal extension with Galois group $G = \text{Gal}(L/k)$.

\[ \pi_C(x, L/k) := \# \left\{ p \subset \mathcal{O}_k : p \text{ unramified in } L, \left[ \frac{L/k}{p} \right] = C, \text{Nm}_{k/Q} p \leq x \right\} \]
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$L$
\[ \text{Gal}(L/k) \cong G \]

- $p$ is a prime ideal in $\mathcal{O}_k$ which is unramified in $L$. 

$k$
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Let $L/k$ be a normal extension with Galois group $G = \text{Gal}(L/k)$.

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- $p$ is a prime ideal in $\mathcal{O}_k$ which is unramified in $L$.
- $\left[ \frac{L/k}{p} \right]$ is the Artin symbol, which denotes the fixed, targeted conjugacy class $C$ within $G$. 
Chebotarev Density Theorem (1922)

\[ \pi_C(x; L/k) \sim \frac{|C|}{|G|} \text{Li}(x), \quad x \to \infty \]
Effective Chebotarev Density Theorem
(Lagarias & Odlyzko 1975)

\[ \pi_C(x; \frac{L}{k}) = \frac{|C|}{|G|} \text{Li}(x) + \text{"error term"}, \quad x \to \infty \]

The error term depends on the zero-free region of the Dedekind zeta-function of \( L \).

\[ \zeta_L(s) := \zeta_k(s) \prod_{\rho \in \hat{G}, \rho \neq \rho_0} L(s, \rho, \frac{L}{k}) \dim \rho \]

Each \( L(s, \rho, \frac{L}{k}) \) is an Artin \( L \)-function.

\[ \xymatrix{ L \ar@{-}[r]_{k} & \text{Gal}(L/k) \cong G \ar@{-}[r]_{Q} & Q } \]
Effective Chebotarev Density Theorem
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\pi_C(x; L/k) = \frac{|C|}{|G|} \text{Li}(x) \ + \ "\text{error term}\”, \quad x \to \infty
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- Each \( L(s, \rho, L/k) \) is an Artin \( L \)-function.
- The product is over the nontrivial irreducible representations of \( G \).
Example of a Dedekind zeta-function $\zeta_L(s)$

Let $k = \mathbb{Q}$ and $G = \text{Gal}(L/\mathbb{Q}) \cong S_3$. 

$\zeta_L(s) =$
Example of a Dedekind zeta-function $\zeta_L(s)$

Let $k = \mathbb{Q}$ and $G = \text{Gal}(L/\mathbb{Q}) \cong S_3$.

$S_3$ has the following Galois representations:

- $\rho_0$ – trivial representation, 1-dimensional
- $\rho_1$ – sign representation, 1-dimensional
- $\rho_2$ – standard representation, 2-dimensional

$$\zeta_L(s) = \zeta(s) \cdot L(s, \rho_1) \cdot L(s, \rho_2)^2$$
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**Example of a Dedekind zeta-function $\zeta_L(s)$**

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**Theorem (Lagarias-Odlyzko, 1975)**

For any fixed conjugacy class $C \subset G$, 

$$|\pi_C(x, L/k) - |C||G| \text{Li}(x)| \leq |C||G| \text{Li}(x^{\beta_0}) + c_1 x \exp(-c_2 n_1^{1/2} \log x)$$

Error term depends on zero-free region of $\zeta_L(s)$ for $x \geq \exp(10 n_L (\log D_L)^2)$, where

- $\beta_0$ is a real, simple exceptional zero of $\zeta_L(s)$;
- $c_1, c_2$ are effectively computable constants.
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for $x \geq \exp(10n_L (\log D_L)^2)$, where

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A Conditional Effective Chebotarev Density Theorem

Let $L/k$ be a normal extension with Galois group $G = \text{Gal}(L/k)$, $D_L = |\text{Disc } L/\mathbb{Q}|$, and $n_L = [L : \mathbb{Q}]$. 

Theorem (Lagarias-Odlyzko, 1975)

If the generalized Riemann hypothesis holds for the Dedekind zeta-function $\zeta_L(s)$, then for any fixed conjugacy class $C \subset G$:

$$\left| \pi_C(x, L/k) - |C||G| \right| \leq C_0 |C||G|x^{1/2} \log(D_L n_L).$$

Error term relies on GRH for $\zeta_L(s)$ for every $x \geq 2$, where $C_0$ is an effectively computable constant.
A Conditional Effective Chebotarev Density Theorem

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Error term relies on GRH for $\zeta_L(s)$.

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Comparing the Theorems (Lagarias-Odlyzko, 1975)

Theorem (Unconditional)

For any fixed conjugacy class $C \subset G$,

$$\left| \pi_C(x, L/k) - \frac{|C|}{|G|} \text{Li}(x) \right| \leq \frac{|C|}{|G|} \text{Li}(x^{\beta_0}) + c_1 x \exp \left( -c_2 n_L^{1/2} (\log x)^{1/2} \right)$$

for $x \geq \exp(10n_L (\log D_L)^2)$. 

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If GRH holds for $\zeta_L(s)$, then for any fixed conjugacy class $C \subset G$,

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for every $x \geq 2$. 

Question: What do a lower threshold and no $\beta_0$ term get you?
Comparing the Theorems  (Lagarias-Odlyzko, 1975)

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COMPARING THE THEOREMS  (Lagarias-Odlyzko, 1975)

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\]

*for* \( x \geq \exp(10n_L (\log D_L)^2) \geq D_L^{10n_L} \).

**Theorem (Conditional)**

*If GRH holds for* \( \zeta_L(s) \), *then for any fixed conjugacy class* \( C \subset G \)

\[
\left| \pi_C(x, L/k) - \frac{|C|}{|G|} \text{Li}(x) \right| \leq C_0 \frac{|C|}{|G|} x^{1/2} \log(D_L x^{n_L}).
\]

*for every* \( x \geq 2 \).
Theorem (Unconditional)

For any fixed conjugacy class \( C \subset G \),

\[
\left| \pi_C(x, L/k) - \frac{|C|}{|G|} \ln(x) \right| \leq \frac{|C|}{|G|} \ln(x^{\beta_0}) + c_1 x \exp \left( -c_2 n_L^{1/2} (\log x)^{1/2} \right)
\]

for \( x \geq \exp(10n_L (\log D_L)^2 \geq D_L^{10n_L} \).

Theorem (Conditional)

If GRH holds for \( \zeta_L(s) \), then for any fixed conjugacy class \( C \subset G \)

\[
\left| \pi_C(x, L/k) - \frac{|C|}{|G|} \ln(x) \right| \leq C_0 \frac{|C|}{|G|} x^{1/2} \log\left( D_L x^{n_L} \right).
\]

for every \( x \geq 2 \).

Want: An unconditional effective CDT with a low threshold on \( x \), no \( \beta_0 \) term, and an acceptable error term.
Skeleton of Theorem (Pierce, T., Wood)

Let $\mathcal{F}(X)$ be a family of fields, where $K \in \mathcal{F}(X)$ have

- fixed degree $n$ over $\mathbb{Q}$
- fixed Galois Group $G = \text{Gal}(\bar{K}/\mathbb{Q})$
- $D_K \leq X$
- a possible ramification restriction on tamely ramified primes;

Suppose it is known that $|\mathcal{F}(X)| \gg X^a$ for some $a > 0$.

Then for at most $O(X^b)$ exceptions, with $b < a$, for fixed $A \geq 2$, we have

$$\left| \pi_{\mathcal{C}}(x, \bar{K}/\mathbb{Q}) - |\mathcal{C}| \right|_{G} \leq |\mathcal{C}| |G| x (\log x)^A$$

where $x \geq \kappa_1 \exp\{\kappa_2 (\log \log D_{\bar{K}})^2\}$, and the $\kappa_i$ depend on $n$, $|G|$, $D_{\bar{K}}$, $a$, $b$, and $A$.

- No $\beta_0$ term.

Can take $x = D_{\eta_{\bar{K}}}$ for $\eta$ small.

We prove most Dedekind zeta-functions in the family satisfy a certain zero-free region.
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Suppose it is known that $|F(X)| \gg X^a$ for some $a > 0$.

Then for at most $O(X^b)$ exceptions, with $b < a$, for fixed $A \geq 2$, we have

$$\left| \pi_C(x, {\tilde{K}}/\mathbb{Q}) - |C|/|G| \right| \leq |C|/|G| x (\log x)^A$$

where $x \geq \kappa_1 \exp\left\{ \kappa_2 (\log \log D_\kappa^{\kappa_3})^2 \right\}$, and the $\kappa_i$ depend on $n$, $|G|$, $D_\kappa$, $a$, $b$, and $A$. No $\beta_0$ term.
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- No $\beta_0$ term. Can take $x = D_{\tilde{K}}^\eta$ for $\eta$ small.

We prove most Dedekind zeta-functions in the family satisfy a certain zero-free region.
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Let $\mathcal{F}(X)$ be a family of fields for which the previous Chebotarev Density Theorem holds. For the nonexceptional fields $K \in \mathcal{F}(X)$, we have

$$|\text{Cl}_K[\ell]| \ll n, \ell, \varepsilon \ D_K^{1/2 \frac{1}{2\ell(n-1)} + \varepsilon}.$$
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Let $\mathcal{F}(X)$ be a family of fields for which the previous Chebotarev Density Theorem holds. For the nonexceptional fields $K \in \mathcal{F}(X)$, we have

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**Question:**
To which families does our Chebotarev Density Theorem apply?
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Ellenberg-Venkatesh

Bhargava

$D_p$ reflection $\ll X^{1/(p-1)}$, $p \geq 5$ $\gg X^{2/(p-1)}$
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OVERVIEW OF ARGUMENT

Ellenberg-Venkatesh

\[ |\text{Cl}_K[\ell]| \ll \ell, n, \varepsilon \quad D_K^{\frac{1}{2} + \varepsilon} M^{-1} \]
Overview of Argument

Ellenberg-Venkatesh

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Effective Chebotarev Density Theorem assuming non-GRH zero-free region
**Overview of Argument**

Ellenberg-Venkatesh

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\[ L = \tilde{K} \]

\[ \text{Gal}(\tilde{K}/\mathbb{Q}) \cong G \]

\[ n \]

\[ \mathbb{Q} \]

\[ \zeta_{\tilde{K}}(s) = \zeta(s) \prod_{\rho \in \hat{G} \atop \rho \neq \rho_0 \text{ irreducible}} L(s, \rho, \tilde{K}/\mathbb{Q})^{\dim \rho} \]
**The zero-free region**

\[ L = \tilde{K} \]

\[ \frac{K}{n} \] \( \text{Gal}(\tilde{K}/\mathbb{Q}) \cong G \)

\[ \frac{\mathbb{Q}}{Q} \]

\[ \zeta_{\tilde{K}}(s) = \zeta(s) \prod_{\substack{\rho \in \hat{G} \\ \rho \neq \rho_0 \text{ irreducible}}} L(s, \rho, \tilde{K}/\mathbb{Q})^{\text{dim } \rho} \]

**Known zero-free region for \( \zeta(s) \):**

\[ \sigma > 1 - \frac{c}{\log^{2/3}(|t| + 2) \log \log^{1/3}(|t| + 3)}. \]
The zero-free region

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**Known zero-free region for** \( \zeta(s) \):

\[ \sigma > 1 - \frac{c}{\log^{2/3}(|t| + 2) \log \log^{1/3}(|t| + 3)} \cdot \]

**Assumed zero-free region for** \( \zeta_{\tilde{K}}(s)/\zeta(s) \):

\[ [1 - \delta, 1] \times [- (\log D_{\tilde{K}})^{2/\delta}, (\log D_{\tilde{K}})^{2/\delta}] \]
\[ \zeta_{\tilde{K}}(s) = \zeta(s) \prod_{\rho \in \hat{G}, \rho \neq \rho_0} L(s, \rho, \tilde{K}/Q)^{\dim \rho} \]
Idea of the proof

- We return to the method of Lagarias-Odlyzko.
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Proving the Chebotarev Density Theorem

Idea of the proof

• We return to the method of Lagarias-Odlyzko.

• We insert our assumed zero-free region for $\zeta_L(s)/\zeta(s)$ at a key point.

• We work delicately to provide both an acceptable effective error term, and a sufficiently small threshold for $x$ depending on $D_L$. 
Theorem (Pierce, T., Wood)

Let \(0 < \delta \leq 1/4\) be a fixed positive constant. For any normal extension of number fields \(L/\mathbb{Q}\) with \([L : \mathbb{Q}] = n_L\) such that \(D_L\) is sufficiently large and \(\zeta_L(s)\) obeys the assumed zero-free region, we have that for any \(A \geq 2\) and any conjugacy class \(C \subset G = \text{Gal}(L/\mathbb{Q})\)

\[
\left| \pi_C(x, L/\mathbb{Q}) - \frac{|C|}{|G|} \text{Li}(x) \right| \leq \frac{|C|}{|G|} \frac{x}{(\log x)^A} \tag{error term depends on assumed zero-free region}
\]

for all

\[
x \geq c_1 \exp \left\{ c_2 (\log \log (D_L^{c_3})^{3/2} \log \log \log (D_L^{c_4}))^{1/2} \right\},
\]

\[
\geq (\log D_L)^{\text{small power}}
\]

where all the constants can be written explicitly.
Bounding $\ell$-torsion without assuming GRH

Ellenberg-Venkatesh

$$|\text{Cl}_K[\ell]| \ll_{\ell,n,\varepsilon} D_K^{\frac{1}{2} + \varepsilon} M^{-1}$$

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Show that within an appropriate family of fields $K$, most $\tilde{\zeta}_K(s)$ obey the zero-free region

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Without assuming GRH, conclude

$$|\text{Cl}_K[\ell]| \ll_{\ell,n,\varepsilon} D_K^{\frac{1}{2} - \frac{1}{2\ell(n-1)} + \varepsilon}$$

for non-exceptional $K$. 


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Let $\pi$ be a cuspidal automorphic representation on $\text{GL}_m(\mathbb{Q})$.

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Key Tool - Zeros of Automorphic $L$-functions

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Define

$$N(\pi; \alpha, T) := \# \text{ of zeros of } L(s, \pi) \text{ such that } \beta > \alpha \text{ and } |\gamma| \leq T.$$
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$$N(\pi; \alpha, T) := \# \text{ of zeros of } L(s, \pi) \text{ such that } \beta > \alpha \text{ and } |\gamma| \leq T.$$ 

Kowalski and Michel have given a bound for $N(\pi; \alpha, T)$ that holds on average for an appropriately defined family of cuspidal automorphic representations.
Theorem (Kowalski & Michel, 2002)

Let \( S(q), q \geq 1 \) be a family of cuspidal automorphic representations satisfying a prescribed set of conditions. Let \( \alpha \geq 3/4 \) and \( T \geq 2 \). Then there exists \( c_0 > 0 \), depending on the family, such that

\[
\sum_{\pi \in S(q)} N(\pi; \alpha, T) \ll T^B q^{c_0 \frac{1-\alpha}{2\alpha-1}}
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for all \( q \geq 1 \) and some \( B \geq 0 \) that depends on the family. The implied constant only depends on the choice of \( c_0 \).
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Applied to $L(s, \pi)$ for $\pi \in S(q)$ $\implies$ a zero-free region of the desired shape that holds for all but a possible zero-density sub-family of $L$-functions
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We wish to apply Kowalski-Michel to $\frac{\zeta_K(s)}{\zeta(s)}$ as $K$ varies over $\mathcal{F}(X)$.

A couple of issues:

1. We are working with Artin $L$-functions, which in general are not known to be automorphic.

2. Kowalski & Michel’s result applies to family of cuspidal automorphic representations. We would like to apply it to a family of isobaric automorphic representations.
\[ \frac{\zeta_{\tilde{K}}(s)}{\zeta(s)} = \prod_{\substack{\rho \in \hat{G} \\
\rho \neq \rho_0 \text{ irreducible}}} L(s, \rho, \tilde{K}/\mathbb{Q})^{d_j}, \quad d_j = \text{deg}(\rho_j). \]

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Assuming the strong Artin conjecture, we have that each \(L(s, \rho, \tilde{K}/\mathbb{Q})\) is automorphic, i.e. we can write

\[
L(s, \rho, \tilde{K}/\mathbb{Q}) = L(s, \pi)
\]

for each \(L(s, \rho, \tilde{K}/\mathbb{Q})\) in our product.
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- We decompose each Dedekind zeta function into a product of cuspidal automorphic $L$-functions.

- We apply the Kowalski-Michel result to the sub-family generated by each factor.
A new obstacle:

In generalizing Kowalski-Michel, we uncover a technical barrier:

– *a priori*, each sub-family could lead to many bad fields for which our Chebotarev Density Theorem does not apply.
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Must define our families of fields to avoid this situation – where potential “bad” elements in each sub-family propagate to create a “large” family of “bad” Dedekind zeta-functions $\zeta_K(s)$. 
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Sketch of new idea

• We transform the problem to counting how often $\tilde{K}_1$ and $\tilde{K}_2$ both contain a particular subfield $F$. This relies on work of Klüners and Nicolae (2016).
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• Then we quantify how many \( K \) can have a particular discriminant.
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Thanks for y’all’s attention!