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On K-trees and Special Classes of K-trees

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ON k-TREES AND SPECIAL CLASSES OF k-TREES

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ABSTRACT

A tree is a connected graph with no cycles. In 1968 Beineke and Pippet introduced the class of generalized trees known as k-trees [3]. In this dissertation, we classify a subclass of k-trees known as tree-like k-trees and show that tree-like k-trees are a common generalization of paths, maximal outerplanar graphs, and chordal planar graphs with toughness exceeding one.

A set I of vertices in a graph G is said to be independent if no pair of vertices of I are incident in G. Let $f_s = f_s(G)$ be the number of independent sets of cardinality s of G. Then the polynomial $I(G;x) = \sum_{s\geq 0}^{\alpha(G)} f_s(G)x^s$ is called the independence polynomial of the graph G. [21]. In this dissertation, all rational roots of the independence polynomials of paths are found, and the exact paths whose independence polynomials have these roots are characterized. Additionally, trees are characterized that have $-1/q$ as a root of their independence polynomials for $1 \leq q \leq 4$. The well known vertex and edge reduction identities for independence polynomials are generalized, and the independence polynomials of k-trees are investigated. Additionally, sharp upper and lower bounds for f_s of maximal outerplanar graphs, i.e. tree-like 2-trees, are shown along with characterizations of the unique maximal outerplanar graphs that obtain these bounds respectively. These results are extensions of the works of Wingard, Song et al., and Alameddine [1].

The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of the graph G are given by: $M_1(G) = \sum_{u\in V(G)} d(u)^2$, and $M_2(G) = \sum_{uv\in E(G)} d(u)d(v)$. The study of the

Zagreb indices M_1 and M_2 have been an active area of research since the report of Gutman and Trinajstić in computational chemistry [23] in 1972. The minimum and maximum M_1 and M_2 values for k-trees are determined, and the unique k-trees that obtain these minimum and maximum values respectively are characterized.

In 2011, Hou, Li, Song, and Wei characterized the Zagreb indices for maximal outerplanar graphs and determined the unique maximal outerplanar graph that obtains minimum M_1 and M_2 values, respectively, as well as maximum M_1 and M_2 values respectively [29]. Select works of Hou et al. are extended to all tree-like k-trees. That is, the maximum M_1 value for tree-like k-trees is determined, and the unique tree-like k-tree that obtains this maximum values respectively is characterized. Additionally, a partial result for the maximum M_2 value for tree-like k-trees is determined, and a conjecture for a full result is presented.

DEDICATION

To Ethan, my son, who is my motivation to become a better person.

LIST OF SYMBOLS

G a graph

V the vertex set of a graph G

E the edge set of a graph G

$G[S]$ the subgraph induced by S

$G - v$ $G[V - v]$

$G - S$ $G[V - S]$

$G - e$ G remove an edge e

$G - F$ G remove a set of edges F

$G \cup \{uv\}$ G add the edge uv

$G \cup H$ the union of G and H

$|G|$ the cardinality of the vertex set of G

$||G||$ the cardinatlity of the edge set of G

$N(v)$ the neighborhood of vertex v

$N[v]$ the closed neighborhood of vertex v

$N(e)$ the neighborhood of edge e

$d(v)$ the degree of v

$\delta(G)$ the minimum degree of G

$\Delta(G)$ the maximum degree of G

T a tree

$d(v, u)$ the minimum distance from v to u in a graph

K_n the complete graph on n vertices

P_n the path on n vertices

S_n the star on n vertices

K_{n_1,n_2} the complete bipartite graph on $n_1 + n_2$ vertices

T_n^k a k-tree on n vertices

$S_1(G)$ the set of simplicial vertices in G

P_n^k the k-path on n vertices

$S_{k,n-k}$ the k-star on n vertices

S_n^k the k-spiral on n vertices

D_n^k the k-diamond on n vertices

$Sh(T_n^k)$ the shell of a k-tree

$\tau(G)$ the toughness of G

$\alpha(G)$ the independence number of G

$f(G)$ the fibonacci number of G

$f_s(G)$ the number of indepedent sets of cardinality s of G

$I(G; x)$ the independence polynomial of G

$M_1(G)$ the first Zagreb index

$M_2(G)$ the second Zagreb index

$T(r)$ a p-descendant or p-ancestor of T

$\{T_i\}_{i=0}^{\beta}$ a genealogy of a tree

$\beta(T_n^k)$ the branching number of T_n^k

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1. Introduction

A graph is chordal if it does not have an induced cycle of length greater than three. A graph is said to be k-degenerate if all of its subgraphs have minimum degree at most k, a concept introduced by Lick and White in 1970 [32], and a graph is maximally k-degenerate if it is k-degenerate and not a spanning subgraph of any other k-degenerate graph. In 1968 Beineke and Pippet introduced the class of generalized trees known as k-trees [3], and these graphs have attracted considerable research as well as many applications [4, 16, 33, 37, 38]. A purpose of this dissertation is to investigate the class of graphs that are both chordal and maximally k-degenerate, and it is shown that a graph is chordal and maximally k-degenerate if and only if it is a k-tree.

A major emphasis of this dissertation is to classify a subclass of k-trees based on a "new" parameter known as the shell of a k-tree, which is a reformation of the $(k+1)$-line graph first introduced in 2006 by Markenzon et al. [33]. The shell gives a way to distinguish k-trees with a particular underlying structure. In particular, two k-tree subclasses known as path-like and tree-like k-trees hold interest. These concepts are introduced in Chapter 2 along with a survey of facts and propositions about k-trees including path-like and tree-like k-trees.

Path-like and tree-like k-trees generalize several commonly studied graph classes including paths, maximal outerplanar graphs, and chordal planar graphs with toughness exceeding one. Thus many results about k-trees may be expanded for tree-like k-trees and the previously listed graph classes. Likewise, results on paths, maximal outerplanar graphs, and chordal planar graphs with toughness exceeding one may generalize to all tree-like k-trees.

A set I of vertices in a graph G is said to be independent if no pair of vertices of I are incident in G. Let $f_s = f_s(G)$ be the number of independent sets of cardinality s of G. Then the polynomial $I(G;x) = \sum_{s\geq 0}^{\alpha(G)} f_s(G)x^s$ is called the independence polynomial (Gutman and Harary [21]), the independent set polynomial (Hoede and Li [26]), or Fibonacci polynomial (Hopkins and Staton [27]) of G. In 1995, Wingard investigated the number of independent sets in trees along with the independence polynomials of trees [44].

In Chapter 3, the works of Wingard are extended by investigating rational roots of the independence polynomials of paths and trees. All rational roots of the independence polynomials of paths are found, and the exact paths whose independence polynomials have these roots are characterized. Additionally trees are characterized that have $-1/q$ as a root of their independence polynomials for $1 \leq q \leq 4$.

Chapter 4 investigates the independence polynomials of k-trees. In 2010, Song, Staton, and Wei generalized select results of Wingard presented in Chapter 3 to k-trees [41], and following their lead a result of Wingard is extended to k-trees in Chapter 4. This result is proven using generalizations of the well known vertex and edge reduction identities for independence polynomials which are also introduced in Chapter 4. Additionally, sharp upper and lower bounds for f_s of maximal outerplanar graphs, i.e. tree-like 2-trees, are shown along with characterizations of the unique maximal outerplanar graphs that obtain these bounds respectively. These results are extensions of the works of Wingard, Song et al., and Alameddine [1].

In 1975, Randić introduced the branching index which later became known as the Randić connectivity index [36]. The Randić connectivity index has been generalized as the general Randić connectivity index and the general zeroth-order Randić connectivity index, where the Zagreb indices appeared as a special case [8]. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of the graph G are given by:

$$M_1(G) = \sum_{u \in V(G)} d(u)^2, \qquad M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

The Zagreb indices M_1 and M_2 have been an active area of research going back to 1972 in the report of Gutman and Trinajstić in computational chemistry [23].

In particular, Das and Gutman in 2004 characterized the Zagreb indices for trees and determined the unique tree that obtains minimum M_1 and M_2 values respectively, as well as maximum M_1 and M_2 values respectively [12, 20]. In Chapter 5, the results of Das and Gutman are generalized to k-trees. That is, the minimum and maximum M_1 and M_2 values for k-trees are determined, and the unique k-trees that obtain these minimum and maximum values respectively are characterized.

In 2011, Hou, Li, Song, and Wei characterized the Zagreb indices for maximal outerplanar graphs and determined the unique maximal outerplanar graph that obtains minimum M_1 and M_2 values respectively, as well as maximum M_1 and M_2 values respectively [29]. In Chapter 6, select works of Hou et al. are extended to all tree-like k-trees. That is, the maximum M_1 value for tree-like k-trees is determined, and the unique tree-like k-tree that obtains this maximum value is characterized. Additionally, a partial result for the maximum M_2 value for tree-like k-trees is determined, and a conjecture for a full result is presented.

Wingard determined that for $s \geq 0$, f_s is minimized among trees by the path, and f_s is maximized by the star. Similarly, Das and Gutman determined that M_i is minimized among trees by the path, and M_i is maximized among trees by the star for $i \in \{1, 2\}$. In Chapter 7, it is shown that for a given tree, it is possible to create a sequence of trees such that f_s (respectively M_1) of a given tree in this sequence is greater than or equal to f_s (respectively M_1) of any previous tree in the sequence for $s \geq 0$.

1.1. **Definitions and Notation.**

The following definitions will be used throughout this dissertation. For definitions not presented here, we refer the reader to Diestel [14].

Definition 1.1. A *graph* G is an ordered pair $G = (V, E)$, where V is a non-empty finite set and E is a collection of unorderd pairs from V. Each element of V is called a vertex and each element of E is called an edge.

Note that in the above definition, graphs are simple and undirected. That is, no edge joins a vertex to itself, no two edges join the same pair of vertices, and edges are not given a direction.

The subgraph $G[S]$ induced by the vertex set $S \subset V(G)$ is the subgraph with vertex set S and edge set $\{uv | u \in S, v \in S, uv \in E(G)\}$. In particular, $G - v$ denotes the induced subgraph $G[V(G) \setminus \{v\}]$ and $G - S$ denotes the induced subgraph $G[V(G) \setminus S]$ for $S \subseteq V(G)$. The graph $G - e$ is the graph resulting from deleting the edge e from G, and $G - F$ is the graph resulting from deleting $F \subseteq E(G)$. Let G and H be two graphs with no common vertex. Then $G \cup H$ is the subgraph with $V(G \cup H) = V(G) \cup V(H)$ and $E(G \cup H) = E(G) \cup E(H)$. Let $u, v \in V(G)$ such that $uv \notin E(G)$. Then $G \cup \{uv\}$ is the graph with vertex set $V(G)$ and edge set $E(G) \cup \{uv\}$. Let $||G||$ denote $|E(G)|$.

Let $v \in V(G)$. Then the neighborhood of v is the set $N(v) = \{u | uv \in E(G)\}$, and $N_H(v)$ denotes the neighborhood of v in the subgraph H. The set $N[v] = \{v\} \cup N(v)$ is called the closed neighborhood of v. The degree of v is defined as $d(v) = |N(v)|$, similarly for a subgraph H, $d_H(v) = |N_H(v)|$. For a graph G, $\delta(G)$ (respectively $\Delta(G)$) denotes the minimum (respectively maximum) degree of G. Let $u, v \in V(G)$. Then $d(u, v)$ is the length of a shortest path connecting u to v in G. Let K_n, P_n, and S_n denote the complete graph, the path, and the star respectively on n vertices, and let K_{n_1,n_2} be the complete bipartite graph on $n_1 + n_2$ vertices.

1.2. k-degenerate Graphs and k-trees.

Definition 1.2. A graph G is called *k-degenerate* if every subgraph H of G is such that $\delta(H) \leq k$.

Note that if G is k-degenerate, then G is $(k+1)$-degenerate. Likewise, if G is k-degenerate and H is any subgraph of G, then H is also k-degenerate. We say that G is maximally k-degenerate if G is k-degenerate and G is not a spanning proper subgraph of any k-degenerate graph.

A vertex in a graph is simplicial if the subgraph induced by its neighborhood is a clique. We say that a vertex v is k-simplicial if $G[N(v)] \cong K_k$. It is commonly known that a chordal graph on at least two vertices contains a simplicial vertex v. Let $G = G_0$, and let $G_i = G_{i-1} - v_i$ for $i \geq 1$. If each v_i is a simplicial vertex in G_{i-1}, then $\{v_1, \ldots, v_n\}$ is a simplicial elimination ordering of the n-vertex graph G. With these defintions, I will define the concept of a k-tree, an idea first introduced by Beineke and Pippet in 1968 [3] and the subject of emphasis for this dissertation.

Definition 1.3. Let T_n^k denote a *k-tree* on n vertices.

(i) The smallest k-tree is the k-clique K_k.

(ii) If T_n^k is a k-tree with n vertices and a new vertex v of degree k is added and joined to the vertices of a k-clique in G, then the larger graph is a k-tree with $n+1$ vertices T_{n+1}^k.

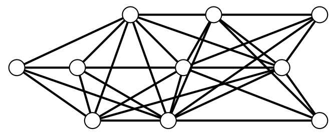


FIGURE 1. A 4-tree on 10 vertices

By the definition of k-trees, it is clear that k-trees are a direct generalization of trees. In fact, trees are k-trees with $k = 1$. The simplicial vertices of a tree are said to be"leaves", and the unique neighbor of a leaf in a tree is said to be the "support vertex" of the leaf.

It was noted by Song in 2010 that k-trees are k-degenerate [40], and clearly k-trees are chordal as well. However, through use of the Principle of Mathematical Induction, we may make a stronger statement.

Theorem 1.4. *Let G be a graph on $n \geq k$ vertices. Then G is a k-tree if and only if G is chordal and maximally k-degenerate.*

Proof. It is clear that if G is a k-tree, then G is chordal and maximally k-degenerate. Suppose that G is chordal and maximally k-degenerate, and suppose that G is smallest such graph that is not a k-tree. Then $n \geq k+2$. As G is a chordal graph on $n \geq 2$, there is a simplicial vertex $v \in V(G)$. Since G is maximally k-degenerate, $\delta(G) = k$. If $|N(v)| \geq k+1$, then $G[N[v]] \cong K_{k+2}$, and so G has a subgraph with $\delta(G) \geq k+1$; a contradiction. Thus $|N(v)| = k$, and $G - v$ is a chordal maximally k-degenerate graph. Hence $G - v$ is a k-tree, and G is formed by attaching a vertex of degree k to a k-clique of $G - v$. Thus G is a k-tree contradicting the assumption that G is not a k-tree. Hence G is a k-tree. $\square$

$d(v_i)$ for the k-path on $k+4 \leq n \leq 2k$ vertices			
i	$1 \leq i \leq n-k-1$	$n-k \leq i \leq k+1$	$k+2 \leq i \leq n$
$d(v_i)$	$k+i-1$	$n-1$	$k+n-i$

$d(v_i)$ for the k-path on $n \geq 2k+1$ vertices			
i	$1 \leq i \leq k$	$k+1 \leq i \leq n-k$	$n-k+1 \leq i \leq n$
$d(v_i)$	$k+i-1$	$2k$	$k+n-i$

TABLE 1. $d(v_i)$ for the k-path on n vertices

Let T_n^k be a k-tree. If $n \geq k+2$, T_n^k has at least two simplicial vertices. If $n = k+1$, then by definition every vertex is k-simplicial. For convention, we say that T_{k+1}^k has one simplicial vertex.

Definition 1.5. Let G_1 be a k-tree, and let S_1 be the set k-simplicial vertices of G_1. For $i \geq 2$, let $G_i = G_{i-1} - S_1(G_{i-1})$. Then S_i denotes the set of k-simplicial vertices of G_i.

Many results throughout this dissertation depend on several particular k-trees. These graphs will now be defined.

Definition 1.6. The *k-path*, P_n^k, has vertex set $\{v_1, \ldots, v_n\}$ where $G[\{v_1, v_2, \ldots, v_k\}] \cong K_k$. For $k+1 \leq i \leq n$, let vertex v_i be adjacent to vertices $\{v_{i-1}, v_{i-2}, \ldots, v_{i-k}\}$.

A helpful characteristic of the k-path P_n^k is that we may order the vertices $v_1, v_2, \ldots, v_n$ such that $P_n^k - \{v_1, \ldots, v_i\}$ is a k-path on $n-i$ vertices for $1 \leq i \leq n-k-1$.

Additionally, the degree of vertex v_i for the k-path may be characterized as follows: for $k+4 \leq n \leq 2k$ and $k \geq 4$, $d(v_i) = \min(k+i-1, n-1, k+n-i)$ and for $n \geq 2k+1$, $d(v_i) = \min(k+i-1, 2k, k+n-i)$. Table 1 shows when these values are reached.

Definition 1.7. The *k-star*, $S_{k,n-k}$, has vertex set $\{v_1, \ldots, v_n\}$ where $G[\{v_1, v_2, \ldots, v_k\}] \cong K_k$ and $N(v_i) = \{v_1, \ldots, v_k\}$ for $k+1 \leq i \leq n$.

Definition 1.8. The *k-spiral*, S_n^k, has vertex set $\{v_1, \ldots, v_n\}$ where $G[\{v_1, v_2, \ldots, v_{k-1}\}] \cong K_{k-1}$, $N(\{v_1, \ldots, v_{k-1}\} \subseteq N(v_i)$ for $k \leq i \leq n$, and $\{v_{i-1}v_i, v_iv_{i+1}\} \subseteq E(S_n^k)$ for $k+1 \leq i \leq n-1$.

Definition 1.9. A *k-diamond* D_n^k has vertex set $V(D_n^k) = \{v_1, v_2, \ldots, v_{k+1}\} \cup \{u_1, \ldots, u_i\}$ for $1 \leq i \leq k+1$ such that $G[\{v_1, v_2, \ldots, v_{k+1}\}] \cong K_{k+1}$ and $N(u_i) = \{v_1, v_2, \ldots, v_{k+1}\} - \{v_i\}$ for all i.

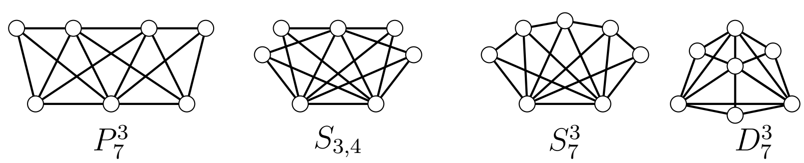


FIGURE 2. The 3-path, 3-star, 3-spiral, and 3-diamond on 7 vertices

2. Tree-like k-trees

A major focal point of the research in this dissertation stems from the ideas that will now be presented. A k-clique in a chordal graph is said to be "bound" if it is contained in more than one $(k+1)$-clique. A k-clique in a k-tree that is not bound is said to be "unbound". The bound and unbound k-cliques of a k-tree help determine the underlying structure of the k-tree, and this structure is referred to as the shell of a k-tree.

Definition 2.1. Let T_n^k be a k-tree. Then *shell* of T_n^k, $Sh(T_n^k)$, is the graph defined as follows:

(i) If X is a $(k+1)$-clique in T_n^k, then X is a vertex in $Sh(T_n^k)$. Hence $V(Sh(T_n^k))$ is the set of $(k+1)$-cliques in T_n^k.

(ii) If X and Y are $(k+1)$-cliques in T_n^k such that $|V(X) \cap V(Y)| = k$, then $XY \in E(Sh(T_n^k))$.

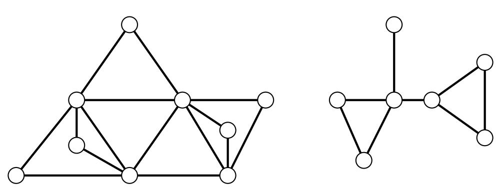


Figure 3. A 2-tree and its shell

We see from this definition that two $(k+1)$-cliques X and Y are adjacent in $Sh(T_n^k)$ if and only if the intersection of X and Y is a bound k-clique.

From the shell of the k-tree, special subclasses of k-trees emerge that may now be defined.

Definition 2.2. The k-tree T_n^k is called *path-like* if $Sh(T_n^k) \cong P_{n-k}$, the path on $n - k$ vertices.

Definition 2.3. The k-tree T_n^k is called *tree-like* if $Sh(T_n^k) \cong T$ where T is a tree.

In 2005, Markenzon, Justel, and Paciornik defined simple-clique k-trees. A k-tree is defined to be a simple-clique k-tree if any bound k-clique is bound by exactly two $(k+1)$-cliques. From this definition, Markenzon et al. introduced the $(k+1)$-line graph for k-trees which is analagous to the shell of the k-tree [33], and they showed that if a k-tree is a simple-clique k-tree, then its $(k+1)$-line graph is a tree. Hence, the simple-clique k-trees of Markenzon et al. are synonymous with tree-like k-trees.

2.1. **Facts and Propositions of k-trees and Tree-like k-trees.**

In this chapter, several facts and propositions about k-trees, in particular path-like and tree-like k-trees, will be noted. These ideas will be used throughout the dissertation and are integral to the study of path-like and tree-like k-trees.

Fact 2.4. *Let T_n^k be a k-tree on n vertices. Then*

(i) $S_1(T_n^k) \neq \emptyset$ *for* $n \geq k+1$,

(i) $S_1(T_n^k)$ *is an independent set for* $n \geq k+2$,

(ii) $S_2(T_n^k) \neq \emptyset$ *for* $n \geq k+3$,

(iii) *every k-clique is contained in a $(k+1)$-clique,*

(iv) T_n^k *is K_{k+2}-free.*

Proposition 2.5. *A k-tree on n vertices has* $\binom{k}{2} + (n-k)k$ *edges.*

Proof. Let $v_1, \ldots, v_n$ be a simplicial elimination ordering, and $G_i = G[\{v_i, \ldots, v_n\}]$ for $1 \leq i \leq n$. Then G_{n-k+1} is a k-clique with $\binom{k}{2}$ edges. For $1 \leq i \leq n-k$, $d_{G_i}(v_i) = k$, and $||G_i|| = ||G_{i+1}|| + k$. Hence $||T_n^k|| = ||G_1|| = ||G_{n-k}|| + (n-k)k = \binom{k}{2} + (n-k)k$. □

In 1992, Fröberg generalized Proposition 2.5 to determine the number of i-cliques in a k-tree for $0 \leq i \leq k$. Define the g-vector of a graph G as $g = (g_1, \ldots, g_{k+1})$ where g_i is the number of i-cliques in G for $1 \leq i \leq k+1$. Fröberg deteremined the g-vector for k-trees.

Theorem 2.6. [16] *For $n \geq 0$, the g-vector of a k-tree on n vertices is as follows:*

$$\left(\binom{k}{1}, \binom{k}{2}, \ldots, \binom{k}{k}, 0\right) + (n-k)\left(\binom{k}{0}, \binom{k}{1}, \ldots, \binom{k}{k-1}, \binom{k}{k}\right).$$

Fröberg also determined that a k-tree may be characterized by the neighborhoods of its vertices.

Theorem 2.7. [16] *Let G be a connected graph on n vertices and $\binom{k}{2} + (n-k)k$ edges. Then G is a k-tree if and only if $G[N(v)]$ is a $(k-1)$-tree for each $v \in V(G)$.*

From Theorem 2.7, if T_n^k is a k-tree with a vertex v such that $d(v) = n-1$, then $T_n^k - v$ is a $(k-1)$-tree. Thus as an extentsion of Theorem 2.7, we state the following theorem.

Theorem 2.8. *Let T_n^k be a k-tree on n vertices and $R = \{v | d(v) = n-1\}$ such that $|R| = r$. Then $T_n^k - R$ is a $(k-r)$-tree.*

Proof. Let $R = \{v_1, \ldots, v_r\}$. Clearly $r \leq k$ as otherwise T_n^k has a K_{k+2} subgraph and is not k-degenerate. From Theorem 2.7, it is clear that $T_n^k - v_1$ is a $(k-1)$-tree and $d_{T_n^k - v_1}(v_i) = n-2$ for $2 \leq i \leq r$. Hence $T_n^k - v_1 - v_2$ is a $(k-2)$-tree. Clearly, $T_n^k - R$ is a $(k-r)$-tree. □

Proposition 2.9. *Let T_n^k be a k-tree, and let X be a k-clique of T_n^k. Then X is bound if and only if X is a cut set.*

Proof. Suppose that X is a bound k-clique, but not a cut set. There are at least two vertices v_1 and v_2 such that $X \subseteq N(v_1) \cap N(v_2)$. If $v_1v_2 \in E(T_n^k)$, then T_n^k has a $(k+2)$-clique. Let P be a shortest v_1, v_2-path in $T_n^k - X$ and v_3 be the vertex on P closest to v_1 such that $N(v_3) \cap X \neq \emptyset$.

If $v_3 = v_2$, then $v_1Pv_3xv_1$ is an induced cycle of length at least four where $x \in V(X)$. If $v_3 \neq v_2$, then $|v_1Pv_3| \geq 3$. Let $x \in N(v_3) \cap X$. Then $v_1Pv_3xv_1$ is an induced cycle of length at least four. This contradicts the fact that T_n^k is chordal. Hence T_n^k is disconnected.

Let X be a cut set, then $T_n^k - X$ has at least two components H_1 and H_2. We may assume that there exists a vertex $v \in V(H_2)$ such that $X \subseteq N(v)$.

Suppose X is not bound, then there is no vertex $u \in V(H_1)$ such that $X \subseteq N(u)$. However T_n^k is k-connected, so there are at least k-edges from H_1 to X. Hence $|H_1| \geq 2$, and there exists $\{u_1', u_2'\} \subseteq V(H_1)$ such that $1 \leq |N(u_1') \cap N(u_2') \cap V(X)| \leq k-2$. That is, there exists $x_1 \in N(u_1') \cap V(X)$ and $x_2 \in N(u_2') \cap V(X)$ such that $x_1 \notin N(u_2'), x_2 \notin N(u_1')$. Of all pairs $\{u_1', u_2'\}$ of $V(H_1)$ meeting these conditions, choose $\{u_1, u_2\}$ such that the smallest u_1, u_2-path P is minimal. Then $x_1u_1Pu_2x_2x_1$ is an induced cycle of at least four. This contradicts the fact that T_n^k is chordal. Thus X is bound. $\square$

In 1974, Rose gave several characterizations of k-trees.

Theorem 2.10. [37] *A graph G is a k-tree if and only if*

(i) *G is connected,*

(ii) *G has a k-clique but no $(k+2)$-clique, and*

(iii) *every minimal x, y separator of G is a k-clique.*

Theorem 2.11. [37] *Let G be a graph on $n \geq k$ vertices such that G has a k-clique but no $(k+2)$-clique and every minimal x, y separator of G is a clique. Then $|E(G)| \leq kn - \binom{k}{2}$ with equality holding if and only if G is a k-tree.*

Theorem 2.12. [37] *A graph G is a k-tree if and only if*

(i) *G is connected,*

(ii) *every minimal x, y separator of G is a k-clique, and*

(iii) $|E(G)| = \binom{k}{2} + (n-k)k$.

Theorem 2.13. [37] *A graph G is a k-tree if and only if*

(i) *G has a k-clique but no $(k+2)$-clique,*

(ii) *every minimal x, y separator of G is a k-clique, and*

(iii) *for all distinct nonadjacent pairs $x, y \in V(G)$, there exists exactly k vertex-disjoint x, y-paths.*

2.2. **Propositions about Path-like and Tree-like k-trees.**

Proposition 2.14. *Let T_n^k be a k-tree, then $Sh(T_n^k)$ is chordal.*

Proof. Suppose $Sh(T_n^k)$ has an induced cycle of length at least four. Then there are at least four $(k+1)$-cliques X_1, X_2, X_3, X_4 such that $|X_i \cap X_{i+1}| = k$ for $1 \leq i \leq 4$ with arithmetic on the indices is modulo 4. Then $X_1 \cap X_2 = Y$ is a bound k-clique and $T_n^k - Y$ is connected; a contradiction. Hence $Sh(T_n^k)$ is chordal. $\square$

Proposition 2.15. *Let T_n^k be a k-tree, then $Sh(T_n^k)$ has $n-k$ vertices.*

Proof. By Theorem 2.6, T_n^k has $n-k$ $(k+1)$-cliques. Hence $Sh(T_n^k)$ has $n-k$ vertices. $\square$

Fact 2.16. *A k-tree T_n^k on $n \geq k_2$ vertices is path-like if and only if $|S_1(T_n^k)| = 2$.*

Fact 2.17. *A k-tree T_n^k with $n \geq k+2$ is path-like if and only if, its vertices may be arranged $v_1, v_2, \ldots, v_n$ so that*

(i) *The vertices $v_1, v_2, \ldots, v_{k+1}$ induce a $(k+1)$-clique.*

(ii) *For each $i \geq k+2$, the vertices $v_1, v_2, \ldots, v_i$ form a path-like k-tree with simplicial vertices v_1 and v_i.*

Such an arrangement of the vertices of a path-like k-tree is called a presentation.

Fact 2.18. *In a presentation of a path-like k-tree T_n^k, $v_i v_{i+1} \in E(T_n^k)$ for each $i < n$. It follows that each path-like k-tree has a spanning path.*

Fact 2.19. *There is a unique tree-like k-tree on n vertices for $k \leq n \leq k+3$.*

Fact 2.20. *A k-tree T_n^k is tree-like if and only if, every bound k-clique is the intersection of exactly two $(k+1)$-cliques.*

Fact 2.20 states that the simple-clique k-trees defined by Markenzon et al. are in fact tree-like k-trees.

Fact 2.21. *Let T_n^k be a tree-like k-tree on $n \geq k+2$ vertices with $v \in S_1(T_n^k)$ and $N(v) = \{u_1, \ldots, u_k\}$. Then $|\cap_{i=1}^k N(u_i)| = 2$.*

Proposition 2.22. *If T_n^k is a tree-like k-tree, then $\Delta(Sh(T_n^k)) \leq k+1$.*

Proof. Suppose that T_n^k is a k-tree such that $\Delta(Sh(T_n^k)) \geq k+2$. Then there is a $(k+1)$-clique X and r $(k+1)$-cliques $X_1, X_2, \ldots, X_r$ such that $r = \Delta(Sh(T_n^k)) \geq k+2$ and $|X \cap X_i| = k$

for all i. Hence there are at least two $(k+1)$-cliques $Y_1, Y_2 \in \{X_1, \ldots, X_r\}$ such that $(Y_1 \cap X) = (Y_2 \cap X)$. Thus $\{XY_1, Y_1Y_2, Y_2X\} \subseteq E(Sh(T_n^k))$, and so $Sh(T_n^k)$ is not a tree. $\square$

Proposition 2.23. *If T_n^k is a tree-like k-tree, then $Sh(T_n^k)$ has $n-k-1$ edges.*

Proof. By Fact 2.15 and the fact that $Sh(T_n^k)$ is a tree, it is clear that $Sh(T_n^k)$ has $n-k-1$ edges. $\square$

Proposition 2.24. *Let T_n^k be a tree-like k-tree, then T_n^k has $nk-(k-1)(k+1)$ k-cliques, where $n-k-1$ are bound and $(k-1)n-(k-2)(k+1)$ are unbound.*

Proof. Let $v_1, \ldots, v_n$ be a simplicial elimination ordering, and $G_i = G[\{v_i, \ldots, v_n\}]$ for $1 \leq i \leq n$. Then G_{n-k-1} is a $(k+1)$-clique with $k+1$ unbound k-cliques and no bound k-cliques. Let $||G||_k$ (respectively $||G||'_k$) be the number of unbound (respectively bound) k-cliques in G for a graph G. For $1 \leq i \leq n-k-1$, $||G_i||_k = ||G_{i-1}||_k + k - 1$ and $||G_i||'_k = ||G_{i-1}||'_k + 1$. Hence $||T_n^k||_k = ||G_n||_k = ||G_{n-k-1}||_k + (n-k-1)(k-1) = (k-1)n-(k-2)(k+1)$, and $||T_n^k||'_k = ||G_n||'_k = ||G_{n-k-1}||'_k + (n-k-1) = n-k-1$. As every k-clique is either bound or unbound there are $n-k-1+(k-1)n-(k-2)(k+1) = nk-(k-1)(k+1)$ k-cliques in T_n^k. $\square$

Theorem 2.25. *Let T_n^k be a tree-like k-tree on $n \equiv (j+1) \mod k$ vertices for $2 \leq j \leq k+1$. Then $|S_1(T_n^k)| \leq \dfrac{k-1}{k}(n-k-1) + \dfrac{j}{k}$.*

Proof. There is a one-to-one correspondence between simplicial vertices of T_n^k and the leaves of the shell of T_n^k, and we will count the number of simplicial vertices by counting the leaves of the shell of T_n^k. Consider $T = Sh(T_n^k)$, and let S_i be the set of vertice of degree i for

$1 \leq i \leq k+1$ and $S = \cup_{i=2}^{k}(S_i)$. Now $|V(T)| = n-k$, and so

$$n - k = |S_1| + |S_{k+1}| + |S|.$$

Suppose $|S| \geq 2$. Then there exists another tree T' such that $|S_1(T)| \leq |S_1(T')|$ and $|S(T')| \leq 1$. We may assume $0 \leq |S(T)| \leq 1$.

If $|S| = 0$ note that T may be formed by starting with $K_{1,k+1}$ and recursively attaching k leaves to a leaf in the previous tree. Thus $n-k \equiv k+2 \equiv 2 \mod k$. Then as $\sum_{v \in V(G)} d(v) = 2|E(G)|$,

$$\begin{aligned}
&|S_1| + (k+1)|S_{k+1}| = 2(n-k-1)\\
&|S_1| + (k+1)(n-k-|S_1|) = 2(n-k-1)\\
&|S_1| = \frac{k-1}{k}(n-k) + \frac{2}{k}\\
&\quad\;\; = \frac{k-1}{k}(n-k) - \frac{k-1}{k} + \frac{k+1}{k}\\
&\quad\;\; = \frac{k-1}{k}(n-k-1) + \frac{j}{k}
\end{aligned}$$

where $j = k+1$.

Suppose $|S| = 1$, and let $v \in S$ such that $d(v) = j$ for some $2 \leq j \leq k$. Then $n-k \equiv k+2+j-1 \equiv (j+1) \mod k$. Then

$$\begin{aligned}
&|S_1| + (k+1)|S_{k+1}| + j|S| = 2(n-k-1)\\
&|S_1| + (k+1)(n-k-|S_1|-1) + j = 2(n-k-1)\\
&|S_1| = \frac{k-1}{k}(n-k) - \frac{k-1-j}{k}\\
&\quad\;\; = \frac{k-1}{k}(n-k-1) + \frac{j}{k}.
\end{aligned}$$

As $|S_1(T)| = |S_1(T_n^k)|$, $|S_1(T_n^k)| \leq \dfrac{k-1}{k}(n-k-1) + \dfrac{j}{k}$ where $n \equiv (j+1) \mod k$ for $2 \leq j \leq k+1$. $\square$

2.3. **Particular Classes of k-trees.**

Fixing k to be $1, 2$ or 3, it becomes clear that tree-like k-trees are particular classes of graphs.

Fact 2.26. *The only tree-like tree (a* 1*-tree) on n vertices is P_n, the path.*

Markenzon et al. verified the following about tree-like k-trees.

Theorem 2.27. [33] *Let G be a graph. Then G is maximal outerplanar if and only if, G is a tree-like* 2*-tree.*

Theorem 2.28. [33] *Let G be a graph with $n > 3$. Then G is a planar* 3*-tree if and only if G is a tree-like* 3*-tree.*

Additionally, for 3-trees, Markenzon et al. found the following.

Theorem 2.29. [33] *Let G be a graph with $n \geq 3$. Then G is a planar* 3*-tree if and only if G is a chordal and maximal planar graph.*

Let $\omega(G)$ denote the number of components of a graph G. A graph G is t-tough if $t \leq \dfrac{|S|}{\omega(G-S)}$ for every subset S of the vertex set $V(G)$ with $\omega(G-S) > 1$. The toughness of G, denoted $\tau(G)$, is the maximum value for t for which G is t-tough.

With adding the condition of toughness exceeding 1 to a tree-like 3-tree, we may restate Theorem 2.29 to all chordal planar graphs.

Theorem 2.30. *Let G be a graph with $\tau(G) > 1$. Then G is chordal planar if and only if, G is a tree-like 3-tree.*

Proof. We need to only show that if G is a chordal planar graph with $\tau(G) > 1$, then G is a tree-like 3-tree. We will proceed by induction on the number of vertices n. If $n = 3$, then $G \cong K_3$. If $n = 4$, then $G \cong K_4$. In both of these cases, G is a tree-like 3-tree.

Suppose that the theorem is true for smaller n, and consider G, a chordal planar graph with $\tau(G) > 1$ on n vertices. Since G is a chordal graph, there is a simplicial vertex v, and since $\tau(G) > 1$ $d(v) = 3$. Let $N(v) = \{u_1, u_2, u_3\}$, and $G[N(v)] = X$ which is a triangle. By induction, $G - v$ is a tree-like 3-tree.

Suppose X is a bound k-clique in $G - v$. Then there are two vertices x_1 and x_2 such that $X \subseteq N(x_i)$ for $i \in \{1, 2\}$. Then G contains a $K_{3,3}$ subgraph with vertex set $\{u_1, u_2, u_3, x_1, x_2, v\}$. Hence G is not planar.

Then X is unbound in $G - v$, and G is a tree-like 3-tree. Thus the theorem holds by the Principle of Mathematical Induction. □

3. Independent Sets of Trees

An independent set in a graph G is a set of pairwise non-adjacent vertices, and the independence number $\alpha(G)$ is the size of a maximum independent set of G. The idea of counting independent sets in graphs was introduced by Prodinger and Tichy in 1982 [35], where they defined, for a graph G, the Fibonacci number $f(G)$ to be the total number of independent sets of G. The Fibonacci number is a parameter of interest to chemists who call it the Merrifield-Simmons index [18, 22, 42]. Let $f_s = f_s(G)$ be the number of independent sets of cardinality s of G. Then the polynomial

$$I(G;x) = \sum_{s\geq 0}^{\alpha(G)} f_s(G)x^s$$

is called the independence polynomial (Gutman and Harary [21]), the independent set polynomial (Hoede and Li [26]), or Fibonacci polynomial (Hopkins and Staton [27]) of G. There are numerous results calculating the Fibonacci number and independence polynomial of classes of graphs [9, 10, 15, 27, 28]. Not only have independence polynomials been related to interesting theoretical problems in graph theory and combinatorics, they have been used in studying statistical physics and combinatorial chemistry. As an example, see [19, 24].

In general, finding the independence polynomial of a graph is a very difficult problem. Most of the literature consists of inequalities and asymptotic results. For more results not given here, we refer to the reader to a thorough survey paper by Levit and Mandrescu [30].

The following propositions are commonly known and are very useful in calculating independence polynomials of graphs.

Proposition 3.1. *Let $G = (V, E)$ with $|V| = n$ and $|E| = m$. Then*

(i) $f_0(G) = 1$,

(ii) $f_1(G) = n$,

(iii) $f_2(G) = \binom{n}{2} - m$,

(iv) $f(G) = f(G - v) + f(G - N[v])$,

(v) $f_s(G) = f_s(G - v) + f_{s-1}(G - N[v])$,

(vi) $I(G; x) = I(G - v; x) + xI(G - N[v]; x)$,

(vii) $f(G) = f(G - e) - f(G - N(e))$,

(viii) $f_s(G) = f_s(G - e) - f_{s-2}(G - N(e))$,

(ix) $I(G; x) = I(G - e; x) - x^2 I(G - N(e); x)$, *and*

(x) *If G is the empty graph, then $I(G; x) = 1$.*

Proposition 3.2. *Let $G, H,$ and J be graphs such that $G = H \cup J$. Then $I(G; x) = I(H; x)I(J; x)$.*

Proposition 3.3. *Let G and H be graphs such that H is a spanning subgraph of G. Then $f_s(G) \leq f_s(H)$.*

3.1. **Results of Wingard.**

In 1995, Wingard researched independence polynomials of trees with an emphasis on roots of the independence polynomial. Among his results are the following:

Theorem 3.4. [44] *Let T be a tree. Then $|I(T; -1)| \leq 1$.*

Lemma 3.5. [44] *If G is a graph with A independent sets of even cardinality and B independent sets of odd cardinality, then $A - B = I(G; -1)$.*

From Lemma 3.5, we see that for a graph G, if $I(G;-1) = 0$, then G has the same number of independent sets of even cardinality as independent sets of odd cardinality. In fact, Wingard characterized exactly when $I(T;-1) = 0$ for trees. First, we state the following definition.

Definition 3.6. If T is a tree, and P is a path in T, then for every vertex v of T, the unique vertex of P of minimal distance from v is called the *nearpoint* of v, denoted $n(v, P)$.

Theorem 3.7. [44] *Let G be a forest. Then $I(G;-1) = 0$ if and only if there is a path $P = \{v_1, \ldots, v_n\}$ in some component T of G where:*

(i) $d(v_1) = d(v_n) = 1$

(ii) $n \equiv 1 \mod 3$

(iii) *for every leaf $v \in V(T) - P$, if $n(v, P) = v_i$ for $i \equiv 1 \mod 3$, then $d(v, v_i) \equiv 0 \mod 3$.*

Wingard also classified which forests have independence polynomials that do not have -1 as a root. The result is determined by a sequence of "reductions". A reduction is carried out by choosing a vertex v which is the neighbor of an end vertex, and removing v and its neighbors from G. The sequence terminates when every remaining component is a star.

Theorem 3.8. [44] *Let G be a forest. If $I(G;-1) \neq 0$ and if k reductions by neighbors of end vertices leaves c components of the type $K_{1,t_i}, t_i \geq 1$ for $1 \leq i \leq c$, then $I(G;-1) = (-1)^{k+c}$.*

3.2. **Roots of Independence Polynomials of Paths and Trees.**

Continuing the work of Wingard, we will now investigate rational roots of independence polynomials of trees. In 1984, Hopkins and Staton gave a characterization of the independence polynomial of the path which is now presented.

Theorem 3.9. [27] *Let P_n be a path on n vertices and $l = \frac{1}{2}(1+\sqrt{1+4x})$. Then $I(P_n;x) = (2l-1)^{-1}(l^{n+2} - (1-l)^{n+2})$.*

From the characterization given by Hopkins and Staton, we may determine all possible rational roots for the independence polynomial of the path. Additionally, we are able to determine which paths have these rational roots for their respective independence polynomials.

Theorem 3.10. *Let P_n be the path on n vertices and c a rational number such that $I(P_n;c) = 0$. Then $c \in \{-1, -\frac{1}{2}, -\frac{1}{3}\}$.*

Proof. The coefficients of $I(P_n;x)$ are all positive. Thus $c < 0$. According to the Rational Root Theorem, $c = -\frac{1}{q}$ for some $q \geq 1$, and so $c \in [-1, 0)$. Let $p = \sqrt{1+4x}$. Then according to Theorem 3.9, $I(P_n;x) = (\frac{1}{p})(\frac{1}{2})^{n+2}((1+p)^{n+2} - (1-p)^{n+2})$.

Thus

$$\begin{aligned}
I(P_n;x) &= \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left((1+p)^{n+2} - (1-p)^{n+2}\right)\\
&= \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left(\sum_{k=0}^{n+2}\binom{n+2}{k}1^{n-k}p^k - \sum_{k=0}^{n+2}\binom{n+2}{k}1^{n-k}(-1)^k p^k\right)\\
&= \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left(\sum_{k=0}^{n+2}\binom{n+2}{k}(p^k - (-1)^k p^k)\right).
\end{aligned}$$

Suppose n is even.

$$\begin{aligned}
I(P_n;x) &= \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left(\sum_{k=0}^{n+2}\binom{n+2}{k}(p^k - (-1)^k p^k)\right)\\
&= \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left((n+2)(2p) + \binom{n+2}{3}(2p^3) + \ldots + \binom{n+2}{n+1}(2p^{n+1})\right)\\
&= \left(\frac{1}{2}\right)^{n+1}\left((n+2) + \binom{n+2}{3}p^2 + \ldots + \binom{n+2}{n+1}p^n\right)
\end{aligned}$$

$$(1) \qquad = \left(\frac{1}{2}\right)^{n+1}\left((n+2) + \binom{n+2}{3}(1+4x) + \ldots + \binom{n+2}{n+1}(1+4x)^{\frac{n}{2}}\right).$$

Suppose n is odd.

$$\begin{aligned}
I(P_n;x) &= \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left(\sum_{k=0}^{n+2}\binom{n+2}{k}(p^k-(-1)^k p^k)\right)\\
&= \left(\frac{1}{p}\right)\left(\frac{1}{2}\right)^{n+2}\left((n+2)(2p)+\binom{n+2}{3}(2p^3)+\ldots+\binom{n+2}{n}(2p^n)+(2p^{n+2})\right)\\
&= \left(\frac{1}{2}\right)^{n+1}\left((n+2)+\binom{n+2}{3}p^2+\ldots+\binom{n+2}{n}p^{n-1}+p^{n+1}\right)
\end{aligned}$$

$$(2) \qquad = \left(\frac{1}{2}\right)^{n+1}\left((n+2)+\binom{n+2}{3}(1+4x)+\ldots+(1+4x)^{\frac{n+1}{2}}\right).$$

For expressions (1) and (2) to be equal to zero, there must be summands of (1) and (2) that are negative. Clearly, this is only possible if $1+4x<0$. Hence $x<-\frac{1}{4}$. Now $I(P_2;x)=1+2x$ and $I(P_4;x)=(1+x)(1+3x)$. Hence $I(P_4;-1)=0$, $I(P_2;-\frac{1}{2})=0$, and $I(P_4;-\frac{1}{3})=0$. Thus if c is a rational root of the independence polynomial of P_n, then $c\in\{-1,-\frac{1}{2},-\frac{1}{3}\}$. □

As the previous theorem states, we have found that the only possible rational roots for independence polynomials of paths are -1, $-\frac{1}{2}$, and $-\frac{1}{3}$. Now, we will demonstrate which paths have these roots for their respective independence polynomials.

Theorem 3.11. *Let P_n be the path on n vertices. Then*

$$I(P_n;-1)=\begin{cases} -1 & \textit{if } n\equiv\{2,3\} \mod 6\\ 0 & \textit{if } n\equiv 1 \mod 3\\ 1 & \textit{if } n\equiv\{5,0\} \mod 6\end{cases}.$$

Proof. Proceed by induction on n. Suppose $1\le n\le 6$. Then by Table 3.2, the theorem holds.

T	$I(T;x)$	$I(T;-1)$	$I(T;-\frac{1}{2})$	$I(T;-\frac{1}{3})$
P_1	$1+x$	0	1/2	2/3
P_2	$1+2x$	-1	0	1/3
P_3	$1+3x+x^2$	-1	$-1/4$	1/9
P_4	$1+4x+3x^2$	0	$-1/4$	0
P_5	$1+5x+6x^2+x^3$	1	$-1/8$	$-1/27$
P_6	$1+6x+10x^2+4x^3$	1	0	$-1/27$
P_7	$1+7x+15x^2+10x^3+4x^4$	0	1/16	$-2/81$
P_8	$1+8x+21x^2+20x^3+5x^4$	-1	1/16	$-1/81$
P_9	$1+9x+28x^2+35x^3+15x^4+x^5$	-1	1/32	$-1/243$
P_{10}	$1+10x+36x^2+56x^3+21x^4+6x^5$	0	0	0

TABLE 2. Independence Polynomials of P_n for $n \leq 10$

Suppose the theorem is true for paths on $1 \leq n' < n$ vertices, and consider P_n and let v be a leaf such that $N(v) = u$. Then by Proposition 3.1,

$$I(P_n;x) = I(P_n - u;x) + xI(P_n - N[u];x)$$

Now $P_n - u$ has two components P_1 and P_{n-2}. Hence

$$I(P_n;x) = I(P_1;x)I(P_{n-2};x) + xI(P_{n-3};x),$$

and thus $I(P_n;-1) = 0 + (-1)I(P_{n-3};-1) = -I(P_{n-3};-1)$.

Suppose $n \equiv 1 \mod 3$. Then $n-3 \equiv 1 \mod 3$, and so by induction $I(P_n;-1) = -I(P_{n-3};-1) = 0$.

Suppose $n \equiv 2(\text{or } 3) \mod 6$. Then $n-3 \equiv 5(\text{or } 0) \mod 6$. Thus by induction $I(P_n; -1) = -I(P_{n-3}; -1) = -(1) = -1$.

Suppose $n \equiv 5(\text{or } 0) \mod 6$. Then $n-3 \equiv 2(\text{or } 3) \mod 6$. Thus by induction $I(P_n; -1) = -I(P_{n-3}; -1) = -(-1) = 1$.

Hence by the Principle of Mathematical Induction, the theorem holds for all n. $\square$

Theorem 3.12. *Let P_n be the path on n vertices. Then*

$$I(P_n; -\frac{1}{2}) = \begin{cases} (-1)(\frac{1}{2}^{\lceil \frac{n}{2} \rceil}) & \textit{if } n \equiv \{3,4,5\} \mod 8 \\ 0 & \textit{if } n \equiv 2 \mod 4 \\ \frac{1}{2}^{\lceil \frac{n}{2} \rceil} & \textit{if } n \equiv \{7,0,1\} \mod 8 \end{cases}.$$

Proof. Proceed by induction on n. Suppose $1 \leq n \leq 8$. Then by Table 3.2, the theorem holds.

Suppose the theorem is true for paths on $1 \leq n' < n$ vertices, and consider P_n and let v be a leaf and $u \in V(T)$ such that $d(v, u) = 2$. Then by Proposition 3.1,

$$I(P_n; x) = I(P_n - u; x) + xI(P_n - N[u]; x)$$

Now $P_n - u$ has two components P_2 and P_{n-3}, and $P_n - N[u]$ has two components P_1 and P_{n-4}. Hence

$$I(P_n; x) = I(P_2; x)I(P_{n-3}; x) + xI(P_1; x)I(P_{n-4}; x),$$

and thus $I(P_n; -\frac{1}{2}) = 0 + -\frac{1}{2}(\frac{1}{2})I(P_{n-4}; -\frac{1}{2}) = -\frac{1}{4}I(P_{n-4}; -\frac{1}{2})$.

Suppose $n \equiv 2 \mod 4$. Then $n - 4 \equiv 2 \mod 4$, and so by induction $I(P_n; -\frac{1}{2}) = -\frac{1}{4}I(P_{n-4}; -\frac{1}{2}) = 0$.

Suppose $n \equiv 3(\text{or } 4,5) \mod 8$. Then $n-4 \equiv 7(\text{or } 0,1) \mod 8$. Thus by induction $I(P_n;-\frac{1}{2}) = -\frac{1}{4}I(P_{n-4};-\frac{1}{2}) = -\frac{1}{4}(\frac{1}{2})^{\lceil\frac{n-4}{2}\rceil} = -\frac{1}{2}^{\lceil\frac{n-4}{2}\rceil+2} = -\frac{1}{2}^{\lceil\frac{n}{2}\rceil}$.

Suppose $n \equiv 7(\text{or } 0,1) \mod 8$. Then $n-4 \equiv 3(\text{or } 4,5) \mod 6$. Thus by induction $I(P_n;-\frac{1}{2}) = -\frac{1}{4}I(P_{n-4};-\frac{1}{2}) = (-\frac{1}{4})(-1)(\frac{1}{2})^{\lceil\frac{n-4}{2}\rceil} = \frac{1}{2}^{\lceil\frac{n-4}{2}\rceil+2} = -\frac{1}{2}^{\lceil\frac{n}{2}\rceil}$.

Hence by the Principle of Mathematical Induction, the theorem holds for all n. □

Theorem 3.13. *Let P_n be the path on n vertices. Then*

$$I(P_n;-\frac{1}{3}) = \begin{cases} (-1)(\frac{2}{3}^{\lceil\frac{n}{2}\rceil}) & \textit{if } n \equiv 7 \mod 12 \\ (-1)(\frac{1}{3}^{\lceil\frac{n}{2}\rceil}) & \textit{if } n \equiv \{5,6,8,9\} \mod 12 \\ 0 & \textit{if } n \equiv 4 \mod 6 \\ \frac{1}{3}^{\lceil\frac{n}{2}\rceil} & \textit{if } n \equiv \{11,0,2,3\} \mod 12 \\ \frac{2}{3}^{\lceil\frac{n}{2}\rceil} & \textit{if } n \equiv 1 \mod 12 \end{cases}.$$

Proof. Proceed by induction on n. Suppose $1 \leq n \leq 10$. Then by Table 3.2, the theorem holds. Now $I(P_{11}) = 1 + 11x + 45x^2 + 84x^3 + 70x^4 + 21x^5 + x^6$, and $I(P_{12};x) = 1 + 12x + 55x^2 + 120x^3 + 126x^4 + 56x^5 + 7x^6$. Thus $I(P_i;-\frac{1}{3}) = 1/729$ for $11 \leq i \leq 12$, and so the theorem holds.

Suppose the theorem is true for paths on $1 \leq n' < n$ vertices, and consider P_n and let v be a leaf and $u \in V(T)$ such that $d(v,u) = 4$. Then by Proposition 3.1,

$$I(P_n;x) = I(P_n - u;x) + xI(P_n - N[u];x)$$

Now $P_n - u$ has two components P_4 and P_{n-5}, and $P_n - N[u]$ has two components P_3 and P_{n-6}. Hence

$$I(P_n;x) = I(P_4;x)I(P_{n-5};x) + xI(P_3;x)I(P_{n-6};x),$$

and thus $I(P_n;-\frac{1}{3}) = 0 + -\frac{1}{3}(\frac{1}{9})I(P_{n-6};-\frac{1}{3}) = -\frac{1}{27}I(P_{n-6};-\frac{1}{3})$.

Suppose $n \equiv 1 \mod 12$. Then $n-6 \equiv 7 \mod 12$, and so by induction $I(P_n;-\frac{1}{3}) = -\frac{1}{27}I(P_{n-6};-\frac{1}{3}) = (-\frac{1}{27})(-1)\frac{2}{3}^{\lceil \frac{n-6}{2} \rceil} = \frac{2}{3}^{\lceil \frac{n-6}{2} \rceil + 3} = \frac{2}{3}^{\lceil \frac{n}{2} \rceil}$.

Suppose $n \equiv 7 \mod 12$. Then $n-6 \equiv 1 \mod 12$, and so by induction $I(P_n;-\frac{1}{3}) = -\frac{1}{27}I(P_{n-6};-\frac{1}{3}) = (-\frac{1}{27})\frac{2}{3}^{\lceil \frac{n-6}{2} \rceil} = -(\frac{2}{3}^{\lceil \frac{n-6}{2} \rceil + 3}) = -(\frac{2}{3}^{\lceil \frac{n}{2} \rceil})$.

Suppose $n \equiv 5(\text{or } 6,8,9) \mod 12$. Then $n-6 \equiv 11(\text{or } 0,2,3) \mod 12$. Thus by induction $I(P_n;-\frac{1}{3}) = -\frac{1}{27}I(P_{n-6};-\frac{1}{3}) = -\frac{1}{27}\frac{1}{3}^{\lceil \frac{n-6}{2} \rceil} = -(\frac{1}{3}^{\lceil \frac{n-6}{2} \rceil + 3}) = -(\frac{1}{3}^{\lceil \frac{n}{2} \rceil})$.

Suppose $n \equiv 11(\text{or } 0,2,3) \mod 12$. Then $n-6 \equiv 5(\text{or } 6,8,9) \mod 12$. Thus by induction $I(P_n;-\frac{1}{3}) = -\frac{1}{27}I(P_{n-6};-\frac{1}{3}) = (-\frac{1}{27})(-1)\frac{1}{3}^{\lceil \frac{n-6}{2} \rceil} = \frac{1}{3}^{\lceil \frac{n-6}{2} \rceil + 3} = \frac{1}{3}^{\lceil \frac{n}{2} \rceil}$.

Suppose $n \equiv 4 \mod 6$. Then $n-6 \equiv 4 \mod 6$, and so by induction $I(P_n;-\frac{1}{3}) = -\frac{1}{27}I(P_{n-6};-\frac{1}{3}) = 0$.

Hence by the Principle of Mathematical Induction, the theorem holds for all n. $\square$

Let $\mathcal{A}_{-1}$ be the family of trees defined as follows:

(i) $P_1 \in \mathcal{A}_{-1}$.

(ii) Let T', T_1, T_2 be trees such that $uv \in E(T')$, $T_1, T_2 \in \mathcal{A}_{-1}$, and $v_i \in V(T_i)$ for $i \in \{1,2\}$, and let T be a tree with $V(T) = V(T') \cup V(T_1) \cup V(T_2)$ and $E(T) = E(T') \cup E(T_1) \cup E(T_2) \cup \{v_1u\} \cup \{vv_2\}$. Then $T \in \mathcal{A}_{-1}$.

In Theorem 3.7, Wingard gave a necessary induced subgraph to guarantee that the independence polynomial of a forest has -1 as a root. With the defintion of $\mathcal{A}_{-1}$, Theorem 3.7 may be restated as follows.

Theorem 3.14. *Let F be a forrest. Then $I(F;-1) = 0$ if and only if F has a component T such that $T \in \mathcal{A}_{-1}$.*

Proof. Let F be a forrest with component T such that $T \in \mathcal{A}_{-1}$ with $V(T)$ and $E(T)$ defined as above. By Proposition 3.2, $I(F;x) = I(H;x)I(T;x)$ where $H \cong F - V(T)$. By Proposition 3.1 $I(T;-1) = I(T-v;-1) - I(T-N[v];-1)$. Now $T-v$ has T_2 as a component, and $T-N[v]$ has T_1 as a component. As $T_1, T_2 \in \mathcal{A}_{-1}$, $I(T;-1) = 0$, and thus $I(F;-1) = 0$.

Suppose that $I(F;-1) = 0$. By induction, we will show that F has a component T such that $T \in \mathcal{A}_{-1}$. If $n \in \{1,2,3,4\}$, then it is routine to check that F has a component $T \in \{P_1, P_4\}$, and thus $T \in \mathcal{A}_{-1}$. Suppose that if $I(F;-1) = 0$ for a forest F on $1 \leq n' < n$ vertices, then F has a component T such that $T \in \mathcal{A}_{-1}$, and let F be a tree on n vertices such that $I(F;-1) = 0$.

Let x be a vertex of degree 1 such that $N(x) = y$. Once again, by Proposition 3.1 $I(F;-1) = I(F-u;-1) - I(F-N[u];-1)$. Now $F-u$ has P_1 as a component, and thus $I(F-u;-1) = 0$. Hence $I(F-N[v];-1) = 0$. By induction, $F-N[v]$ has a component T such that $T \in \mathcal{A}_{-1}$. If T is a component of F, then the theorem is verified. Suppose then that T is not a component of F. Then there is a vertex $z \in V(T)$ such that $zw \in E(F)$ for some $w \in N(y)$. Hence there is a component of F, T'', such that $V(T'') = V(T') \cup V(T_1) \cup V(T_2)$ and $E(T'') = E(T') \cup E(T_1) \cup E(T_2) \cup \{xy\} \cup \{wz\}$ where $T_1 \cong P_1$, $T_2 \cong T$, and $T' \cong G[N[u]-v]$. By definition, $T'' \in \mathcal{A}_{-1}$. By the Principle of Mathematical Induction, the theorem is verified. $\square$

In a similar manner, other rational roots for independence polynomials of trees may be found.

Let $\mathcal{A}_c$ be the family of trees defined as follows:

(i) Let T_c be a smallest tree such that $I(T_c;c) = 0$. Then $T_c \in \mathcal{A}_c$.

(ii) Let T', T_1, T_2 be trees such that $uv \in E(T')$, $T_1, T_2 \in \mathcal{A}_c$, and $v_i \in V(T_i)$ for $i \in \{1, 2\}$, and let T be a tree with $V(T) = V(T') \cup V(T_1) \cup V(T_2)$ and $E(T) = E(T') \cup E(T_1) \cup E(T_2) \cup \{v_1 u\} \cup \{v v_2\}$. Then $T \in \mathcal{A}_c$.

Theorem 3.15. *Let $T \in \mathcal{A}_c$. Then $I(T; c) = 0$.*

Proof. Proceed by induction on $|V(T)|$. The smallest tree in $\mathcal{A}_c$ is T_c. In this case, the theorem holds. Let $T \in \mathcal{A}_c$, and suppose that for trees in $\mathcal{A}_c$ on fewer than $|V(T)|$ the theorem holds. As $T \in \mathcal{A}_c$, there exists trees T', T_1, and T_2 such that $V(T) = V(T') \cup V(T_1) \cup V(T_2)$, and $E(T) = E(T') \cup E(T_1) \cup E(T_2) \cup \{v_1 u\} \cup \{v v_2\}$ where $uv \in E(T')$, $T_1, T_2 \in \mathcal{A}_c$, and $v_i \in V(T_i)$ for $i \in \{1, 2\}$. By Proposition 3.1

$$I(T; x) = I(T - v; x) + xI(T - N[v]; x).$$

Now $T - v$ has T_2 as a component, and $T - N[v]$ has T_1 as a component. By induction, as $T_1, T_2 \in \mathcal{A}_c$, $I(T - v; c) = I(T - N[v]; c) = 0$. Thus $I(T; c) = I(T - v; c) + cI(T - N[v]; c) = 0$. Thus by the Principle of Mathematical Induction, if $T \in \mathcal{A}_c$, then $I(T; c) = 0$. □

By Theorem 3.15, to classify a family of trees whose independence polynomials have c as a root one simply has to find a minimal example of a tree that has c as a root of its independence polynomial. In this sense, $\mathcal{A}_c$ is characterized by T_c. Thus by Theorem 3.12 and Theorem 3.13, $\mathcal{A}_{-\frac{1}{2}}$ and $\mathcal{A}_{-\frac{1}{3}}$ quickly follow.

Theorem 3.16. *Let T_c be a smallest tree such that $I(T_c; c) = 0$. Then*

(i) $T_{-\frac{1}{2}} \cong P_2$,

(ii) $T_{-\frac{1}{3}} \cong P_4$.

As stated, the only rational roots for the independence polynomials of the path are -1, $-\frac{1}{2}$, and $-\frac{1}{3}$. The smallest examples of a tree in the families of trees $\mathcal{A}_{-1}$, $\mathcal{A}_{-\frac{1}{2}}$, and $\mathcal{A}_{-\frac{1}{3}}$ are all paths. It may seem that -1, $-\frac{1}{2}$, and $-\frac{1}{3}$ are the only possible rational roots for independence polynomials of trees. However, that is not the case as we will now demonstrate. Let $T_{7,7}$ be the tree in Figure 4.

Theorem 3.17. *Let T_c be a smallest tree such that $I(T_c; c) = 0$. Then $T_{-\frac{1}{4}} \cong T_{7,7}$.*

Proof. The independence polynomial of $T_{7,7}$ is $I(T_{7,7}; x) = (1 + 2x)^2(1 + x)^2 + x(1 + x)^2 = (1+x)^2(1+5x+4x^2) = (1+x)^3(1+4x)$. Hence $I(T_{7,7}; -\frac{1}{4}) = 0$. It is easy to verify that for a tree $T \ncong T_{7,7}$ on $1 \leq n \leq 7$ vertices that $I(T; -\frac{1}{4}) \neq 0$. □

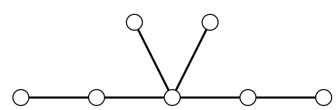


FIGURE 4. $T_{7,7}$

The question of what possible rational roots exist for the independence polynomials of paths is closed. However, Theorem 3.17 raises the question as to what are the possible rational roots of independence polynomials of trees. If $T_{-\frac{1}{q}}$ exists for $q \geq 5$ exists, it would be interesting to determine such trees.

Additionally, $I(T; -1) = 0$ if and only if $T \in \mathcal{A}_{-1}$. It would be interesting to determine whether or not “if and only if” statements can be made for $\mathcal{A}_{-\frac{1}{2}}$, $\mathcal{A}_{-\frac{1}{3}}$, and $\mathcal{A}_{-\frac{1}{4}}$ as well.

4. Independent Sets of k-trees and tree-like k-trees

As discussed in Chapter 3, Wingard determined numerous results in regards to independence polynomials of trees. It is then a natural train of thought to generalize the results of Wingard to independence polynomials of k-trees.

4.1. The Results of Song et al.

In 2010, Song, Staton, and Wei characterized independence polynomials for certain classes of k-trees and k-tree related graphs. Among their results, they found the following.

Theorem 4.1. [41] *For the k-path P_n^k, the following are true:*

(i) $\alpha(P_n^k) = \lfloor \dfrac{n+1}{k+1} \rfloor$;

(ii) *If* $1 \leq s \leq \alpha(P_n^k)$, *then* $f_s(P_n^k) = f_s(P_{n-1}^k) + f_{s-1}(P_{n-k-1}^k)$;

(iii) *If* $0 \leq s \leq \alpha(P_n^k)$, *then* $f_s(P_n^k) = \binom{n-k(s-1)}{s}$;

(iv) $I(P_n^k; x) = \sum_{s=0}^{\alpha(P_n^k)} \binom{n-k(s-1)}{s} x^s$.

Theorem 4.2. [41] *For the k-star $S_{k,n-k}$, the following are true:*

(i) $\alpha(S_{k,n-k}) = n - k$;

(ii) $f_s(S_{k,n-k}) = \binom{n-k}{s}, s \geq 2$;

(iii) $I(S_{k,n-k}; x) = kx + (1+x)^{n-k}$.

Theorem 4.3. [40] *For the k-spiral S_n^k, the following are true:*

(i) $\alpha(S_n^k) = \lfloor \frac{n-k+2}{2} \rfloor$;

(ii) $f_s(S_n^k) = \binom{n+2-k-s}{s}, s \geq 2$;

(iii) $I(S_n^k;x) = 1 + nx + \sum_{s=2}^{\lfloor \frac{n-k+2}{2} \rfloor} \binom{n+2-k-s}{s} x^s$.

Theorem 4.4. [41] *Let G be a k-degenerate graph on n vertices. For $2 \leq s \leq \alpha(G)$, the following are true:*

(i) $\binom{n-k(s-1)}{s} \leq f_s(G)$;

(ii) $f_s(G) \leq \binom{n-k}{s}$ *if G is maximum k-degenerate.*

Theorem 4.5. [41] *If $I(G;x) = I(S_{k,n-k};x)$ for a graph G of order $n \geq k+1$, then $G \cong S_{k,n-k}$.*

Theorem 4.6. [41] *If T_n^k is a k-tree with $I(T_n^k;x) = I(P_n^k;x)$ and $\alpha(T_n^k) \geq 3$, then $T_n^k \cong P_n^k$.*

As an extension of the previous results of Song et al., it is not difficult to obtain a similar result for the k-diamond.

Theorem 4.7. *Let D_n^k be the k-diamond on n vertices. Then*

(i) $\alpha(D_n^k) = n - k - 1$;

(ii) $f_2(D_n^k) = \binom{n-k-1}{2} + (2k + 2 - n)$;

(iii) $f_s(D_n^k) = \binom{n-k-1}{s}$ *for* $s \geq 3$;

(iv) $I(D_n^k;x) = 1 + nx + (\binom{n-k-1}{2} + (2k+2-n))x^2 + \sum_{s=3}^{n-k-1} \binom{n-k-1}{s} x^s$.

Along with the results of Wingard listed in Chapter 3, Wingard determined, in what we refer to as Wingard's bound, sharp bounds of the function values of independence polynomials of trees obtained at $x = -1$, which is now presented.

Theorem 4.8 (Wingard's Bound)**.** [44] *Let T be a tree. Then $|I(T;-1)| \leq 1$.*

We seek to generalize Wingard's Bound to k-degenerate graphs and thus k-trees. We will give Lemma 4.10 which generalizes Proposition 3.1(vi) to vertex sets. This formula may be useful for the study of independence polynomials. As an application, we use Lemma 4.10 to give Theorem 4.11 which generalizes Wingard's Bound to the k-path. In Section 4.3, we give Lemma 4.13 which generalizes Proposition 3.1(ix) to edge sets. Through use of Lemma 4.13 we give Theorem 4.14 which generalizes Wingard's Bound to all k-degenerate graphs. Though the result of Theorem 4.14 covers the result of Theorem 4.11, both approaches are useful.

4.2. **Wingard's Bound for the k-path and k-star.**

As mentioned, Song et al. demonstrated that, for a k-degenerate graph G, the lower bound for $f_s(G)$ is obtained uniquely for the class of k-trees by the k-path, and the upper bound for $f_s(G)$ is obtained uniquely for maximal k-degenerate graphs by the k-star [41]. In this sense, the k-path and k-star are extremal cases for the number of independent sets among k-trees.

We will now generalize Wingard's Bound to the k-path and k-star. First, we introduce some lemmas.

Lemma 4.9. *Let K_n be the clique on n vertices. Then $|I(K_n; -\frac{1}{k})| < 1$ for $n \leq k$.*

Proof. $I(K_n; x) = 1 + nx$. Hence $I(K_n; -\frac{1}{k}) = 1 - \frac{n}{k} < 1$. □

Lemma 4.10. *Let G be a graph on n vertices, and let j be an integer, $1 \leq j \leq n$. Let $S_j = \{u_1, \ldots, u_j\} \subseteq V(G)$, and define $S_i = \{u_1, \ldots, u_i\}$ and $G_i = G - S_i$ for $1 \leq i \leq j$.*

Then

$$I(G;x) = I(G_j;x) + xI(G_{j-1} - N[u_j];x) + xI(G_{j-2} - N[u_{j-1}];x)$$
$$+ \ldots + xI(G_1 - N[u_2];x) + xI(G - N[u_1];x).$$

Proof. We will proceed by induction on $|S_j|$. If $j = 1$, then by Proposition 3.1(vi) $I(G;x) = I(G - v_1;x) + xI(G - N[v_1];x)$. Suppose the statement is true for vertex sets with cardinality less than j. Then by induction,

$$I(G;x) = I(G_{j-1};x) + xI(G_{j-2} - N[v_{j-1}];x) + \ldots + xI(G - N[v_1];x).$$

By Proposition 3.1(vi) $I(G_{j-1};x) = I(G_{j-1} - u_j;x) + xI(G_{j-1} - N_{G_{j-1}}[u_j];x)$, and note that $N_{G_{j-1}}[u_j] \subseteq N[u_j]$. Hence $G_{j-1} - N_{G_{j-1}}[u_j] \cong G_{j-1} - N_G[u_j]$. Thus,

$$\begin{aligned} I(G;x) =& I(G_{j-1};x) + xI(G_{j-2} - N[v_{j-1}];x) \\ & + xI(G_{j-3} - N[v_{j-2}];x) + \ldots + xI(G - N[v_1];x) \\ =& I(G_{j-1} - u_j;x) + xI(G_{j-1} - N[u_j];x) + xI(G_{j-2} - N[u_{j-1}];x) \\ & + \ldots + xI(G_1 - N[v_2];x) + xI(G - N[v_1];x) \\ =& I(G_j;x) + xI(G_{j-1} - N[v_j];x) + xI(G_{j-2} - N[v_{j-1}];x) \\ & + \ldots + xI(G_1 - N[v_2];x) + xI(G - N[v_1];x). \end{aligned}$$

Hence, the lemma is true for vertex sets of cardinality j for $1 \leq j \leq n$. $\square$

As an application of Lemma 4.10, we will now generalize Wingard's Bound to the k-path.

Theorem 4.11. *Let P_n^k be the k-path on $n \geq k$ vertices and $k \geq 2$. Then*

$$|I(P_n^k; -\frac{1}{k})| < 1.$$

Proof. We will proceed by induction on n. If $n = k$, P_n^k is a k-clique, and the theorem is true by Lemma 4.9. If $n = k+1$, then $P_n^k \cong K_{k+1}$. Thus $I(P_n^k; -\frac{1}{k}) = 1 - (k+1)/k = 1 - 1 - \frac{1}{k}$.

Suppose the theorem is true for k-paths with less than $n \geq k+2$ vertices, and consider P_n^k on n vertices with $v_1, v_2, \ldots, v_n$ ordered according to a presentation. Let $u_i = v_{k+i}$ for $1 \leq i \leq r$ where $r = \min(k, n-k)$, and define $U_i = \{u_1, \ldots, u_i\}$, $G_i = P_n^k - U_i$ for $1 \leq i \leq r$, and $G_0 = P_n^k$. Then by Proposition 3.1(x) and Lemma 4.10,

$$\begin{aligned} I(P_n^k; x) =& I(G_r; x) + xI(G_{r-1} - N[u_r]; x) + xI(G_{r-2} - N[u_{r-1}]; x) \\ & + \ldots + xI(G_1 - N[u_2]; x) + xI(P_n^k - N[u_1]; x), \end{aligned} \tag{3}$$

and this summation has $r+1$ summands on the right hand side.

According to the definition of a k-path $G[U_r] \cong K_r$, and hence $U_r \subseteq N[u_i]$ for $1 \leq i \leq r$. Now $G[\{v_1, \ldots, v_k\}] \cong K_k$ is a component of G_r, and so $I(G_r; -\frac{1}{k}) = 0$. Also, by the structure of the k-path, each graph $G_i - N[u_{i+1}]$ for $0 \leq i \leq r-1$ has at most two components J_i and H_i such that $V(J_i) \subseteq \{v_1, \ldots, v_k\}$ and $V(H_i) \subseteq V(P_n^k) - (\{v_1, \ldots, v_k\} \cup U_r)$. Hence each component J_i and H_i are congruent to either cliques of smaller size than k, the empty graph, or a k-path on fewer than n vertices. Hence by Lemma 4.9, Proposition 3.1, and induction, $|I(G_i - N[u_{i+1}]; -\frac{1}{k})| \leq 1$ for $0 \leq i \leq r-1$ with equality holding if and only if $G_i - N[u_{i+1}]$ is the empty graph. However, if $G_i - N[u_{i+1}]$ is the empty graph for all i, $0 \leq i \leq r-1$, then $\{v_1, \ldots, v_k\} \subseteq N[u_{i+1}]$ for $0 \leq i \leq r-1$. Then $r = 1$, as $u_1 \notin N[u_2]$. However, $r \geq 2$, so there is a j such that $G_j - N[u_{j+1}]$ is not the empty graph. Thus

$$\begin{aligned} |I(P_n^k; -\frac{1}{k})| \leq & |I(G_r; -\frac{1}{k})| + |\frac{1}{k}||I(G_{r-1} - N[u_r]; -\frac{1}{k})| \\ & + \ldots |\frac{1}{k}||I(G_1 - N[u_2]; -\frac{1}{k})| + |\frac{1}{k}||I(G - N[u_1]; -\frac{1}{k})| \\ < & 0 + r(\frac{1}{k}) \leq 1. \end{aligned}$$

Therefore, by the Principle of Mathematical Induction, $|I(P_n^k; -\frac{1}{k})| < 1$. $\square$

Using Song's characterization of the independence polynomial of the k-star given in Theorem 4.2, we can easily verify the following theorem.

Theorem 4.12. *Let $S_{k,n-k}$ be the k-star on n vertices and $k \geq 2$. Then*

$$|I(S_{k,n-k}; -\frac{1}{k})| < 1.$$

4.3. **Wingard's Bound for k-degenerate Graphs.**

In this section, we seek to generalize Theorem 4.11 and Theorem 4.12 by investigating Wingard's Bound to k-degenerate graphs. Though Lemma 4.10 is not sufficient to do so, we will introduce Lemma 4.13, a generalization of Proposition 3.1(ix) that will be useful to generalize Wingard's Bound to k-degenerate graphs.

Lemma 4.13. *Let G be a graph with $v \in V(G)$ where $\{u_1, \ldots, u_r\} \subseteq N(v)$ for some $1 \leq r \leq d(v)$. Let $e_i = vu_i$, $E_i = \{e_1, \ldots, e_i\}$, and $G_i' = G - E_i$ for $1 \leq i \leq r$. Then*

$$\begin{aligned} I(G;x) =& I(G_r';x) - x^2 I(G_{r-1}' - N_{G_{r-1}'}(e_r);x) - x^2 I(G_{r-2}' - N_{G_{r-2}'}(e_{r-1});x) \\ & - \ldots - x^2 I(G_1' - N_{G_1'}(e_2);x) - x^2 I(G - N_G(e_1);x). \end{aligned}$$

Proof. We will proceed by induction on r. If $r = 1$, by Proposition 3.1 we have the identity $I(G;x) = I(G - e_1;x) - x^2 I(G - N(e_1);x)$. Suppose the statement is true for $1 \leq r' < r$. By induction,

$$\begin{aligned} I(G;x) =& I(G_{r-1}';x) - x^2 I(G_{r-2}' - N_{G_{r-2}'}(e_{r-1});x) \\ & - x^2 I(G_{r-3}' - N_{G_{r-3}'}(e_{r-2});x) - \ldots - x^2 I(G_1' - N_{G_1'}(e_2);x) \\ & - x^2 I(G - N_G(e_1);x). \end{aligned}$$

Now by Proposition 3.1, $I(G'_{r-1};x) = I(G'_{r-1} - e_r;x) - x^2 I(G'_{r-1} - N_{G'_{r-1}}(e_r);x)$. Thus

$$\begin{aligned} I(G;x) =& I(G'_{r-1} - e_r;x) - x^2 I(G'_{r-1} - N_{G'_{r-1}}(e_r);x) \\ & - \ldots - x^2 I(G'_1 - N_{G'_1}(e_2);x) - x^2 I(G - N_G(e_1);x) \\ =& I(G'_r;x) - x^2 I(G'_{r-1} - N_{G'_{r-1}}(e_r);x) - x^2 I(G'_{r-2} - N_{G'_{r-2}}(e_{r-1});x) \\ & - \ldots - x^2 I(G'_1 - N_{G'_1}(e_2);x) - x^2 I(G - N_G(e_1);x). \end{aligned}$$

Hence, the lemma is true by the Principle of Mathematical Induction. □

We will now, with the help of Lemma 4.13, generalize Wingard's Bound to k-degenerate graphs and thus k-trees.

Theorem 4.14. *Let G be a k-degenerate graph on $n \geq 1$ vertices with $k \geq 2$. Then $|I(G; -\frac{1}{k})| < 1$.*

Proof. We will proceed by induction on n. If $n = 1$, then $I(G;x) = 1 + x$. We see, then, that $I(G; -\frac{1}{k}) = \frac{k-1}{k}$.

Suppose the theorem is true for k-degenerate graphs of order less than n, and consider G, a k-degenerate graph on n vertices. As G is k-degenerate, $\delta \leq k$. Choose $v \in V(G)$ such that $d(v) = \delta$ and v is incident to edges $e_1, e_2, \ldots, e_\delta$. Let $E_i = \{e_1, \ldots, e_i\}$ and $G'_i = G - E_i$ for $1 \leq i \leq \delta$.

Then by Lemma 4.13,

$$\begin{aligned} I(G;x) =& I(G'_\delta;x) - x^2 I(G'_{\delta-1} - N_{G'_{\delta-1}}(e_\delta);x) - x^2 I(G'_{\delta-2} - N_{G'_{\delta-2}}(e_{\delta-1};x) \\ & - \ldots - x^2 I(G'_1 - N_{G'_1}(e_2);x) - x^2 I(G - N(e_1);x), \end{aligned} \tag{4}$$

and the right hand side has $\delta + 1 \leq k + 1$ summands.

Now G'_δ has a component of order one, and the other components of G'_δ are k-degenerate graphs. So $|I(G'_\delta; -\frac{1}{k})| < (1-\frac{1}{k})(1) = \frac{k-1}{k}$. Also each of the components of $G'_i - N_{G'_i}(e_{i+1})$ for $1 \leq i \leq \delta - 1$ and $G - N(e_1)$ is either the empty graph or a k-degenerate graph on at least one vertex and on fewer than n vertices.

Hence, by applying the induction hypothesis and (4),

$$\begin{aligned}
|I(G;-\frac{1}{k})| \leq & |I(G'_\delta;-\frac{1}{k})| + |(-\frac{1}{k})^2||I(G'_{\delta-1} - N_{G'_{\delta-1}}(e_\delta);-\frac{1}{k})| \\
& + |(-\frac{1}{k})^2||I(G'_{\delta-2} - N_{G'_{\delta-2}}(e_{\delta-1});-\frac{1}{k})| + \dots \\
& + |(-\frac{1}{k})^2||I(G'_1 - N_{G'-1}(e_2);-\frac{1}{k})| + |(-\frac{1}{k})^2||I(G - N_G(e_1);-\frac{1}{k})| \\
< & \frac{k-1}{k} + |\frac{1}{k^2}||1| + \dots + |\frac{1}{k^2}||1| \\
= & \frac{k-1}{k} + \delta\frac{1}{k^2} \leq \frac{k-1}{k} + k\frac{1}{k^2} = 1.
\end{aligned}$$

Therefore, by the Principle of Mathematical Induction, $|I(G;-\frac{1}{k})| < 1$ for G a k-degenerate graph. □

Corollary 4.15. *Let T^k_n be a k-tree on n vertices and $k \geq 2$. Then $|I(T^k_n;-\frac{1}{k})| < 1$.*

Proof. All k-trees are k-degenerate. □

We note that Wingard's bound is achieved when $k = 1$. In particular, there are examples of trees such that $|I(T;-1)| = 1$; for example $S_{1,n-1}$. However, for k-degenerate graphs with $k \geq 2$, Wingard's Bound is strict.

4.4. **The Fibonacci Number of Maximal Outerplanar Graphs.**

As has been mentioned, Song et al. determined that $f_s(S_{k,n-k})$ is a strict upper bound among k-trees for $s \geq 3$ that is uniquely obtained by $S_{k,n-k}$. For $n \geq k+3$, $S_{k,n-k}$ is not tree-like. Thus for tree-like k-trees, we seek a stricter upper bound of f_s for $s \geq 0$ than the one provided by Song et al..

In 1998, Alameddine determined sharp bounds of the Fibonacci number of maximal outerplar graphs and characterized the unique maximal outerplar graphs that obtained these bounds. He found the following:

Theorem 4.16. [1] *Let G be a maximal outerplanar graph on $n \geq 3$ vertices. Then $f(G) \geq f(P_n^2)$, and equality is reached if and only if $G \cong P_n^2$.*

Theorem 4.17. [1] *Let G be a maximal outerplanar graph on $n \geq 3$ vertices. Then $f(G) \leq f(S_n^2)$, and equality is reached if and only if $G \cong S_n^2$.*

We note for $n = 6$, $f(S_6^2) = f(D_6^2) = 14$, and thus Theorem 4.17 is not complete. We will demonstrate a revision of the results of Alameddine including this special case through investigating lower and upper bounds of the coefficients of $I(G;x)$, $f_s(G)$ for $s \geq 0$. Additionally, we will classify the unique graphs that obtain these bounds.

As the k-path is a tree-like k-tree, it is clear by Theorem 4.4 and Theorem 4.6 that the lower bound of f_s for maximal outerplar graphs for $s \geq 3$ immediately follows by the results of Song et al.. with $k = 2$. We only need to consider the upper bound.

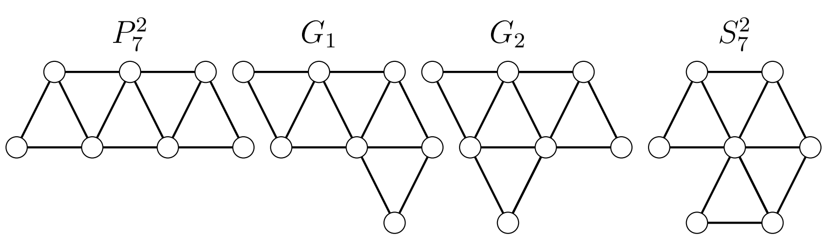


FIGURE 5. Maximal outerplanar graphs on $n = 7$ vertices

Theorem 4.18. *Let G be a maximal outerplanar graph on $n \geq 6$ vertices. Then for all $s \geq 3$,*

$$f_s(G) \leq \binom{n-s}{s},$$

and equality holds if and only if $G \in \{S_6^2, D_6^2\}$ for $n = 6$ and $G \cong S_n^2$ for $n \geq 7$.

Proof. Suppose $n = 6$. Then $G \in \{P_6^2, S_6^2, D_6^2\}$. Let $n = 7$. Then $G \in \{P_7^2, G_1, G_2, S_7^2\}$ as pictured in Figure 5. Routine calculations show that for $n \in \{6, 7\}$, $\alpha(G) \leq 3$, $f_3(P_6^2) = 0$, $f_3(D_6^2) = f_3(S_6^2) = 1 = \binom{6-3}{3}$, $f_3(P_7^2) = 1$, $f_3(G_1) = 2$, $f_3(G_2) = 3$, and $f_3(S_7^2) = 4 = \binom{7-3}{3}$. Thus the theorem holds for $n \in \{6, 7\}$.

Suppose that for maximal outerplanar graphs on $7 \leq n' < n$ vertices the theorem holds, and let G be a maximal outerplanar graph on $n \geq 8$ vertices. Let $v \in V(G)$ such that $d(v) = 2$ and $N(v) = \{u_1, u_2\}$. By Proposition 3.1(iv)

$$f_s(G) = f_s(G - v) + f_{s-1}(G - N[v]), \tag{5}$$

and as $G - v$ is a maximal outerplanar graph by induction, $f_s(G - v) \leq f_s(S_{n-1}^2) = \binom{n-1-s}{s}$.

Now G has a hamiltonian cycle C that passes through all of the unbound edges of G. Thus $u_1 v u_2$ is a segment of C, and so $G - N[v]$ has a spanning path on $n - 3$ vertices, namely $C - N[v]$. By Proposition 3.1, $f_{s-1}(G - N[v]) \leq f_{s-1}(P_{n-3}) = \binom{n-3+1-(s-1)}{s-1} = \binom{n-1-s}{s-1}$.

Thus by induction and (5),

$$\begin{aligned} f_s(G) =& f_s(G - v) + f_{s-1}(G - N[v]) \\ \leq& f_s(S_{n-1}^2) + f_{s-1}(P_{n-3}) \\ =& \binom{n-1-s}{s} + \binom{n-1-s}{s-1} \\ =& \binom{n-s}{s}, \end{aligned}$$

and for $s \geq 3$ equality holds if and only if $G - v \cong S^2_{n-1}$ and $G - N[v] \cong P_{n-3}$, i.e. $G \cong S^2_n$. $\square$

As a corollary, we obtain the following modified result of Alameddine.

Corollary 4.19. *Let G be a maximal outerplanar graph on $n \geq 6$ vertices such that $I(G;x) = I(S^2_n;x)$. Then, if $n = 6$, $G \in \{D^2_6, S^2_6\}$, and if $n \geq 7$, $G \cong S^2_n$.*

Corollary 4.20. *Let G be a maximal outerplanar graph on $n \geq 6$ vertices. Then $f(G) \leq f(S^2_n)$. Equality is reached if and only if $G \in \{D^2_6, S^2_6\}$ when $n = 6$ and if and only if $G \cong S^2_n$ when $n \geq 7$.*

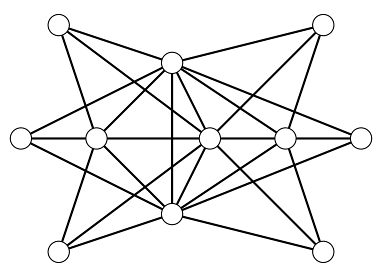


FIGURE 6. A tree-like 3-tree on 11 vertices G

4.5. Independent Sets of Cardinality s in Chordal Planar Graphs with Toughness Exceeding 1.

It should be noted that for the general $k \geq 3$, there is a tree-like k-tree T^k_n such that $f_s(T^k_n) > \binom{n-k(s-1)}{s}$ for some $s \geq 0$. As an example, $f_6(G) = 1$ for the tree-like 3-tree in Figure 6 whereas $f_6(S^3_{11}) = 0$.

As mentioned in Chapter 2, a graph is a tree-like 3-trees with toughness exceeding 1 if and only if it is a chordal planar graph with toughness exceeding 1. Let $\mathcal{L}$ be the set of chordal planar graphs with toughness exceeding 1. Though there are tree-like k-trees with

$k \geq 3$ with f_s greater than $\binom{n-k(s-1)}{s}$ for some $s \geq 0$, for the class $\mathcal{L}$ we state the following theorem.

Theorem 4.21. *Let $G \in \mathcal{L}$ on $n \geq 7$ vertices. Then for all $s \geq 3$,*

$$f_s(G) \leq \binom{n-1-s}{s},$$

and equality holds if and only if $G \in \{S_7^3, D_7^3\}$ for $n = 7$ and $G \cong S_n^3$ for $n \geq 8$.

In order to prove Theorem 4.21, we must first show that for $G \in \mathcal{L}$ with $v \in S_1(T)$, $G - N[v]$ has a spanning path which will now be presented.

Lemma 4.22. *Let T_n^k be a k-tree on $n \geq k$ vertices such that $e = xy \in E(T_n^k)$. Then there exists a simplicial elimination ordering $v_1, \ldots, v_n$ of T_n^k such that $x = v_n$ and $y = v_{n-1}$.*

Proof. The vertices x and y are in a k-clique D in G. Let $V(D) = \{x_1, \ldots, x_k\}$, and let $x = x_k$ and $y = x_{k-1}$. For $n \geq 5$, there exists a simplicial vertex $v \notin V(D)$. Let $v_1 \in S_1(T_n^k) - V(D)$, and $v_i \in S_1(T_n^k - \{v_1, \ldots, v_{i-1}\}) - V(D)$ for $1 \leq i \leq n-k$. Then $V(T_n^k - \{v_1, \ldots, v_{n-k}\}) = V(D)$. Then without loss of generality $x_i = v_{n-k-i}$ for $1 \leq i \leq k$. Hence $x = v_n$ and $y = v_{n-1}$. $\square$

Theorem 4.23. *Let $G \in \mathcal{L}$ on $n \geq 4$ vertices, and let $e \in E(G)$. Then G has a hamiltonian cycle passing through e.*

Proof. Let $e = xy$, and let $v_1, v_2, \ldots, v_n$ be a simplicial elimination ordering such that $x = v_n$ and $y = v_{n-1}$. Let $G_0 \cong T_n^k$, $G_i \cong G - \{v_1, \ldots, v_i\}$ for $1 \leq i \leq n-3$. Note then that $e \in E(G_i)$ for $0 \leq i \leq n-3$. Then $v_i \in S_1(G_{i-1})$, and so $D_i = G[N_{G_{i-1}}(v_i)] \cong K_3$ for $1 \leq i \leq n-3$. We say that the 3-clique D is active in G_i if D is unbound in G_i but bound in

G. As G is tree-like, it is clear that if D is bound, there are exactly two vertices u_1 and u_2 such that $V(D) \subseteq N(u_i)$ for $1 \leq i \leq 2$. Then clearly D_i is active in G_{i-1} for $1 \leq i \leq n-3$. A good cycle pair of G_i is an ordered pair (C, f) where C is a hamiltonian cycle of G_i, and f is an injection that maps every active 3-clique D of G_i onto an edge $f(D)$ such that $f(D) \in E(C \cap D) - e$. We claim that for every $i \in \{0, \ldots, n-4\}$, there is a good cycle pair (C_i, f_i) of G_i. We will proceed by induction on $n - i$.

Suppose $i = 4$. Then $G_{n-4} \cong K_4$ with vertex set $\{v_n, v_{n-1}, v_{n-2}, v_{n-3}\}$. Then G_{n-4} has four 3-cliques D'_0, D'_1, D'_2, and D'_3 such that $V(D'_j) = \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\} - \{v_{n-j}\}$ for $0 \leq j \leq 3$. At most three of these cliques are active as otherwise G has toughness at most 1. Let $C_{n-4} = v_n v_{n-1} v_{n-2} v_{n-3} v_n$, and define f_{n-4} as in Table 4.5. Then f_{n-4} is an injection between all active 3-cliques of G_{n-4} and edges of C_{n-4}. Thus G_{n-4} has a good cycle pair (C_{n-4}, f_{n-4}).

Suppose that G_{n-i} has a good cycle pair (C_{n-i}, f_{n-i}) for $4 \leq i < r \leq n$, and consider G_{n-r}. Suppose D_{n-r} is bound in G_{n-r}. Then there are two vertices $\{u_1, u_2\} \in V(G_{n-r})$ such that $V(D_{n-r}) \subseteq N_{G_{n-r}}(u_i)$ for $1 \leq i \leq 2$. Then $V(D_{n-r}) \subseteq (N_{G_{n-r-1}}(u_1) \cap N_{G_{n-r-1}}(u_2) \cap$

Suppose D'_j is not active	$f_{n-4}(D'_{j+1})$	$f_{n-4}(D'_{j+2})$	$f_{n-4}(D'_{j+3})$
D'_0	$x_{n-2}x_{n-3}$	$x_{n-3}x_n$	$x_{n-1}x_{n-2}$
D'_1	$x_{n-3}x_n$	$x_{n-1}x_{n-2}$	$x_{n-2}x_{n-3}$
D'_2	$x_{n-1}x_{n-2}$	$x_{n-2}x_{n-3}$	$x_{n-3}x_n$
D'_3	$x_{n-1}x_{n-2}$	$x_{n-2}x_{n-3}$	$x_{n-3}x_n$

TABLE 3. f_{n-4}

$N_{G_{n-r-1}}(v_{n-r}))$, and so G is not tree-like. Thus D_{n-r} is unbound in G_{n-r} and bound in G_{n-r-1}. Hence D_{n-r} is active in G_{n-r}.

By the inductive hypothesis there is a hamiltonian cycle pair C_{n-r} passing through e and a injection f_{n-r} of G_{n-r} mapping all active 3-cliques of G_{n-r} to an edge of $C_{n-r} - e$. Let $V(D_{n-r}) = \{a, b, c\}$ and $f_{n-r}(D_{n-r}) = ab$. Then, by induction $ab \neq e$, and as D_{n-r} is bound in G_{n-r-1} D_{n-r} is not active in G_{n-r-1}. There are exactly three new 3-cliques in G_{n-r-1}: C_a, C_b, and C_c such that $V(C_a) = \{v_{n-r}, b, c, \}$, $V(C_b) = \{v_{n-r}, c, a\}$, and $V(C_c) = \{v_{n-r}, a, b\}$. If all three of these 3-cliques are active in G_{n-r-1}, then $G - \{v_{n-r}, a, b, c\}$ has four components. Thus, in this case G is at most 1-tough. Thus at most two of C_a, C_b, and C_c are active in G_{n-r-1}. We may assume that C_c is not active in G_{n-r-1}.

Let C_{n-r-1} be the hamiltonian cycle of G_{n-r-1} obtained from C_{n-r} by replacing the edge ab by the path $av_{n-r}b$, and define f_{n-r-1} as follows. Let D be an active 3-clique of G_{n-r-1}. If D is a subgraph of G_{n-r}, then D is an active 3-clique of G_{n-r}. Let $f_{n-r-1}(D) = f_{n-r}(D)$. For $z \in \{a, b\}$ and $z' \in \{a, b\} - z$, let $f_{n-r-1}(C_z) = v_{n-r}z'$ if C_z is an active 3-clique of G_{n-r-1}. Thus G_{n-r-1} has a hamiltonian cycle C_{n-r-1} that passes through e and an injection f_{n-r-1} that maps every active 3-clique to an edge $f_{n-r-1}(D)$ such that $f_{n-r-1}(D) \in E(C_{n-r-1}) - e$. That is, G_{n-r-1} has a good cycle pair.

Thus by the Principle of Mathematical Induction, G has a hamiltonian cycle that passes through e. $\square$

With use of Theorem 4.23, we are now able to prove Theorem 4.21.

Proof. If $n = 7$, then $G \in \{P_7^3, S_7^3, D_7^3\}$. If $n = 8$, then $G \in \{P_8^3, G_1, G_2, G_3, G_4, S_8^3\}$ where G_1, G_2, G_3, and G_4 are the graphs in Figure 7. It is routine to deduce that if $n = 8$, then $\alpha(G) \leq 3$ and $f_3(P_8^3) = 0$, $f_3(G_1) = 2$, $f_3(G_2) = 3$, $f_3(G_3) = 2$, $f_3(G_4) = 1$, and $f_3(S_8^3) = 4$.

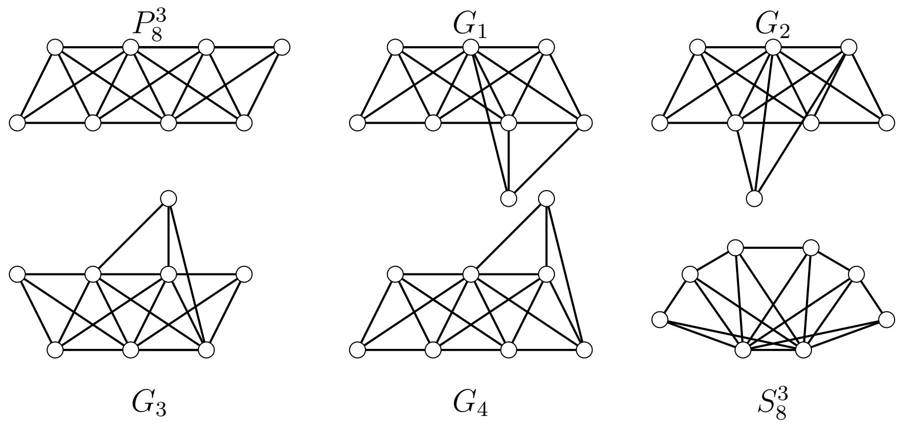


FIGURE 7. Tree-like 3-trees on 8 vertices with $\tau > 1$

Proceed by induction on n. Suppose that the theorem holds for chordal planar graphs with toughness exceeding 1 on $8 \leq n' < n$ vertices, and consider $G \in \mathcal{L}$ on n vertices. As G is a tree-like 3-tree, there exists $v \in S_1(G)$. Then by Proposition 3.1, $f_s(G) = f_s(G-v) + f_{s-1}(G - N[v])$ for $s \geq 1$.

Now $G - v \in \mathcal{L}$ on $n-1$ vertices, and so by induction $f_s(G-v) \leq \binom{n-1+2-3-s}{s} = \binom{n-2-s}{s}$ for $s \geq 0$ with equality holding if and only if $G - v \cong S^3_{n-1}$. As $G[N(v)]$ is a bound k-clique, $G - N(v)$ has two components; one being v. Hence $G - N[v]$ is connected. By Proposition 3.3, $f_s(G - N[v])$ for $s \geq 0$ is maximized when $E(G - N[v])$ is minimal i.e. $G - N[v]$ is a tree.

Suppose that $G - N[v]$ is a tree with at least 3 leaves. Then there is a vertex $u \in V(G - N[v])$ such that $d_{G-N[v]}(u) \geq 3$. However, in this case, $G - (N(v) \cup \{u\})$ has at least four components. Thus $\tau(G) \leq 1$, a contradiction. Hence if $G - N[v]$ is a tree, then $G - N[v] \cong P_{n-4}$. Thus $f_s(G - N[v]) \leq f_s(P_{n-4}) = \binom{n-4+1-s}{s} = \binom{n-3-s}{s}$ for $s \geq 0$.

Thus

$$\begin{aligned} f_s(G) =& f_s(G-v) + f_{s-1}(G - N[v]) \\ \leq& \binom{n-2-s}{s} + \binom{n-3-(s-1)}{s-1} \end{aligned}$$

$$=\binom{n-2-s}{s}+\binom{n-2-s}{s-1}$$
$$=\binom{n-1-s}{s},$$

for $s \geq 0$ with equality holding if and only if $G - v \cong S^3_{n-1}$ and $G - N[v] \cong P_{n-4}$. Hence equality holds if and only if $G \cong S^3_n$. $\square$

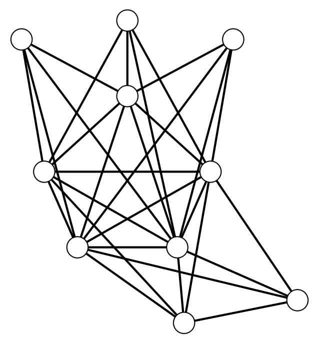


FIGURE 8. A tree-like 4-tree on 10 vertices with toughness exceeding 1

Now Theorem 4.21 can not be directly generalized to tree-like k-trees for $k \geq 4$. For example, the 4-tree in Figure 8 is tree-like with toughness exceeding 1 on 10 vertices, and f_4 of this 4-tree is 2 while $f_4(S^4_{10}) = 1$.

4.6. Independent Sets of Cardinality s of Path-like k-trees.

Though is is not clear how to generalize Theorem 4.4 to tree-like k-trees for the general k, we may demonstrate a stricter upper bound of f_s for $s \geq 3$ for path-like k-trees. We will now demonstrate this strict upper bound in a similar manner to Theorem 4.21.

Lemma 4.24. *Let T^k_n be a path-like k-tree on n vertices and $v \in S_1(T^k_n)$. Then $T^k_n - N[v]$ has a spanning path.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be a simplicial elmination ordering such that $v = v_n$ and $\{v_{n-1}, \ldots, v_{n-k-1}\} = N(v)$. As $G[N[v]] \cong K_{k+1}$, such an ordering exists. Let $G_0 \cong T^k_n$ and $G_i \cong$

$T_n^k - \{v_1, \dots, v_i\}$ for $1 \leq i \leq n$. Clearly v_{i+1} is simplicial in G_i for $1 \leq i \leq n-k-2$. If $v_{i+1} \notin N(v_i)$, then $S_1(G_i) = \{v, v_{i+1}, v_i\}$. Hence $v_{i+1}v_i \in E(T_n^k)$ for $1 \leq i \leq n-k-2$. It immediately follows that $T_n^k - N[v]$ has a spanning path. □

Theorem 4.25. *Let T_n^k be a path-like k-tree on n vertices. Then for $s \geq 3$,*

$$\binom{n-k(s-1)}{s} \leq f_s(T_n^k) \leq \binom{n+2-k-s}{s},$$

with the left inequality holding if and only if $T_n^k \cong P_n^k$, and the right inequality holding if and only $T_n^k \cong S_n^k$.

Proof. The lower bound was shown to be true by Song et. al [41]. Thus, we only need to show the upper bound. Proceed by induction on n. If $k \leq n \leq k+3$, then T_n^k is unique, and the theorem is true. Suppose that for path-like k-trees on $k \leq n' < n$ vertices, the theorem holds, and consider T_n^k, a path-like k-tree on n vertices.

Let $v_1, v_2, \dots, v_n$ be a presentation of T_n^k. Then $v_n \in S_1(T_n^k)$. By Proposition 3.1, for $s \geq 1$

$$f_s(T_n^k) = f_s(T_n^k - v_n) + f_{s-1}(T_n^k - N[v_n]).$$

Now $T_n^k - v_n$ is a path-like k-tree on $n-1$ vertices. Hence, by induction $f_s(T_n^k - v_n) \leq \binom{n+1-k-s}{s}$ for $s \geq 0$ with equality holding if and only if $T_n^k \cong S_n^k$.

Now by Lemma 4.24, $T_n^k - N[v_n]$ has a spanning path. As $G[N[v_n]]$ is a bound $(k+1)$-clique and $v_n \in S_1(T_n^k)$, it is clear that $T_n^k - N[v_n]$ is connected. Thus by Proposition 3.3 $f_s(T_n^k - N[v_n]) \leq \binom{n-k+1-s}{s}$ for $s \geq 0$ with equality holding if and only if $T_n^k - N[v_n] \cong P_{n-k-1}$.

Thus,

$$\begin{aligned} f_s(T_n^k) =& f_s(T_n^k - v_n) + f_{s-1}(T_n^k - N[v_n]) \\ \leq& \binom{n+1-k-s}{s} + \binom{n-k+1-(s-1)}{s-1} \end{aligned}$$

$$= \binom{n-k+2-s}{s},$$

with equality holding if and only if $T_n^k - v_n \cong S_{n-1}^k$ and $T_n^k - N[v_n] \cong P_{n-k-1}$. Thus equality holds if and only if $T_n^k \cong S_n^k$, and by the Principle of Mathematical Induction the theorem holds. $\square$

5. The Zagreb Indices of k-trees

In 1975, Randić introduced the branching index which later became known as the Randić connectivity index [36]. The Randić connectivity index is mostly used as a molecular discriptor in computational chemistry describing nonempirical quantitative structure-property relationships and quantitative structure-activity relationships [17]. However, mathematicians have also expressed interest in the Randić connectivity index [6].

The Randić connectivity index has been generalized as the general Randić connectivity index and the general zeroth-order Randić connectivity index, where the Zagreb indices appeared as a special case [8]. The first Zagreb index $M_1(G)$ and the second Zagreb index $M_2(G)$ of the graph G are given by:

$$M_1(G) = \sum_{u \in V(G)} d(u)^2, \qquad M_2(G) = \sum_{uv \in E(G)} d(u)d(v).$$

The Zagreb indices M_1 and M_2 have been an active area of research going back to 1972 in the report of Gutman and Trinajstić in computational chemistry [23].

In regards to the Zagreb indices, there are two classical problems which have attracted the attention of researchers for some time:

(i) How $M_1(G)$ (respectively $M_2(G)$) depends on the structure of G.

(ii) Given a set of graphs $\mathcal{G}$, find upper and lower bounds for $M_1(G)$ and $M_2(G)$ of graphs in $\mathcal{G}$ and characterize the graphs in which the maximal (respectively minimal) M_1 and M_2 values are attained.

There have been numerous studies in the literature of the properties of Zagreb indices of given graph classes [11, 13, 29, 31]. In particular, Das and Gutman in 2004 characterized the Zagreb indices for trees and determined the unique tree that obtains minimum M_1 and M_2 values respectively, as well as maximum M_1 and M_2 values respectively.

Theorem 5.1. [12, 20] *Let T be any tree of order n. Then*

(i) $4n - 6 \leq M_1(T) \leq n^2 - n$, *the left equality holds if and only if $T \cong P_n$, and the right equality holds if and only if $T \cong S_n$.*

(ii) $4n - 8 \leq M_2(T) \leq n^2 - 2n + 1$, *the left equality holds if and only if $T \cong P_n$ and the right equality holds if and only if $T \cong S_n$.*

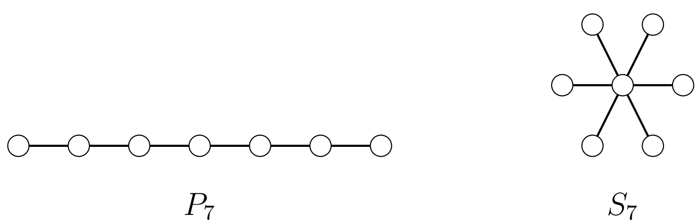


FIGURE 9. The path and star on 7 vertices

In 2011, Hou et al. characterized the Zagreb indices for maximal outerplanar graphs and determined the unique maximal outerplanar graph that obtains minimum M_1 and M_2 values respectively, as well as maximum M_1 and M_2 values respectively. Hou et al. found the following:

Theorem 5.2. [29] *Let G be a maximal outerplaner graph on $n \geq 4$ vertices. Then*

(i) $M_1(G) \geq 16n - 38$, *with equality holding if and only if $G \cong P_n^2$.*

(ii) $M_2(G) \geq 32n - 100$, *with equality holding if and only if $G \cong P_n^2$.*

Theorem 5.3. [29] *Let G be a maximal outerplanar graph on $n \geq 4$ vertices. Then*

(i) *When $n = 6$, $M_1(G) \leq 60$ with equality if and only if $G \cong S_6^2$ or D_6^2.*

(ii) *When $n \neq 6$, $M_1(G) \leq n^2 + 7n - 18$ with equality if and only if $G \cong S_n^2$.*

Theorem 5.4. [29] *Let G be a maximal outerplanar graph on $n \geq 4$ vertices.*

(i) *When $n = 6$, $M_2(G) \leq 96$ with equality if and only if $G \cong D_6^2$.*

(ii) *When $n \neq 6$, $M_2(G) \leq 3n^2 + n - 19$ with equality if and only if $G \cong S_n^2$.*

As k-trees are a generalization of trees and maximal outerplanar graphs, it is a natural connection to generalize the results of Das and Gutman, as well as the results of Hou et al., to the broader class of k-trees.

5.1. **Some Lemmas.**

In this section, we give some lemmas that are critical in subsequent sections. For the remainder of this chapter, let T_n^k be a k-tree on n vertices, and let $v \in S_1(T_n^k)$ such that $N(v) = U = \{u_1, \ldots, u_k\}$. Then $T_n^k - v$ is a k-tree. Let $V(T_n^k) = \{v\} \cup U \cup X \cup Y$ where $X = N(U) - N[v]$ and $Y = V(T_n^k) - X - N[v]$. Let $|X| = l$ and $X = \{x_1, \ldots, x_l\}$. Then $l \geq \min(n - k - 1, k)$. Arrange the vertices of X such that $x_i \in U$ for $1 \leq i \leq j$ and $|N(x_i) \cap U| \geq |N(x_{i+1}) \cap U|$ for $j + 1 \leq i \leq l - 1$.

If $n \geq k + 2$, then $|S_1(T_n^k)| \geq 2$. Thus if $n \geq k + 2$, there exists $v' \in S_1(T_n^k) - v$. Choose v' such that $|N(v') \cap U| = t$ is as small as possible, and let $N(v') = U' = \{u'_1, \ldots, u'_k\}$. Arrange the vertices of U' such that $u'_i \in U$ for $1 \leq i \leq t$ and $|N(u'_i) \cap U| \geq |N(u'_{i+1}) \cap U|$ for $t + 1 \leq i \leq k$. Let $f : U' \to \mathbb{N}$ where $f(u'_i) = \begin{cases} 0 & \text{if } u'_i \in U \\ |N(u'_i) \cap U| & \text{if } u'_i \notin U \end{cases}$. Let $d^*(v_j)$

(respectively $d_*(v_j)$) be the degree obtained by vertex v_j of a presentation of P_n^k (respectively P_{n-1}^k).

Then we may state the following lemmas.

Lemma 5.5. *Let T_n^k be a k-tree on $n \geq k+3$ vertices, and let $v \in S_1$ where $N(v) = \{u_1, \ldots, u_k\}$. Then:*

(i) $\sum_{i=1}^{k} d(u_i) \geq 2kn - \frac{1}{2}(k(k+5)) - \frac{1}{2}((n-1)(n-2))$ *for* $k+3 \leq n \leq 2k$*;*

(ii) $\sum_{i=1}^{k} d(u_i) \geq k^2 + \frac{1}{2}(k(k+1))$ *for* $n \geq 2k+1$.

Equality is reached if and only if $G[\cup_{i=1}^{k} N[u_i]] \cong P_r^k$, $r = \min(n, 2k+1)$.

Proof. We will proceed by induction on n. There are two k-trees on $k+3$ vertices: P_{k+3}^k and $S_{k,3}$. If $T_n^k \cong P_{k+3}^k$, then $\sum_{i=1}^{k} d(u_i) = k^2 + 2k - 1$. If $T_n^k \cong S_{k,3}$, then $\sum_{i=1}^{k} d(u_i) = k^2 + 2k$, and so the lemma holds. Suppose the lemma is true for $T_{n'}^k$ with $k+3 < n' < n$, and consider T_n^k. Clearly for the simplicial vertex $v' \neq v$,

$$\sum_{i=1}^{k} d(u_i) = \sum_{i=1}^{k} d_{T_n^k - v'}(u_i) + |U' \cap U|. \tag{6}$$

Suppose $k+4 \leq n \leq 2k$ which implies $k \geq 4$. Then $l = n-k-1$ and $|Y| = 0$. Hence $v' \in X$, and without loss of generality let $v' = x_l$. Thus $|N(x_l) \cap U| \neq \emptyset$. As $k+4 \leq n \leq 2k$, $3 \leq l \leq k-1$ and so $k-(l-1) \leq |N(x_l) \cap U| \leq k$. Thus $2k-n+2 \leq |N(x_l) \cap U| \leq k$. By induction and (6),

$$\begin{aligned}\sum_{i=1}^{k} d(u_i) =& \sum_{i=1}^{k} d_{T_n^k - x_l}(u_i) + |N(x_l) \cap U| \\ \geq& 2k(n-1) - \frac{1}{2}(k(k+5)) - \frac{1}{2}(n-2)(n-3) + 2k - n + 2 \\ =& 2kn - \frac{1}{2}(k(k+5)) - \frac{1}{2}(n-1)(n-2)\end{aligned}$$

with equality holding if and only if $T_n^k - x_l \cong G[\cup_{i=1}^k N_{T_n^k - x_l}[u_i]] \cong P_{n-1}^k$ and $|N(x_l) \cap U| = 2k - n + 2 = k - (l-1)$. In this case $X - x_l \subseteq N(x_l)$, $x_{l-1} \in S_2(T_n^k)$, and so $N(x_l) \subseteq N[x_{l-1}]$. Thus $(N(x_l) \cap U) \subseteq (N(x_{l-1}) \cap U)$. Hence $T_n^k \cong G[\cup_{i=1}^k N[u_i]] \cong P_n^k$.

Suppose $n = 2k+1 > k+3$, then $l = k$ and $|Y| = 0$. Thus without loss of generality let $v' = x_k$. Hence $1 \leq |N(x_k) \cap U| \leq k$. Then by induction and (6)

$$\begin{aligned}\sum_{i=1}^{k} d(u_i) =& \sum_{i=1}^{k} d_{T_n^k - x_k}(u_i) + |N(x_k) \cap U| \\ \geq& 2k(n-1) - \frac{1}{2}(k(k+5)) - \frac{1}{2}(n-2)(n-3) + 1 \\ =& 4k^2 - \frac{1}{2}(k(k+5)) - \frac{1}{2}(4k^2 - 6k + 2) + 1 \\ =& k^2 + \frac{1}{2}(k^2 + k)\end{aligned}$$

with equality holding if and only if $T_n^k - x_k \cong G[\cup_{i=1}^k N_{T_n^k - x_k}[u_i]] \cong P_{2k}^k$ and $|N(x_k) \cap U| = 1$, i.e. $T_{2k+1}^k \cong G[\cup_{i=1}^k N[u_i]] \cong P_{2k+1}^k$.

Suppose $n \geq 2k+2 > k+3$. Then $l \geq k$ and $|Y| \geq 0$. If $|Y| = 0$, then $|N(v') \cap U| \geq 1$. If $|Y| \geq 1$, then $|N(v') \cap U| = 0$, and so by induction and (6)

$$\begin{aligned}\sum_{i=1}^{k} d(u_i) =& \sum_{i=1}^{k} d_{T_n^k - v'}(u_i) + |N(v') \cap U| \\ \geq& k^2 + \frac{1}{2}(k^2 + k)\end{aligned}$$

with equality holding if and only if $G[\cup_{i=1}^k N_{T_n^k - v'}[u_i]] \cong P_{2k+1}^k$ and $|N(v') \cap U| = 0$. Hence $G[\cup_{i=1}^k N[u_i]] \cong P_{2k+1}^k$.

Hence by the Principle of Mathematical Induction, the lemma holds. □

Lemma 5.6. *Let G be a k-degenerate graph on $n \geq k+1$ vertices, and let $v \in V(G)$ such that $d(v) = \delta(G)$ and $N(v) = \{u_1, \ldots, u_{\delta(G)}\}$. Then $\sum_{i=1}^{\delta(G)} d(u_i) \leq k(n-1)$, and equality holds if and only if $G \cong S_{k,n-k}$.*

Proof. Clearly $\delta(G) \leq k$ and $d(u_i) \leq n-1$ for $1 \leq i \leq \delta(G)$. Thus $\sum_{i=1}^{\delta(G)} d(u_i) \leq k(n-1)$ with equality reached if and only if $\delta(G) = k$ and $d(u_i) = n-1$ for $1 \leq i \leq k$. Furthermore, equality is reached if and only if $V(G) - N[v]$ is independent as otherwise G has a K_{k+2} subgraph and thus not k-degenerate. That is, for $x \in V(G) - N[v]$, x is a k-simplicial vertex. Thus equality is reached if and only if $G \cong S_{k,n-k}$. □

Let $N_0(u_i) = N(u_i) - N[v]$, and let

$$\Psi(T_n^k; v) = \sum_{x \in N_0(u_1)} d(x) + \sum_{x \in N_0(u_2)} d(x) + \ldots + \sum_{x \in N_0(u_k)} d(x).$$

Then for $n \geq k+2$ and $v' \in S_1(T_n^k) - v$,

$$\begin{aligned} \Psi(T_n^k; v) =& \Psi(T_n^k - v'; v) + d(v')t + \sum_{i=1}^{k} f(u_i') \\ =& \Psi(T_n^k - v'; v) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U|. \end{aligned} \tag{7}$$

With this in mind, we may state the following lemmas:

Lemma 5.7. *Let T_n^k be a k-tree on $n \geq k+5$ vertices and $v \in S_1(T_n^k)$. Then $\Psi(T_n^k; v) \geq \sum_{i=1}^{l} (k+1-i) d(x_i)$ where $d(x_i) = d^*(v_{i+k+1})$ with respect to a presentation of P_n^k. Furthermore equality holds if and only if $G[N(N_0(u_1)) \cup N(N_0(u_2)) \cup \ldots \cup N(N_0(u_k))] \cong P_r^k$ where $r = \min(n, 3k+1)$.*

Proof. We will proceed by induction on n. Note that

$$\sum_{i=1}^{l} (k+1-i) d^*(v_{i+k+1}) = \sum_{i=k+2}^{l+k+1} (2k+2-i) d^*(v_i).$$

Now $d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U|$ is a summand with k summands with at least t summands of value k and at most $k-t$ summands of value at most $k-1$. It is clear then that $d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U|$ is minimized when t is minimized.

Suppose $n = k+5$. Then $t \geq \max(0, k-3)$, and by (7) $\Psi(T^k_{k+5}; v) = \Psi(T^k_{k+5} - v'; v) + kt + \sum_{i=1}^{k} f(u'_i)$. Suppose $k \in \{1, 2\}$. Then $l \geq k$ and $t \geq 0$, and $\Psi(T^k_{k+5}; v)$ is minimized when $t = 0$, and so clearly $\sum_{i=1}^{k} f(u'_i) \geq k-1$. Hence by Table 1,

$$\begin{aligned}\Psi(T^k_{k+5}; v) =& \Psi(T^k_{k+5} - v'; v) + kt + \sum_{i=1}^{k} f(u'_i) \\ \geq& 3k^2 - 1 + 0 + k - 1 \\ =& \sum_{i=k+2}^{2k+1} (2k+2-i) d^*(v_i).\end{aligned}$$

Suppose $n = k+5$ and $k \geq 3$. Then $t \geq k-3$. If $t = k-3$, then $T^k_{k+5} \cong P^k_{k+5}$. That is, $\Psi(T^k_n; v)$ is minimized when $T^k_{k+5} \cong P^k_{k+5}$. Hence $\Psi(T^k_{k+5}; v) \geq \Psi(P^k_{k+5}; v) = \sum_{i=k+2}^{l+k+1} (2k+2-i) d^*(v_i)$ with equality holding if and only if $T^k_{k+5} \cong P^k_{k+5}$.

Suppose that the lemma is true for T^k_m where $k+5 \leq m < n$ and consider T^k_n. Let $|X - v'| = l'$. Now $kt + \sum_{i=1}^{k} |N(u'_i) \cap U|$ is minimized when $|N(v') \cap Y|$ is as large as possible and t is as small as possible.

Suppose $k+6 \leq n \leq 2k+1$. Then $l = n-k-1$, $|Y| = 0$, and $l' = l-1$. As $n \leq 2k+1$, $d_*(v_{i+k+1}) = d^*(v_{i+k+1}) - 1$ for $1 \leq i \leq l'$. Now $t \geq k - (n-1-(k+1)) = 2k+2-n$. If $t = 2k+2-n$, then $|N(u'_i) \cap U|$ is minimized when $u'_j \in N(u'_i)$ for $t+1 \leq j < i$. Hence $|N(u'_i) \cap U| \geq k+t+1-i$ for $t+1 \leq i \leq k$. Thus

$$\begin{aligned}\sum_{i=t+1}^{k} |N(u'_i) \cap U| \geq \sum_{i=t+1}^{k} (k+t-1-i) =& \sum_{i=2k+3-n}^{k} (3k+3-n-i) \\ =& \sum_{i=k+2}^{n-1} (2k+2-i)\end{aligned}$$

with equality holding if and only if $T^k_n \cong P^k_n$. Hence by induction and (7),

$$\Psi(T^k_n; v) = \Psi(T^k_n - v'; v) + d(v')t + \sum_{i=t+1}^{k} |N(u'_i) \cap U|$$

$$
\begin{aligned}
&\geq \Psi(P_{n-1}^k; v) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U| \\
&= \sum_{i=1}^{l-1}(k+1-i)d_*(v_{i+k+1}) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U| \\
&= \sum_{i=k+2}^{n-1} (2k+2-i)(d^*(v_i)-1) + d^*(v_n)(2k+2-n) \\
&\quad + \sum_{i=k+2}^{n-1} (2k+2-i) \\
&= \sum_{i=k+2}^{n} (2k+2-i)d^*(v_i)
\end{aligned}
$$

with equality holding if and only if $T_n^k \cong P_n^k$.

Suppose $k+6 \leq 2k+2 \leq n \leq 3k+1$. Then $k \leq l' \leq l \leq l'+1$. As $Y \cap (\{v\} \cup U \cup X) = \emptyset$, $|N(v') \cap Y| \leq n-1-(2k+1) = n-(2k+2)$ and $t \geq 0$ with equality if and only if $T_n^k \cong P_n^k$. Hence $\Psi(T_n^k; v)$ is minimized when $|N(v') \cap Y| = n-(2k+2)$ and $t = 0$. That is, $u_i' \in Y$ for $k-(n-(2k+2)) = 3k+3-n \leq i \leq k$ and $|N(u_i') \cap U| \geq 3k+3-n-i$ for $1 \leq i \leq 3k+2-n$. In this case, $d^*(v_i) = d_*(v_i)$ for $k+2 \leq i \leq n-k$, and

$$
\sum_{i=t+1}^{k} |N(u_i') \cap U| = \sum_{i=1}^{3k+2-n} (3k+3-n-i) = \sum_{i=n-k+1}^{2k+2} (2k+2-i).
$$

Hence by induction and (7),

$$
\begin{aligned}
\Psi(T_n^k; v) &= \Psi(T_n^k - v'; v) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U| \\
&\geq \Psi(P_{n-1}^k; v) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U| \\
&= \sum_{i=1}^{l'}(k+1-i)d_*(v_{i+k+1}) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U| \\
&\geq \sum_{i=k+2}^{n-k} (2k+2-i)d^*(v_i) + \sum_{i=n-k+1}^{2k+2} (2k+2-i)(d^*(v_i)-1)
\end{aligned}
$$

$$+ d^*(v_n)(2k+2-n) + \sum_{i=n-k-1}^{2k+2}(2k+2-i)$$

$$= \sum_{i=k+2}^{n}(2k+2-i)d^*(v_i)$$

with equality holding if and only if $T_n^k \cong P_n^k$.

Suppose $k+6 \leq 3k+2 \leq n$. Let $G' = T_n^k - v'$. Then by induction and (7),

$$\begin{aligned}\Psi(T_n^k; v) =& \Psi(G'; v) + kt + \sum_{i=t+1}^{k}|N(u_i') \cap U| \\ \geq& \sum_{i=k+2}^{2k+1}(2k+2-i)d_*(v_i) + kt + \sum_{i=t+1}^{k}|N(u_i') \cap U|\end{aligned}$$

with equality holding if and only if $G[N(N_o(u_1)) \cup N(N_o(u_2)) \cup \ldots$ $\cup N(N_o(u_k))] \cong G[N_{G'}(N_o(u_1)) \cup N_{G'}(N_o(u_2)) \cup \ldots \cup N_{G'}(N_o(u_k))] \cong P_{3k+1}^k$. Note that $t \geq 0$ and $|N(v') \cap Y| \leq k$ with both equalities holding if and only if $G[N(N_o(u_1))\cup N(N_o(u_2)) \cup \ldots \cup N(N_o(u_k))] \cong P_{3k+1}^k$, and note that if $|N(v') \cap Y| = k$ then $\sum_{i=1}^{k}|N(u_i') \cap U| = 0$. Hence

$$\Psi(T_n^k; v) \geq \sum_{i=k+2}^{2k+1}(2k+2-i)d^*(v_i)$$

with equality holding if and only if $G[N(N_o(u_1)) \cup N(N_o(u_2)) \cup \ldots \cup N(N_o(u_k))] \cong P_{3k+1}^k$.

Hence, by the Principle of Mathematical Induction, $\Psi(T_n^k; v) \geq \sum_{i=1}^{l}(k+1-i)d^*(v_{i+k+1})$, and equality is reached if and only if $G[N(N_0(u_1)) \cup N(N_0(u_2)) \cup \ldots \cup N(N_0(u_k))] \cong P_r^k$ where $r = \min(n, 3k+1)$. □

Lemma 5.8. *Let T_n^k be a k-tree on $n \geq k+4$ vertices and $v \in S_1(T_n^k)$ with $N(v) = U = \{u_1, \ldots, u_k\}$. Then:*

(i) $\Psi(T_n^k; v) \geq \frac{1}{6}(n-k-1)(2nk+5n-n^2+5k^2-5k-6)$ *for* $k+4 \leq n \leq 2k$

(ii) $\Psi(T_n^k; v) \geq \frac{1}{6}(n^3 - 9n^2 - 6n^2 + 27nk^2 + 36nk + 6n - 21k^3 - 24k^2 - 33k - 6)$ *for* $2k+1 \leq n \leq 3k$

(iii) $\Psi(T_n^k; v) \geq k^3 + k^2$ *for* $n \geq \max(5, 3k+1)$.

And for $n \geq k+5$*, equality is reached if and only if* $G[N(N_0(u_1)) \cup N(N_0(u_2)) \cup \ldots \cup N(N_0(u_k))] \cong P_r^k$ *where* $r = \min(n, 3k+1)$.

Proof. First consider the two k-trees on $k+3$ vertices: P_{k+3}^k and $S_{k,3}$. It is routine to determine that $\Psi(P_{k+3}^k; v) = \Psi(S_{k,3}; v) = 2k^2$.

Suppose $n = k+4$, and suppose $k \in \{1, 2\}$. Then $T_{k+4}^k - v' \in \{P_{k+3}^k, S_{k,3}\}$, $l \geq k$, $t \geq 0$, and $\sum_{i=1}^k f(u_i') \leq k^2 - 1$. Hence by (7)

$$\begin{aligned}\Psi(T_{k+4}^k; v) =& \Psi(T_{k+4}^k - v'; v) + kt + \sum_{i=1}^{k} f(u_i') \\ \geq& 2k^2 + k^2 - 1 = 2k^2 + k^2 - k + 1.\end{aligned}$$

Note that when $k = 1$ and $n = k+4$, $3k^2 - 1 = k^3 + k^2$, and when $k = 2$ and $n = k+4$, $3k^2 - 1 = \frac{1}{6}(n^3 - 9n^2 - 6n^2 + 27nk^2 + 36nk + 6n - 21k^3 - 24k^2 - 33k - 6)$, and so the lemma holds.

Suppose that $n = k+4$ and $k \geq 3$. Then $l = 3$ and $|Y| = 0$. Hence there exists $v' \in S_1(T_n^k) \cap X$, and $k-2 \leq t \leq k$. If $t = k-2$ (respectively $t = k$), then $T_{k+4}^k \cong P_{k+4}^k$ ($T_{k+4}^k \cong S_{k,4}$) and $\Psi(T_{k+4}^k; v) = 3k^2 - 1$ ($\Psi(T_{k+4}^k; v) = 3k^2$). Suppose that $t = k-1$. Then $T_{k+4}^k \in \{G_1, G_2, G_3\}$ where G_1, G_2, and G_3 are defined as follows. Let G_1 and G_2 be k-trees such that $G_i - v \cong P_{k+3}^k$ for $1 \leq i \leq 2$, and let $x \in S_1(G_i - v')$ for $1 \leq i \leq 2$ such that $x \in S_1(G_1)$ and $x \notin S_1(G_2)$. Let G_3 be a k-tree such that $G_3 - v' \cong S_{k,3}$, but $G_3 \ncong S_{k,4}$. Then $\Psi(G_1; v) = 3k^2 + k - 1$, $\Psi(G_2; v) = 3k^2 - 1$, and $\Psi(G_3; v) = 3k^2 + k - 1$. By these calculations, we see that $\Psi(T_{k+4}^k; v) \geq 3k^2 - 1$. Note if $k = 3$, then $n = 7 = 2k+1$ and

$3k^2-1=k(n-2k-1)(4k+2-n)+\frac{1}{6}(n-3k-1)(n^2-5n-9k^2+3k+6)$. If $k \geq 4$, then $3k^2-1=\frac{1}{6}(n-k-1)(2nk+5n-n^2+5k^2-5k-6)$.

Suppose that $n \geq k+5$. Then by Lemma 5.7, $\Psi(T_n^k;v) \geq \sum_{i=k+2}^{l+k+1}(2k+2-i)d^*(v_i)$. According to Table 1,

(i) $\Psi(T_n^k;v) \geq \frac{1}{6}(n-k-1)(2nk+5n-n^2+5k^2-5k-6)$ for $k+4 \leq n \leq 2k$

(ii) $\Psi(T_n^k;v) \geq \frac{1}{6}(n^3-9n^2-6n^2+27nk^2+36nk+6n-21k^3-24k^2-33k-6)$ for $2k+1 \leq n \leq 3k$

(iii) $\Psi(T_n^k;v) \geq k^3+k^2$ for $n \geq 3k+1$.

Furthermore, by Lemma 5.7, for $n \geq k+5$ equality is reached if and only if $G[N(N_0(u_1)) \cup N(N_0(u_2)) \cup \ldots \cup N(N_0(u_k))] \cong P_r^k$ where $r = \min(n, 3k+1)$. $\square$

Let $\mathcal{J}^k$ be the set of k-trees T_n^k with a vertex $v \in S_1(T_n^k)$ and vertex set $P = \{p | p \in V(T_n^k), |N(p) \cap N(v)| = k\}$ such that $V(T_n^k) - N[v] - P$ is an independent set. Then we may state the following lemma.

Lemma 5.9. *Let T_n^k be a k-tree on $n \geq k+1$ vertices. Then $\Psi(T_n^k;v) \leq (n-k-1)k^2$ with equality holding if and only if $T_n^k \in \mathcal{J}^k$.*

Proof. Let $P = \{p_1, \ldots, p_r\} = \{p \in V(T_n^k) || N(p) \cap U| = k\}$ and $Q = \{q_1, \ldots, q_s\} = V(T_n^k) - P - N[v]$. Order the vertices of Q such that $|N(q_i) \cap P| \geq |N(q_{i+1}) \cap P|$ for $1 \leq i \leq t-1$. Then $|P| + |Q| = r + s = n-k-1$.

Proceed by induction on $|Q| = s$. If $s = 0$, then $\Psi(T_n^k;v) = rk^2 = (n-k-1)k^2$ as for any $p \in P$, $p \in N(u_i)$ for $1 \leq i \leq k$. Suppose that for k-trees with $|Q| = s'$ such that $0 < s' < s$, $\Psi(T_n^k;v) \leq (n-k-1)k^2$ with equality holding if and only if $T_n^k \in \mathcal{J}^k$; consider T_n^k with $|Q| = s$.

As $|Q| \neq 0$, there exists $v' \in S_1(T_n^k) \cap Q$. Let $N(v') = U' = \{u'_1, \ldots, u'_k\}$. Arrange the vertices of U' such that $u'_i \in U$ for $1 \leq i \leq t$ and $|N(u'_i) \cap U| \geq |N(u'_{i+1}) \cap U|$ for $t+1 \leq i \leq k$. Then $T_n^k - v'$ is a k-tree with $|V(T_n^k - v') - P - N[v]| = s - 1$. By induction, $\Psi(T_n^k - v'; v) \leq (n-k-2)k^2$ with equality holding if and only if $T_n^k - v' \in \mathcal{J}^k$.

Now $|N(u'_i) \cap U| \leq k-1$ for $t+2 \leq i \leq k$. Hence $k|U' \cap U| + \sum_{i=t+1}^{k} |N(u'_i) \cap U|$ is maximized when $|U' \cap U|$ is maximized. By (7),

$$\Psi(T_n^k; v) = \Psi(T_n^k - v'; v) + k|U' \cap U| + \sum_{i=t+1}^{k} |N(u'_i) \cap U|.$$

Suppose $T_n^k \notin \mathcal{J}^k$, then $|U' \cap U| \leq k-1$. If $|U' \cap U| = k-1$, then $|N(u'_k) \cap U| \leq k-1$ otherwise $T_n^k \in \mathcal{J}^k$, and so if $|U' \cap U|$, then $k|U' \cap U| + \sum_{i=t+1}^{k} |N(u'_i) \cap U| \leq k(k-1) + k - 1 = k^2 - 1$. Hence

$$\begin{aligned}\Psi(T_n^k; v) &= \Psi(T_n^k - v'; v) + k|U' \cap U| + \sum_{i=t+1}^{k} |N(u'_i) \cap U| \\ &\leq (n-k-2)k^2 + k(k-1) + k - 1 = (n-k-1)k^2 - 1.\end{aligned}$$

Suppose then that $T_n^k \in \mathcal{J}^k$, then $T_n^k - v' \in \mathcal{J}^k$, $|U' \cap U| = k$, and $\sum_{i=t+1}^{k} |N(u'_i) \cap U| = 0$ as $t = k$. Hence

$$\begin{aligned}\Psi(T_n^k; v) =& \Psi(T_n^k - v'; v) + k|U' \cap U| \\ \leq& (n-k-2)k^2 + k^2 = (n-k-1)k^2.\end{aligned}$$

Hence by induction, the lemma holds. □

5.2. **The Zagreb Indices of the k-path and the k-star.**

The following lemmas may be deduced through fairly routine calculations by induction on n.

Lemma 5.10. *Let P_n^k be the k-path on $n \geq k+3$ vertices. Then*

$$M_1(P_n^k) = 2nk(n-2) - \frac{1}{3}(n(n-1)(n-2)) - \frac{1}{3}(k(k+1)(2k-5))$$

$$\textit{for } k+3 \leq n \leq 2k \textit{ and } k \geq 3,$$

$$M_1(P_n^k) = 4nk^2 - \frac{1}{3}(k(10k-1)(k+1)) \textit{ for } n \geq \max(4, 2k+1).$$

Lemma 5.11. *Let $S_{k,n-k}$ be the k-star on $n \geq k+1$ vertices. Then $M_1(S_{k,n-k}) = n^2k + (k^2-2k)n - k^3 + 1$.*

For the second Zagreb indices, we have the following:

Lemma 5.12. *Let P_n^k be the k-path on $n \geq k+3$ vertices. Then*

$$M_2(P_n^k) = \frac{1}{2}(k^4 + 9k^3 + 12k^2 - 8k + 2), \textit{ for } n = k+3,$$

$$M_2(P_n^k) = \frac{1}{24}((10-4k)n^3 - n^4 + (54k^2 - 18k - 23)n^2 -$$

$$(44k^3 + 66k^2 - 54k - 14)n + 7k^4 + 38k^3 + 5k^2 - 26k)$$

$$\textit{for } k+4 \leq n \leq 2k,$$

$$M_2(P_n^k) = \frac{1}{24}(n^4 - (12k+6)n^3 + (54k^2 + 54k + 11)n^2 -$$

$$(12k^3 + 162k^2 + 66k + 6)n - (25k^4 - 70k^3 - 109k^2 - 14k))$$

$$\textit{for } 2k+1 \leq n \leq 3k-1,$$

$$M_2(P_n^k) = \frac{1}{12}(48nk^3 - 53k^4 - 46k^3 + 5k^2 - 2k) \textit{ for } n \geq \max(5, 3k).$$

Proof. We will proceed by induction on n. By simple calculations, $M_2(P_{k+3}^k) = \frac{1}{2}(k^4 + 9k^3 + 12k^2 - 8k + 2)$, and the lemma holds true. Suppose that for k-paths of an order smaller than $n \geq k+4$ the lemma holds, and consider P_n^k. Let T_n^k be a k-tree on n vertices, and let $v \in S_1(T_n^k)$ with $N(v) = \{u_1, \ldots, u_k\}$. Let $G' \cong T_n^k - v$, which is a k-tree. Note that $d_{G'}(u_i) = d(u_i) - 1$ for $1 \le i \le k$ and $\sum_{u_iu_j, i \ne j}[(d_{G'}(u_i)+1)(d_{G'}(u_j)+1) - d_{G'}(u_i)d_{G'}(u_j)] = \sum_{u_iu_j, i\ne j}(d_{G'}(u_i)+d_{G'}(u_j)+1) = \sum_{u_iu_j,i\ne j}(d(u_i)+d(u_j)-1) = \sum_{u_iu_j,i\ne j}(d(u_i)+d(u_j)) - \binom{k}{2}$. Thus

$$\begin{aligned}
M_2(T_n^k) =& M_2(G') + d(v)\left(\sum_{i=1}^k d(u_i)\right) + \left(\sum_{i=1}^k (d(u_i) - d_{G'}(u_i))\right)\left(\sum_{x \in N_o(u_i)} d(x))\right) \\
&+ \sum_{u_iu_j, i\ne j}[(d_{G'}(u_i)+1)(d_{G'}(u_j)+1) - d_{G'}(u_i)d_{G'}(u_j)] \\
=& M_2(G') + k\sum_{i=1}^k d(u_i) + \Psi(T_n^k; v) + \sum_{u_iu_j,i\ne j}(d(u_i)+d(u_j)) - \binom{k}{2} \\
=& M_2(G') + k\sum_{i=1}^k d(u_i) + \Psi(T_n^k; v) + (k-1)\sum_{i=1}^k d(u_i) - \binom{k}{2} \\
=& M_2(G') + (2k-1)\sum_{i=1}^k d(u_i) + \Psi(T_n^k; v) - \binom{k}{2}, \qquad (8)
\end{aligned}$$

and so for P_n^k,

$$M_2(P_n^k) = M_2(P_{n-1}^k) + (2k-1)\sum_{i=1}^k d(u_i) + \Psi(P_n^k; v) - \binom{k}{2}. \qquad (9)$$

As a special case, consider when $k = 1$ and $n = 5$. In this case, clearly $M_2(P_n^k) = 12 = \frac{1}{12}(48nk^3 - 53k^4 - 46k^3 + 5k^2 - 2k)$.

Let $f_1 = 4n^3 + 12n^2k - 36n^2 - 108nk^2 + 24nk + 80n + 44k^3 + 120k^2 - 68k - 48$ and $f_2 = -4n^3 + 36n^2k + 24n^2 - 108nk^2 - 144nk - 44n - 12k^3 - 216k^2 - 132k - 24$.

Suppose that $k + 4 \le n \le 2k$ which implies $k \ge 4$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$M_2(P_n^k) = M_2(P_{n-1}^k) + (2k-1)\sum_{i=1}^k d(u_i) + \Psi(P_n^k; v) - \binom{k}{2}$$

$$
\begin{aligned}
=&M_2(P^k_{n-1}) + (2k-1)(2kn - \frac{1}{2}(k(k+5) + (n-1)(n-2))\\
&+ \frac{1}{6}(n-k-1)(2nk + 5n - n^2 + 5k^2 - 5k - 6) + \frac{1}{2}(k(k-1))\\
=&M_2(P^k_{n-1}) - \frac{1}{24}f_1\\
=&\frac{1}{24}(-(n-1)^4 - (4k-10)(n-1)^3 + (54k^2 - 18k - 23)(n-1)^2\\
&- (44k^3 + 66k^2 - 54k - 14)(n-1) + 7k^4 + 38k^3 + 5k^2 - 26k)\\
&- \frac{1}{24}f_1\\
=&\frac{1}{24}(-(n^4 - 4n^3 + 6n^2 - 4n + 1) - (4k-10)(n^3 - 3n^2 + 3n - 1)\\
&+ (54k^2 - 18k - 23)(n^2 - 2n + 1) - (44k^3 + 66k^2 - 54k\\
&- 14)(n-1) + 7k^4 + 38k^3 + 5k^2 - 26k) - \frac{1}{24}f_1\\
=&\frac{1}{24}(-n^4 - (4k-10)n^3 + (54k^2 - 18k - 23)n^2 - (44k^3 + 66k^2\\
&- 54k - 14)n + 7k^4 + 38k^3 + 5k^2 - 26k) + \frac{1}{24}f_1 - \frac{1}{24}f_1\\
=&\frac{1}{24}((10-4k)n^3 - n^4 + (54k^2 - 18k - 23)n^2 -\\
&(44k^3 + 66k^2 - 54k - 14)n + 7k^4 + 38k^3 + 5k^2 - 26k).
\end{aligned}
$$

Suppose that $n = 2k+1 \geq k+4$ which implies $k \geq 3$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$
\begin{aligned}
M_2(P^k_{2k+1}) =&M_2(P^k_{2k}) + (2k-1)\sum_{i=1}^{k} d(u_i) + \Psi(P^k_{2k+1}; v) - \binom{k}{2}\\
=&M_2(P^k_{2k}) + (2k-1)(k^2 + \frac{1}{2}(k(k+1))\\
&- \frac{1}{6}(5k + 6k^2 + k^3 - (3k + 3k^2)(2k+1)) - \frac{1}{2}(k(k-1))
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{24}(-(2k)^4-(4k-10)(2k)^3+(54k^2-18k-23)(2k)^2-(44k^3+66k^2\\
&-54k-14)2k+7k^4+38k^3+5k^2-26k)+\frac{1}{24}(72k^3-24k^2)\\
&-\frac{1}{24}(20k+24k^2+4k^3)+\frac{1}{24}(24k^3+36k^2+12k)\\
=&\frac{1}{8}(29k^4+2k^3+3k^2-2k)\\
=&\frac{1}{24}(n^4-(12k+6)n^3+(54k^2+54k+11)n^2-\\
&(12k^3+162k^2+66k+6)n-(25k^4-70k^3-109k^2-14k)).
\end{aligned}
$$

Suppose that $2k+2 \leq n \leq 3k-1$ which implies $k \geq 3$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$
\begin{aligned}
M_2(P_n^k) =&M_2(P_{n-1}^k)+(2k-1)\sum_{i=1}^{k} d(u_i)+\Psi(P_n^k;v)-\binom{k}{2}\\
=&M_2(P_{n-1}^k)+(2k-1)(k^2+\frac{1}{2}k(k+1))+\frac{1}{6}(n^3-9n^2\\
&-6n^2+27nk^2+36nk+6n-21k^3-24k^2-33k-6)\\
&-\frac{1}{2}(k(k-1))\\
=&M_2(P_{n-1}^k)-\frac{1}{24}f_2\\
=&\frac{1}{24}((n-1)^4-(12k+6)(n-1)^3+(54k^2+54k+11)(n-1)^2\\
&-(12k^3+162k^2+66k+6)(n-1)-25k^4+70k^3+109k^2+14k)\\
&-\frac{1}{24}f_2\\
=&\frac{1}{24}((n^4-4n^3+6n^2-4n+1)-(12k+6)(n^3-3n^2+3n-1)\\
&+(54k^2+54k+11)(n^2-2n+1)-(12k^3+162k^2+66k\\
&+6)(n-1)-25k^4+70k^3+109k^2+14k)-\frac{1}{24}f_2
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{1}{24}(n^4-(12k+6)n^3+(54k^2+54k+11)n^2-(12k^3\\
&+162k^2+66k+6)n-25k^4+70k^3+109k^2+14k)+\frac{1}{24}f_2-\frac{1}{24}f_2\\
=&\frac{1}{24}(n^4-(12k+6)n^3+(54k^2+54k+11)n^2-\\
&(12k^3+162k^2+66k+6)n-(25k^4-70k^3-109k^2-14k)).
\end{aligned}
$$

Suppose that $n = 3k \geq k+4$ which implies $k \geq 2$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$
\begin{aligned}
M_2(P^k_{3k+1}) =&M_2(P^k_{3k})+(2k-1)\sum_{i=1}^{k} d(u_i)+\Psi(P^k_{3k+1};v)-\binom{k}{2}\\
=&M_2(P^k_{3k-1})+(2k-1)(k^2+\frac{1}{2}k(k+1))+-k(1-k)(2+k)\\
&+(\frac{1}{6}(-6-3k+9k^2+15k-9k^2)-\frac{1}{2}(k(k-1))\\
=&M_2(P^k_{3k-1})+4k^3-1\\
=&\frac{1}{24}((3k-1)^4-(12k+6)(3k-1)^3+(54k^2+54k+11)(3k-1)^2\\
&-(12k^3+162k^2+66k+6)(3k-1)-(25k^4-70k^3-109k^2\\
&-14k))+4k^3-1\\
=&\frac{1}{12}(91k^4-46k^3+5k^2-2k)\\
=&\frac{1}{12}(48nk^3-53k^4-46k^3+5k^2-2k).
\end{aligned}
$$

Suppose $n \geq \max(6, 3k+1)$. Then, by (9), Lemma 5.5, and Lemma 5.8,

$$
\begin{aligned}
M_2(P^k_n) =&M_2(P^k_{n-1})+(2k-1)\sum_{i=1}^{k} d(u_i)+\Psi(P^k_n;v)-\binom{k}{2}\\
=&M_2(P^k_{n-1})+(2k-1)(k^2+\frac{1}{2}k(k+1))+k^3+k^2-\frac{1}{2}(k(k-1))
\end{aligned}
$$

$$
\begin{aligned}
&= M_2(P^k_{n-1}) + 4k^3 \\
&= \frac{1}{12}(48(n-1)k^3 - 53k^4 - 46k^3 + 5k^2 - 2k) + 4k^3 \\
&= \frac{1}{12}(48nk^3 - 53k^4 - 46k^3 + 5k^2 - 2k).
\end{aligned}
$$

Thus by the principle of mathmatical induction, the lemma is verified. □

The following lemma follows from direct calculation and can be easily verified through induction.

Lemma 5.13. *Let $S_{k,n-k}$ be the k-star on $n \geq k+1$ vertices. Then*

$$M_2(S_{k,n-k}) = \frac{1}{2}((3k^2 - k)n^2 - (2k^3 + 4k^2 - 2k)n + k(2k-1)(k+1)).$$

5.3. **Sharp Upper and Lower Bounds for M_1 of k-trees.**

In this section, we determine the upper and lower bounds of M_1 of k-trees, and the corresponding extremal graphs are characterized.

Theorem 5.14. *Let T^k_n be a k-tree on $n \geq k$ vertices. Then $M_1(P^k_n) \leq M_1(T^k_n)$, and equality is reached if and only if $T^k_n \cong P^k_n$.*

Proof. For $k \leq n \leq k+1$, $T^k_n \cong K_n$, and $M_1(K_n) = n(n-1)^2$. Note that in this case, $K_n \cong P^k_n$. If $n = k+2$, then $T^k_n \cong P^k_{k+2}$, which is a k-clique bound by two simplicial vertices. Hence $M_1(P^k_{k+2}) = M_1(T^k_{k+2})$. Suppose $n = k+3$. Then $T^k_{k+3} \in \{P^k_{k+3}, S_{k,3}\}$. By routine calculations, $M_1(P^k_{k+3}) = k^3 + 7k^2 + 4k - 2$ and $M_1(S_{k,3}) = k^3 + 7k^2 + 4k$, and so the lemma holds.

We now use induction on $n \geq k+4$. If $T^k_n \cong P^k_n$, we are done. Suppose, then, that $T^k_n \ncong P^k_n$, and let $v \in S_1(T^k_n)$ be such that $N(v) = \{u_1, \ldots, u_k\}$ and $d(u_1) + \ldots + d(u_k)$ is as small as possible. Consider $G' = T_n^k - v$. By the choice of v, if $T_n^k \ncong P_n^k$ then $G' \ncong P_{n-1}^k$. Now,

$$\begin{aligned}M_1(T_n^k) =& M_1(G') + (d(v))^2 + \big((d(u_1)^2 - (d_{G'}(u_1))^2\big) + \dots \\ &+ \big((d(u_k)^2 - (d_{G'}(u_k))^2\big) \\ =& M_1(G') + k^2 + \big((d(u_1)^2 - (d(u_1) - 1)^2\big) + \dots \\ &+ \big((d(u_k)^2 - (d(u_k) - 1)^2\big).\end{aligned}$$

Thus,

$$M_1(T_n^k) = M_1(G') + k^2 + 2\sum_{i=1}^{k} d(u_i) - k. \tag{10}$$

Suppose $k + 4 \leq n \leq 2k$ which implies $k \geq 4$. Then from (10), Lemma 5.5, and Lemma 5.10,

$$\begin{aligned}M_1(T_n^k) =& M_1(G') + k^2 + 2\sum_{i=1}^{k} d(u_i) - k \\ >& M_1(P_{n-1}^k) + k^2 + 4kn - k(k+5) - (n-1)(n-2) - k \\ =& 2k(n-1)(n-3) - \frac{1}{3}((n-1)(n-2)(n-3) + 3(n-1)(n-2)) \\ &- \frac{1}{3}(k(k+1)(2k-5)) + k^2 + 4kn - k(k+5) - k \\ =& 2nk(n-2) - \frac{1}{3}(n(n-1)(n-2)) - \frac{1}{3}(k(k+1)(2k-5)) = M_1(P_n^k).\end{aligned}$$

Suppose $n = 2k + 1 > k + 3$, which implies $k \geq 3$. Then from (10), Lemma 5.5, and Lemma 5.10,

$$\begin{aligned}M_1(T_n^k) =& M_1(G') + k^2 + 2\sum_{i=1}^{k} d(u_i) - k \\ >& M_1(P_{n-1}^k) + k^2 + 2k^2 + k(k+1) - k\end{aligned}$$

$$
\begin{aligned}
&=2k(n-1)(n-3) - \frac{1}{3}(n-1)(n-2)(n-3) - \frac{1}{3}k(k+1)(2k-5) \\
&\quad + 4k^2 \\
&=2k(2k)(2k-2) - \frac{1}{3}(2k(2k-1)(2k-2)) - \frac{1}{3}(2k^3 - 3k^2 - 5k) + 4k^2 \\
&=8k^3 + 4k^2 - \frac{1}{3}(10k^3 + 9k^2 + k) \\
&=4nk^2 - \frac{1}{3}k(10k-1)(k+1) = M_1(P_n^k).
\end{aligned}
$$

Suppose $n \geq 2k+2$. Then from (10), Lemma 5.5, and Lemma 5.10,

$$
\begin{aligned}
M_1(T_n^k) =& M_1(G') + k^2 + 2\sum_{i=1}^{k} d(u_i) - k \\
>& M_1(P_{n-1}^k) + k^2 + 2k^2 + k(k+1) - k \\
=& 4(n-1)k^2 - \frac{1}{3}(k(10k-1)(k+1)) + 4k^2 \\
=& 4nk^2 - \frac{1}{3}k(10k-1)(k+1) = M_1(P_n^k).
\end{aligned}
$$

Thus, the theorem is true for all $n \geq k$ by the Principle of Mathematical Induction. □

Theorem 5.15. *Let G be k-degenerate on $n \geq k$ vertices. Then $M_1(G) \leq M_1(S_{k,n-k})$ with equality holding if and only if $G \cong S_{k,n-k}$.*

Proof. We will proceed by induction on n. If $n \in \{k, k+1\}$, then $M_1(G) \leq M_1(K_n)$ with equality holding if and only if $G \cong K_n$. Note $K_n \cong S_{k,n-k}$ in this case. Suppose that the theorem holds for k-degenerate graphs of order smaller than n and consider G, a k-degenerate graph on n vertices. Let $v \in V(G)$ such that $d(v) = \delta$ with $N(v) = \{u_1, \ldots, u_\delta\}$ and $G' = G - v$. Then G' is k-degenerate. Hence by induction, (10), Lemma 5.6, and

Lemma 5.11,

$$\begin{aligned}M_1(G) =&M_1(G') + k^2 + 2\sum_{i=1}^{\delta} d(u_i) - k\\ \leq&M_1(S_{k,n-1-k}) + k^2 + 2\sum_{i=1}^{\delta} d(u_i) - k\\ \leq&(n-1)^2k + (k^2-2k)(n-1) - k^3 + 1 + k^2 + 2k(n-1) - k\\ =&n^2k + (k^2-2k)n - k^3 + 1 = M_1(S_{k,n-k}).\end{aligned}$$

Here equality holds if and only if $\delta(G) = k$, $\sum_{i=1}^{\delta(G)} d(u_i) = k(n-1)$ and $G' \cong S_{k,n-1-k}$ i.e. $G \cong S_{k,n-k}$. □

Since all k-trees are k-degenerate, the following corollary is immediate.

Corollary 5.16. *Let T_n^k be a k-tree on n vertices. Then $M_1(T_n^k) \leq M_1(S_{k,n-k})$ with equality holding if and only if $T_n^k \cong S_{k,n-k}$.*

5.4. **Sharp Upper and Lower Bounds for M_2 for k-trees.**

In this section, we determine upper and lower bounds of M_2 for k-trees. Also, the corresponding extremal graphs will be characterized.

Theorem 5.17. *Let T_n^k be a k-tree on $n \geq k+3$ vertices. Then $M_2(P_n^k) \leq M_2(T_n^k)$, and equality is reached if and only if $T_n^k \cong P_n^k$.*

Proof. We will proceed by induction on n. There are just two k-trees on $k+3$ vertices, which are P_{k+3}^k and $S_{k,n-k}$. By simple calculations, $M_2(P_{k+3}^k) = \frac{1}{2}(k^4 + 9k^3 + 12k^2 - 8k + 2)$ and $M_2(S_{k,n-k}) = \frac{1}{2}(k^4 + 9k^3 + 12k^2 - 4k)$. Thus, the theorem holds true. Suppose that for k-trees of an order smaller than n the theorem holds, and consider T_n^k. We may assume that

$T_n^k \ncong P_n^k$. Choose $v \in S_1(T_n^k)$ with $N(v) = \{u_1, \ldots, u_k\}$ such that $\sum_{i=1}^k d(u_i)$ is minimal. Then $G' \cong T_n^k - v$, a k-tree, is not isomorphic to P_{n-1}^k by choice of v. Hence by (8),

$$M_2(T_n^k) = M_2(G') + (2k-1)\sum_{i=1}^k d(u_i) + \Psi(T_n^k; v) - \binom{k}{2}.$$

Suppose as a special case that $n = 5$ and $k = 1$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12

$$\begin{aligned} M_2(T_5^1) =& M_2(G') + d(u_1) + \Psi(T_5^1; v) \\ >& M_2(P_4^1) + 2 + 2 = 12 = M_2(P_5^1), \end{aligned}$$

as can be verified in the proof of Lemma 5.12.

Suppose then that $k + 4 \le n \le 2k$ which implies $k \ge 4$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$\begin{aligned} M_2(T_n^k) =& M_2(G') + (2k-1)\sum_{i=1}^k d(u_i) + \Psi(T_n^k; v) - \binom{k}{2} \\ >& M_2(P_{n-1}^k) + (2k-1)(2kn - \frac{1}{2}(k(k+5) + (n-1)(n-2)) \\ &+ \frac{1}{6}(n-k-1)(2nk + 5n - n^2 + 5k^2 - 5k - 6) + \frac{1}{2}(k(k-1)) \\ =& M_2(P_n^k), \end{aligned}$$

as can be verified in the proof of Lemma 5.12.

Suppose that $n = 2k + 1 \ge k + 4$ which implies $k \ge 3$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$\begin{aligned} M_2(T_n^k) =& M_2(G') + (2k-1)\sum_{i=1}^k d(u_i) + \Psi(T_n^k; v) - \binom{k}{2} \\ >& M_2(P_{2k}^k) + (2k-1)(k^2 + \frac{1}{2}k(k+1)) \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{6}(5k+6k^2+k^3-(3k+3k^2)(2k+1))-\frac{1}{2}(k(k-1)) \\
=&\frac{1}{8}(29k^4+2k^3+3k^2-2k)=M_2(P^k_{2k+1}),
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.

Suppose that $2k+2 \leq n \leq 3k-1$ which implies $k \geq 3$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$
\begin{aligned}
M_2(T^k_n) =&M_2(G')+(2k-1)\sum_{i=1}^{k} d(u_i)+\Psi(T^k_n;v)-\binom{k}{2} \\
>&M_2(P^k_{n-1})+(2k-1)(k^2+\frac{1}{2}k(k+1))+\frac{1}{6}(n^3-9n^2 \\
&-6n^2+27nk^2+36nk+6n-21k^3-24k^2-33k-6) \\
&-\frac{1}{2}(k(k-1)) \\
=&M_2(P^k_n),
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.

Suppose that $n=3k \geq k+4$ which implies $k \geq 2$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$
\begin{aligned}
M_2(T^k_n) =&M_2(G')+(2k-1)\sum_{i=1}^{k} d(u_i)+\Psi(T^k_n;v)-\binom{k}{2} \\
>&M_2(P^k_{3k-1})+(2k-1)(k^2+\frac{1}{2}(k(k+1))+-k(1-k)(2+k) \\
&+(\frac{1}{6}(-6-3k+9k^2+15k-9k^2)-\frac{1}{2}(k(k-1)) \\
=&M_2(P^k_{3k}),
\end{aligned}
$$

as can be verified in the proof of Lemma 5.12.

Suppose $n \geq \max(6, 3k+1)$. Then, by (9), Lemma 5.5, Lemma 5.8, and Lemma 5.12,

$$\begin{aligned}
M_2(T_n^k) =& M_2(G') + (2k-1)\sum_{i=1}^{k} d(u_i) + \Psi(T_n^k; v) - \binom{k}{2} \\
>& M_2(P_{n-1}^k) + (2k-1)(k^2 + \frac{1}{2}k(k+1)) + k^3 + k^2 - \frac{1}{2}(k(k-1)) \\
=& M_2(P_n^k),
\end{aligned}$$

as can be verified in the proof of Lemma 5.12.

Thus by the Principle of Mathematical Induction, $M_2(P_n^k) \leq M_2(T_n^k)$ with equality holding if and only if $T_n^k \cong P_n^k$. $\square$

Theorem 5.18. *Let T_n^k be a k-tree on $n \geq k$ vertices. Then $M_2(T_n^k) \leq M_2(S_{k,n-k})$ with equality holding if and only if $T_n^k \cong S_{k,n-k}$.*

Proof. We will proceed by induction on n. If $n \in \{k, k+1\}$, then $M_2(T_n^k) \leq M_2(K_n)$ with equality holding if and only if $G \cong K_n$. Note $K_n \cong S_{k,n-k}$ in this case. Suppose that the theorem holds for k-trees of smaller order and consider T_n^k, a k-tree on n vertices. Let $v \in S_1(T_n^k)$ with $N(v) = \{u_1, \ldots, u_k\}$ and $G' = T_n^k - v$, which is a k-tree. By (8),

$$M_2(T_n^k) = M_2(G') + (2k-1)\sum_{i=1}^{k} d(u_i) + \Psi(T_n^k; v) - \binom{k}{2}.$$

Thus by induction, Lemma 5.6, Lemma 5.9, and Lemma 5.13, we have

$$\begin{aligned}
M_2(T_n^k) =& M_2(G') + (2k-1)\sum_{i=1}^{k} d(u_i) + \Psi(T_n^k; v) - \binom{k}{2} \\
\leq& M_2(S_{k,n-1-k}) + (2k-1)(nk-k) + (n-k-1)k^2 - \frac{1}{2}(k-1)k
\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{2}((3k^2-k)(n-1)^2-(2k^3+4k^2-2k)(n-1)+k(2k-1)(k+1))\\
&+(2k-1)(nk-k)+(n-k-1)k^2-\frac{1}{2}(k-1)k\\
=&M_2(S_{k,n-k})+2nk^2-2k^2-nk+k+\frac{1}{2}(-6nk^2+3k^2+2nk-k)\\
&+\frac{1}{2}(2k^3+4k^2-2k)+nk^2-k^3-k^2+\frac{1}{2}(k-k^2)\\
=&M_2(S_{k,n-k}).
\end{aligned}
$$

Here equality is obtained if and only if $G' \cong S_{k,n-1-k}$ and $T_n^k \in \mathcal{J}^k$. Hence equality holds when $T_n^k \cong S_{k,n-k}$. Thus the theorem holds by the Principle of Mathematical Induction. $\square$

The upper bound for M_1 values given in Theorem 5.15 applies to k-degenerate graphs, a generalization of k-trees. However the proof techniques presented here are not sufficient to demonstrate similar results for a lower bound of M_1 values of maximally k-degenerate graphs and an upper bound of M_2 values of k-degenerate graphs. It may be interesting to show that for a maximally k-degenerate graph G and a k-degenerate graph G', $M_i(P_n^k) \leq M_i(G)$ for $1 \leq i \leq 2$ and $M_2(G') \leq M_2(S_{k,n-k})$.

6. The Zagreb Indices of Tree-Like k-trees

In 2010 Hou et al. characterized the Zagreb indices for maximal outerplanar graphs and determined the unique maximal outerplanar graph that obtains minimum M_1, M_2 values respectively, as well as maximum M_1, M_2 values respectively. As mentioned in Chapter 5, they determined the following:

Theorem 6.1. [29] *Let G be a maximal outerplaner graph on $n \geq 4$ vertices. Then*

(i) $M_1(G) \geq 16n - 38$, *with equality holding if and only if* $G \cong P_n^2$.

(ii) $M_2(G) \geq 32n - 100$, *with equality holding if and only if* $G \cong P_n^2$.

Theorem 6.2. [29] *Let G be a maximal outerplanar graph on $n \geq 4$ vertices. Then*

(i) *When* $n = 6$, $M_1(G) \leq 60$ *with equality if and only if* $G \cong S_6^2$ *or* D_6^2.

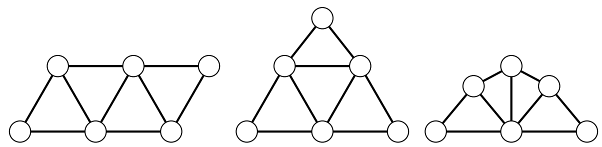
(ii) *When* $n \neq 6$, $M_1(G) \leq n^2 + 7n - 18$ *with equality if and only if* $G \cong S_n^2$.

Theorem 6.3. [29] *Let G be a maximal outerplanar graph on $n \geq 4$ vertices.*

(i) *When* $n = 6$, $M_2(G) \leq 96$ *with equality if and only if* $G \cong D_6^2$.

(ii) *When* $n \neq 6$, $M_2(G) \leq 3n^2 + n - 19$ *with equality if and only if* $G \cong S_n^2$.

It has been shown that a graph G is a maximal outerplanar graph if and only if G is a tree-like 2-tree. By making this connection, it is a natural question to generalize the works of Hou et al. to tree-like k-trees. In Chapter 5, we deduced sharp upper and lower bounds of M_1 and M_2 for k-trees and showed that the k-path (respectively the k-star) uniquely obtains the sharp lower bound (respectively the sharp upper bound) of M_1 and M_2.



P_6^2 D_6^2 S_6^2

FIGURE 10. The 2-path, 2-diamond, and 2-star on 6 vertices

As the k-path is tree-like, it is clear that the sharp lower bounds of M_1 and M_2 for tree-like k-trees are obtained uniquely by the k-path. Hence, to generalize the results of Hou et al., we need to only consider upper bounds of M_1 of M_2 for tree-like k-trees.

6.1. **Some Lemmas.**

In this section, we give some lemmas that will be relied upon in subsequent sections.

Define $\mathcal{G}_n^k$ to be the class of tree-like k-trees as follows: Let $T_n^k \in \mathcal{G}_n^k$. Then there exists a vertex $v \in S_1(T_n^k)$ such that for any vertex $x \in V(T_n^k) - v$, $|N(x) \cap N(v)| \geq k-1$.

Lemma 6.4. *Let T_n^k be a tree-like k-tree on $n \geq k+2$ vertices and $v \in S_1(T_n^k)$ with $N(v) = \{u_1, \ldots, u_k\}$. Then $\sum_{i=1}^k d(u_i) \leq (k-1)(n-1)+(k+1)$, with equality holding when $T_n^k \in \mathcal{G}_n^k$.*

Proof. As $v \in S_1(T_n^k)$, $G[N(v)] \cong K_k$. By Fact 2.21 $|\cap_{i=1}^k N(u_i)| = 2$, so we may assume that $\{v, x\} = \cap_{i=1}^k N(u_i)$ where $x \neq v$. There are $n-(k+2)$ vertices in $V' = V(T_n^k) - \{v, x, u_1, u_2, \ldots, u_k\}$. It is clear to see that $\sum_{i=1}^k d(u_i)$ attains maximality if and only if for every $y \in V'$, $|N(y) \cap \{v_1, \ldots, v_k\}| = k-1$.

Now $\sum_{i=1}^k d(u_i) = \sum_{u \in V(T_n^k)} |N(v) \cap N(u)|$. Thus $\sum_{i=1}^k d(u_i) = |N(v)| + |N(v) \cap N(x)| + \sum_{i=1}^k |N(v) \cap N(u_i)| + \sum_{y \in V'} |N(v) \cap N(y)| \leq 2k + k(k-1) + (n-k-2)(k-1) = (k-$

$1)(n-1)+(k+1)$, and equality holds if and only if $|N(v)\cap N(y)|=k-1$ for each $y\in V'$. In other words, equality holds if and only if $T_n^k\in\mathcal{G}_n^k$. $\square$

The following lemmas demonstrate the Zagreb indices for the specific tree-like k-trees, the k-star and k-diamond and may be deduced through routine calculations.

Lemma 6.5. *Let S_n^k the k-spiral on n vertices. Then*

$$M_1(S_n^k)=(k-1)n^2+(k^2+3)n-(k^3+k^2+2k+2).$$

Lemma 6.6. *For $k+1\leq n\leq 2k+2$, $M_1(D_n^k)=M_1(S_n^k)$.*

Proof. The k-diamond D_n^k is only defined for $k+1\leq n\leq 2k+2$. Let $V(D_n^k)=\{v_1,\dots,v_{k+1}\}$ $\cup\{u_1,\dots,u_j\}$ for some $j\in\{1,\dots,k+1\}$ where $n=k+1+j$, and $G[\{v_1,\dots,v_{k+1}\}]\cong K_{k+1}$. For any $u\in S_1(D_n^k)$, there exists a unique vertex $v\in V(\{v_1,\dots,v_k\})$ such that $v\notin N(u)$. Without loss of generality, $v_i\notin N(u_i)$ for $1\leq i\leq j$. Thus, $d(v_i)=n-2$ for $1\leq i\leq j$. That is, there are $j=n-(k+1)$ simplicial vertices, $n-(k+1)$ vertices of $\{v_1,\dots,v_k\}$ of degree $n-2$, and $k+1-(n-(k+1))=2k+2-n$ vertices of $\{v_1,\dots,v_k\}$ of degree $n-1$. Hence

$$\begin{aligned}M_1(D_n^k)=&(n-(k+1))k^2+(n-(k+1))(n-2)^2+(2k+2-n)(n-1)^2\\=&kn^2-n^2+nk^2+3n-k^3-k^2-2k-2\\=&(k-1)n^2+(k^2+3)n-(k^2+2)(k+1)=M_1(S_n^k).\end{aligned}$$

$\square$

Lemma 6.7. *Let S_n^k be the k-spiral on n vertices. Then*

$$M_2(S_n^k)=(k^2-1)n^2-(k^3-k^2-k-3)n-(3k^2+2k+3)+\binom{k-1}{2}(n-1)^2.$$

Lemma 6.8. *Let D_n^k be the k-diamond on n vertices. Then*

$$M_2(D_n^k) = \frac{1}{2}((3k^2 - 3k + 1)n^2 - (2k^3 - 6k + 3)n - 4k^2 - 2k + 2).$$

For the remainder of the chapter, let T_n^k be a tree-like k-tree such that $v \in S_1(T_n^k)$ and $N(v) = U = \{u_1, \ldots, u_k\}$. Let $N_0(u_i) = N(u_i) - N[v]$, and let

$$\Psi(T_n^k; v) = \sum_{x \in N_0(u_1)} d(x) + \sum_{x \in N_0(u_2)} d(x) + \ldots + \sum_{x \in N_0(u_k)} d(x).$$

Let $v' \in S_1(T_n^k) - v$ and $N(v') = \{u_1', \ldots, u_k'\}$. Arrange the vertices of $N(v')$ such that $u_i' \in N(v)$ for $1 \leq i \leq t$ and $|N(v) \cap N(u_{i-1}'| \geq |N(v) \cap N(u_i')|$ for $t+1 \leq i \leq k$. Then for $n \geq k+2$ and $v' \in S_1(T_n^k) - v$,

$$\Psi(T_n^k; v) = \Psi(T_n^k - v'; v) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U|. \tag{11}$$

As mentioned in Chapter 5, $d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U|$ is a summand with k summands with at least t summands of value k and at most $k-t$ summands of value at most $k-1$. It is clear then that $d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U|$ is maximized when t is maximized.

Lemma 6.9. *Let T_n^k be a tree-like k-tree on $k+1 \leq n \leq 2k+2$ vertices. Then $\Psi(T_n^k; v) \leq k^2(n-k-1)$ with equality holding if and only if $T_n^k \cong D_n^k$.*

Proof. Proceed by induction on the number of vertices. Suppose $n = k+1$, then $T_n^k \cong K_{k+1}$. Then for $v \in S_1(T_n^k)$, $\Psi(T_n^k; v) = 0$, and thus the theorem holds. Suppose that the theorem is true for tree-like k-trees on $k+1 \leq n' < n$ vertices, and consider T_n^k, a tree-like k-tree on n vertices.

Let $v' \in S_1(T_n^k) - v$. Then by (11),

$$\Psi(T_n^k; v) = \Psi(T_n^k - v'; v) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U|.$$

As T_n^k is tree-like $t \leq k-1$. If $t = k-1$, then $\sum_{i=t+1}^{k} |N(u_i') \cap U| = |N(u_k') \cap U| \leq k$. Also by induction, $\Psi(T_n^k - v'; v) \leq \Psi(D_{n-1}^k; v) = k^2(n-k-2)$. Hence

$$\begin{aligned}\Psi(T_n^k; v) =& \Psi(T_n^k - v'; v) + d(v')t + \sum_{i=t+1}^{k} |N(u_i') \cap U| \\ \leq& \Psi(D_{n-1}^k; v) + k(k-1) + k \\ =& k^2(n-k-2) + k^2 \\ =& k^2(n-k-1),\end{aligned}$$

with equality holding if and only if $T_n^k - v' \cong D_{n-1}^k$, $t = k-1$, and $|N(u_k') \cap N(v)| = k$. Hence equality holds if and only if $T_n^k \cong D_n^k$. By the Principle of Mathematical Induction, the theorem holds. $\square$

6.2. **Sharp Upper Bounds of M_1 of Tree-like k-trees.**

In this section, we determine the upper bounds of M_1 of tree-like k-trees, and the corresponding extremal graphs are characterized.

Theorem 6.10. *Let T_n^k be a tree-like k-tree on $n \geq k+1$ vertices. Then $M_1(T_n^k) \leq (k-1)n^2 + (k^2+3)n - (k^2+2)(k+1)$. Equality is reached if and only if $T_n^k \in \{D_n^k, S_n^k\}$. In particular, if $n \geq 2k+3$, then equality is reached if and only if $T_n^k \cong S_n^k$.*

Proof. Proceed by induction on n. If $n = k+1$, then $T_n^k \cong K_{k+1}$, and $M_1(K_{k+1}) = (k+1)k^2 = 2k^2+k^3-k^2 = 2k^2+(k-1)k^2 = 2k^2+(k-1)(n-1)^2+(n-(k+1))(k+1)^2$. Suppose $n = k+2$. Then T_n^k is a k-clique bound by two simplicial vertices. Hence $M_1(T_n^k) = 2k^2+k(k+1)^2$, and it may be verified that $(k-1)n^2+(k^2+3)n-(k^3+k^2+2k+2) = k^3+2k^2+2k^2+k = 2k^2+k(k+1)^2$.

Suppose the theorem is true for tree-like k-trees on $k+1 \leq n' < n$ vertices, and consider T_n^k vertices. Let $v \in S_1(T_n^k)$ where $N(v) = \{u_1, \ldots, u_k\}$. Let $G' = T_n^k - v$. By Lemma 6.5

and the inductive hypothesis,

$$\begin{aligned}
M_1(G') \leq & M_1(S^k_{n-1}) \\
= & (k-1)(n-1)^2 + (k^2+2)(k+1) \\
= & (k-1)n^2 + (5-2k+k^2)n - (k^3+2k^2+k+6).
\end{aligned}$$

Now $d_{G'}(u_i) = d(u_i) - 1$ for $1 \leq i \leq k$ and $d_{G'}(x) = d(x)$ for all $x \in V - \{u_1, \ldots, u_k\}$. Then by Lemma 6.4,

$$\begin{aligned}
M_1(T^k_n) = & M_1(G') + (d(v))^2 + \big((d(u_1))^2 - (d_{G'}(u_1))^2\big) + \ldots + \big((d(u_k))^2 - (d_{G'}(u_k))^2\big) \\
= & M_1(G') + \big((d(u_1))^2 - (d(u_1)-1)^2\big) + \ldots + \big((d(u_k))^2 - (d(u_k)-1)^2\big) \\
= & M_1(G') + k^2 + 2[d(u_1) + \ldots + d(u_k)] - k \\
\leq & M_1(S^k_{n-1}) + k^2 - k + 2\left((k-1)(n-1) + (k+1)\right) \\
= & (k-1)n^2 + (5-2k+k^2)n - (k^3+2k^2+k+6) + k^2 - k + 2kn - 2n + 4 \\
= & (k-1)n^2 + (k^2+3)n - (k^3+k^2+2k+2).
\end{aligned}$$

Here, equality is reached only when $G' \cong S^k_{n-1}$ or $G' \cong D^k_{n-1}$, and $T^k_n \in \mathcal{G}^k_n$. If $G' \cong S^k_{n-1}$ and $T^k_n \in \mathcal{G}^k_n$, then clearly $T^k_n \cong S^k_n$.

Suppose $G' \cong D^k_{n-1}$. If $G' \cong D^k_{2k+2}$ and $T^k_n \in \mathcal{G}^k_n$, then T^k_n is not tree-like. Hence $n \leq 2k+2$, and clearly $T^k_n \cong D^k_n$. $\square$

6.3. **Sharp Upper Bounds of M_2 for Tree-like k-trees.**

In this section, we determine upper bounds of M_2 for tree-like k-trees on less than $2k+2$ vertices and characterize the corresponding extremal graphs. Additionally, we state a conjecture for sharp upper bounds of M_2 for tree-like k-trees on at least $2k+3$ vertices.

Theorem 6.11. *Let T_n^k be a tree-like k-tree on $k+1 \leq n \leq 2k+2$ vertices. Then $M_2(T_n^k) \leq M_2(D_n^k)$ with equality holding if and only if $T_n^k \cong D_n^k$.*

Proof. Proceed by induction on n. If $n = k+1, k+2$. Suppose that for tree-like k-trees on $k+1 \leq n' < n \leq 2k+2$ vertices, and let T_n^k be a tree-like k-tree on $n \leq 2k+2$ vertices. Let $v \in S_1(T_n^k)$ and $N(v) = \{u_1, \ldots, u_k\}$. From Chapter 5, we know that

$$M_2(T_n^k) = M_2(T_n^k - v) + (2k-1)\sum_{i=1}^{k} d(u_i) + \Psi(T_n^k; v) - \binom{k}{2}. \tag{12}$$

Thus by Lemma 6.8, Lemma 6.4, and Lemma 6.9

$$\begin{aligned}
M_2(T_n^k) &\leq M_2(D_{n-1}^k) + (2k-1)((k-1)(n-1)+k+1) + k^2(n-k-1) - \binom{k}{2} \\
&= M_2(D_{n-1}^k) + \frac{1}{2}(6nk^2 - 6nk + 2n - 2k^3 - 3k^2 + 9k - 4) \\
&= \frac{1}{2}(3k^2 - 3k + 1)(n^2 - 2n + 1) - (2k^3 - 6k + 3)(n-1) - 4k^2 - 2k + 2) \\
&+ \frac{1}{2}(6nk^2 - 6nk + 2n - 2k^3 - 3k^2 + 9k - 4) \\
&= \frac{1}{2}(3k^2 - 3k + 1)n^2 - (2k^3 - 6k + 3)n - 4k^2 - 2k + 2) \\
&+ \frac{1}{2}((3k^2 - 3k + 1)(-2n + 1) + (2k^3 - 6k + 3)) + \\
&+ \frac{1}{2}(6nk^2 - 6nk + 2n - 2k^3 - 3k^2 + 9k - 4) \\
&= M_2(D_n^k) - \frac{1}{2}(6nk^2 - 6nk + 2n - 2k^3 - 3k^2 + 9k - 4) \\
&+ \frac{1}{2}(6nk^2 - 6nk + 2n - 2k^3 - 3k^2 + 9k - 4) \\
&= M_2(D_n^k).
\end{aligned}$$

Hence, by the Principle of Mathematical Induction, the theorem holds. $\square$

The k-diamond is only defined for $k \leq n \leq 2k+2$ vertices. We strongly believe that for $n \geq 2k+3$, the k-spiral uniquely obtains the strong upper bound for M_2 among tree-like

k-trees. However, the techniques presented here are not sufficient to prove that such is the case. Instead we state the following conjecture.

Conjecture 6.12. *Let T_n^k be a tree-like k-tree on n vertices such that $T_n^k \ncong D_n^k$. Then $M_2(T_n^k) \le M_2(S_n^k)$ with equality holding if and only if $T_n^k \cong S_n^k$.*

We believe that the ideas presented in Chapter 7, once generalized to k-trees, will provide a framework to prove Conjecture 6.12.

7. Tree Genealogies

Continuing his investigation of the independence polynomials of trees in 1995, Wingard determined sharp lower and upper bounds of f_s for trees for $s \geq 0$ on n vertices and characterized the unique trees that obtain these bounds.

Theorem 7.1. [44] *Let T_n be a tree with n vertices. Then for any $s \geq 2$, $\binom{n-s+1}{s} \leq f_s(T_n) \leq \binom{n-1}{s}$.*

Theorem 7.2. [44] *Let T_n be a tree with $n \geq 2s$ vertices for $s \geq 3$. If $f_s(T_n) = \binom{n-s+1}{s}$, then $T_n \cong P_n$.*

Theorem 7.3. [44] *Let T_n be a tree with n vertices. If $f_s(T_n) = \binom{n-1}{s}$ for $3 \leq s \leq n-1$, then $T_n \cong S_n$.*

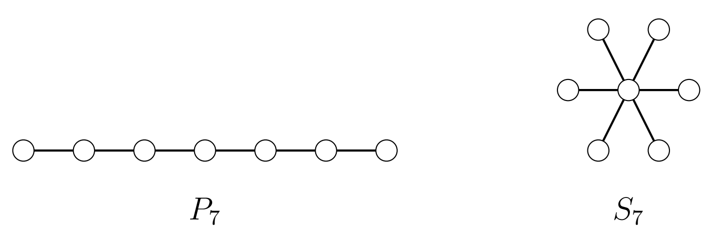


Figure 11. The path and star on 7 vertices

As stated in Chapter 5, Das and Gutman characterized the Zagreb indices for trees and determined the unique tree that obtains minimum M_1 and M_2 values respectively, as well as maximum M_1 and M_2 values respectively in 2004.

Theorem 7.4. [12, 20] *Let T be any tree of order n. Then*

(i) $4n-6 \leq M_1(T) \leq n^2-n$, *the left equality holds if and only if $T \cong P_n$, and the right equality holds if and only if $T \cong S_n$.*

(ii) $4n-8 \leq M_2(T) \leq n^2-2n+1$, *the left equality holds if and only if $T \cong P_n$ and the right equality holds if and only if $T \cong S_n$.*

From Wingard and Das and Gutman we deduce that P_n and S_n can be thought of as the extremal trees in regards to f_s, M_1, and M_2.

Let T be a tree. Define a starring triple r to be $r = \{v, u, x\}$ where $v \in S_1(T)$, $u \in N(v)$, and $x \in V(T) - \{v, u\}$. Let $R_1(T)$ be the set of starring triples of T. For $r \in R_1(T)$, $T(r)$ is the tree with $V(T(r)) = V(T)$ and $E(T(r)) = (E(T) \cup \{vx\}) - \{vu\}$. Let $g_1 : R_1 \to \mathbb{Z}$ be such that for $r = \{v, u, x\} \in R_1(T)$, $g_1(r) = d_T(x) - d_T(u)$. If $g_1(r) \geq 0$, then $T(r)$ is said to be a 1-descendant of T. If $g_1(r) \leq -2$, then $T(r)$ is said to be a 1-ancestor of T.

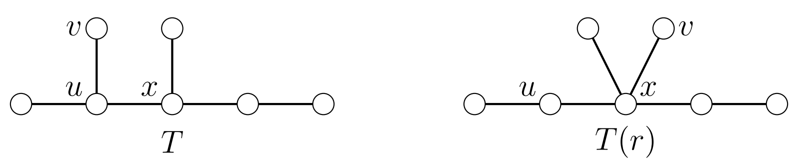


FIGURE 12. A tree T and $T(r)$

Theorem 7.5. *Let T and T' be trees. Then T is an* 1*-ancestor of T' if and only if T' is a* 1*-descendant of T.*

Proof. Let $r = \{v, u, x\}$ be a starring triple of T, and suppose $g_1(r) \geq 0$. Then $T(r)$ is a 1-descendant of T, and $r' = \{v, x, u\}$ is a starring triple of $T(r)$. Note that $T(r)(r') \cong T$. Now $d_{T(r)}(u) = d_T(u) - 1$ and $d_{T(r)}(x) = d_T(x) + 1$. Thus $g_1(r') = d_{T(r)}(u) - d_{T(r)}(x) =$

$d_T(u) - d_T(x) - 2 \leq -2$. Thus T is a 1-ancestor of $T(r)$. The argument is reversible to show that if $T(r)$ is a 1-ancestor of T, then T is a 1-descendant of $T(r)$. $\square$

It is easy to see that any tree $T \ncong S_n$ has a 1-descendant. Hence there is a sequence of trees $\{T_i\}_{i=0}^{\beta}$ such that $T_0 \cong T$, $T_\beta \cong S_n$, and T_{i+1} is a 1-descendant of T_i for $i \leq 0 \leq \beta - 1$. Then we say that T_i is a 1^i-descendant of T for $1 \leq i \leq \beta$.

Now there are trees that have no 1-ancestor as in Figure 13. Hence there is a sequence of trees $\{T_i\}_{i=0}^{\beta_2}$ such that $T_0 \cong T$, $T_{\beta_2} \cong T'$ where T' is a tree that has no 1-ancestor, and T_{i+1} is a 1-ancestor of T_i for $0 \leq i \leq \beta_2$.

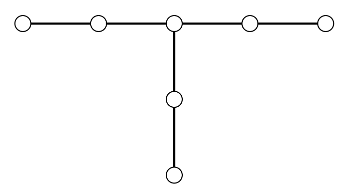


FIGURE 13. A tree with no 1-ancestor

For the tree T, we may generalize the starring triple as follows: Let $r = \{v, u, x\}$ be such that

(i) $v \in S_1(T)$

(ii) the vu-path P is of length p,

(iii) for any $y \in V(P) - \{v, u\}$, $d(y) = 2$,

(iv) $x \in V(T) - V(P)$.

Let $R_p(T)$ be the set of such triples. For $r \in R_p(T)$, $T(r)$ is the tree with $V(T(r)) = V(T)$ and $E(T(r)) = (E(T) \cup \{y'x\}) - \{y'u\}$ where $y' \in N(u) \cap V(P)$. Define $g_p : R_p \to \mathbb{Z}$ such that $g_p(r) = d_T(x) - d_T(u)$. If $g_p(r) \geq 0$, then $T(r)$ is said to be a p-descendant of T. If $g_p(r) \leq -2$, then $T(r)$ is said to be a p-ancestor of T.

Theorem 7.6. *Let T and T' be trees. Then T is a p-ancestor of T' if and only if T' is a p-descendant of T.*

Proof. Let $r = \{v, u, x\} \in R_p(T)$, and suppose $g_p(r) \geq 0$. Then $T(r)$ is a p-descendant of T, and $r' = \{v, x, u\} \in R_p(T(r))$. Note that $T(r)(r') \cong T$. Now $d_{T(r)}(u) = d_T(u) - 1$ and $d_{T(r)}(x) = d_T(x) + 1$. Thus $g_p(r') = d_{T(r)}(u) - d_{T(r)}(x) = d_T(u) - d_T(x) - 2 \leq -2$. Thus T is a p-ancestor of $T(r)$. The argument is reversable to show that if $T(r)$ is a p-ancestor of T, then T is a p-descendant of $T(r)$. $\square$

Let T and T' be trees, and suppose that T' is the p_1-descendant of a p_2-descendant of T. Then we say that T' is a p_1, p_2-descendant of T, and T is a p_2, p_1-ancestor of T'. Suppose that $p_1 = p_2$. Then T' is a p_1^2-descendant of T, and T is a p_1^2-ancestor of T'.

Theorem 7.7. *Let $T \ncong P_n$ be a tree on n vertices. Then P_n is a $p_1^{i_1}, p_2^{i_2}, \ldots, p_j^{i_j}$-ancestor of T for some $j \geq 1$.*

Proof. All we must show is that for any tree with more than two leaves, there exists a p-ancestor of T for some $p \geq 1$ such that this p-ancestor has fewer leaves than T. Let $v \in S_1(T)$. As $T \ncong P_n$, there is a vertex u such that $d(u) \geq 3$, and let $S = \{u | d(u) \geq 3\}$. Choose $u \in S$ such that $d(v, u) < d(v, u')$ for all $u' \in S - u$, and let P be the vu-path in T. Additionally, let $y' \in N(u) \cap V(P)$ and $x \in S_1(T) - v$. Clearly $x \notin V(P)$. Thus $r = \{v, u, x\} \in R_p(T)$ where p is the length of P, and $g_p(r) = d(x) - d(u) \leq 1 - 3 = -2$. Thus $T(r)$ is a p-ancestor of T.

If $|S_1(T(r))| \geq 3$, then $T(r) \ncong P_n$, and so $T(r)$ has a p_1-ancestor $T'(r)$ with fewer leaves than $T(r)$. Hence $T'(r)$ is a pp_1-ancestor of T. A reiteration of this process yields a tree with two leaves, i.e. P_n, that is a $p_1^{i_1}, p_2^{i_2}, \ldots, p_j^{i_j}$-ancestor of T for some $j \geq 1$. $\square$

By Theorem 7.7, we see that for a given tree T, there is a sequences of trees $\{T_i\}_{i=0}^{\beta_3}$ such that $T_0 \cong T$, $T_{\beta_3} \cong P_n$, and T_{i+1} is a p_{i+1}-ancestor of T_i for $0 \leq i \leq \beta_3$. We may now construct a sequence of trees that we define as a genealogy.

Definition 7.8. Let T be a tree on n vertices. Then the sequence of trees on n vertices $\{T_i\}_{i=0}^{\beta}$ satisfying

(i) $T_0 \cong P_n$,

(ii) $T_{\beta_2} \cong T$, for some β_2, $0 \leq \beta_2 \leq \beta$,

(iii) T_{i+1} is a p_{i+1}-descendant of T_i for $0 \leq i \leq \beta - 1$,

(iv) $T_\beta \cong S_n$,

is said to be a *genealogy of* T.

The definition of a genealogy of a tree says that for a given tree, T, there is a sequence of trees starting with P_n and ending with S_n such that T is a member of this sequence. Additionally, given a tree T_i in this sequence, T_{i+1} is a p_{i+1}-descendant for $0 \leq i \leq \beta - 1$. In the subsequent sections, we will show that a genealogy of a tree creates a partial ordering of trees with respect to f_s and M_1.

7.1. **Independent Sets of a Tree and Its Descendants.**

By investigating the relationship between a tree and other trees in a genealogy of that tree, we may generalize Theorem 7.1.

First define the family $\mathcal{F}$ of trees as follows; let $T \in \mathcal{F}$. Then $V(T) = \{u_1, \ldots, u_{n_1}\}$ $\cup\{v_1, \ldots, v_{n_2}\}$ where $v_i \in S_1(T)$ for $1 \leq i \leq n_2$, $G[\{u_1, \ldots, u_{n_1}\}] \cong P_{n_1}$, $N(v_i) \subseteq \{u_1, \ldots, u_{n_1}\}$ for $1 \leq i \leq n_2$. Thus we may state the following lemma.

Lemma 7.9. *Let T be a tree in $\mathcal{F}$ such that $r = \{u_1, v_1, v_{n_1}\} \in R_1(T)$ and $g_1(r) \geq 0$. Then $f_s(T) \leq f_s(T(r))$ for $s \geq 0$.*

Proof. For the vertex set S, let $f_{s,S}(T)$ denote the number of independent sets of cardinality s in T containing S, and let $f_{s,\bar{S}}(T)$ denote the number of independent sets of cardinality s in T not containing S. Then $f_s(T) = f_{s,S}(T) + f_{s,\bar{S}}(T)$.

Let $v_1 = v$, $u_1 = u$, and $u_{n_1} = x$. Then $r = \{v, u, x\}$, and let I be an independent set of T. If $\{v, x\} \nsubseteq I$, then I is an independent set of $T(r)$. Thus $f_{s,\overline{\{v,x\}}}(T) \leq f_{s,\overline{\{v,x\}}}(T(r))$ for $s \geq 0$.

Let I be an independent set of T such that $\{v, x\} \subseteq I$. Then I is not independent in $T(r)$. Note that as $T(r)$ is a 1-descendant of T, $d(x) \geq d(u)$, and thus $|N(x)| \geq |N(u)|$. Also it is clear that for any subset S' of $N(x)$, $S' \nsubseteq I$. Let $S = (N(u) - v) \cap I$. Then as $|S| \leq |N(u)| - 1$ there exists at least one set $S' \subseteq N(x)$ such that $|S'| = |S|$ and $x' \in S_1(T)$ for all $x' \in S'$. Hence there exists at least one independent set I' of $T(r)$ such that $I' = (I - (\{x\} \cup S)) \cup (\{u\} \cup S')$ for some $S' \subseteq N(x)$ such that $|S| = |S'|$ and $x' \in S_1(T)$ for all $x' \in S'$. Note that $|I'| = |I|$, and I' is not an independent set of T. Thus $f_{s,\{v,x\}}(T) \leq f_{s,\{v,x\}}(T(r))$ for $s \geq 0$.

Then $f_s(T) = f_{s,\{v,x\}}(T) + f_{s,\overline{\{v,x\}}}(T) \leq f_{s,\{v,x\}}(T(r)) + f_{s,\overline{\{v,x\}}}(T(r)) = f_s(T(r))$ for $s \geq 0$. $\square$

Now Theorem 7.1 and Theorem 7.3 of Wingard may now be extended.

Theorem 7.10. *Let T be a tree, and $r \in R_1(T)$. If $T(r)$ is a 1-descendant of T, then $f_s(T) \leq f_s(T(r))$ for $s \geq 0$.*

Proof. Proceed by induction on the number of vertices n. There is nothing to show for $1 \leq n \leq 3$. Suppose that $n = 4$, then $T \in \{P_4, S_4\}$. Now P_4 is an ancestor of S_4, and $f_s(P_4) \leq f_s(S_4)$ for $s \geq 0$. Suppose that for trees T and T' on $4 \leq n' < n$ vertices, such that T' is a descendant of T, $f_s(T) \leq f_s(T')$, and consider T a tree on n vertices.

Let $r = \{v, u, x\} \in R$ such that $g_1(r) \geq 0$. Then $T(r)$ is a descendant of T, and let P be the ux-path in T. Suppose that there for every $v' \in S_1(T) - v$ with support vertex u', $u' \in V(P)$. Then $T \in \mathcal{F}$, and by Lemma 7.9 $f_s(T) \leq f_s(T(r))$. Thus we may assume that there exists $v' \in S_1(T) - v$ with support vertex u' such that $u' \notin V(P)$. By the vertex reduction identity,

$$f_s(T) = f_s(T - v') + f_{s-1}(T - N[v'])$$

$$f_s(T(r)) = f_s(T(r) - v') + f_{s-1}(T(r) - N[v']).$$

As $v' \notin \{v, u, x\}$, $T(r) - v'$ is a descendant of $T - v'$. Thus, by induction $f_s(T - v') \leq f_s(T(r) - v')$. Now $T - N[v']$ and $T(r) - N[v']$ are forrests on l connected components. Also there are $l - 1$ connected components of $T - N[v']$, H_i for $1 \leq i \leq l - 1$ and $l - 1$ connected components of $T(r) - N[v']$, $H(r)_i$ for $i \leq i \leq l - 1$ such that $\cup_i^{l-1} H_i \cong \cup_i^{l-1} H(r)_i$. Then $\Pi_i^{l-1} f_s(H_i) = \Pi_i^{l-1} f_s(H(r)_i)$. Let $H \cong T - \cup_i^{l-1} H_i$ and $H(r) \cong T(r) - \cup_i^{l-1} H(r)_i$. Then $f_s(T - N[v']) = (\Pi_i^{l-1} f_s(H_i)) f_s(H)$, and $f_s(T(r) - N[v']) = (\Pi_i^{l-1} f_s(H(r)_i)) f_s(H(r))$ for $s \geq 0$.

Note that $\{v, u, x\} \subseteq V(H)$, $\{v, u, x\} \subseteq V(H(r))$, and $H - v \cong H(r) - v$. That is, $H(r)$ is a 1-descendant of H. Then, by induction $f_s(H) \leq f_s(H(r))$ for $s \geq 0$. Hence $f_s(T) = f_s(T - v') + f_{s-1}(T - N[v']) = f_s(T - v') + (\Pi_i^{l-1} f_{s-1}(H_i)) f_{s-1}(H) \leq f_s(T(r) - v') + (\Pi_i^{l-1} f_{s-1}(H(r)_i)) f_{s-1}(H(r)) = f_s(T(r) - v') + f_{s-1}(T(r) - N[v']) = f_s(T(r))$. Thus by induction, the theorem holds. $\square$

Theorem 7.11. *Let T be a tree and T' be a 1^j-descendant of T for some $j \geq 1$. Then $f_s(T) \leq f_s(T')$ for $s \geq 0$.*

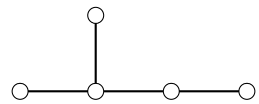


FIGURE 14. T_5

Theorem 7.12. *Let T and T' be trees on n vertices, and let T' be a p-descendant of T for some $p \geq 1$. Then $f_s(T) \leq f_s(T')$ for $s \geq 0$.*

Proof. We will show that for any p-ancestor T' of a given tree T, $f_s(T') \leq f_s(T)$ for $s \geq 0$ by induction on n. There is only one tree on $1 \leq n \leq 3$ vertices. Consider $n = 4$, then $T' \cong P_4$ and $T \cong S_4$, and the theorem holds. Suppose $n = 5$. Then either $T' \cong P_5$ and $T \in \{T_5, S_5\}$, or $T' \cong T_5$ and $T \cong S_5$ where T_5 is the tree pictured in Figure 14. In either case, the theorem clearly holds.

Suppose that if T' is a tree on $1 \leq n' < n$ vertices and is a p-ancestor of another tree T for some $p \geq 1$, then $f_s(T') \leq f_s(T)$ for $s \geq 0$. Let T be a tree on n vertices, and let T' be a p-ancestor of T. Then there exists $r = \{v, u, x\} \in R_p(T)$ such that $g_p(r) \leq -2$ and $T(r) \cong T'$. Let P be the v,u-path in T of length p, and let $y \in N(u) \cap V(P)$. By Proposition 3.1,

$$\begin{aligned} f_s(T) =& f_s(T - v) + f_{s-1}(T - N_T[v]) \\ f_s(T') =& f_s(T' - v) + f_{s-1}(T' - N_{T'}[v]). \end{aligned} \tag{13}$$

Suppose that $p = 1$, then by Theorem 7.10 $f_s(T) \geq f_s(T')$ for $s \geq 0$. Suppose that $p = 2$, then $T - v$ is a 1-descendant of $T' - v$, and $T - N_T[v] \cong T' - N_{T'}[v]$. Hence, by (13), for

$s \geq 0$,

$$f_s(T) = f_s(T - v) + f_{s-1}(T - N_T[v]) \geq f_s(T' - v) + f_{s-1}(T' - N_{T'}[v] = f_s(T').$$

If $p \geq 3$, then $T' - v$ is a $(p-1)$-ancestor of $T - v$, and so by induction $f_s(T-v) \geq f_s(T'-v)$ for $s \geq 0$. Also, $f_s(T - N_T[v]) \geq f_s(T' - N_{T'}[v])$ for $s \geq 0$ by induction as $T' - N_{T'}[v]$ is a $(p-2)$-ancestor of $T - N_T[v]$. Hence, by (13),

$$f_s(T) = f_s(T - v) + f_{s-1}(T - N_T[v]) \geq f_s(T' - v) + f_{s-1}(T' - N_{T'}[v]) = f_s(T'),$$

for $s \geq 0$, and so the theorem holds by the Principle of Mathematical Induction. □

If T is a p_1-descendant of T', and T' is a p_2-descendant of T'', then $f_s(T'') \leq f_s(T') \leq f_s(T)$ for $s \geq 0$. By the transitive property, the following theorem immediately follows.

Theorem 7.13. *Let T and T' be trees on n vertices such that T' is a $p_1^{i_1}, p_2^{i_2}, \ldots, p_j^{i_j}$-descendant of T. Then $f_s(T) \leq f_s(T')$ for $s \geq 0$.*

By Theorem 7.10 and Theorem 7.12, we may state the following theorem.

Theorem 7.14. *Let T be a tree on n vertices and $\{T_i\}_{i=0}^{\beta}$ be a genealogy of T. Then $f_s(T_i) \leq f_s(T_{i+1})$ for $0 \leq i \leq \beta - 1$ and $s \geq 0$.*

Thereom 7.14 is an extension of Theorem 7.1 as for any tree T we may find a sequence of trees $\{T_i\}_{i=0}^{\beta}$ such that $f_s(T_i) \leq f_s(T_{i+1})$ for $0 \leq i \leq \beta - 1$ and $s \geq 0$. Thus, a genealogy of T along with f_s for $s \geq 0$ yields a partial ordering of a set of trees on n vertices.

7.2. **Comparing p-descendants of a Tree.**

By Theorem 7.10 and Theorem 7.12, we are able to generate a partial ordering of trees such that f_s of a tree in this ordering is at least as large as f_s of the previous tree in the ordering for $s \geq 0$. It is not difficult to show that a genealogy of a tree is not unique. Now we will consider the set of p^1-ancestors and p^1-descendants of a given tree and investigate f_s values of trees in this set for $s \geq 0$.

Theorem 7.15. *Let T be a tree with starring triples $r_i = \{v_i, u_i, x\} \in R_p(T)$ for $i \in \{1, 2\}$ such that $d(u_2) < d(u_1)$. Then $f_s(T(r_1)) \leq f_s(T(r_2))$ for $s \geq 0$.*

Proof. Consider $T(r_1)$. Then $r = \{v_2, u_2, u_1\} \in R_p(T(r_1))$, and $g_p(r) = d_{T(r_1)}(u_1) - d_{T(r_1)}(u_2) \geq 0$. By Theorem 7.10 and Theorem 7.12, $f_s(T(r_1)) \leq f_s((T(r_1))(r))$ for $s \geq 0$. We claim that $(T(r_1)(r)) \cong T(r_2)$. Let $P_i = v_i \ldots y_i u_i$ be the $v_i u_i$-path in T for $i \in \{1, 2\}$. Note that $T(r_2) - \{y_1 u_1\} \cong (T(r_1))(r) - \{y_2 u_1\}$, and $P_1 - u_1 \cong P_2 - u_2$. Hence $V(T(r_2)) \cong V((T(r_1))(r))$ and $E(T(r_2)) \cong E((T(r_1))(r))$. Thus the claim is true, and so $T(r_1)$ is an p-ancestor of $T(r_2)$. Hence, $f_s(T(r_1)) \leq f_s(T(r_2))$ for $s \geq 0$. $\square$

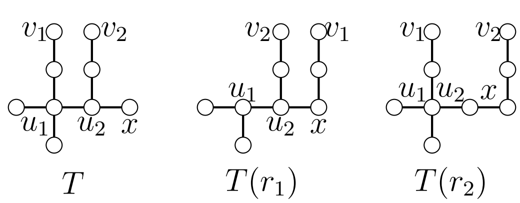


FIGURE 15. T, $T(r_1)$, $T(r_2)$

Theorem 7.16. *Let T be a tree with starring triples $r_i = \{v, u, x_i\} \in R_p(T)$ for $i \in \{1, 2\}$ such that $d(x_1) < d(x_2)$. Then $f_s(T(r_1)) \leq f_s(T(r_2))$.*

Proof. Consider $r = \{v, x_1, x_2\} \in R_p(T(r_1))$. Then as $d_{T(r_1)}(x_2) - d_{T(r_1)}(x_1) \geq 0$, $T(r_1)(r)$ is a p-descendant of $T(r_1)$. Hence by Theorem 7.10 and Theorem 7.12, $f_s(T(r_1)) \leq f_s(T(r_1)(r))$ for $s \geq 0$. Let $P = v \ldots yu$ be the uv-path in T. Note that $V(T(r_2)) = V(T(r_1))$ and $E(T(r_2)) = (E(T(r_1)) - \{yx_1\}) \cup \{yx_2\}$. Hence $T(r_2)$ is a p-descendant of $T(r_1)$, namely $T(r_1)(r)$. □

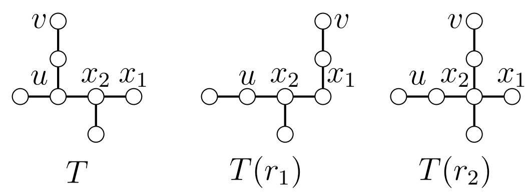


FIGURE 16. T', $T'(r_1)$, $T'(r_2)$

It has been shown that for a tree T with starring triple $r \in R_p(T)$ such that $g_p(r) \geq 0$, $f_s(T) \leq f_s(T(r))$ for $s \geq 0$. Additionally, if $g_p(r) \leq -2$, $f_s(T) \geq f_s(T(r))$ for $s \geq 0$. However, if $g_p(r) = -1$, the relationship between $f_s(T)$ and $f_s(T(r))$ for $s \geq 0$ is inconclusive. It would be interesting to investigate what parameters determine that $f_s(T) \leq f_s(T(r))$ for $g_p(r) = -1$ and $s \geq 0$.

7.3. **The First Zagreb Index of a Tree and Its Descendants.**

In the same way that Theorem 7.10 and Theorem 7.12 extend the works of Wingard and Theorem 7.1. A genealogy of a tree also extends the works of Das and Gutman and Theorem 7.4.

Theorem 7.17. *Let T be a tree and $r \in R_p(T)$. Then $M_1(T(r)) = M_1(T) + 2g_p(r) + 2$.*

Proof. Suppose that $p = 1$, and let $r = \{v, u, x\} \in R_1(T)$. Then

$$\begin{aligned} M_1(T) =& M_1(T - v) + d_T(v)^2 + (d_T(u)^2 - d_{T-v}(u)^2) \\ =& M_1(T - v) + 1 + d_T(u)^2 - (d_T(u) - 1)^2 \\ =& M_1(T - v) + 2d_T(u). \end{aligned}$$

Similarly,

$$\begin{aligned} M_1(T(r)) =& M_1(T(r) - v) + d_{T(r)}(v)^2 + (d_{T(r)}(x)^2 - d_{T(r)-v}(x)^2) \\ =& M_1(T(r) - v) + 1 + d_{T(r)}(x)^2 - (d_{T(r)}(x) - 1)^2 \\ =& M_1(T(r) - v) + 2d_{T(r)}(x). \end{aligned}$$

Note that $T - v \cong T(r) - v$, and $d_T(x) = d_{T(r)}(x) - 1$. Then

$$\begin{aligned} M_1(T(r)) =& M_1(T) + 2d_T(x) + 2 - 2d_T(u) \\ =& M_1(T) + 2g_1(r) + 2. \end{aligned}$$

Suppose that $p \geq 2$, and let $r = \{v, u, x\} \in R_p(T)$, and let P be the v, u-path of T of lenght p. Let $y \in N(u) \cap V(P)$ and $y' \in N(y) \cap V(P) - y$. If $p = 2$, then

$$\begin{aligned} M_1(T) =& M_1(T - y) + d_T(y)^2 + (d_T(u)^2 - d_{T-y}(u)^2) + (d_T(y')^2 - d_{T-y}(y')^2) \\ =& M_1(T - y) + 4 + d_T(u)^2 - (d_T(u) - 1)^2 + 1 \\ =& M_1(T - y) + 2d_T(u) + 4. \end{aligned}$$

Similarly,

$$\begin{aligned} M_1(T(r)) =& M_1(T(r) - v) + d_{T(r)}(v)^2 + (d_{T(r)}(x)^2 - d_{T(r)-v}(x)^2) + (d_{T(r)}(y')^2 - d_{T(r)-y}(y')^2) \\ =& M_1(T(r) - v) + 4 + d_{T(r)}(x)^2 - (d_{T(r)}(x) - 1)^2 + 1 \\ =& M_1(T(r) - v) + 2d_{T(r)}(x) + 4. \end{aligned}$$

Note that $T - v \cong T(r) - v$, and $d_T(x) = d_{T(r)}(x) - 1$. Then

$$\begin{aligned} M_1(T(r)) =& M_1(T) + 2d_T(x) + 2 - 2d_T(u) \\ =& M_1(T) + 2g_2(r) + 2. \end{aligned}$$

Suppose that $l \geq 3$, then

$$\begin{aligned} M_1(T) =& M_1(T - y) + d_T(y)^2 + (d_T(u)^2 - d_{T-y}(u)^2) + (d_T(y')^2 - d_{T-y}(y')^2) \\ =& M_1(T - y) + 4 + d_T(u)^2 - (d_T(u) - 1)^2 + 3 \\ =& M_1(T - y) + 2d_T(u) + 6. \end{aligned}$$

Similarly,

$$\begin{aligned} M_1(T) =& M_1(T(r) - v) + d_{T(r)}(v)^2 + (d_{T(r)}(x)^2 - d_{T(r)-v}(x)^2) + (d_{T(r)}(y')^2 - d_{T(r)-y}(y')^2) \\ =& M_1(T(r) - v) + 4 + d_{T(r)}(x)^2 - (d_{T(r)}(x) - 1)^2 + 3 \\ =& M_1(T(r) - v) + 2d_{T(r)}(x) + 6. \end{aligned}$$

Note that $T - v \cong T(r) - v$, and $d_T(x) = d_{T(r)}(x) - 1$. Then

$$\begin{aligned} M_1(T(r)) =& M_1(T) + 2d_T(x) + 2 - 2d_T(u) \\ =& M_1(T) + 2g_p(r) + 2. \end{aligned}$$

□

Corollary 7.18. *Let T be a tree and $r \in R_p(T)$. Then*

(i) $M_1(T) < M_1(T(r))$ *if* $g_p(r) \geq 0$,

(ii) $M_1(T) = M_1(T(r))$ *if* $g_p(r) = -1$,

(iii) $M_1(T) > M_1(T(r))$ *if* $g_p(r) \leq -2$.

Thus for a given tree T, a genealogy of T gives a sequence of trees such that the M_1 value of a tree in the sequence is larger than the M_1 value of any previous tree in this sequence. Hence, a genealogy of T along with M_1 provides a partial ordering of a set of trees on n vertices.

It should be noted that for $p \geq 1$, g_p is not sufficient to build a sequence of trees such that the M_2 value of a tree in this sequence is larger that the M_2 value of any previous tree in this sequence. However, we believe that a similar function may be defined to generate such a sequence. It would be interesting to determine such a function and consequently determine a partial ordering of a set of trees with respect to M_2.

8. Potential Research in the Future

In Chapter 2, it was shown that maximal outerplanar graphs are tree-like 2-trees, and chordal planar graphs with toughness exceeding 1 are tree-like 3-trees with toughness exceeding 1. It would be interesting to classify graphs that are tree-like k-trees for $k \geq 4$.

The shell of a k-tree was introduced in Chapter 2, and it allowed us to define classes of k-trees such as path-like and tree-like k-trees. We say a clique is "maximal" if it is not contained in a larger clique. Thus for a k-tree, a $(k+1)$-clique is maximal. The shell may be generalized for any graph G as follows.

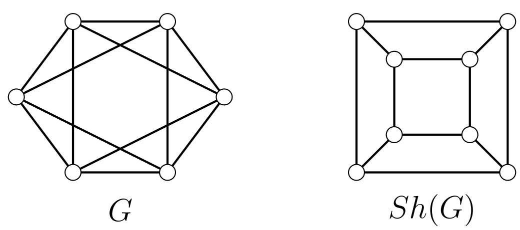


Figure 17. A graph G and its shell

Definition 8.1. Let G be graph. Then the *shell* of the graph G, $Sh(G)$, is a graph such that

(i) if X is a maximal clique, then $X \in V(Sh(G))$,

(ii) if X and Y are maximal cliques of size r_1 and r_2 respectfully, and $|V(X) \cap V(Y)| = \min(r_1, r_2) - 1$, then $XY \in E(Sh(G))$.

With this modified definition of the shell, it would be interesting to investigate the shells of graphs other than k-trees.

In Chapter 3, we defined families of trees $\mathcal{A}_c$ such that the independence polynomial of any tree in such a family has c as a rational root. The family $\mathcal{A}_{-1}$ was determined to be unique, and families $\mathcal{A}_{-1/2}$, $\mathcal{A}_{-1/3}$, and $\mathcal{A}_{-1/4}$ were characterized. However it would be interesting to verify that $T \in \mathcal{A}_i$ if and only if $I(T;i) = 0$ for $i \in \{-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}\}$.

It was shown that if $I(P_n;c) = 0$, then $c \in \{-1, -\frac{1}{2}, -\frac{1}{3}\}$. It would be interesting to determine the set of rational numbers $\mathcal{C}$ such that, for the tree T, $I(T;c) = 0$ if and only if $c \in \mathcal{C}$.

Wingard determined that, for the graph G, if $I(G;-1) = 0$, then G has the same number of independent sets of even cardinality as independent sets of odd cardinality. However, we were unsuccessful to find in the literature any significance to other rational roots of the independence polynomial of a graph. It would be an interesting question to ask what a given root of a graph's independence polynomial implies about the graph itself.

It is a natural parallel to generalize results about trees to k-trees. With this in mind, it would be interesting to investigate rational roots of independence polynomials of k-trees. Can the results about rational roots of the independence polynomials of paths be extended to the k-path or to path-like k-trees? Can families of k-trees be defined similarly to the families of trees defined in Chapter 3? There are many intriguing questions of this nature about the class of k-trees.

In Chapter 4, Wingard's bound, $|I(T;-1)| \leq 1$, was generalized to k-degenerate graphs, and thus k-trees. We determined that for the k-degenerate graph G, $|I(G;-\frac{1}{k})| \leq 1$. However, we may state the following conjecture.

Conjecture 8.2. *Let G be a maximally k-degenerate graph and $k \geq 2$. Then $|I(G;-\frac{1}{k})| > 0$.*

In Chapter 4, the works of Alameddine were extended by showing a strict upper bound of f_s of tree-like 2-trees for $s \geq 0$ that is uniquely obtained by the 2-spiral. Additionally, it was shown that for tree-like 3-trees with toughness exceeding 1, the strict upper bound of f_s is uniquely obtained by the 3-spiral for $s \geq 0$. It was also conjectured that for path-like k-trees, the strict upper bound of f_s is uniquely obtained by the k-spiral. In addition to verifying this conjecture, three other questions naturally follow:

(i) What is the strict upper bound of f_s for tree-like k-trees for $s \geq 0$?

(ii) What is the strict upper bound of f_s for tree-like k-trees with toughness exceeding 1, $k \geq 3$, and $s \geq 0$?

(iii) What tree-like k-trees obtain these upper bounds?

Lower and upper bounds of the Zagreb indices for k-trees were demonstrated in Chapter 5 along with the unique k-trees that obtain these bounds for both M_1 and M_2. Furthermore, a strict upper bound of M_1-values for k-degenerate graphs was determined along with the k-degenerate graph that obtains this bound. The lower bound of the Zagreb indices for k-degenerate graphs is trivially zero as the empty graph is k-degenerate for $k \geq 0$. However, it would be interesting to deduce a strict lower bound of the Zagreb indices for maximally k-degenerate graphs. It is reasonable to think that this lower bound is obtained by the k-path, though maybe not uniquely. Likewise, it would be interesting to determine a strict upper bound of M_2-values for k-degenerate graphs and characterize the k-degenerate graphs that obtain this upper bound. It is again reasonable to believe that this bound is obtained by the k-star.

In Chapter 6, an upper bound of the first Zagreb index for tree-like k-trees was demonstrated along with the unique tree-like k-trees that obtain this bound. A strict upper bound of the second Zagreb index for tree-like k-trees was partially solved. Conjecture 6.12 was presented, and it would be interesting to verify this unverified statement that the k-spiral uniquely obtains a strict upper bound for M_2 values among tree-like k-trees.

Genealogies of trees were introduced in Chapter 7, and it was shown that a genealogy of a tree helps provide a sequence of trees $\{T_i\}_{i=0}^{\beta}$ such that f_s and M_1 are increasing as i increases. It would be interesting to find a similar construction for M_2 of trees.

There is some difficulty in generalizing starring triples of trees to k-trees. If, however, the starring triples of a tree can be generalized to starring triples of k-trees, then a genealogy of a k-tree may be defined. If a genealogy of a k-tree can be successfully defined, then questions stated about finding a strict upper bound of f_s of tree-like k-trees may be found. Given a tree-like k-tree, can we find a sequence of tree-like k-trees such that f_s is increasing according to this sequence? This seems to be a reasonable question, and the graph in this sequence with the greatest index might obtain an upper bound of f_s for tree-like k-trees. Similarly, a genealogy of a k-tree may provide the correct structure to verify Conjecture 6.12.

It would also be interesting to determine what other topological indices, such as the toughness, behave in a way similar to f_s and M_1 in a genealogy of a tree.

A graph is said to be "hamiltonian" if it contains a cycle that passes through all of its vertices. Hamiltonicity has been a major area of research, and a common approach to questions of hamiltonicity is to examine a graph through its toughness. In 1973, Chv$\acute{a}$tal conjectured that there exists a number t such that all t-tough graphs are hamiltonian. From the definition of toughness, it is clear that a cycle of length at least four is exactly 1-tough. It is thus clearly necessary for a hamiltonian graph to be 1-tough. For many years, it was thought that all 2-tough graphs are hamiltonian. However, this has been found to be untrue.

Theorem 8.3. [2] *For every $\epsilon > 0$, there exists a $(\frac{9}{4} - \epsilon)$-tough graph containing no hamiltonian path.*

Chen, Jacobson, Kézdy, and Lehel proved Chvátal's conjecture for chordal graphs, and Böhme, Harant, and Tkáč solved the conjecture for chordal planar graphs with toughness exceeding 1.

Theorem 8.4. [7] *Every 18-tough chordal graph is hamiltonian.*

Theorem 8.5. [5] *Every chordal planar graph with toughness exceeding* 1 *is hamiltonian.*

In 2003, Broersma, Xiong, and Yoshimoto addressed hamiltonicity of k-trees.

Theorem 8.6. [4] *If $T_n^k \neq K_2$ is a $(\frac{k+1}{3})$-tough k-tree ($k \geq 2$), then T_n^k is hamiltonian.*

Shook and Wei studied the hamiltonicity of k-trees through a parameter called the branch number, $\beta(T_n^k)$. Let the edge e be contractible in T_n^k if the graph resulting in contracting e is a k-tree. The branch number may be calculated by $\beta(T_n^k) = |S_1(T_n^k)| + |A(T_n^k)| + k - n$ where A is the set of contractible edges in T_n^k. Their result is a direct generalization of the result of Broersma.

Theorem 8.7. [38] *For $k > 1$, if T_n^k is a k-tree with $\beta(T_n^k) \leq k$, then T_n^k is hamiltonian.*

Theorem 8.8. [38] *If $T_n^k \neq K_2$ is a $(\frac{k+1}{3})$-tough k-tree ($k \geq 2$), then $\beta(T_n^k) \leq 2$.*

For the class of tree-like k-trees, Broersma's bound on the toughness may be tightened. By making the connection that chordal planar graphs with toughness exceeding 1 are tree-like 3-trees with toughness exceeding 1, we state the following conjecture which is a direct generalization of Theorem 8.5.

Conjecture 8.9. *Let T_n^k be a tree-like k-tree with toughness exceeding 1 and $k \geq 3$. Then T_n^k is hamiltonian.*

Even for the case of $k = 4$, Conjecture 8.9 is a difficult question.

The connection between trees and k-trees is very interesting, and there are plenty of questions surrounding trees, k-trees, and tree-like k-trees. There is plenty of opportunity to propose and ask questions in regards to these graphs.

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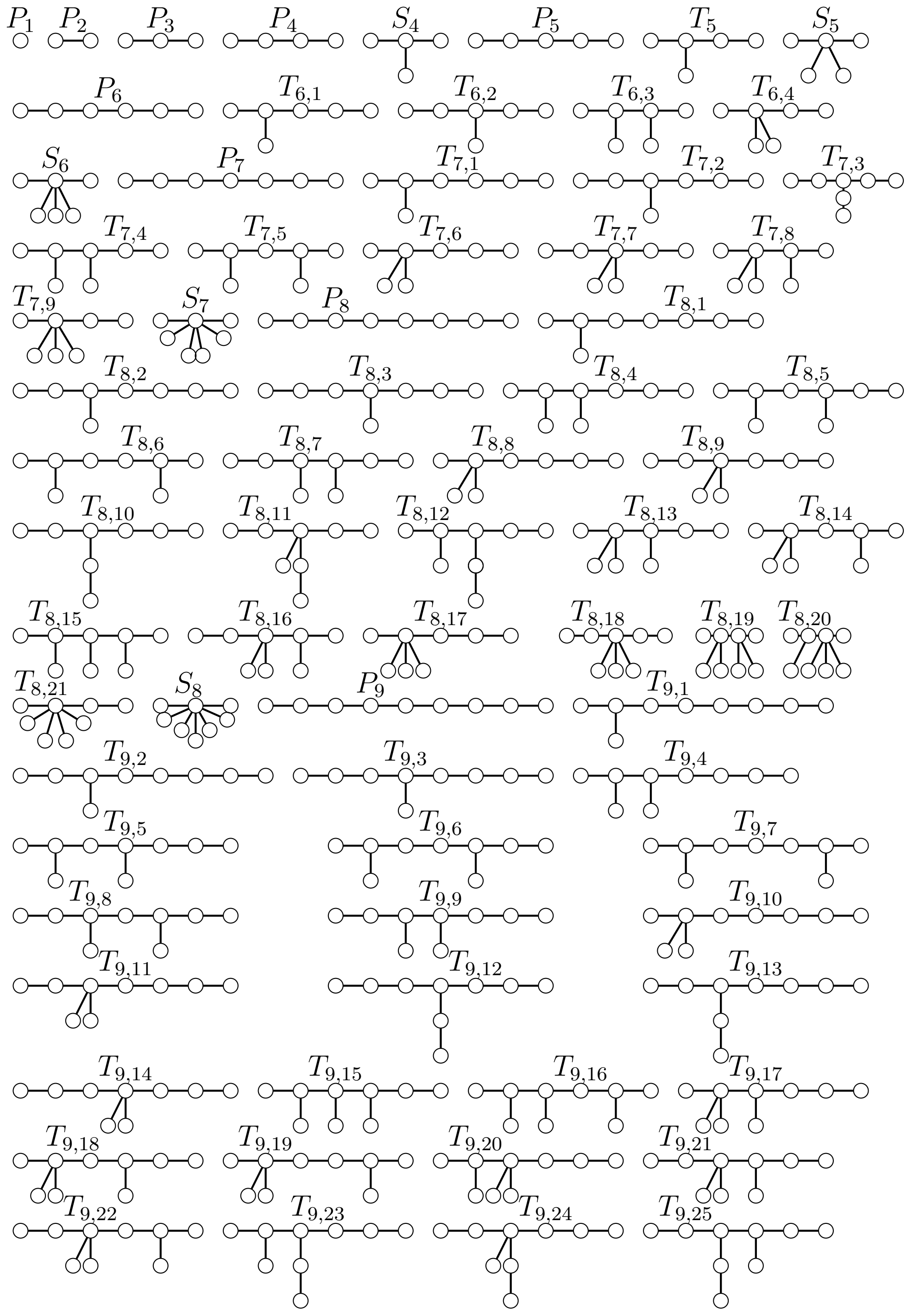
LIST OF APPENDICES

Appendix A: Trees on $1 \leq n \leq 10$ vertices


P_1 P_2 P_3 P_4 S_4 P_5 T_5 S_5
P_6 $T_{6,1}$ $T_{6,2}$ $T_{6,3}$ $T_{6,4}$
S_6 P_7 $T_{7,1}$ $T_{7,2}$ $T_{7,3}$
$T_{7,4}$ $T_{7,5}$ $T_{7,6}$ $T_{7,7}$ $T_{7,8}$
$T_{7,9}$ S_7 P_8 $T_{8,1}$
$T_{8,2}$ $T_{8,3}$ $T_{8,4}$ $T_{8,5}$
$T_{8,6}$ $T_{8,7}$ $T_{8,8}$ $T_{8,9}$
$T_{8,10}$ $T_{8,11}$ $T_{8,12}$ $T_{8,13}$ $T_{8,14}$
$T_{8,15}$ $T_{8,16}$ $T_{8,17}$ $T_{8,18}$ $T_{8,19}$ $T_{8,20}$
$T_{8,21}$ S_8 P_9 $T_{9,1}$
$T_{9,2}$ $T_{9,3}$ $T_{9,4}$
$T_{9,5}$ $T_{9,6}$ $T_{9,7}$
$T_{9,8}$ $T_{9,9}$ $T_{9,10}$
$T_{9,11}$ $T_{9,12}$ $T_{9,13}$
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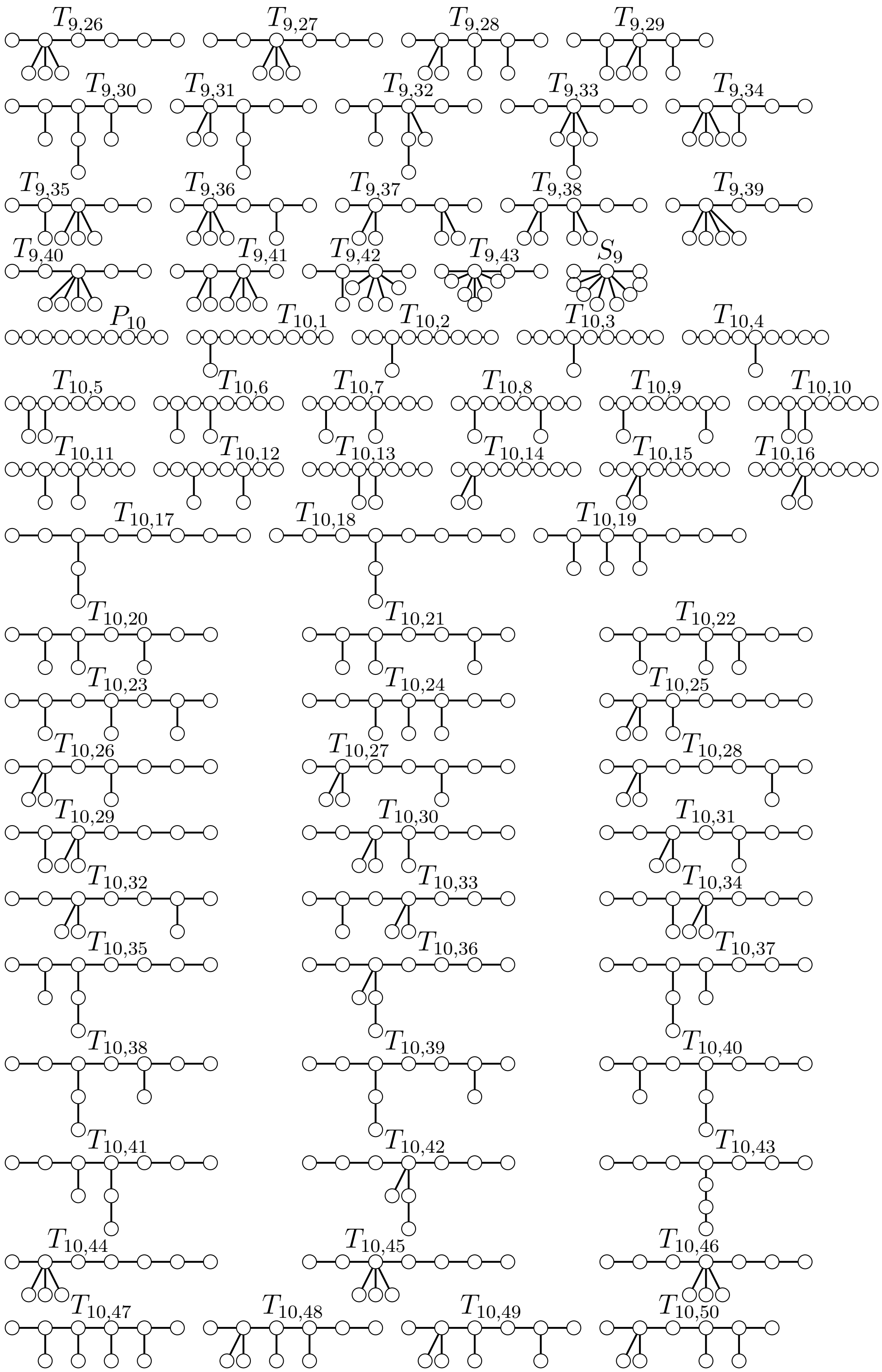
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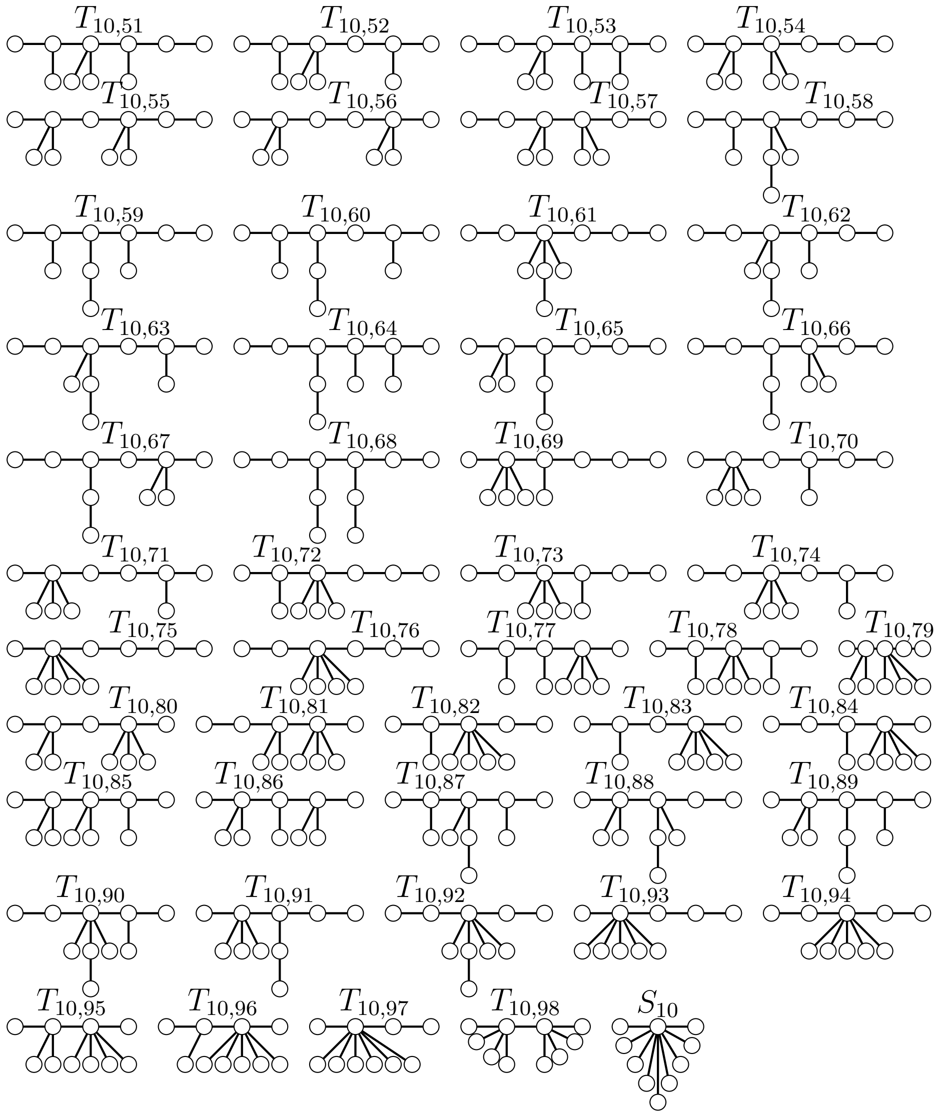
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$T_{10,75}$ $T_{10,76}$ $T_{10,77}$ $T_{10,78}$ $T_{10,79}$
$T_{10,80}$ $T_{10,81}$ $T_{10,82}$ $T_{10,83}$ $T_{10,84}$
$T_{10,85}$ $T_{10,86}$ $T_{10,87}$ $T_{10,88}$ $T_{10,89}$
$T_{10,90}$ $T_{10,91}$ $T_{10,92}$ $T_{10,93}$ $T_{10,94}$
$T_{10,95}$ $T_{10,96}$ $T_{10,97}$ $T_{10,98}$ S_{10}

Appendix B: Independence Polynomials of Trees on $1 \leq n \leq 10$ vertices

$1 \leq n \leq 6$

1 $I(P_1; x) = 1 + x$

2 $I(P_2; x) = 1 + 2x$

3 $I(P_3; x) = 1 + 3x + x^2$

4 $I(P_4; x) = 1 + 4x + 3x^2$

5 $I(S_4; x) = 1 + 4x + 3x^2 + x^3$

6 $I(P_5; x) = 1 + 5x + 6x^2 + x^3$

7 $I(T_5; x) = 1 + 5x + 6x^2 + 2x^3$

8 $I(S_5; x) = 1 + 5x + 6x^2 + 4x^3 + x^4$

9 $I(P_6; x) = 1 + 6x + 10x^2 + 4x^3$

10 $I(T_{6,1}; x) = 1 + 6x + 10x^2 + 5x^3 + x^4$

11 $I(T_{6,2}; x) = 1 + 6x + 10x^2 + 5x^3$

12 $I(T_{6,3}; x) = 1 + 6x + 10x^2 + 6x^3 + x^4$

13 $I(T_{6,4}; x) = 1 + 6x + 10x^2 + 7x^3 + 2x^4$

14 $I(S_6; x) = 1 + 6x + 10x^2 + 10x^3 + 5x^4 + x^5$

$n = 7$

1 $I(P_7; x) = 1 + 7x + 15x^2 + 10x^3 + x^4$

2 $I(T_{7,1}; x) = 1 + 7x + 15x^2 + 11x^3 + 3x^4$

3 $I(T_{7,2}; x) = 1 + 7x + 15x^2 + 11x^3 + 2x^4$

4 $I(T_{7,3}; x) = 1 + 7x + 15x^2 + 11x^3 + x^4$

5 $I(T_{7,4}; x) = 1 + 7x + 15x^2 + 12x^3 + 3x^4$

6 $I(T_{7,5}; x) = 1 + 7x + 15x^2 + 12x^3 + 5x^4 + x^5$

7 $I(T_{7,6}; x) = 1 + 7x + 15x^2 + 13x^3 + 6x^4 + x^5$

8 $I(T_{7,7};x) = 1 + 7x + 15x^2 + 13x^3 + 4x^4$

9 $I(T_{7,8};x) = 1 + 7x + 15x^2 + 14x^3 + 6x^4 + x^5$

10 $I(T_{7,9};x) = 1 + 7x + 15x^2 + 16x^3 + 9x^4 + 2x^5$

11 $I(S_7;x) = 1 + 7x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6$

$n = 8$

1 $I(P_8;x) = 1 + 8x + 21x^2 + 20x^3 + 5x^4$

2 $I(T_{8,1};x) = 1 + 8x + 21x^2 + 21x^3 + 8x^4 + x^5$

3 $I(T_{8,2};x) = 1 + 8x + 21x^2 + 21x^3 + 7x^4$

4 $I(T_{8,3};x) = 1 + 8x + 21x^2 + 21x^3 + 7x^4 + x^5$

5 $I(T_{8,4};x) = 1 + 8x + 21x^2 + 22x^3 + 9x^4 + x^5$

6 $I(T_{8,5};x) = 1 + 8x + 21x^2 + 22x^3 + 10x^4 + 2x^5$

7 $I(T_{8,6};x) = 1 + 8x + 21x^2 + 22x^3 + 11x^4 + 2x^5$

8 $I(T_{8,7};x) = 1 + 8x + 21x^2 + 22x^3 + 8x^4$

9 $I(T_{8,8};x) = 1 + 8x + 21x^2 + 23x^3 + 13x^4 + 3x^5$

10 $I(T_{8,9};x) = 1 + 8x + 21x^2 + 23x^3 + 11x^4 + 2x^5$

11 $I(T_{8,10};x) = 1 + 8x + 21x^2 + 21x^3 + 6x^4$

12 $I(T_{8,11};x) = 1 + 8x + 21x^2 + 23x^3 + 9x^4$

13 $I(T_{8,12};x) = 1 + 8x + 21x^2 + 22x^3 + 8x^4 + x^5$

14 $I(T_{8,13};x) = 1 + 8x + 21x^2 + 24x^3 + 13x^4 + 3x^5$

15 $I(T_{8,14};x) = 1 + 8x + 21x^2 + 24x^3 + 16x^4 + 6x^5 + x^6$

16 $I(T_{8,15};x) = 1 + 8x + 21x^2 + 23x^3 + 11x^4 + 2x^5$

17 $I(T_{8,16};x) = 1 + 8x + 21x^2 + 24x^3 + 12x^4 + 2x^5$

18 $I(T_{8,17};x) = 1 + 8x + 21x^2 + 26x^3 + 19x^4 + 7x^5 + x^6$

19 $I(T_{8,18};x) = 1 + 8x + 21x^2 + 26x^3 + 16x^4 + 4x^5$

20 $I(T_{8,19};x) = 1 + 8x + 21x^2 + 26x^3 + 17x^4 + 6x^5 + x^6$

21 $I(T_{8,20};x) = 1 + 8x + 21x^2 + 27x^3 + 19x^4 + 7x^5 + x^6$

22 $I(T_{8,21};x) = 1 + 8x + 21x^2 + 30x^3 + 25x^4 + 11x^5 + 2x^6$

23 $I(S_8;x) = 1 + 8x + 21x^2 + 35x^3 + 35x^4 + 21x^5 + 7x^6 + x^7$

$n = 9$

1 $I(P_9;x) = 1 + 9x + 28x^2 + 35x^3 + 15x^4 + x^5$

2 $I(T_{9,1};x) = 1 + 9x + 28x^2 + 36x^3 + 19x^4 + 4x^5$

3 $I(T_{9,2};x) = 1 + 9x + 28x^2 + 36x^3 + 18x^4 + 2x^5$

4 $I(T_{9,3};x) = 1 + 9x + 28x^2 + 36x^3 + 18x^4 + 3x^5$

5 $I(T_{9,4};x) = 1 + 9x + 28x^2 + 37x^3 + 21x^4 + 4x^5$

6 $I(T_{9,5};x) = 1 + 9x + 28x^2 + 37x^3 + 22x^4 + 7x^5 + x^6$

7 $I(T_{9,6};x) = 1 + 9x + 28x^2 + 37x^3 + 22x^4 + 5x^5$

8 $I(T_{9,7};x) = 1 + 9x + 28x^2 + 37x^3 + 23x^4 + 7x^5 + x^6$

9 $I(T_{9,8};x) = 1 + 9x + 28x^2 + 37x^3 + 21x^4 + 4x^5$

10 $I(T_{9,9};x) = 1 + 9x + 28x^2 + 37x^3 + 20x^4 + 3x^5$

11 $I(T_{9,10};x) = 1 + 9x + 28x^2 + 38x^3 + 26x^4 + 9x^5 + x^6$

12 $I(T_{9,11};x) = 1 + 9x + 28x^2 + 38x^3 + 24x^4 + 6x^5$

13 $I(T_{9,12};x) = 1 + 9x + 28x^2 + 36x^3 + 17x^4 + 2x^5$

14 $I(T_{9,13};x) = 1 + 9x + 28x^2 + 36x^3 + 17x^4 + x^5$

15 $I(T_{9,14};x) = 1 + 9x + 28x^2 + 38x^3 + 24x^4 + 8x^5 + x^6$

16 $I(T_{9,15};x) = 1 + 9x + 28x^2 + 38x^3 + 23x^4 + 5x^5$

17 $I(T_{9,16};x) = 1 + 9x + 28x^2 + 38x^3 + 25x^4 + 8x^5 + x^6$

18 $I(T_{9,17};x) = 1 + 9x + 28x^2 + 39x^3 + 27x^4 + 9x^5 + x^6$

19 $I(T_{9,18};x) = 1 + 9x + 28x^2 + 39x^3 + 29x^4 + 12x^5 + 2x^6$

20 $I(T_{9,19};x) = 1 + 9x + 28x^2 + 39x^3 + 30x^4 + 12x^5 + 2x^6$

21 $I(T_{9,20};x) = 1 + 9x + 28x^2 + 39x^3 + 26x^4 + 8x^5 + x^6$

22 $I(T_{9,21};x) = 1 + 9x + 28x^2 + 39x^3 + 25x^4 + 6x^5$

23 $I(T_{9,22};x) = 1 + 9x + 28x^2 + 39x^3 + 28x^4 + 11x^5 + 2x^6$

24 $I(T_{9,23};x) = 1 + 9x + 28x^2 + 37x^3 + 20x^4 + 4x^5$

25 $I(T_{9,24};x) = 1 + 9x + 28x^2 + 38x^3 + 22x^4 + 4x^5$

26 $I(T_{9,25};x) = 1 + 9x + 28x^2 + 37x^3 + 19x^4 + 2x^5$

27 $I(T_{9,26};x) = 1 + 9x + 28x^2 + 41x^3 + 35x^4 + 16x^5 + 3x^6$

28 $I(T_{9,27};x) = 1 + 9x + 28x^2 + 41x^3 + 32x^4 + 13x^5 + 2x^6$

29 $I(T_{9,28};x) = 1 + 9x + 28x^2 + 40x^3 + 30x^4 + 12x^5 + 2x^6$

30 $I(T_{9,29};x) = 1 + 9x + 28x^2 + 40x^3 + 28x^4 + 9x^5 + x^6$

31 $I(T_{9,30};x) = 1 + 9x + 28x^2 + 38x^3 + 23x^4 + 7x^5 + x^6$

32 $I(T_{9,31};x) = 1 + 9x + 28x^2 + 39x^3 + 26x^4 + 9x^5 + x^6$

33 $I(T_{9,32};x) = 1 + 9x + 28x^2 + 39x^3 + 24x^4 + 5x^5$

34 $I(T_{9,33};x) = 1 + 9x + 28x^2 + 41x^3 + 29x^4 + 8x^5$

35 $I(T_{9,34};x) = 1 + 9x + 28x^2 + 42x^3 + 35x^4 + 16x^5 + 3x^6$

36 $I(T_{9,35};x) = 1 + 9x + 28x^2 + 42x^3 + 33x^4 + 13x^5 + 2x^6$

37 $I(T_{9,36};x) = 1 + 9x + 28x^2 + 42x^3 + 39x^4 + 22x^5 + 7x^6 + x^7$

38 $I(T_{9,37};x) = 1 + 9x + 28x^2 + 41x^3 + 37x^4 + 21x^5 + 7x^6 + x^7$

39 $I(T_{9,38};x) = 1 + 9x + 28x^2 + 41x^3 + 31x^4 + 12x^5 + 2x^6$

40 $I(T_{9,39};x) = 1 + 9x + 28x^2 + 45x^3 + 45x^4 + 26x^5 + 8x^6 + x^7$

41 $I(T_{9,40};x) = 1 + 9x + 28x^2 + 45x^3 + 41x^4 + 20x^5 + 4x^6$

42 $I(T_{9,41};x) = 1 + 9x + 28x^2 + 44x^3 + 40x^4 + 22x^5 + 7x^6 + x^7$

43 $I(T_{9,42};x) = 1 + 9x + 28x^2 + 46x^3 + 45x^4 + 26x^5 + 8x^6 + x^7$

44 $I(T_{9,43};x) = 1 + 9x + 28x^2 + 50x^3 + 55x^4 + 36x^5 + 13x^6 + 2x^7$

45 $I(S_9;x) = 1 + 9x + 28x^2 + 56x^3 + 70x^4 + 56x^5 + 28x^6 + 8x^7 + x^8$

$n = 10$

1 $I(P_{10};x) = 1 + 10x + 36x^2 + 56x^3 + 35x^4 + 6x^5$

2 $I(T_{10,1};x) = 1 + 10x + 36x^2 + 57x^3 + 40x^4 + 12x^5 + x^6$

3 $I(T_{10,2};x) = 1 + 10x + 36x^2 + 57x^3 + 39x^4 + 9x^5$

4 $I(T_{10,3};x) = 1 + 10x + 36x^2 + 57x^3 + 39x^4 + 10x^5 + x^6$

5 $I(T_{10,4};x) = 1 + 10x + 36x^2 + 57x^3 + 39x^4 + 10x^5$

6 $I(T_{10,5};x) = 1 + 10x + 36x^2 + 58x^3 + 43x^4 + 13x^5 + x^6$

7 $I(T_{10,6};x) = 1 + 10x + 36x^2 + 58x^3 + 44x^4 + 17x^5 + 3x^6$

8 $I(T_{10,7};x) = 1 + 10x + 36x^2 + 58x^3 + 44x^4 + 16x^5 + 2x^6$

9 $I(T_{10,8};x) = 1 + 10x + 36x^2 + 58x^3 + 44x^4 + 15x^5 + 2x^6$

10 $I(T_{10,9};x) = 1 + 10x + 36x^2 + 58x^3 + 45x^4 + 18x^5 + 3x^6$

11 $I(T_{10,10};x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 11x^5$

12 $I(T_{10,11};x) = 1 + 10x + 36x^2 + 58x^3 + 43x^4 + 14x^5 + 2x^6$

13 $I(T_{10,12};x) = 1 + 10x + 36x^2 + 58x^3 + 43x^4 + 12x^5$

14 $I(T_{10,13};x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 12x^5 + x^6$

15 $I(T_{10,14};x) = 1 + 10x + 36x^2 + 59x^3 + 49x^4 + 22x^5 + 4x^6$

16 $I(T_{10,15};x) = 1 + 10x + 36x^2 + 59x^3 + 47x^4 + 17x^5 + 2x^6$

17 $I(T_{10,16};x) = 1 + 10x + 36x^2 + 59x^3 + 47x^4 + 19x^5 + 3x^6$

18 $I(T_{10,17};x) = 1 + 10x + 36x^2 + 57x^3 + 38x^4 + 7x^5$

19 $I(T_{10,18};x) = 1 + 10x + 36x^2 + 57x^3 + 38x^4 + 8x^5$

20 $I(T_{10,19};x) = 1 + 10x + 36x^2 + 59x^3 + 46x^4 + 16x^5 + 2x^6$

21 $I(T_{10,20};x) = 1 + 10x + 36x^2 + 59x^3 + 47x^4 + 17x^5 + 2x^6$

22 $I(T_{10,21};x) = 1 + 10x + 36x^2 + 59x^3 + 58x^4 + 19x^5 + 3x^6$

23 $I(T_{10,22};x) = 1 + 10x + 36x^2 + 59x^3 + 47x^4 + 18x^5 + 3x^6$

24 $I(T_{10,23};x) = 1 + 10x + 36x^2 + 59x^3 + 49x^4 + 24x^5 + 7x^6 + x^7$

25 $I(T_{10,24};x) = 1 + 10x + 36x^2 + 59x^3 + 45x^4 + 13x^5$

26 $I(T_{10,25};x) = 1 + 10x + 36x^2 + 60x^3 + 51x^4 + 22x^5 + 4x^6$

27 $I(T_{10,26};x) = 1 + 10x + 36x^2 + 60x^3 + 53x^4 + 28x^5 + 8x^6 + x^7$

28 $I(T_{10,27};x) = 1 + 10x + 36x^2 + 60x^3 + 53x^4 + 25x^5 + 5x^6$

29 $I(T_{10,28};x) = 1 + 10x + 36x^2 + 60x^3 + 54x^4 + 28x^5 + 8x^6 + x^7$

30 $I(T_{10,29};x) = 1 + 10x + 36x^2 + 60x^3 + 50x^4 + 20x^5 + 3x^6$

31 $I(T_{10,30};x) = 1 + 10x + 36x^2 + 60x^3 + 49x^4 + 18x^5 + 2x^6$

32 $I(T_{10,31};x) = 1 + 10x + 36x^2 + 60x^3 + 51x^4 + 22x^5 + 4x^6$

33 $I(T_{10,32};x) = 1 + 10x + 36x^2 + 60x^3 + 52x^4 + 23x^5 + 4x^6$

34 $I(T_{10,33};x) = 1 + 10x + 36x^2 + 60x^3 + 52x^4 + 27x^5 + 8x^6 + x^7$

35 $I(T_{10,34};x) = 1 + 10x + 36x^2 + 60x^3 + 49x^4 + 19x^5 + 3x^6$

36 $I(T_{10,35};x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 12x^5 + x^6$

37 $I(T_{10,36};x) = 1 + 10x + 36x^2 + 59x^3 + 45x^4 + 13x^5$

38 $I(T_{10,37};x) = 1 + 10x + 36x^2 + 58x^3 + 41x^4 + 10x^5 + x^6$

39 $I(T_{10,38};x) = 1 + 10x + 36x^2 + 58x^3 + 42x^4 + 11x^5$

40 $I(T_{10,39};x) = 1 + 10x + 36x^2 + 58x^3 + 43x^4 + 13x^5 + x^6$

41 $I(T_{10,40};x) = 1 + 10x + 36x^2 + 58x^3 + 43x^4 + 15x^5 + 2x^6$

42 $I(T_{10,41};x) = 1 + 10x + 36x^2 + 58x^3 + 41x^4 + 10x^5$

43 $I(T_{10,42};x) = 1 + 10x + 36x^2 + 59x^3 + 45x^4 + 15x^5 + 2x^6$

44 $I(T_{10,43};x) = 1 + 10x + 36x^2 + 57x^3 + 38x^4 + 9x^5 + x^6$

45 $I(T_{10,44};x) = 1 + 10x + 36x^2 + 62x^3 + 61x^4 + 35x^5 + 10x^6 + x^7$

46 $I(T_{10,45};x) = 1 + 10x + 36x^2 + 62x^3 + 58x^4 + 29x^5 + 6x^6$

47 $I(T_{10,46};x) = 1 + 10x + 36x^2 + 62x^3 + 58x^4 + 32x^5 + 9x^6 + x^7$

48 $I(T_{10,47};x) = 1 + 10x + 36x^2 + 60x^3 + 50x^4 + 20x^5 + 3x^6$

49 $I(T_{10,48};x) = 1 + 10x + 36x^2 + 61x^3 + 54x^4 + 25x^5 + 5x^6$

50 $I(T_{10,49};x) = 1 + 10x + 36x^2 + +61x^3 + 56x^4 + 29x^5 + 8x^6 + x^7$

51 $I(T_{10,50};x) = 1 + 10x + 36x^2 + 61x^3 + 57x^4 + 31x^5 + 9x^6 + x^7$

52 $I(T_{10,51};x) = 1 + 10x + 36x^2 + 61x^3 + 52x^4 + 21x^5 + 3x^6$

53 $I(T_{10,52};x) = 1 + 10x + 36x^2 + 61x^3 + 55x^4 + 28x^5 + 8x^6 + x^7$

54 $I(T_{10,53};x) = 1 + 10x + 36x^2 + 61x^3 + 53x^4 + 23x^5 + 4x^6$

55 $I(T_{10,54};x) = 1 + 10x + 36x^2 + 62x^3 + 57x^4 + 29x^5 + 8x^6 + x^7$

56 $I(T_{10,55};x) = 1 + 10x + 36x^2 + 62x^3 + 61x^4 + 37x^5 + 13x^6 + 2x^7$

57 $I(T_{10,56};x) = 1 + 10x + 36x^2 + 62x^3 + 63x^4 + 38x^5 + 13x^6 + 2x^7$

58 $I(T_{10,57};x) = 1 + 10x + 36x^2 + 62x^3 + 55x^4 + 24x^5 + 4x^6$

59 $I(T_{10,58};x) = 1 + 10x + 36x^2 + 60x^3 + 48x^4 + 17x^5 + 2x^6$

60 $I(T_{10,59};x) = 1 + 10x + 36x^2 + 59x^3 + 45x^4 + 15x^5 + 2x^6$

61 $I(T_{10,60};x) = 1 + 10x + 36x^2 + 59x^3 + 47x^4 + 19x^5 + 3x^6$

62 $I(T_{10,61};x) = 1 + 10x + 36x^2 + 62x^3 + 55x^4 + 24x^5 + 4x^6$

63 $I(T_{10,62};x) = 1 + 10x + 36x^2 + 60x^3 + 47x^4 + 14x^5$

64 $I(T_{10,63};x) = 1 + 10x + 36x^2 + 60x^3 + 50x^4 + 21x^5 + 4x^6$

65 $I(T_{10,64};x) = 1 + 10x + 36x^2 + 59x^3 + 45x^4 + 14x^5 + x^6$

66 $I(T_{10,65};x) = 1 + 10x + 36x^2 + 60x^3 + 50x^4 + 22x^5 + 4x^6$

67 $I(T_{10,66};x) = 1 + 10x + 36x^2 + 60x^3 + 48x^4 + 17x^5 + 2x^6$

68 $I(T_{10,67};x) = 1 + 10x + 36x^2 + 60x^3 + 52x^4 + 25x^5 + 5x^6$

69 $I(T_{10,68};x) = 1 + 10x + 36x^2 + 58x^3 + 40x^4 + 8x^5$

70 $I(T_{10,69};x) = 1 + 10x + 36x^2 + 63^3 + 62x^4 + 35x^5 + 10x^6 + x^7$

71 $I(T_{10,70};x) = 1 + 10x + 36x^2 + 63x^3 + 65x^4 + 41x^5 + 14x^6 + 2x^7$

72 $I(T_{10,71};x) = 1 + 10x + 36x^2 + 63x^3 + 66x^4 + 41x^5 + 14x^6 + 2x^7$

73 $I(T_{10,72};x) = 1 + 10x + 36x^2 + 63x^3 + 60x^4 + 32x^5 + 9x^6 + x^7$

74 $I(T_{10,73};x) = 1 + 10x + 36x^2 + 63x^3 + 59x^4 + 29x^5 + 6x^6$

75 $I(T_{10,74};x) = 1 + 10x + 36x^2 + 63x^3 + 63x^4 + 38x^5 + 13x^6 + 2x^7$

76 $I(T_{10,75};x) = 1 + 10x + 36x^2 + 66x^3 + 75x^4 + 51x^5 + 19x^6 + 3x^7$

77 $I(T_{10,76};x) = 1 + 10x + 36x^2 + 66x^3 + 71x^4 + 45x^5 + 15x^6 + 2x^7$

78 $I(T_{10,77};x) = 1 + 10x + 36x^2 + 64x^3 + 66x^4 + 41x^5 + 14x^6 + 2x^7$

79 $I(T_{10,78};x) = 1 + 10x + 36x^2 + 64x^3 + 62x^4 + 33x^5 + 9x^6 + x^7$

80 $I(T_{10,79};x) = 1 + 10x + 36x^2 + 65x^3 + 66x^4 + 39x^5 + 13x^6 + 2x^7$

81 $I(T_{10,80};x) = 1 + 10x + 36x^2 + 65x^3 + 75x^4 + 57x^5 + 28x^6 + 8x^7 + x^8$

82 $I(T_{10,81};x) = 1 + 10x + 36x^2 + 65x^3 + 67x^4 + 41x^5 + 14x^6 + 2x^7$

83 $I(T_{10,82};x) = 1 + 10x + 36x^2 + 67x^3 + 72x^4 + 45x^5 + 15x^6 + 2x^7$

84 $I(T_{10,83};x) = 1 + 10x + 36x^2 + 67x^3 + 80x^4 + 61x^5 + 29x^6 + 8x^7 + x^8$

85 $I(T_{10,84};x) = 1 + 10x + 36x^2 + 67x^3 + 75x^4 + 51x^5 + 19x^6 + 3x^7$

86 $I(T_{10,85};x) = 1 + 10x + 36x^2 + 63x^3 + 60x^4 + 32x^5 + 9x^6 + x^7$

87 $I(T_{10,86};x) = 1 + 10x + 36x^2 + 63x^3 + 63x^4 + 38x^5 + 13x^6 + 2x^7$

88 $I(T_{10,87};x) = 1 + 10x + 36x^2 + 61x^3 + 51x^4 + 20x^5 + 3x^6$

89 $I(T_{10,88};x) = 1 + 10x + 36x^2 + 62x^3 + 55x^4 + 25x^5 + 5x^6$

90 $I(T_{10,89};x) = 1 + 10x + 36x^2 + 61x^3 + 54x^4 + 28x^5 + 8x^6 + x^7$

91 $I(T_{10,90};x) = 1 + 10x + 36x^2 + 63x^3 + 57x^4 + 25x^5 + 4x^6$

92 $I(T_{10,91};x) = 1 + 10x + 36x^2 + 63x^3 + 61x^4 + 35x^5 + 10x^6 + x^7$

93 $I(T_{10,92};x) = 1 + 10x + 36x^2 + 66x^3 + 67x^4 + 36x^5 + 8x^6$

94 $I(T_{10,93};x) = 1 + 10x + 36x^2 + 71x^3 + 90x^4 + 71x^5 + 34x^6 + 9x^7 + x^8$

95 $I(T_{10,94};x) = 1 + 10x + 36x^2 + 71x^3 + 85x^4 + 61x^5 + 24x^6 + 4x^7$

96 $I(T_{10,95};x) = 1 + 10x + 36x^2 + 73x^3 + 91x^4 + 71x^5 + 34x^6 + 9x^7 + x^8$

97 $I(T_{10,96};x) = 1 + 10x + 36x^2 + 72x^3 + 90x^4 + 71x^5 + 34x^6 + 9x^7 + x^8$

98 $I(T_{10,97};x) = 1 + 10x + 36x^2 + 77x^3 + 105x^4 + 91x^5 + 49x^6 + 15x^7 + 2x^8$

99 $I(T_{10,98};x) = 1 + 10x + 36x^2 + 68x^3 + 78x^4 + 58x^5 + 28x^6 + 8x^7 + x^8$

100 $I(S_{10};x) = 1 + 10x + 36x^2 + 84x^3 + 126x^4 + 126x^5 + 84x^6 + 36x^7 + 9x^8 + x^9$

Appendix C: The Zagreb Indices of Trees on $1 \leq n \leq 10$ vertices

T	M_1	M_2	T	M_1	M_2	T	M_1	M_2
P_1	0	0	$T_{7,7}$	28	28	$T_{8,17}$	38	36
P_2	2	1	$T_{7,8}$	30	30	$T_{8,18}$	38	39
P_3	6	4	$T_{7,9}$	34	32	$T_{8,19}$	38	40
P_4	10	8	S_7	42	36	$T_{8,20}$	40	41
S_4	12	9	P_8	26	24	$T_{8,21}$	46	44
P_5	14	12	$T_{8,1}$	28	26	S_8	56	49
T_5	16	14	$T_{8,2}$	28	27	P_9	30	28
S_5	20	16	$T_{8,3}$	28	27	$T_{9,1}$	32	30
P_6	18	16	$T_{8,4}$	30	30	$T_{9,2}$	32	31
$T_{6,1}$	20	18	$T_{8,5}$	30	29	$T_{9,3}$	32	31
$T_{6,2}$	20	19	$T_{8,6}$	30	28	$T_{9,4}$	34	34
$T_{6,3}$	22	21	$T_{8,7}$	30	31	$T_{9,5}$	34	33
$T_{6,4}$	24	22	$T_{8,8}$	32	30	$T_{9,6}$	34	33
S_6	30	25	$T_{8,9}$	32	32	$T_{9,7}$	34	32
P_7	22	20	$T_{8,10}$	28	28	$T_{9,8}$	34	34
$T_{7,1}$	24	22	$T_{8,11}$	32	34	$T_{9,9}$	34	35
$T_{7,2}$	24	23	$T_{8,12}$	30	31	$T_{9,10}$	36	34
$T_{7,3}$	24	24	$T_{8,13}$	34	35	$T_{9,11}$	36	36
$T_{7,4}$	26	26	$T_{8,14}$	34	32	$T_{9,12}$	32	32
$T_{7,5}$	26	24	$T_{8,15}$	32	33	$T_{9,13}$	32	32
$T_{7,6}$	28	26	$T_{8,16}$	34	36	$T_{9,14}$	36	36

T	M_1	M_2	T	M_1	M_2	T	M_1	M_2
$T_{9,15}$	36	38	$T_{9,36}$	44	42	$T_{10,12}$	38	38
$T_{9,16}$	36	36	$T_{9,37}$	42	40	$T_{10,13}$	38	39
$T_{9,17}$	38	39	$T_{9,38}$	42	46	$T_{10,14}$	40	38
$T_{9,18}$	38	37	$T_{9,39}$	50	48	$T_{10,15}$	40	40
$T_{9,19}$	38	36	$T_{9,40}$	50	52	$T_{10,16}$	40	40
$T_{9,20}$	38	40	$T_{9,41}$	48	52	$T_{10,17}$	36	36
$T_{9,21}$	38	41	$T_{9,42}$	52	54	$T_{10,18}$	36	36
$T_{9,22}$	38	38	$T_{9,43}$	60	58	$T_{10,19}$	40	42
$T_{9,23}$	34	35	S_9	72	64	$T_{10,20}$	40	41
$T_{9,24}$	36	38	P_{10}	34	32	$T_{10,21}$	40	40
$T_{9,25}$	34	36	$T_{10,1}$	36	34	$T_{10,22}$	40	41
$T_{9,26}$	42	40	$T_{10,2}$	36	35	$T_{10,23}$	40	39
$T_{9,27}$	42	43	$T_{10,3}$	36	35	$T_{10,24}$	40	43
$T_{9,28}$	40	42	$T_{10,4}$	36	35	$T_{10,25}$	42	43
$T_{9,29}$	40	44	$T_{10,5}$	38	38	$T_{10,26}$	42	41
$T_{9,30}$	36	38	$T_{10,6}$	38	37	$T_{10,27}$	42	41
$T_{9,31}$	38	40	$T_{10,7}$	38	37	$T_{10,28}$	42	40
$T_{9,32}$	38	42	$T_{10,8}$	38	37	$T_{10,29}$	42	44
$T_{9,33}$	42	46	$T_{10,9}$	38	36	$T_{10,30}$	42	45
$T_{9,34}$	44	46	$T_{10,10}$	38	39	$T_{10,31}$	42	43
$T_{9,35}$	44	48	$T_{10,11}$	38	38	$T_{10,32}$	42	42

T	M_1	M_2	T	M_1	M_2	T	M_1	M_2
$T_{10,33}$	42	42	$T_{10,54}$	46	50	$T_{10,75}$	54	52
$T_{10,34}$	42	45	$T_{10,55}$	46	46	$T_{10,76}$	54	56
$T_{10,35}$	38	39	$T_{10,56}$	46	44	$T_{10,77}$	50	53
$T_{10,36}$	40	42	$T_{10,57}$	46	52	$T_{10,78}$	50	57
$T_{10,37}$	38	40	$T_{10,58}$	42	46	$T_{10,79}$	52	59
$T_{10,38}$	38	39	$T_{10,59}$	40	43	$T_{10,80}$	52	50
$T_{10,39}$	38	38	$T_{10,60}$	40	41	$T_{10,81}$	52	58
$T_{10,40}$	38	38	$T_{10,61}$	46	50	$T_{10,82}$	56	62
$T_{10,41}$	38	40	$T_{10,62}$	42	47	$T_{10,83}$	56	54
$T_{10,42}$	40	42	$T_{10,63}$	42	44	$T_{10,84}$	56	59
$T_{10,43}$	36	36	$T_{10,64}$	40	43	$T_{10,85}$	48	54
$T_{10,44}$	46	44	$T_{10,65}$	42	44	$T_{10,86}$	48	51
$T_{10,45}$	46	47	$T_{10,66}$	42	46	$T_{10,87}$	44	50
$T_{10,46}$	46	47	$T_{10,67}$	42	42	$T_{10,88}$	46	52
$T_{10,47}$	42	45	$T_{10,68}$	38	41	$T_{10,89}$	44	47
$T_{10,48}$	44	47	$T_{10,69}$	48	50	$T_{10,90}$	48	55
$T_{10,49}$	44	45	$T_{10,70}$	48	47	$T_{10,91}$	48	51
$T_{10,50}$	44	44	$T_{10,71}$	48	46	$T_{10,92}$	54	60
$T_{10,51}$	44	49	$T_{10,72}$	48	52	$T_{10,93}$	64	62
$T_{10,52}$	44	46	$T_{10,73}$	48	53	$T_{10,94}$	64	67
$T_{10,53}$	44	48	$T_{10,74}$	48	49	$T_{10,95}$	60	66

T	M_1	M_2	T	M_1	M_2	T	M_1	M_2
$T_{10,96}$	66	69						
$T_{10,97}$	76	74						
$T_{10,98}$	58	65						
S_{10}	90	81						

Vita

John Wheless Estes was born in Morristown Tennessee on March 27th 1985, the son of Michael Earl Estes and Suellen Wheless Estes. His family moved three times before finally settling in Blue Mountain Mississippi for the purpose of starting a church, Life Connection. There he volunteered time with construction projects, musical endeavors, and the leading of teenage groups.

John graduated from Ripley High School in 2003, after which he attented Oral Roberts University where he served as a floor chaplain and as a community outreach team leader. He graduated in 2007 with a Bachelors of Science degree in Mathematics and a Spanish Major.

May of that year, John married his high school sweetheart, Stephany Rangel, and the two began their life together in Oxford, Mississippi.

In the fall of 2007, John started as a full time graduate student in the Department of Mathematics at the University of Mississippi, and during his time there taught a variety of undergraduate mathematics courses. In May 2009, John obtained his Masters of Science in Mathematics under the supervision of Dr. William Staton.

On October 21st 2011, Stephany gave birth to the Estes' first son, Ethan. He weighed 6 lbs 11 oz.

In the latter part of 2012, the Estes family will be moving to Jackson Mississippi where John will start his professional career as a professor of mathematics at Belhaven University.