[Management Services: A Magazine of Planning, Systems, and Controls](https://egrove.olemiss.edu/mgmtservices)

Volume 1 | Number 3 Article 8

7-1964

Breakeven for individual Products, Plants, and Sales Territories

William L. Ferrara

If fixed as well as variable costs are linked with varying products, plants, and territories, then it is possible, with linear programing, to determine the optimum product mix in a multiplant, multiproduct company—

BREAKEVEN FOR INDIVIDUAL PRODUCTS, PLANTS, AND SALES TERRITORIES

by William L. Ferrara
Pennsylvania State University

A VIRTUALLY untouched problem in the literature of accounting is that of breakeven analysis for parts of the firm. More often than not discussions of breakeven analysis revolve around the firm as a whole and take a three-line (fixed costs, variable costs, and revenues) approach to the complex multiproduct, multiplant, and multiterritory firms of today. This three-line approach can be useful as a general indicator of cost-volume-profit relationships within a firm. However, the three-line approach cannot yield specific information about the cost-volume-profit relationships of individual products, individual plants, and individual sales territories. In order to obtain precise information that can be used to make specific decisions about individual products, plants, and territories, breakeven analysis must be considered for these parts of a firm.

The goal of this paper is the development of an approach to deriving breakeven data for individual products produced in one or more plants and sold in one or more territories. A by-product of the approach developed is the not too startling conclusion that linear programing is a most appropriate technique to use when trying to determine the optimum product mix in the complex multiproduct, multiplant, multiterritory firm of today.

Some basic assumptions

Two basic assumptions of this paper are these:

1. Costs can be segregated into fixed and variable categories.
2. Many fixed costs can be identified with individual products, plants, and territories.

Certainly, there can be disagreement on methods of segregating costs into fixed and variable com-

---

This article is adapted from materials to be included in *Management Accounting for Profit Control,* Second Edition, by I. Wayne Keller and William L. Ferrara, which will be published by McGraw-Hill Book Company, Inc., next year.

ponents. However, for our purposes we shall assume that the costs can be and are segregated using techniques similar to those discussed in NAA Accounting Practice Report Number 10 entitled "Separating and Using Costs as Fixed and Variable."*

Identifying or tracing fixed costs to individual products, plants, and territories should present no problem even if the accounting system has not been designed to implement this identification process. Salaries of plant and territorial personnel are fixed costs that are traceable to plants and territories. Straight-line depreciation on plant or territory equipment can also be traced to individual plants and territories. Finally, in a product-line organization it would not be difficult to conceive of many fixed costs traceable to specific product lines and at least some fixed costs traceable to specific products within a line.

The cost allocation problem

A problem that automatically pops up when parts of the firm are considered is cost allocation. Substantial amounts of factory overhead, distribution costs, and administrative costs are not really traceable to individual products, manufacturing plants, and even sales territories. These nontraceable costs are normally fixed costs such as factory administrative costs and general administrative costs. A useful method of dealing with these nontraceable costs (often considered common or joint costs) is to forget about allocating them to parts of the firm since for most cost analyses there really is no need to allocate them. Omission of such allocations may seem strange, but it is quite realistic (as will be illustrated) for purposes of making decisions and measuring performance via breakeven analysis for parts of the firm.

In order to illustrate the breakeven concept for parts of the firm and the consequences of the nonallocation of nontraceable costs, the following types of companies will be considered:

1. One Product—One Plant
2. Two Products—One Plant
3. One Product—Two Plants
4. Two Products—Two Plants
5. One Product—One Plant—Two Territories
6. Two Products—One Plant—Two Territories
7. Two Products—Two Plants—Two Territories

One product-one plant company

In a company with a single product produced in a single plant, all costs are traceable to the product and to the plant. Thus, there is no problem of allocation. With the facts given in Exhibit 1, above, the contribution margin per unit and the breakeven point can be calculated as shown in Exhibit 1.

Two product-one plant company

In a two product—one plant situation some costs will not be traceable to products. These are the common fixed costs. The fixed costs that are traceable to each product can be described as direct fixed

*This is a summary of practice published by the National Association of Accountants in June, 1960.

EXHIBIT 1

ONE PRODUCT—ONE PLANT

Fixed costs	$265,000
Variable costs	$4.00 per unit
Sales price	$8.50 per unit

Contribution per unit = $8.50 — 4.00 = $4.50

$$\text{Breakeven point} = \frac{265{,}000}{4.50} = 58{,}888 \text{ units}$$

EXHIBIT 2

TWO PRODUCTS — ONE PLANT

Breakeven Data

	Product A	Product B
Direct fixed costs	$100,000	$120,000
Variable costs	$3.00 per unit	$4.00 per unit
Sales prices	$6.00 per unit	$8.50 per unit
Common fixed costs	$45,000	

Breakeven Calculations

	Product A	Product B
Contribution per unit	$3.00	$4.50
Breakeven to cover direct fixed costs	$\frac{100{,}000}{3} = 33{,}333$ units	$\frac{120{,}000}{4.50} = 26{,}667$ units

costs. With the data given, breakeven calculations can be made as shown in Exhibit 2, page 39.

In order to cover the common fixed costs of $45,000, there are many possible combinations of sales of Products A and B. Algebraically, one would express the combinations of A and B that would cover the $45,000 as:

$$3\ A + 4.5\ B = 45{,}000$$

Note that the 3 and the 4.5 are the profit contributions per unit of A and B, respectively. The only limits on the solutions to the above equation would be the possibility of limited production capacity and/or limited sales potential.

One product-two plant company

With one product and two plants some of the fixed costs will not be traceable to plants. Ordinarily these common fixed costs would be the costs of general administration. The data given can be used to calculate breakeven points as shown in Exhibit 3 on this page.

To cover the common fixed costs of $25,000 there are many possible combinations of the output of Plants I and II. Algebraically, these combinations can be expressed in the following form:

$$4\ I + 4.5\ II = 25{,}000$$

The 4 and the 4.5 represent the profit contributions of products produced in Plants I and II. Limits on the possible solutions to the above equations would be the possibility of limited production capacity in Plant I and/or II.

EXHIBIT 3

ONE PRODUCT — TWO PLANTS

Breakeven Data

	Plant I	Plant II
Direct fixed costs	$110,000	$135,000
Variable costs	$4.50 per unit	$4.00 per unit
Sales prices	$8.50 per unit	$8.50 per unit
Common fixed costs	$25,000	

Breakeven Calculations

	Plant I	Plant II
Contribution per unit	$4.00	$4.50
Breakeven to cover direct fixed costs	$\frac{110{,}000}{4} = 27{,}500$ units	$\frac{135{,}000}{4.50} = 30{,}000$ units

Two product-two plant company

In a more complex situation with two products and two plants there arise three layers of common fixed costs. These layers represent the costs common to products A and B in Plant I and in Plant II and the costs common to the entire operation of all products and all plants. Breakeven data and breakeven calculations to illustrate the two product—two plant situation are shown in Exhibit 4 below.

The subscripts in the equations

EXHIBIT 4

TWO PRODUCTS — TWO PLANTS

Breakeven Data

	Plant I Product A	Plant I Product B	Plant II Product A	Plant II Product B
Direct fixed costs	$45,000	$50,000	$60,000	$50,000
Variable costs per unit	$3.00	$4.50	$3.50	$4.50
Sales prices per unit	$6.00	$8.50	$6.00	$8.50
Fixed costs common to products	$20,000		$40,000	
Fixed costs common to total operations	$60,000			

Breakeven Calculations

	Plant I Product A	Plant I Product B	Plant II Product A	Plant II Product B
Contribution per unit	$3.00	$4.00	$2.50	$4.00
Breakeven to cover direct fixed costs	$\frac{45{,}000}{3.00} = 15{,}000$	$\frac{50{,}000}{4.00} = 12{,}500$	$\frac{60{,}000}{2.50} = 24{,}000$	$\frac{50{,}000}{4.00} = 12{,}500$

Breakeven to cover common fixed costs

$20,000 costs common to Plant I

Any combination of $3A_1 + 4B_1 = 20{,}000$

$40,000 costs common to Plant II

Any combination of $2.5A_2 + 4B_2 = 40{,}000$

$60,000 costs common to total operations

Any combination of $3A_1 + 4B_1 + 2.5A_2 + 4B_2 = 60{,}000$

in Exhibit 4 designate the plants from which products originate. Limits on the possible number of combinations of A and B to cover common fixed costs are the limits of production capacity and sales potential.

One product-one plant-two territory company

When sales territories are considered in a breakeven analysis, there arises the possibility of fixed costs common to the sales territories as well as the possibility of dealing with variable costs segregated by sales and production. The data and calculations in Exhibit 5 on this page illustrate how such a situation might be handled.

As in previous illustrations, the numbers in the equations in Exhibit 5 represent the profit contributions

EXHIBIT 5

ONE PRODUCT — ONE PLANT — TWO TERRITORIES

Breakeven Data

	Territory I		Territory II	Plant
Direct fixed costs	$40,000		$57,750	$150,000
Variable costs per unit	$1.50		$1.75	$2.00
Sales prices per unit	$8.50		$9.00	
Common fixed costs				
Common to both territories		$15,000		
Common to all operations				$20,000

Breakeven Calculations

	Territory I	Territory II
Contribution per unit (Sales price minus all variable costs)	$5.00	$5.25
Breakeven to cover direct fixed costs of each territory	$\frac{40,000}{5} = 8,000$	$\frac{57,750}{5.25} = 11,000$

Breakeven to cover other fixed costs

$150,000 direct fixed costs of production

Any combination of $5\ \text{I} + 5.25\ \text{II} = 150,000$

$15,000 costs common to both territories

Any combination of $5\ \text{I} + 5.25\ \text{II} = 15,000$

$20,000 costs common to all operations

Any combination of $5\ \text{I} + 5.25\ \text{II} = 20,000$

EXHIBIT 6

TWO PRODUCT — ONE PLANT — TWO TERRITORIES

Breakeven Data

	Eastern Territory		Western Territory		Plant	
	Product A	Product B	Product A	Product B	Product A	Product B
Direct fixed costs	$15,000	$10,000	$30,000	$10,000	$50,000	$55,000
Variable costs per unit	$2.00	$1.50	$2.50	$2.00	$1.00	$2.00
Sales prices per unit	$6.00	$8.50	$6.50	$9.00		
Common fixed costs						
Common to products	$15,000		$20,000		$45,000	
Common to territories		$15,000				
Common to all operations				$30,000		

Breakeven Calculations

	Eastern Territory		Western Territory	
	Product A	Product B	Product A	Product B
Contribution per unit (Sales price minus all variable costs)	$3.00	$5.00	$3.00	$5.00
Breakeven to cover direct fixed costs in each territory	$\frac{15,000}{3} = 5,000$	$\frac{10,000}{5} = 2,000$	$\frac{30,000}{3} = 10,000$	$\frac{10,000}{5} = 2,000$

Breakeven to cover other fixed costs

$50,000 direct production costs for Product A

Any combination of $3A_E + 3A_W = 50,000$

$55,000 direct production costs for Product B

Any combination of $5B_E + 5B_W = 55,000$

$45,000 production costs common to both products

Any combination of $3A_E + 3A_W + 5B_E + 5B_W = 45,000$

$20,000 Western Territory costs common to both products

Any combination of $3A_W + 5B_W = 20,000$

$15,000 Eastern Territory costs common to both products

Any combination of $3A_E + 5B_E = 15,000$

$15,000 costs common to both territories

Any combination of $3A_E + 5B_E + 3A_W + 5B_W = 15,000$

$30,000 costs common to all operations

Any combination of $3A_E + 5B_E + 3A_W + 5B_W = 30,000$

EXHIBIT 7

PLANT AND TERRITORY COST STRUCTURE

	Plant I		Plant II		Eastern Territory		Western Territory	
	A	B	A	B	A	B	A	B
Direct fixed costs	$30,000	$40,000	$20,000	$15,000	$15,000	$10,000	$30,000	$10,000
Variable costs	$1.00/hr.	$2.00/hr.	$1.50/hr.	$1.00/hr.	$2.00	$1.50	$2.50	$2.00
Common fixed costs								
Products	$20,000		$10,000		$15,000		$20,000	
Plants		$15,000						
Territories						$15,000		
Total firm				$10,000				

EXHIBIT 8

SALES POTENTIALS, SELLING PRICES, AND TRANSPORTATION COSTS

	Eastern Territory		Western Territory	
Product	A	B	A	B
Sales prices per unit	$6.00	$8.50	$6.50	$9.00
Sales potential (units)	60,000	80,000	50,000	40,000
Transportation costs per unit				
From Plant I	$ .50	$ .50	$1.00	$1.00
From Plant II	$1.00	$1.00	$ .50	$ .50

EXHIBIT 9

PRODUCTION HOUR REQUIREMENTS AND AVAILABILITIES

	Plant I		Plant II	
	A	B	A	B
Required Production Hours Per Unit				
Special purpose equipment	.5	1	1	1.5
or				
General purpose equipment	1.5	1	1.5	1.5
Available Production Hours				
Special purpose equipment	40,000	60,000	20,000	10,000
General purpose equipment—can produce A or B	40,000		10,000	

EXHIBIT 10

PROFIT CONTRIBUTION PER UNIT
USING SPECIAL PURPOSE EQUIPMENT

	Product A				Product B			
Sold in	Eastern Territory		Western Territory		Eastern Territory		Western Territory	
Produced in	Plant I	Plant II	Plant I	Plant II	Plant I	Plant II	Plant I	Plant II
Sales prices	$6.00	$6.00	$6.50	$6.50	$8.50	$8.50	$9.00	$9.00
Variable costs								
Production	.50	1.50	.50	1.50	2.00	1.50	2.00	1.50
Distribution	2.00	2.00	2.50	2.50	1.50	1.50	2.00	2.00
Transportation	.50	1.00	1.00	.50	.50	1.00	1.00	.50
Contribution per unit of sales	$3.00	$1.50	$2.50	$2.00	$4.50	$4.50	$4.00	$5.00

EXHIBIT 11

PROFIT CONTRIBUTION PER UNIT USING GENERAL PURPOSE EQUIPMENT

	Product A				Product B			
Sold in	Eastern Territory		Western Territory		Eastern Territory		Western Territory	
Produced in	Plant I	Plant II	Plant I	Plant II	Plant I	Plant II	Plant I	Plant II
Sales prices	$6.00	$6.00	$6.50	$6.50	$8.50	$8.50	$9.00	$9.00
Variable costs								
Production	1.50	2.25	1.50	2.25	2.00	1.50	2.00	1.50
Distribution	2.00	2.00	2.50	2.50	1.50	1.50	2.00	2.00
Transportation	.50	1.00	1.00	.50	.50	1.00	1.00	.50
Contribution per unit of sales	$2.00	$ .75	$1.50	$1.25	$4.50	$4.50	$4.00	$5.00

per unit. Limits on the possible combinations to achieve the various levels of breakeven are the sales potential in each territory and the production capacity of the plant.

Two product—one plant—two territory company

The two product—one plant—two territory situation is quite similar to the preceding illustration. Actually, the only differences are the extra layers of common fixed costs. The data in Exhibit 6, page 41, illustrate the basic problem and its solution.

Subscripts in all of the equations in Exhibit 6 refer to the territories in which products are sold. The limits on the possible combinations to achieve the various levels of breakeven again are the sales potential in each territory and the production capacity of the plant.

Review of preceding illustrations

At this point the reader should begin to question the utility of a breakeven chart that attempts to express the multidimensional character of a business with only three lines, i.e., revenue, fixed costs, and variable costs. When a business is taken apart plant by plant, product by product, and territory by territory, it becomes evident that there is a breakeven point related to each and every layer of direct fixed costs and common fixed costs. The existence of multiple breakeven points may be frustrating to some people, but others see in it the key issue in cost-volume-profit relationships, that is, the use of production and sales facilities to produce the optimum profit. This optimum profit is produced by obtaining the optimum product mix, that is, the combination of products A and B that will produce the greatest profit consistent with the other financial and nonfinancial goals of the enterprise.

In order to deal with the problem of optimum product mix, the preceding illustrations will be enlarged to a two product—two plant—two territory situation. Furthermore, specific consideration will be given to sales potentials, production hours available on special purpose and general purpose equipment, production time requirements for each product, and transportation costs. Exhibits 7 through 11, pages 42-43, contain the basic data for this illustration.

Two product—two plant—two territory company

The reader will avoid confusion for himself in Exhibits 7 through 11 if he keeps in mind that variable costs are expressed as costs per unit of product for territorial costs and as costs per production hour for production costs. Admittedly, costs could vary with other measures of activity, but use of additional categories of variable costs would make the analysis more complicated than is necessary to illustrate the basic ideas involved.

Since there are two ways (using general purpose or special purpose equipment) of producing products A and B, two territories in which to sell each product, and two plants that could produce each product, there is no possibility of calculating a specific breakeven point for the fixed costs shown in Exhibit 7. All that is possible is to produce a list of the ways of combining the production and sales possibilities that could yield a breakeven point for each layer of fixed costs. This list is shown as Exhibit 12, page 44.

Before examining Exhibit 12 the reader should realize that the superscripts S and G refer to special and general purpose equipment, respectively. The subscripts E and W refer to the Eastern and Western Territory, respectively, and the subscripts 1 and 2 refer to Plants I and II, respectively. Thus:

$3A^{S}_{E1}$ = the profit contribution of Product A produced on special purpose equipment in Plant I and sold in the Eastern Territory.

$4B^{G}_{W2}$ = the profit contribution of Product B produced on general purpose equipment in Plant II and sold in the Western Territory.

Even with only a cursory examination of Exhibit 12 two points should be clear:

1. The various combinations of Products A and B that could cover the various layers of fixed costs are restricted by the sales potentials shown in Exhibit 8 and the available production hours shown in Exhibit 9.

EXHIBIT 12

Combinations of Products Able to Cover Various Layers of Fixed Costs

$30,000 of direct production costs for Product A in Plant I
Any combination of $3A_{E1}^{S} + 2A_{E1}^{G} + 2.5A_{W1}^{S} + 1.5A_{W1}^{G} = 30{,}000$

$40,000 of direct production costs for Product B in Plant I
Any combination of $4.5B_{E1}^{S} + 4.5B_{E1}^{G} + 4B_{W1}^{S} + 4B_{W1}^{G} = 40{,}000$

$20,000 of direct production costs for Product A in Plant II
Any combination of $1.5A_{E2}^{S} + .75A_{E2}^{G} + 2A_{W2}^{S} + 1.25A_{W2}^{G} = 20{,}000$

$15,000 of direct production costs for Product B in Plant II
Any combination of $4.5B_{E2}^{S} + 4.5B_{E2}^{G} + 5B_{W2}^{S} + 5B_{W2}^{G} = 15{,}000$

$15,000 of direct territory costs for Product A in Eastern Territory
Any combination of $3A_{E1}^{S} + 2A_{E1}^{G} + 1.5A_{E2}^{S} + .75A_{E2}^{G} = 15{,}000$

$10,000 of direct territory costs for Product B in Eastern Territory
Any combination of $4.5B_{E1}^{S} + 4.5B_{E1}^{G} + 4.5B_{E2}^{S} + 4.5B_{E2}^{G} = 10{,}000$

$30,000 of direct territory costs for Product A in Western Territory
Any combination of $2.5A_{W1}^{S} + 1.5A_{W1}^{G} + 2A_{W2}^{S} + 1.25A_{W2}^{G} = 30{,}000$

$10,000 of direct territory costs for Product B in Western Territory
Any combination of $4B_{W1}^{S} + 4B_{W1}^{G} + 5B_{W2}^{S} + 5B_{W2}^{G} = 10{,}000$

$20,000 of production costs common to Products A and B in Plant I
Any combination of $3A_{E1}^{S} + 2A_{E1}^{G} + 2.5A_{W1}^{S} + 1.5A_{W1}^{G} + 4.5B_{E1}^{S} + 4.5B_{E1}^{G} + 4.50B_{W1}^{S} + 4.50B_{W1}^{G} = 20{,}000$

$10,000 of production costs common to products A and B in Plant II
Any combination of $1.5A_{E2}^{S} + .75A_{E2}^{G} + 2A_{W2}^{S} + 1.25A_{W2}^{G} + 4.5B_{E2}^{S} + 4.5B_{E2}^{G} + 5B_{W2}^{S} + 5B_{W2}^{G} = 10{,}000$

$15,000 of territory costs common to Products A and B in Eastern Territory
Any combination of $3A_{E1}^{S} + 2A_{E1}^{G} + 1.5A_{E2}^{S} + .75A_{E2}^{G} + 4.5B_{E1}^{S} + 4.5B_{E1}^{G} + 4.5B_{E2}^{S} + 4.5B_{E2}^{G} = 15{,}000$

$20,000 of territory costs common to Products A and B in Western Territory
Any combination of $2.5A_{W1}^{S} + 1.5A_{W1}^{G} + 2A_{W2}^{S} + 1.25A_{W2}^{G} + 4B_{W1}^{S} + 4B_{W1}^{G} + 5B_{W2}^{S} + 5B_{W2}^{G} = 20{,}000$

$15,000 of production costs common to both Products and Plants
Any combination of $3A_{E1}^{S} + 2A_{E1}^{G} + 1.5A_{E2}^{S} + .75A_{E2}^{G} + 4.5B_{E1}^{S} + 4.5B_{E1}^{G} + 4.5B_{E2}^{S} + 4.5B_{E2}^{G} + 2.5A_{W1}^{S} + 1.5A_{W1}^{G} + 2A_{W2}^{S} + 1.25A_{W2}^{G} + 4B_{W1}^{S} + 4B_{W1}^{G} + 5B_{W2}^{S} + 5B_{W2}^{G} = 15{,}000$

$15,000 of territory costs common to both ?roducts and Territories
Any combination of $3A_{E1}^{S} + 2A_{E1}^{G} + 1.5A_{E2}^{S} + .75A_{E2}^{G} + 4.5B_{E1}^{S} + 4.5B_{E1}^{G} + 4.5_{E2}^{S} + 4.5B_{E2}^{G} + 2.5A_{W1}^{S} + 1.5A_{W1}^{G} + 2A_{W2}^{S} + 1.25A_{W2}^{G} + 4B_{W1}^{S} + 4B_{W1}^{G} + 5B_{W2}^{S} + 5B_{W2}^{G} = 15{,}000$

$10,000 of costs common to all operations
Any combination of $3A_{E1}^{S} + 2A_{E1}^{G} + 1.5A_{E2}^{S} + .75A_{E2}^{G} + 4.5B_{E1}^{S} + 4.5B_{E1}^{G} + 4.5B_{E2}^{S} + 4.5B_{E2}^{G} + 2.5A_{W1}^{S} + 1.5A_{W1}^{G} + 2A_{W2}^{S} + 1.25A_{W2}^{G} + 4B_{W1}^{S} + 4B_{W1}^{G} + 5B_{W2}^{S} + 5B_{W2}^{G} = 10{,}000$

2. Exhibit 12 is quite complicated and perhaps overstresses the idea of covering fixed costs. What should be stressed is *the combination* of Products A and B which will produce the optimum profit as opposed to *the many combinations* which could cover the various layers of fixed costs.

Each of these points will now be discussed.

Sales potentials and capacity

Both sales potentials and available production capacity restrict the possible combinations of Products A and B that could cover the various layers of fixed costs.

Sales of Product A would probably not exceed the 60,000-unit potential in the Eastern Territory and the 50,000-unit potential in the Western Territory. In the case of Product B the Eastern Territory unit potential is 80,000 while the Western Territory unit potential is 40,000. See Exhibit 8, page 42, for these potentials.

The special purpose equipment of Plant I can produce a maximum of 80,000 units of A and 60,000 units of B. The general purpose equipment of Plant I could produce maximums of either 26,666 units of A or 40,000 units of B. In Plant II special purpose equipment can produce a maximum of 20,000 units of A and 6,666 units of B. General purpose equipment in Plant II could be used to produce a maximum of either 26,666 units of A or 6,666 units of B. Each of these production possibilities can be determined by dividing the total of available production hours by the required production hours per unit, both of which are found in Exhibit 9, page 42.

Optimum combination

Considering all of the above data, how does one approach the determination of that combination of production and sales which will yield the optimum profit? In this case the problem is easily solved since the profit contribution per

EXHIBIT 13

DETERMINATION OF THE OPTIMUM COMBINATION OF PRODUCTS

Plant I Production on Special Equipment

	Product A	Product B
Units produced using full capacity	80,000	60,000
Disposal of units produced to most profitable territories*		
Eastern Territory	60,000	60,000
Western Territory	20,000	

Plant II Production on Special Equipment

	Product A	Product B
Units produced using full capacity	20,000	6,666
Disposal of units produced to most profitable territories*		
Western Territory	20,000	6,666

Summary of Sales Potential and Its Fulfillment

	Eastern Territory		Western Territory	
	A	B	A	B
Potential sales	60,000	80,000	50,000	40,000
Sales potential filled by special purpose equipment				
Plant I	60,000	60,000	20,000	
Plant II			20,000	6,666
Remaining potential to be filled by general purpose equipment	0	20,000	10,000	33,333

*Units are sent to territories in the order that will produce the greatest contribution per unit as shown in Exhibits 10 and 11.

unit is greater when special purpose equipment is used than when general purpose equipment is used. Thus, as long as there is capacity on special purpose equipment it will be used. When the special purpose equipment capacity is used up, then the less profitable general purpose equipment will be used. Since there is not enough production capacity on the general purpose equipment to produce at the potential sales level, the available hours on the general purpose equipment will be used on those products that produce the greatest return per production hour as shown in Exhibit 13 on this page.

As was mentioned previously, the general purpose equipment capacity is less than the remaining sales potential. This makes hours of general purpose equipment capacity a scarce resource and the optimum usage of a scarce resource is in accordance with its contribution. Thus, the fulfillment of the remaining sales potential by general purpose equipment should be based upon a contribution *per hour of production time available* and not a contribution *per unit* of Product A or Product B.

The contribution per hour is calculated by dividing the contribution per unit by the required production hours per unit as shown in Exhibit 14 on this page.

Based on these calculations of contribution per production hour the remaining sales potential would be filled by the use of general pur-

EXHIBIT 14

CALCULATION OF CONTRIBUTION PER HOUR OF PRODUCTION TIME AVAILABLE

	Product A				Product B			
	Eastern Territory		Western Territory		Eastern Territory		Western Territory	
	Plant I	Plant II	Plant I	Plant II	Plant I	Plant II	Plant I	Plant II
Contribution per unit of sales (Exhibit 11)	$2.00	$ .75	$1.50	$1.25	$4.50	$4.50	$4.00	$5.00
Required production hours	1.5	1.5	1.5	1.5	1	1.5	1	1.5
Contribution per production hour	$1.33	$ .50	$1.00	$ .83	$4.50	$3.00	$4.00	$3.33

pose equipment as shown in Exhibit 15 on this page.

Review of last illustration

Fortunately, in this illustration the optimum combination of products can be derived rather easily by using the contribution data in Exhibits 10 and 11. The reason is that the contribution per sales unit is less when general purpose equipment is used for all production and sales possibilities.

If, in any particular situation, the above condition does not hold, one may be forced to express the entire problem in algebraic form. The algebra involved is known as linear algebra since it forms a set of linear equations.

The use of algebra in solving optimum product mix problems is seen as a necessity if some products have greater contributions when produced on general purpose equipment. For example, one might consider using general purpose equipment to the extent of its capacity followed by the use of special purpose equipment when contributions per unit are $5.00 and $4.50 per unit, respectively, for Product A. However, this may not produce an optimum if Product B can be produced on general purpose equipment at a contribution of $4.75 and on special purpose equipment at a contribution of $4.95 and there is enough special purpose equipment to produce all needs for Product A. As the number of such combinations increases, the optimum product mix becomes less and less observable. Ultimately, there is no alternative but the use of algebra.

In this case, the preceding problem can be expressed in algebraic form using the notation of Exhibit 12, page 44.

The algebraic expression of the problem is shown in Exhibit 16 on this page.

In the algebraic expression of the problem there is one equation that expresses the objective, that is, to maximize the total profit contribution. This equation is known as the *objective function*. The numbers in the objective function represent the profit contributions for the products produced in various plants and sold in various territories, represented by the letters in the equation.

All the other equations represent restrictions of sales potential and production capacity. These restrictions are all expressed in the form of inequalities since sales potentials and production capacity represent upper limits, and both sales and production could be equal to or less than potential in the optimum solution to the set of eleven equations.

Solving these equations means that we desire to derive values for the A's and B's that will produce the maximum total profit contribution. An interesting and distressing point concerning solutions to such a set of equations is that there are situations that will yield no solution, that will yield one solution,

EXHIBIT 15

USE OF GENERAL PURPOSE EQUIPMENT

	Eastern Territory A	Eastern Territory B	Western Territory A	Western Territory B
Remaining potential to be filled by general purpose equipment	0	20,000	10,000	33,333
Filled by general purpose equipment				
Plant I		20,000		20,000
Plant II				6,666
Unfilled potential — capacity not available	0	0	10,000	6,667

EXHIBIT 16

OPTIMUM PRODUCT MIX PROBLEM

Objective

$$\text{Maximize } 3A_{E1}^{S} + 4.5B_{E1}^{S} + 1.5A_{E2}^{S} + 4.5B_{E2}^{S} + 2.5A_{W1}^{S} + 4B_{W1}^{S} + 2A_{W2}^{S} + 5B_{W2}^{S} + 2A_{E1}^{G} + 4.5B_{E1}^{G} + .75A_{E2}^{G} + 4.5B_{E2}^{G} + 1.5A_{W1}^{G} + 4B_{W1}^{G} + 1.25A_{W2}^{G} + 5B_{W2}^{G}$$

Restrictions on Objective

Sales Potential

$$A_{E1}^{S} + A_{E2}^{S} + A_{E1}^{G} + A_{E2}^{G} \leq 60{,}000 \quad \text{Eastern Territory}$$
$$A_{W1}^{S} + A_{W2}^{S} + A_{W1}^{G} + A_{W2}^{G} \leq 50{,}000 \quad \text{Western Territory}$$
$$B_{E1}^{S} + B_{E2}^{S} + B_{E1}^{G} + B_{E2}^{G} \leq 80{,}000 \quad \text{Eastern Territory}$$
$$B_{W1}^{S} + B_{W2}^{S} + B_{W1}^{G} + B_{W2}^{G} \leq 40{,}000 \quad \text{Western Territory}$$

Available Production Hours

Plant I

$$.5A_{E1}^{S} + .5A_{W1}^{S} \leq 40{,}000$$
$$B_{E1}^{S} + B_{W1}^{S} \leq 60{,}000$$
$$1.5A_{E1}^{G} + 1.5A_{W1}^{G} + B_{E1}^{G} + B_{W1}^{G} \leq 40{,}000$$

Plant II

$$A_{E2}^{S} + A_{W2}^{S} \leq 20{,}000$$
$$1.5B_{E2}^{S} + 1.5B_{W2}^{S} \leq 10{,}000$$
$$1.5A_{E2}^{G} + 1.5A_{W2}^{G} + 1.5B_{E2}^{G} + 1.5B_{W2}^{G} \leq 10{,}000$$

and even that will yield many solutions. In our case, we desire the solution that maximizes the profit contribution.

Where there is no solution, there is no maximum; where there is one solution, it must be the maximum; where there are many solutions, the maximum must be sought out. Seeking out the maximum may be a long and tedious process since one would never know that he had the maximum if all possible solutions were not tested.

Fortunately, there has been developed a routine technique for deriving the maximum solution (if a solution to the equations exists). This technique, known as the "simplex method," can easily be programed on electronic computers. This process of using linear algebra and the simplex method of solving linear equations have been given the imposing title "Linear Programing."

The role of fixed costs

The above decision on optimum product mix ignored the fixed cost factors since the decision related only to those factors that changed because of the decision. That is, variable costs were assumed to be the only costs relevant to the decision.

In a more exacting situation than that illustrated above the assumption concerning the irrelevance of fixed costs will have to be modified since there are many possibilities for specific types of fixed costs to change with the decision made. One need only consider such items as salaries and service department costs, which can be reduced if production capacity is not utilized. Actually, it would be preferable to consider more specific and precise breakdowns of costs than the simple fixed-variable breakdown even if fixed and variable costs are subcategorized into production costs, distribution costs, and transportation costs. There are more degrees of cost fixity and variability than can be allowed for in a simple cost breakdown of fixed and variable.

Another reason why fixed costs cannot be ignored in a real situation is that if the optimum product mix is such that some elements of capacity are not needed, the information should be brought to the fore. In the preceding two product—two plant—two territory example all plant capacity was utilized, but some elements of territorial capacity were not utilized. There were 10,000 units of A and 6,667 units of B that could have been sold in the Western Territory, but production capacity was not available.

With this in mind, the decision maker should consider the possibility of reducing the fixed costs of the Western Territory which the company cannot take advantage of. These fixed costs may be sales promotion and advertising costs or the salaries of sales personnel who are not really needed. On the other hand, the company could also consider the possibility of increasing the fixed production costs in order to provide the additional production capacity necessary to meet sales potentials. Either way the reader must realize that fixed costs cannot be ignored for purposes of decision making and, furthermore, that information on various layers of direct and common fixed costs, as illustrated above, can be very useful to the decision maker.

The uses of linear programing

Linear programing can be used in those many decision-making situations where the objective is to maximize or even minimize a certain value such as revenue or costs. The problem should be expressed algebraically in as precise a manner as possible. A possible objection is that the equations are linear and straight lines may not fully express the cost and revenue functions of a business situation. However, one should remember that a linear equation is probably as close an approximation to real life as the estimates of cost and revenue that will be used in the equations. Why, therefore, should we ask for more precision in the algebra when the precision of the cost and revenue data is probably of the same approximate level of exactitude? Another point to remember is that within the usual range of relevance to specific decisions the cost and revenue functions approximate a straight line.

With linear programing, breakeven analysis proves to be much more useful. In fact, linear programing stretches the cost-volume-profit relationships inherent in breakeven analysis into a fairly realistic quantitative approach to the incremental cost and revenue concepts of micro-economics.

Much work has already been done by individual firms in the area of linear programing. Those who have yet to learn of it are missing out on one of the most valuable quantitative tools developed in recent years.

There is no doubt that more businessmen and accountants should begin to consider the possibility of using linear programing to express cost-volume-profit relationships and to derive the optimum combination of cost, volume, and profit. One need not worry about the size of the equations or the number of equations since computers are readily available to use the simplex method of solving linear equations. Furthermore, there is the possibility that the number of factors and equations could be fewer in some situations than in the illustration presented in this article.