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The Legendre Transform of the Holst Action

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THE LEGENDRE TRANSFORM OF THE HOLST ACTION

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Abstract

I start with a short introduction on 3+1 ADM form and the tetrad form of General Relativity, then I review the Legendre transform of the Einstein-Hilbert action and the Palatini action. The Holst action is a generalization of the Palatini action by including a topological term. I derive Ashtekar's connection form directly from this action by doing the Legendre transformation rather than by a canonical transformation in the usual phase space. This is done in both Remmanian signature with half-flat connection and Lorentz signature with general Barbero-Immirzi parameter.

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Chapter 1
INTRODUCTION

1.1 Overview

In the last century, General Relativity (GR) and Quantum Mechanics (QM) have obtained remarkable success in their own range of applicability. Quantum mechanics describes the microscopic world of nuclear force, atoms and molecules in which GR effects are negligible, while GR describes the macroscopic world, such as the motion of binary black holes, galaxy structures, or the universe evolution itself, in which QM effects are negligible. However, there are several important issues we can not neglect so far. Firstly, the two theories are based on contradictory assumptions. GR, which is deterministic and continuous, theoretically can predict exact trajectories for particles given initial conditions and energy distributions, while quantum mechanics can only predict probabilities for them. Followed by this is that, we do not know how to get the spacetime of a quantum particle system. Also, Quantum Field Theory (QFT) as a successful application of QM is based on a fixed background spacetime, which works well when the energy scale is small. However, once the particle's energy is at the Planck scale, the spacetime is dramatically dynamical, we can not apply the current QFT anymore, and therefore we can't make any predictions using current theory. The other important issue for GR is the singularity problem, as we know that under the current GR the center of a black hole and the beginning the the universe lead to singularities. The existence of these singularities is a sign that GR itself is not complete and it can be replaced by some other more fundamental theory. The solution to all these problems is widely believed to be the theory of Quantum Gravity.

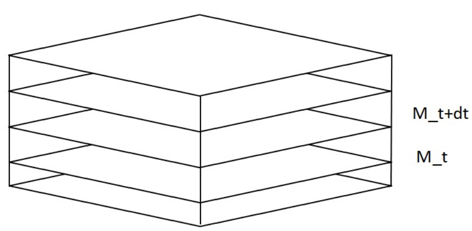
To quantize gravity, one may suggest to follow the ways the other three forces are being quantized. However, since gravity is fundamentally different from the others, the metric tensor that we use to describe the gravity field also plays the role of spacetime. This means that the spacetime itself is dynamical and there is no background anymore. When we quantize gravity, we are also quantizing the spacetime. This is quite different from the other forces, where we can assume an almost static background spacetime. Consequently, to quantize gravity we have to find a new way, which should be background independent.

Loop quantum gravity, which was first proposed by Ashtekar in 1986 [2, 3, 4], is one of this kind of approaches. It has achieved encouraging results in the past 20 years. In this frame, the geometrical area and volume are being quantized and the black hole entropy is derived to be the classical result plus a correction term. In this paper we will focus on the classical part of this approach and start from a classical action which was proposed by Holst [10], to derive the Hamiltonian of the theory.

In the following, I will first review some basic facts about the ADM formalism and tetrad formalism, then start with the Holst action in the next chapter.

1.2 ADM formalism

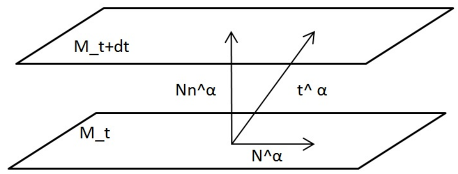
For a thorough treatment, go to reference [1] or the Appendix E of [21]. We assume that the spacetime manifold $(\mathcal{M}, g_{\mu\nu})$ is globally hyperbolic (although the signature of the metric $g_{\mu\nu}$ we are considering in this chapter is Lorentzian, the quantization technique in the next few chapters also applies to the Euclidean spacetime [4]), so the topological structure is of the form $\mathcal{M} \cong R \times M$, where M is a compact 3-manifold with Euclidean signature. We can foliate such a spacetime with a one parameter family of hypersurfaces M_t, where t is a global time function.



Let n^α be a unit vector normal to the hypersurface, $n^\alpha n_\alpha = \sigma$, where σ is the sign of the determinant of the metric tensor $g_{\mu\nu}$ (for the Lorentzian signature $\sigma = -1$, Euclidean signature $\sigma = 1$). The induced metric on the hypersurface is $q_{\mu\nu} = g_{\mu\nu} - \sigma n_\mu n_\nu$, with

$q_{\mu\nu}n^{\mu} = 0$. The extrinsic curvature of the hypersurface is given by $K_{ab} = q_a^{\alpha} q_b^{\beta} \nabla_{\alpha} n_{\beta} = \frac{1}{2}\mathcal{L}_n q_{ab}$, where ∇_{α} is the 4-d covariant derivative operator defined by the metric $g_{\mu\nu}$.

Let t^{α} be a vector satisfying $t^{\alpha}\nabla_{\alpha} t = 1$. We can decompose it as $t^{\alpha} = Nn^{\alpha} + N^{\alpha}$, where N is the lapse function and N^{α} is the shift vector on the hypersurface M.



We have $n^{\mu}N_{\mu} = 0$. The vector field t^{α} can be interpreted as representing the time flow throughout spacetime. By moving forward over time, the spatial metric is evolving from $q_{ab}(t = 0)$ to $q_{ab}(t)$. In this way, we can view the manifold $(\mathcal{M}, g_{\mu\nu})$ as the time development of the spatial metric on the hypersurface. This will be obvious once we finish the Legendre transformation. Choosing the natural basis adapted to coordinates [9], the metric is given by $\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + q_{ab}(\mathrm{d}x^a + N^a\mathrm{d}t)(\mathrm{d}x^b + N^b\mathrm{d}t)$.

In this paper, we use Greek letters to label spacetime indices, Latin letters from the beginning of the alphabet (a, b, c) are used when indices are projected on the hypersurface. For an arbitrary spacetime tensor $T^{\alpha\cdot\cdot\beta}{}_{\gamma\cdots\delta}$, we can decompose it by contracting with q_{α}^{a} and n^{μ},

$$T^{\alpha\cdot\cdot\beta}{}_{\gamma\cdots\delta} = T^{\mu\cdot\cdot\nu}{}_{\lambda\cdot\cdot\rho}\, g_{\mu}^{\alpha}\cdots g_{\nu}^{\beta} g_{\beta}^{\lambda}\cdots g_{\gamma}^{\lambda}\cdots g_{\delta}^{\rho} = T^{\mu\cdot\cdot\nu}{}_{\lambda\cdot\cdot\rho}(q_{\mu}^{\alpha} + \sigma n^{\alpha}n_{\mu})\cdots(q_{\delta}^{\rho} + \sigma n^{\rho}n_{\delta}) \quad (1.1)$$

Those contracted with q_a^{α} are projected on the hypersurface M_t,

$$T^{a\cdots b}{}_{c\cdots d} := q_{\alpha}^{a}\,\cdots\, q_{\beta}^{b}\, q_{c}^{\gamma}\cdots\, q_{d}^{\delta}\, T^{\alpha\cdot\cdot\beta}{}_{\gamma\cdots\delta}, \quad (1.2)$$

while those contracted with n^{μ}, such as $T^{\alpha\cdot\cdot\beta}{}_{\gamma\cdots\delta}\, n_{\alpha}\cdots n^{\delta}$, are orthogonal to the hypersurface.

For those tensors that have only spatial indices, we define the covariant derivative operator on the hypersurface as

$${}^{3}\nabla_{c}T^{a_1\cdots a_k}{}_{b_1\cdots b_l} := q_{c_1}^{a_1}\cdots q_{b_l}^{d_l} q_{c}^{\lambda}\nabla_{\lambda}T^{c_1\cdots c_k}{}_{d_1\cdots d_l} \quad (1.3)$$

If ∇_μ is compatible with the metric $g_{\alpha\beta}$, then ${}^3\nabla_a$ is compatible with the induced metric q_{cd} (proof can be found in Chapter 10 of [21]). Although the definition of this derivative operator is based on tensors on the hypersurface, it can also act on spacetime indices. They are related by

$$\begin{aligned} q_a^\alpha\, q_\beta^b\, \nabla_\alpha\, T^\beta &= q_a^\alpha\, q_\beta^b\, \nabla_\alpha\, (T^\gamma g_\gamma^\beta) \\ &= q_a^\alpha\, q_c^b\, \nabla_\alpha\, (T^\gamma q_\gamma^c) + q_a^\alpha\, q_\beta^b\, \nabla_\alpha\, (T^\gamma\, \sigma\, n^\beta\, n_\gamma) \\ &= {}^3\nabla_a\, T^b + \sigma\; n_\gamma\, T^\gamma\, \bar{K}_a{}^b, \end{aligned} \tag{1.4}$$

where we have applied $q_\beta^b\, n^\beta = 0$ and the definition the extrinsic curvature $\bar{K}_a{}^b = q_a^\alpha\, q_\beta^b\, \nabla_\alpha n^\beta$ in the last step.

The curvature associated with this derivative operator is given by

$${}^3\Omega_{ab}{}^{cd}\, T_d := 2^3\nabla_{[a}\nabla_{d]}\, T^c. \tag{1.5}$$

This curvature is related to the spacetime curvature by the Gauss-Codazzi equation (proof can be found in Chapter 12 of [13]):

$${}^3\Omega_{abc}{}^d = q_a^\alpha\, q_b^\beta\, q_c^\gamma\, q_\delta^d\, \Omega_{\alpha\beta\gamma}{}^\delta - \bar{K}_{ac}\bar{K}_b{}^d + \bar{K}_{bc}\bar{K}_a{}^d. \tag{1.6}$$

We will derive a similar equation in Chapter 2 to include internal indices.

1.3 Tetrad formalism

For a 4-d manifold with a general coordinate basis ($\hat{e}_{(\mu)} = \partial\, / \partial\, x^\mu$), at each point we can always find an orthonormal basis, $\hat{e}_{(I)}$ [8, 21]. The two tetrads are related by

$$\hat{e}_{(\mu)} = e_\mu{}^I(x)\, \hat{e}_{(I)}, \quad \hat{e}_{(I)} = e^\mu{}_I(x)\, \hat{e}_{(\mu)}, \tag{1.7}$$

where the tetrad $e_\mu{}^I(x)$ forms an $n\times n$ invertible matrix with inverse $e^\mu{}_I(x)$. Since the new basis is orthonormal, we have

$$g_{\mu\nu}(x)\, e^\mu{}_I(x)\, e^\nu{}_J(x) = \eta_{IJ}, \tag{1.8}$$

where η_{IJ} is the fixed metric of signature (–,+,+,+) for Lorentzian spacetime and (+,+,+,+) for Euclidean spacetime. We are free to rotate this new tetrad as long as the basis vectors are orthonormal, the gauge group is then SO(4) for Riemannian signature and SO(3, 1) for Lorentz signature. We use V to label this internal space with fixed metric. The corresponding indices are denoted by Latin letters from the middle of the alphabet $(I, J, K, \cdots)$. The metric η_{IJ} can be used to raise or lower these internal indices just like $g_{\mu\nu}$ is used to raise and lower the spacetime indices. The alternating tensor ϵ_{IJKL} on V is related to the alternating tensor on spacetime as $\epsilon_{IJKL} = \epsilon_{\alpha\beta\gamma\delta} e^\alpha{}_I\, e^\beta{}_J\, e^\lambda{}_K\, e^\delta{}_L$.

Tensor components can be written in general coordinates or in an orthonormal basis, or even a combination of them. We can extend the covariant derivative operator to include internal indices,

$$D_\mu T^{\nu I} = \nabla_\mu T^{\nu I} + \omega_\mu{}^I{}_J T^{\nu J}, \tag{1.9}$$

where ∇_μ only acts on spacetime indices and treats internal indices as scalars, and $\omega_\mu{}^{IJ}$ is a Lie-algebra so(3, 1) valued one-form. Usually we take D_μ compatible with η_{IJ}, and then the connection one form is antisymmetric, $\omega_\mu{}^{IJ} = \omega_\mu{}^{[IJ]}$. There are two curvatures associated with this extended covariant derivative operator,

$$\Omega_{\mu\nu}{}^{IJ} T_J := 2D_{[\mu}D_{\nu]}T^I \tag{1.10}$$

$$\Omega_{\mu\nu}{}^{\alpha\beta} T_\beta := 2D_{[\mu}D_{\nu]}T^\alpha = 2\nabla_{[\mu}\nabla_{\nu]}T^\alpha\,. \tag{1.11}$$

The second one is identified with the curvature in a general spacetime coordinate system as we introduced in the previous section. The first one can be written equivalently

$$\Omega_{\mu\nu}{}^{IJ} := 2\partial_{[\mu}\omega_{\nu]}{}^{IJ} + 2\omega_{[\mu}{}^{IK}\omega_{\nu]K}{}^{J}. \tag{1.12}$$

If D_μ is compatible with the tetrad, $D_\beta e^I_\alpha = 0$, together with the condition, $D_\mu \eta_{IJ} = 0$, we obtain $D_\mu g_{\alpha\beta} = 0$. Consequently, the connection is identified with the spin connection given by $\omega_\mu{}^{IJ} = -e^{\nu J}\nabla_\mu e_{\nu I}$, where ∇_μ is associated with the Levi-Civita connection. In this case the two curvatures are related as

$$\Omega_{\mu\nu}{}^{IJ} = \Omega_{\mu\nu}{}^{\alpha\beta} e^I_\alpha e^J_\beta. \tag{1.13}$$

We will use them in chapter 2 to prove that the Palatini action reproduces the Einstein-Hilbert action.

Chapter 2

THE LEGENDRE TRANSFORMATION

In this chapter, I will give a short review of the Einstein-Hilbert action and the Palatini action, then discuss the Holst action in detail.

2.1 The Einstein-Hilbert action

General Relativity can be derived from the Einstein-Hilbert action,

$$S_H = \int_{\mathcal{M}} \mathrm{d}^4x \, \sqrt{|g|} \, \Omega, \tag{2.1}$$

where Ω is the Ricci scalar of the metric $g_{\mu\nu}$, given by $\Omega = g^{\mu\alpha} \, g^{\nu\beta} \Omega_{\mu\nu\alpha\beta}$. The connection is determined by the metric, so this is a second-order action. By varying it with respect to the metric $g_{\mu\nu}$, we can get the vacuum Einstein equation

$$\Omega_{\mu\nu} - \tfrac{1}{2} \Omega \, g_{\mu\nu} = 0. \tag{2.2}$$

By a 3+1 decomposition, this action turns out to be [1, 21, 16]:

$$S_H = \int \mathrm{d}t \int_M \mathrm{d}^3x \sqrt{q} \, N[\, {}^3\Omega + \bar{K}_{ab} \, \bar{K}^{ab}] + \bar{K}^2, \tag{2.3}$$

where the extrinsic curvature is related to the time derivative of q_{ab} by

$$\bar{K}_{ab} = \tfrac{1}{2} \, N^{-1} [\dot{q}_{ab} - {}^3\nabla_a N_b - {}^3\nabla_b N_a]. \tag{2.4}$$

The Hamiltonian is

$$
\begin{aligned}
H &= \int \mathrm{d}t \int_M \mathrm{d}^3x\sqrt{q}\{\, N\,[-{}^3\Omega + q^{-1}\pi^{ab}\pi_{ab} - \tfrac{1}{2}q^{-1}\pi^2\,] \,-2N_b\,[D_a(q^{-1/2})\pi^{ab}] \\
&\quad + 2D_a(q^{-1}N_b\,\pi^{ab})\},
\end{aligned}
\tag{2.5}
$$

where $\pi = \pi^a{}_a$, and $\bar{K}_{ab} = \frac{1}{2}\,N\,[\,\dot{q}_{ab} - {}^3\nabla_a\,N_b - {}^3\nabla_b\,N_a\,]$ with $\dot{q}_{ab} = q_a^c\,q_b^d\,\pounds_t h_{cd}$ [21]. The only variable that has time derivative is the metric on the hypersurace q_{ab}. The momentum canonically conjugate to it is $\pi_{ab} = \sqrt{q}\,(\bar{K}^{ab} - \bar{K}\,q^{ab})$, so the canonical variables are (q_{ab}, π_{ab}). There are no time derivatives of N, N^a, so the momenta canonically conjugate to them vanish. They are not dynamical variables. We will take them as Lagrangian multipliers. The corresponding coefficients are the scalar constraint and the vector constraint,

$$
-{}^3\Omega + q^{-1}\pi^{ab}\pi_{ab} - \tfrac{1}{2}\,q^{-1}\pi^2 = 0 \tag{2.6}
$$

$$
{}^3\nabla_a\,(q^{-1}\pi^{ab}) = 0. \tag{2.7}
$$

Since the Hamiltonian constraint is a complicated and non-polynomial function of the metric, it runs into troubles when people want to go further to get a canonical quantum theory.

2.2 The Hilbert-Palatini action

Instead of taking the connection as a function of the metric, the Hilbert-Palatini action considers the metric and the connection as independent variables. By using the tetrad form, the action is given by

$$
S_P(e,\omega) = \int_{\mathcal{M}} \epsilon_{IJKL}\,e^I\,\wedge\,e^J\,\wedge\Omega^{KL}\,, \tag{2.8}
$$

where the curvature with internal indices is given by equation (1.11) or (1.09). By varying this action with respect to the connection, we get that the connection is actually determined by the tetrad,

$$
\mathrm{d}e + \omega\,\wedge e = 0\,. \tag{2.9}
$$

The other equation of motion, obtained by varying the action with respect to the metric, is

$$\tilde{\epsilon}^{\alpha\beta\lambda\delta}\,\epsilon_{IJKL}\,e_\beta^J\,\Omega_{\lambda\delta}{}^{KL}=0\,. \qquad (2.10)$$

By contracting it with $e^{\mu I}$ and using equation (1.13), we get $G^{\alpha\mu}=\Omega^{\alpha\mu}-\frac{1}{2}\,\Omega\,g^{\alpha\mu}$, which is the vacuum Einstein field equation. The Hilbert-Palatini action reproduces the same classical results as the Einstein-Hilbert action.

With the tetrad introduced in this way, the theory has more structures now. By following the same procedure as in the previous section, we do the 3+1 decomposition [16]

$$\begin{aligned} S_P &= -\int \mathrm{d}t \int_M \mathrm{d}^3x\,[\,\tfrac{1}{2}\frac{N}{k}\,\mathrm{tr}\,(\tilde{e}^a\tilde{e}^b\Omega_{ab})+\tfrac{1}{2}N^a\mathrm{tr}(\tilde{e}^b\Omega_{ab}) \\ &\quad +\tfrac{1}{2}\tilde{e}^a{}_{IJ}\mathcal{L}_t\omega_a{}^{IJ}+\tfrac{1}{2}\,({}^3D_a\,\tilde{e}^a{}_{IJ})(t\cdot\omega^{IJ})\,], \end{aligned} \qquad (2.11)$$

where $\Omega_{cd}{}^{IJ}=q_c^\lambda\,q_d^\delta\,\Omega_{\lambda\delta}{}^{IJ}$, $\tilde{e}^a{}_{IJ}:=\frac{1}{2}\epsilon_{IJKL}\tilde{\epsilon}^{abc}\,e_\beta^K\,e_\lambda^L\,q_b^\beta\,q_c^\lambda$. There are 3 Lagrange multipliers now, N, N^a, $t\cdot\omega^{IJ}$. The only dynamical variable is $\tilde{e}^a{}_{IJ}$, the momentum is actually the connection $\omega_a{}^{IJ}$. Thus the canonical pair of conjugate variables is $(\tilde{e}^a{}_{IJ},\omega_a{}^{IJ})$.

It is now straightforward to get the Hamiltonian form from this action. It is the sum of three constraints, each multiplied by a Lagrangian multiplier. Those constraints are given by

$$\begin{aligned} \mathrm{tr}(\tilde{e}^a\tilde{e}^b\,\Omega_{ab}) &= 0 \\ \mathrm{tr}(\tilde{e}^b\Omega_{ab}) &= 0 \\ {}^3D_a\tilde{e}^a{}_{IJ} &= 0. \end{aligned} \qquad (2.12)$$

Compare to the Einstein-Hilbert action, there are 3 constraints now. The first two are the scalar constraint and the vector constraint. They are much simpler and are both polynomial in the fundamental variables. By introducing the tetrad formalism, which is Lorentz invariant, we have Gauss constraint here. This is very similar as that in the Yang-Mills theory. Thus we may be able to use some technique from there to quantize gravity. However, as we calculate the Poisson brackets between them, we find that there exists a second class constraint [16]. Once it is solved and plugged back into the action, we are led back to the Einstein-Hilbert action. The reference to connection dynamics is lost, our original difficulty still exists.

2.3 The Holst action

The Holst action [10] for gravity is the sum of the Palatini action $S_{\rm P}(e,\omega)$ and a term containing the Barbero-Immirzi parameter γ [5], which can take any fixed non-zero real value. Written in tetrad form,

$$\begin{aligned} S_{\rm H}(e,\omega) &:= S_{\rm P}(e,\omega) - \frac{1}{2k\gamma}\int_{\mathcal{M}} e^I \wedge e^J \wedge \Omega_{IJ} \\ &= \frac{1}{4k}\int_{\mathcal{M}} \epsilon_{IJKL}\, e^I \wedge e^J \wedge \Omega^{KL} - \frac{1}{2k\gamma}\int_{\mathcal{M}} e^I \wedge e^J \wedge \Omega_{IJ}, \end{aligned} \tag{2.13}$$

which is a functional of a tetrad $e_\alpha{}^I$ and an SO(3,1) connection $\omega_{\mu I}{}^J$. The equation of motion [4] obtained by varying the action with respect to the connection is the same as that for the Palatini action,

$$\mathrm{d}e + \omega \wedge e = 0\,. \tag{2.14}$$

If we plug this connection into the second term of the Holst action, it vanishes because of the Bianchi identity. Thus the Holst action also reproduce the same classical results as the Einstein-Hilbert action [14]. The second term is just a topological term, γ is analogous to the θ parameter in the Yang-Mills theory. By working out the symplectic structure, it shows that γ induces a canonical transformation on phase space: $\tilde{e}^a{}_{IJ}$ stays the same, $\omega_\mu{}^{IJ} \to \frac{1}{2}\left(\omega_\mu{}^{IJ} - \frac{\gamma}{2}\epsilon^{IJ}{}_{KL}\omega_\mu{}^{KL}\right)$. The new constraints can be worked out from ADM form by this transformation [5, 19]. However, we will not do that way in this paper. We will perform the Legendre transform from the Holst action and derive all the constraints directly from this action.

In the following, we will discuss half-flat connections in the Riemannian signature first, in which case γ is either i or$-i$. In the second part, we consider the general case, in which γ can take any non-zero real value. It applies to both Riemmanian and Lorentz signatures.

2.3.1 Riemannian signature with half-flat connection

In this part, we set $\sigma = 1$. We introduce the half-flat connection [2, 3, 12]

$$\omega^{(+)}{}_{IJ} := \tfrac{1}{2}(\omega_{IJ} - \frac{\gamma}{2}\epsilon_{IJ}{}^{KL}\omega_{KL})\,, \tag{2.15}$$

so that $\omega^{(+)}$ is the anti-self dual part of ω if $\gamma = 1$ and the self dual part, if $\gamma = -1$. Similarly, we have:

$$\Sigma^{IJ}_{(+)} := \tfrac{1}{2}(e^I \wedge e^J - \frac{\gamma}{2}\epsilon^{IJ}{}_{KL}\, e^K \wedge e^L)\,. \tag{2.16}$$

If we replace ω with $\omega^{(+)}$ in the Holst action, we get

$$\begin{aligned} S_{(H)}(e, \omega^{(+)}) &= -\frac{1}{k\gamma}\int_{\mathcal{M}} \Sigma^{IJ}_{(+)} \wedge \Omega^{(+)}{}_{IJ} \\ &= -\frac{1}{4k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{g}\Sigma_{(+)\alpha\beta}{}^{IJ}\, \Omega^{(+)}_{\lambda\delta IJ}\, \epsilon^{\alpha\beta\lambda\delta}, \end{aligned} \tag{2.17}$$

where $\Omega^{(+)}{}_{IJ}$ is both the (anti)self dual part of Ω_{IJ} and the curvature of $\omega^{(+)}{}_{IJ}$ [12]. By using $g^\alpha{}_\beta = q^\alpha{}_\beta - n^\alpha n_\beta$, we carry out the 3+1 decomposition

$$\begin{aligned} S_{(H)}(e, \omega^{(+)}) &= -\frac{1}{4k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{g}\epsilon^{\alpha\beta\lambda\delta}\, \Sigma_{(+)\alpha\beta}{}^{IJ}\, \Omega^{(+)}_{\mu\nu IJ}\, (q^\mu{}_\lambda - n^\mu n_\lambda)(q^\nu{}_\delta - n^\nu\, n_\delta) \\ &= \frac{1}{4k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{g}\epsilon^{\alpha\beta\lambda\delta}\, \Sigma_{(+)\alpha\beta}{}^{IJ}\, \Omega^{(+)}_{\mu\nu IJ}\, q^\mu{}_\lambda\, q^\nu{}_\delta \\ &\quad - \frac{1}{2k\gamma}\int \mathrm{d}t \int_M \mathrm{d}^3x\sqrt{q}N\Sigma_{(+)\alpha\beta}{}^{IJ}\Omega^{(+)}_{\mu\nu IJ}\epsilon^{\alpha\beta\delta}\, n^\mu\, q^\nu{}_\delta\,, \end{aligned} \tag{2.18}$$

where $\epsilon^{\alpha\beta\lambda\delta}n_\lambda = \epsilon^{\alpha\beta\delta}$ has been used. For convenience, we label the first term as $S_{(H1)}$ and the second one as $S_{(H2)}$. We will deal with the second term first. Using $n^\mu = \frac{1}{N}(t^\mu - N^\mu)$,

$$S_{(H2)} = -\frac{1}{2k\gamma}\int \mathrm{d}t \int_M \mathrm{d}^3x\sqrt{q}\, \Sigma_{(+)\alpha\beta}{}^{IJ}\epsilon^{\alpha\beta\delta}\, [\, \Omega^{(+)}_{\mu\nu IJ}\, t^\mu - N^\mu\, \Omega^{(+)}_{\mu\nu IJ}\,]\, q^\nu_\delta\,. \tag{2.19}$$

Since

$$\begin{aligned} t^\mu\,\Omega^{(+)}_{\mu\nu IJ}\,q^\nu_\delta &= t^\mu\,\nabla_\mu(\omega^{(+)})_\nu q^\nu_\delta + \omega_\mu \nabla_\nu(t^\mu) q^\nu_\delta - \nabla_\nu\,(t\cdot\omega^{(+)}_{IJ})q^\nu_\delta \\ &\quad - \omega^{(+)}_{\nu JK} t^\mu \omega^{(+)K}_{\mu I}\,q^\nu_\delta - \omega^{(+)K}_{\nu I}\,t^\mu\,\omega^{(+)}_{\mu KJ} q^\nu_\delta \\ &= q^\nu_\delta\,\pounds_t(\omega^{(+)}_{\nu IJ}) - q^\nu_\delta \nabla_\nu(t^\mu\,\omega^{(+)}_{\mu IJ})\,, \end{aligned} \tag{2.20}$$

introducing

$$P^a{}_{IJ} := -\frac{1}{2k\gamma}\epsilon^{abc}\Sigma^{(+)}_{bcIJ},$$

we have now

$$S_{(H2)} = \int \mathrm{d}t \int_M \mathrm{d}^3x\,[\,P^a{}_{IJ}(\mathcal{L}_t A_a{}^{IJ} - {}^3\nabla(t\cdot\omega^{(+)})_{IJ}) - N^a\,P^b{}_{IJ}F_{abIJ}\,]\,, \tag{2.21}$$

where we have used $(a, b, c, ...)$ to label the spatial indices, and

$$A_a{}^{IJ} := q^\alpha_a \omega_\alpha{}^{IJ}, \quad F_{ab}{}^{IJ} = q^\alpha_a\,q^\beta_b\,\Omega^{(+)IJ}_{\alpha\beta} = 2\mathrm{d}_{[a}A_{b]}{}^{IJ} + A_a{}^{IK} \wedge\, A_{bK}{}^J. \tag{2.22}$$

Applying integration by parts on the second term, neglecting the surface term, we obtain

$$S_{(H2)} = \int \mathrm{d}t \int_M \mathrm{d}^3x\,[\,P^a{}_{IJ}\mathcal{L}_t A_a{}^{IJ} + (t\cdot\omega_{IJ})^3\nabla_a\,P^a{}_{IJ} - N^a\,P^b{}_{IJ}\Omega_{abIJ}\,]\,. \tag{2.23}$$

For $S_{(H1)}$, by using $\Omega^{(+)}_{\mu\nu IJ} = \frac{1}{2}\epsilon_{IJKL}\Omega^{(+)KL}_{\mu\nu}$ and $\epsilon^{\alpha\beta\lambda\delta}\epsilon_{IJKL}e^I_\alpha\,e^J_\beta = 4\,e^\lambda_K\,e^\delta_L$ we have now

$$S_{(H1)} = -\frac{1}{2k\gamma}\int \mathrm{d}t \int_M \mathrm{d}^3x N\,\sqrt{q}\,(\,e^c_M\,e^d_N - \frac{\gamma}{2}\epsilon_{IJMN}e^{cI}\,e^{dJ}\,)\,F_{cd}{}^{MN}\,. \tag{2.24}$$

Since $e^c_N = \frac{1}{2\sqrt{q}}\epsilon_{NBG}\epsilon^{cab}\, e^B_a\, e^J_b$, we can rewrite the term above as

$$\begin{aligned}
& \frac{-1}{2k\gamma} N\sqrt{q}\, e^c_N\, e^d_M\, F_{cdN}{}^M \\
= \;& -\frac{k^2\gamma^2}{2k\gamma} N\,\sqrt{q}\,\big(\,\frac{-1}{2k\,\gamma\sqrt{q}}\epsilon^{NBI}\epsilon^{cab}\, e_{aB}\, e_{bI}\,\big)\,\big(\,\frac{-1}{2k\,\gamma\sqrt{q}}\epsilon_{MJQ}\epsilon^{dgl} e^J_g\, e^Q_l\,\big)\, F_{cdN}{}^M \\
= \;& -\frac{k\gamma}{2}\frac{N}{\sqrt{q}} P^c{}_{BI} P^{dJQ}\epsilon^{NBI}\epsilon_{MJQ} F_{cdN}{}^M \\
= \;& -\frac{2k\gamma\, N}{\sqrt{q}} P^c{}_{MI} P^{dIN} F_{cdN}{}^M. \qquad (2.25)
\end{aligned}$$

Repeating the same work for the second term in $S_{(H1)}$, we have

$$S_{H1} = -k\gamma \int \mathrm{d}t \int_M \mathrm{d}^3x\, \frac{N}{\sqrt{q}} [P\wedge P]^{(+)cd}_{MN} F_{cd}{}^{MN}\,. \qquad (2.26)$$

Applying $[P\wedge P]^{(+)} = P^{(+)}\wedge\, P^{(+)}$ and putting all terms above together, we have

$$\begin{aligned}
S_{(H)} \;=\; & \int \mathrm{d}t \int_M \mathrm{d}^3x\,\big[\, P^a{}_{IJ}\,\mathcal{L}_t\, A_a{}^{IJ} + (t\cdot\omega_{IJ})^3 \nabla_a P^a{}_{IJ} - N^a P^b{}_{IJ}\Omega_{abIJ} \\
& - \frac{k\gamma N}{\sqrt{q}} P^{(+)c}{}_N{}^J\, P^{(+)d}{}_J{}^M\, F_{cd}{}^{MN}\,\big]. \qquad (2.27)
\end{aligned}$$

The Hamiltonian is given by the sum of the last three terms. The canonical pair of conjugate variables is $(A_a{}^{IJ}, P^a{}_{IJ})$. Similarly to the Palatini action, there are three constraints and they are polynomial in the fundamental variables. This simplifies the quantization a lot. However, since the variables are complex now, we have to apply reality conditions to get physical results. However, the reality conditions become non-polynomial. To avoid this extra work, Barbero introduced real canonical variables, which are equivalent to setting γ to be real in the Holst action.

2.3.2 General Barbero-Immirzi parameter γ

This part applies to both the Riemannian signature and the Lorentz signature, so $\sigma = \pm 1$. In the Riemannian case, the gauge group is SO(4), while for the Lorentz case it is SO(3,1), which is not compact. We use compact groups in quantum theory, so we will reduce this group to SO(3). First we write down $S_{\rm H}$ with explicit spacetime indices

$$\begin{aligned} S_{\rm H}(e,\omega) &= \frac{1}{8k}\int_{\mathcal{M}} \epsilon_{IJKL}\, e_\alpha{}^I e_\beta{}^J\, \Omega_{\lambda\delta}{}^{KL}\, \tilde{\epsilon}^{\alpha\beta\lambda\delta}\, {\rm d}^4x - \frac{1}{4k\gamma}\int_{\mathcal{M}} e_\alpha{}^I\, e_\beta{}^J\, \Omega_{\lambda\delta IJ}\, \tilde{\epsilon}^{\alpha\beta\lambda\delta}\, {\rm d}^4x \\ &= \frac{1}{8k}\int_{\mathcal{M}} {\rm d}^4x\, 4\,\sigma\, |g|^{1/2}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\lambda\delta}{}^{KL} - \frac{1}{4k\gamma}\int_{\mathcal{M}} {\rm d}^4x\, |g|^{1/2}\, \epsilon^{IJKL}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\lambda\delta IJ}, \end{aligned}$$

where $\tilde{\epsilon}^{\alpha\beta\lambda\delta}\,\epsilon_{IJKL}\, e_\alpha{}^I\, e_\beta{}^J = 4\,\sigma\,|g|^{1/2}\, e^{[\lambda}{}_K\, e^{\delta]}{}_L$ and $\tilde{\epsilon}^{\alpha\beta\lambda\delta}\, e_\alpha{}^I\, e_\beta{}^J = |g|^{1/2}\,\epsilon^{IJKL}\, e^\lambda{}_K\, e^\delta{}_L$ have been used.

Proof of the 1st equation: if we multiply the equation on both sides by $e_\mu{}^K\, e_\nu{}^L$,then

$$\begin{aligned} RHS &= 4\,\sigma\,|g|^{1/2}\,\delta^{[\lambda}_\mu\,\delta^{\delta]}_\nu \\ LHS &= \tilde{\epsilon}^{\alpha\beta\lambda\delta}\,\epsilon_{IJKL}\, e_\alpha{}^I\, e_\beta{}^J\, e_\mu{}^K\, e_\nu{}^L \\ &= \tilde{\epsilon}^{\alpha\beta\lambda\delta}\,\epsilon_{\alpha\beta\mu\nu} \\ &= 2!2!\sigma\,|g|^{1/2}\,\delta^{[\lambda}_\mu\,\delta^{\delta]}_\nu \\ &= RHS. \end{aligned} \tag{2.28}$$

Proof of the 2nd equation: multiply the equation we just proved above on both sides by ϵ^{PQKL}, then

$$\begin{aligned} LHS &= \tilde{\epsilon}^{\alpha\beta\lambda\delta}\, 2!2!\,\sigma\,\delta^{[P}_I\,\delta^{Q]}_J\, e_\alpha{}^I\, e_\beta{}^J \\ &= \tilde{\epsilon}^{\alpha\beta\lambda\delta}\, 2!2!\,\sigma\, e_\alpha{}^P\, e_\beta{}^Q \\ RHS &= 4\,\sigma\,|g|^{1/2}\,\epsilon^{PQKL}\, e^\lambda{}_K\, e^\delta{}_L. \end{aligned} \tag{2.29}$$

If we divided 4σ on both sides, we can get the desired equation.

To carry out the 3+1 decomposition as introduced in section 1, we rewrite the metric as

$g^\mu{}_\nu = q^\mu{}_\nu + \sigma\, n^\mu\, n_\nu$. We have

$$
\begin{aligned}
S_{\mathrm{H}}(e,\omega) &= \frac{1}{2k}\int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, |g|^{1/2}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta}{}^{KL}\, g^\alpha{}_\lambda\, g^\beta{}_\delta \\
&\quad - \frac{1}{4k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, |g|^{1/2}\, \epsilon^{IJKL}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta IJ}\, g^\alpha{}_\lambda\, g^\beta{}_\delta \\
&= \frac{1}{2k}\int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, |g|^{1/2}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta}{}^{KL}\, (q^\alpha{}_\lambda + \sigma\, n^\alpha\, n_\lambda)(q^\beta{}_\delta + \sigma\, n^\beta\, n_\delta) \\
&\quad - \frac{1}{4k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, |g|^{1/2}\, \epsilon^{IJKL}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta IJ}\, (q^\alpha{}_\lambda + \sigma\, n^\alpha\, n_\lambda)(q^\beta{}_\delta + \sigma\, n^\beta\, n_\delta).
\end{aligned}
$$

Since $\Omega_{\alpha\,\beta}{}^{KL}\, n^\alpha\, n^\beta\; = 0$ due to the antisymmetry in the first two indices, two terms above will vanish. We are left with

$$
\begin{aligned}
S_{\mathrm{H}}(e,\omega) &= \frac{1}{2k}\int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, |g|^{1/2}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta}{}^{KL}\, (q^\alpha{}_\lambda\, q^\beta{}_\delta + \sigma\, q^\alpha{}_\lambda\, n^\beta\, n_\delta + \sigma\, q^\beta{}_\delta n^\alpha\, n_\lambda) \\
&\quad - \frac{1}{4k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, |g|^{1/2}\, \epsilon^{IJKL}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta IJ}\, (q^\alpha{}_\lambda\, q^\beta{}_\delta + \sigma\, q^\alpha{}_\lambda\, n^\beta\, n_\delta + \sigma\, q^\beta{}_\delta n^\alpha\, n_\lambda).
\end{aligned}
$$

For $e^\lambda{}_K\, e^\delta{}_L\, \Omega^{KL}{}_{\alpha\beta}\, q^\alpha{}_\lambda\, n^\beta\, n_\delta$, since $\alpha\beta$ are antisymmetrized, $\lambda\delta$ are also antisymmetrized, so if we switch α and β, λ and δ at the same time in $q^\alpha{}_\lambda\, n^\beta\, n_\delta$, we will have

$$
e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta}{}^{KL}\, q^\alpha{}_\lambda\, n^\beta\, n_\delta = e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta}{}^{KL}\, q^\beta{}_\delta\, n^\alpha\, n_\lambda. \tag{2.30}
$$

Plugging this into the above equation, we have

$$
\begin{aligned}
S_{\mathrm{H}}(e,\omega) &= \frac{1}{2k}\int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, |g|^{1/2}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta}{}^{KL}\, (q^\alpha{}_\lambda\, q^\beta{}_\delta + \sigma\, 2\, n^\alpha\, n_\lambda\, q^\beta{}_\delta) \\
&\quad - \frac{1}{4k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, |g|^{1/2}\, \epsilon^{IJKL}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta IJ}\, (q^\alpha{}_\lambda\, q^\beta{}_\delta + \sigma 2\, n^\alpha\, n_\lambda\, q^\beta{}_\delta) \\
&= \frac{1}{2k}\int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, N\sqrt{q}\, e^\lambda{}_K\, e^\delta{}_L\, \Omega_{\alpha\beta}{}^{KL}\, q^\alpha{}_\lambda\, q^\beta{}_\delta \\
&\quad + \frac{1}{k}\int_{\mathcal{M}} \mathrm{d}^4x\, N\sqrt{q}\, e^\delta{}_L\, n^\alpha\, n_K\, \Omega_{\alpha\beta}{}^{KL} q^\beta{}_\delta \\
&\quad - \frac{1}{4k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, N\sqrt{q}\, e^\lambda{}_K\, e^\delta{}_L\, \epsilon^{IJKL}\, \Omega_{\alpha\beta IJ} q^\alpha{}_\lambda\, q^\beta{}_\delta \\
&\quad - \frac{\sigma}{2k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, N\sqrt{q}\, e^\delta{}_L\, n^\alpha\, \epsilon^{IJKL}\, n_K\, \Omega_{\alpha\beta IJ} q^\beta{}_\lambda.
\end{aligned} \tag{2.31}
$$

We will label them separately with (S_{H1}, S_{H2}, S_{H3}, S_{H4}) from now on. In the following we will first deal with $S_{H1}+S_{H3}$, then with $S_{H2}+S_{H4}$.

Since the tetrad we introduced earlier, ${e_\alpha}^I$, has 6 more degrees of freedom than the original metric tensor $g_{\mu\nu}$, we are free to choose a gauge here. Thus we can fix an internal vector field $n^I=(1,0,0,0)$ with $n_I\,n^I=\sigma$. The partial derivative operator ∂_μ annihilates both n^I and $\bar\eta^{IJ}$. The orthogonal 3-dimensional subspace is $V_\perp$ with metric $\eta_{KL}=\bar\eta_{KL}-\sigma\,n_K\,n_L$, which is equivalent to $\eta_{kl}=q^K_k\,q^L_l\,\bar\eta_{KL}$, where $q^K_k=\bar\eta^K_k-\sigma n^K n_k$. Also, there is a natural antisymmetric tensor on $V_\perp$, $\epsilon_{ijk}=q^I_i\,q^J_j\,q^K_k\,n^L\,\epsilon_{LIJK}$, induced by ϵ_{IJKL} on V. The co-frame compatible with this choice is $n^\alpha=n^I\,{e^\alpha}_I$, where n^α is the normal vector in space time. Consequently, ${e_\alpha}^I$ naturally defines a triad ${e_a}^i:={e_\alpha}^I\,q^i_I\,q^\alpha_a$. The metric on M is now $q_{ab}={e_a}^i\,e^j_b\,\eta_{ij}$.

From D_α, we define a derivative operator on the hypersurface,

$$ {}^3D_a\,T^{b\cdot\cdot c}i..j:=q^b_\beta\cdots q^c_\lambda q^\alpha_a q^i_I\cdots q^j_J D_\alpha T^{\beta\cdot\cdot\lambda IJ}. \tag{2.32}$$

The corresponding connection one-form is ${\omega_a}^{ij}=q^\alpha_a q^i_I q^j_J {\omega_\alpha}^{IJ}$. Since dim(SO(3)) is 3, we can define an SO(3) Lie algebra-valued connection one-form as

$$\Gamma^i_a:=-\tfrac{1}{2}{\epsilon^i}_{jk}{\omega_\alpha}^{jk}=\tfrac{1}{2}q^\alpha_a q^i_I {\epsilon^{IJ}}_{KL}n_J{\omega_\alpha}^{KL}. \tag{2.33}$$

If D_α is compatible with the tetrad e^I_β, then ${}^3D_a$ is compatible with the triad e^i_b. For any tensor T^j on the hypersurface, the curvature of ${}^3D_a$ is defined by

$$ {}^3{\Omega_{ab}}^i{}_k\,T^k:=2\,{}^3D_{[a}{}^3D_{b]}T^i. \tag{2.34}$$

We have

$$\begin{aligned} {}^3\,D_a{}^3D_b\,T^i &= {}^3D_a(q^\beta_b\,q^i_I D_\beta T^I)\\ &= q^\alpha_a q^\delta_b q^i_J D_\alpha(q^\beta_\delta q^J_K D_\beta T^K)\\ &= q^\alpha_a q^\beta_b q^i_K D_\alpha D_\beta T^K-\sigma n^\beta \bar K_{ab} q^i_K D_\beta T^K-\sigma n_K K^i_a q^\beta_b D_\beta T^K, \end{aligned} \tag{2.35}$$

where $\bar{K}_{ab} = q_a^\alpha q_b^\beta \nabla_\alpha n_\beta$ is the extrinsic curvature. In the third step we have used the fact that

$$
\begin{aligned}
q_a^\alpha q_b^\delta D_\alpha(q_\delta^\beta) &= q_a^\alpha q_b^\delta \nabla_\alpha(g_\delta^\beta - \sigma n^\beta n_\delta) \\
&= -\sigma q_a^\alpha q_b^\delta n^\beta \nabla_\alpha n_\delta \\
&= -\sigma n^\beta \bar{K}_{ab}
\end{aligned}
\tag{2.36}
$$

and

$$
\begin{aligned}
q_a^\alpha q_J^i D_\alpha q_K^J &= q_a^\alpha q_J^i D_\alpha(\eta_K^J - \sigma n^J n_K) \\
&= -\sigma q_a^\alpha q_J^i n_K D_\alpha n^J \\
&= -\sigma q_a^\alpha q_J^i n_K(\partial_\alpha n^J + \omega_\alpha{}^{JM} n_M) \\
&= -\sigma n_K K_a^i,
\end{aligned}
\tag{2.37}
$$

where $K_a^i := q_a^\alpha q_I^i \omega_\alpha{}^{IJ} n_J$. If the equation of motion (2.14) holds, K_a^i can be identified with the extrinsic curvature, $K_a^i = \bar{K}_{ab} e^{bi}$.

Now, going back to the curvature equation, we see that the term including the extrinsic curvature vanishes when antisymmetrized over a and b. Furthermore, we have

$$q_b^\beta n_K D_\beta T^K = q_b^\beta D_\beta(n_K T^K) - q_b^\beta T^K D_\beta n_K = -q_b^\beta \omega_{\beta K}{}^J n_J T^K. \tag{2.38}$$

Putting all these results together, we have

$${}^3\Omega_{ab}{}^i{}_k T^k = q_a^\alpha q_b^\beta q_M^i \Omega_{\alpha\beta}{}^M{}_K T^K + \sigma K_a^i q_b^\beta \omega_{\beta K}{}^J n_J T^K - \sigma K_b^i q_a^\beta \omega_{\beta K}{}^J n_J T^K. \tag{2.39}$$

We note that this result should hold for all vectors of the form $T^k = q_K^k T^K$. We obtain

$${}^3\Omega_{ab}{}^{ij} = q_a^\alpha q_b^\beta q_I^i q_J^j \Omega_{\alpha\beta}{}^{IJ} + 2\sigma K_{[a}^i K_{b]}^j. \tag{2.40}$$

Similarly we have

$$
\begin{aligned}
q_a^\alpha q_b^\beta q_I^i \, \Omega_{\alpha\beta}{}^{IJ} \, n_J &= 2q_a^\alpha q_b^\beta q_I^i D_{[\alpha} D_{\beta]} n^I \\
&= 2q_{[a}^\alpha q_{b]}^\beta q_I^i D_\alpha (\partial_\beta n^I + \omega_\beta{}^{IJ} n_J) \\
&= 2q_{[a}^\alpha q_{b]}^\beta q_I^i D_\alpha (\omega_\beta{}^{IJ} n_J) \\
&= 2\,{}^3D_{[a} (q_{b]}^\beta q_I^i \omega_\beta{}^{IJ} n_J) \\
&= 2\,{}^3D_{[a} \, K_{b]}^i,
\end{aligned}
\tag{2.41}
$$

where n^I is annihilated by the partial derivative since we fixed $n^I = (1,0,0,0)$ by choosing the time gauge, and we used the definition of ${}^3D_a$ in the fourth line and the definition of K_a^i in the last line.

Now we are ready to go back to our Holst action, equation (2.29). First we deal with the first term and the third one together,

$$
\begin{aligned}
S_{H1} + S_{H3} &= \frac{1}{2k} \int_{\mathcal{M}} \mathrm{d}^4 x \, \sigma \, N \sqrt{q} \, e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} q_\lambda^\alpha \, q_\delta^\beta \, \eta_M^K \, \eta_N^L \\
&\quad - \frac{1}{4k\gamma} \int_{\mathcal{M}} \mathrm{d}^4 x \, N \sqrt{q} \, \epsilon_{IJ}{}^{KL} \, e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} \, q_\lambda^\alpha \, q_\delta^\beta \, \eta_M^I \, \eta_N^J \\
&= \frac{1}{2k} \int_{\mathcal{M}} \mathrm{d}^4 x \, \sigma \, N \sqrt{q} \, e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} q_\lambda^\alpha \, q_\delta^\beta \, (q_M^K + \sigma \, n^K \, n_M)(q_N^L + \sigma \, n^L \, n_N) \\
&\quad - \frac{1}{4k\gamma} \int_{\mathcal{M}} \mathrm{d}^4 x \, N \sqrt{q} \, \epsilon_{IJ}{}^{KL} \, e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} \, q_\lambda^\alpha \, q_\delta^\beta \, (q_M^I + \sigma \, n^I \, n_M)(q_N^J + \sigma \, n^J \, n_N) \\
&= \frac{1}{2k} \int_{\mathcal{M}} \mathrm{d}^4 x \, \sigma \, N \sqrt{q} \, e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} q_\lambda^\alpha \, q_\delta^\beta \, q_M^K \, q_N^L \\
&\quad - \frac{2}{4k\gamma} \int_{\mathcal{M}} \mathrm{d}^4 x \, N \sqrt{q} \, \epsilon_{IJ}{}^{KL} \, e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} \, q_\lambda^\alpha \, q_\delta^\beta \, q_M^I \, \sigma \, n^J \, n_N.
\end{aligned}
\tag{2.42}
$$

From the 1st line to the 2nd, we used the definition of the induced metric. In the last line, since $e^\delta{}_L \, n^L q_\delta^\beta = n^\delta q_\delta^\beta = 0$, we have $e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} q_\lambda^\alpha \, q_\delta^\beta \, q_M^K \, \sigma \, n^L \, n_N = 0$. Two terms vanish due to the antisymmetry of the curvature tensor as $e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} q_\lambda^\alpha \, q_\delta^\beta \, n^K \, n_M \, n^L \, n_N = 0$, $\epsilon_{IJ}{}^{KL} \, e^\lambda{}_K \, e^\delta{}_L \, \Omega_{\alpha\beta}{}^{MN} \, q_\lambda^\alpha \, q_\delta^\beta \, n^I \, n_M \, n^J \, n_N = 0$; Another term vanishes because we can't project all four indices of ϵ^{IJKL} on the orthogonal 3 dimenstional space $V_{\perp}$,

$$\epsilon_{IJ}{}^{KL}\, e^{\lambda}{}_{K}\, e^{\delta}{}_{L}\, \Omega_{\alpha\beta}{}^{MN}\, q^{\alpha}_{\lambda}\, q^{\beta}_{\delta}\, q^{I}_{M}\, q^{J}_{N} = 0. \tag{2.43}$$

Now applying the two formulas (2.40 and 2.41) we derived earlier, we obtain

$$\begin{aligned} S_{H1} + S_{H3} \;=\; & \int \mathrm{d}t \int_M \mathrm{d}^3x \frac{\sigma N \sqrt{q}}{2k}\, e^a_k\, e^b_l ({}^3\Omega_{ab}{}^{kl} - 2\sigma K^k_{[a} K^l_{b]}) \\ & - \int \mathrm{d}t \int_M \mathrm{d}^3x \frac{\sigma N \sqrt{q}}{2k\gamma}\, e^a_k\, e^b_l \epsilon^{kl}{}_{iJ} n^J\, 2^3 D_{[a} K^i_{b]}. \end{aligned} \tag{2.44}$$

We define a new connection $A^i_a := \Gamma^i_a - \sigma\, \gamma\, K^i_a$, with the corresponding derivative operator $\mathcal{D}_a$. We have

$${}^3D_a\, P^i_b = \mathcal{D}_a\, P^i_b + \sigma\gamma\epsilon^i{}_{mn} K^m_a\, P^n_b, \tag{2.45}$$

and there will be two curvatures associated to${}^3D_a$ and $\mathcal{D}_a$ separately on the hypersurface,

$${}^3\Omega_{ab}{}^k = 2\, \mathrm{d}_{[a}\, \Gamma_{b]}{}^k + \epsilon^k{}_{mn}\, \Gamma^m_a \Gamma^n_b \tag{2.46}$$

$${}^3F_{ab}{}^k = 2\mathrm{d}_{[a} A^k_{b]} + \epsilon^k{}_{mn} A^m_a A^n_b. \tag{2.47}$$

They are related as

$${}^3\Omega_{ab}{}^k = F_{ab}{}^k + 2\sigma\gamma^3 D_{[a} K^k_{b]} - \gamma^2 \epsilon^k{}_{mn} K^m_a\, K^n_b \tag{2.48}$$

$${}^3\Omega_{ab}{}^{ij} = -\epsilon^{ij}{}_k\, {}^3R_{ab}{}^k = -\epsilon^{ij}{}_k\, {}^3\Omega_{ab}{}^k - 2\sigma\gamma\epsilon^{ij}{}_k {}^3D_{[a} K^k_{b]} + 2\gamma^2 K^i_{[a}\, K^j_{b]}. \tag{2.49}$$

We define a new parameter $P^d{}_l := 2k\gamma^{-1}\, e^j{}_a\, e^k{}_b\, \eta^{dab}\, \epsilon_{ljk}$, where η^{dab} is the metric-independent Levi-Civita density on M. Notice that $e^d{}_l = \frac{1}{2\sqrt{q}}\, e^j{}_a\, e^k{}_b\, \eta^{dab}\, \epsilon_{ljk}$. Then we can replace

$\frac{1}{k\gamma}\sqrt{q}\,e^d{}_l$ with $P^d{}_l$. With this new element we have

$$\begin{aligned}
S_{H1}+S_{H3} &= \int \mathrm{dt} \int_M \mathrm{d}^3x \frac{\sigma N k\gamma^2}{2\sqrt{q}}\, P^a_k\, P^b_l(-\epsilon^{kl}{}_i\, F^i{}_{ab} - 2\sigma\gamma\epsilon^{kl}{}_i\, {}^3D_{[a}K^i_{b]} + 2\sigma^2\gamma^2 K^k_{[a}K^l_{b]} \\
&\quad - 2\sigma\, K^k_{[a}K^l_{b]}) - \int \mathrm{dt} \int_M \mathrm{d}^3x \frac{\sigma N k\gamma}{2\sqrt{q}}\, P^a_k\, P^b_l \epsilon^{kl}{}_{iJ} n^J\, 2\, {}^3D_{[\lambda}K^i_{b]} \\
&= -\int \mathrm{dt} \int_M \mathrm{d}^3x\, \frac{\sigma N k\gamma^2}{2\sqrt{q}}\, P^a_k\, P^b_l(\epsilon^{kl}{}_i\, F^i_{ab} + (\sigma-\gamma^2)2K^k_{[a}\, K^l_{b]}) \\
&\quad - \int \mathrm{dt} \int_M \mathrm{d}^3x\, \sigma^2 N \frac{k\gamma}{\sqrt{q}}\, P^a_k\, P^b_l(\gamma^2-\sigma)\epsilon^{kl}{}_i\, {}^3D_{[a}K^i_{b]}.
\end{aligned} \tag{2.50}$$

Instead of using K^i_a here, we can rewrite equation (2.46) above as

$${}^3\Omega_{ab}{}^k - F_{ab}{}^k = 2\sigma\gamma\, {}^3D_{[a}K^k_{b]} - \gamma^2\epsilon^k{}_{mn}K^m_a\, K^n_b, \tag{2.51}$$

and plug it in (2.48), and we obtain now

$$S_{H1}+S_{H3} = \int \mathrm{dt} \int_M \mathrm{d}^3x \sigma\, N\, \frac{k}{2\sqrt{q}} P^a_k\, P^b_l\, \epsilon^{kl}{}_i[(\sigma-\gamma^2)\, {}^3\Omega_{ab}{}^i - \gamma\, F_{ab}{}^i], \tag{2.52}$$

which is of the form in formula (13) in [10, 15].

Next, we consider the other two terms in the action by using $n^\alpha = N^{-1}(t^\alpha - N^\alpha)$,

$$\begin{aligned}
S_{H2}+S_{H4} &= \frac{1}{k}\int_{\mathcal{M}} \mathrm{d}^4x\, N\sqrt{q}\, e^\delta{}_L\, n^\alpha\, n_K\, \Omega_{\alpha\beta}{}^{KL} q^\beta{}_\delta \\
&\quad - \frac{\sigma}{2k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, N\sqrt{q}\, e^\delta{}_L\, n^\alpha\, \epsilon_{IJ}{}^{KL}\, n_K\, \Omega_{\alpha\beta}{}^{IJ} q^\beta{}_\lambda \\
&= \frac{1}{k}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{q}\, e^\delta{}_L\, n_K\, t^\alpha \Omega_{\alpha\beta}{}^{KR} q^\beta{}_\delta\, q^L_R && A \\
&\quad - \frac{1}{k}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{q}\, N^a\, e^\delta_L\, q^\beta_\delta\, q^\alpha_a\, \Omega_{\alpha\beta}{}^{KR}\, n_K\, q^L_R && B \\
&\quad - \frac{\sigma}{2k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{q}\, e^\delta_L\, \epsilon_{IJ}{}^{KL} n_K\, q^\beta_\delta\, t^\alpha\, \Omega_{\alpha\beta}{}^{IJ} && C \\
&\quad + \frac{\sigma}{2k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\, \sqrt{q} N^a\, e^\delta_L \epsilon_{IJ}{}^{KL} n_K\, q^\beta_\delta\, q^\alpha_a\, \Omega_{\alpha\beta}{}^{IJ} q^L_R. && D
\end{aligned} \tag{2.53}$$

We will consider $B+D$ and $A+C$ in the following.

$$
\begin{aligned}
B+D &= -\frac{1}{k}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{q}\,N^a\,e^\delta_L\,q^\beta_\delta\,q^\alpha_a\,\Omega_{\alpha\beta}{}^{KR}\,n_K\,q^L_R \\
&\quad+\frac{\sigma}{2k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{q}N^a\,e^\delta_L\epsilon_{IJ}{}^{KR}n_K\,q^\beta_\delta\,q^\alpha_a\,\Omega_{\alpha\beta}{}^{MN}q^L_R\,q^I_M\,q^J_N \\
&= \frac{1}{k}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{q}\,N^a\,e^b_l\,2\,{}^3D_{[a}K^l_{b]} \\
&\quad+\frac{\sigma}{2k\gamma}\int_{\mathcal{M}} \mathrm{d}^4x\,\sqrt{q}N^a\,e^\delta_L\epsilon_{ij}{}^{KR}n_K\,q^l_R\,({}^3R_{a\delta}{}^{ij}-2\sigma\,K^i_{[a}K^j_{\delta]}) \\
&= \int_{\mathcal{M}} \mathrm{d}^4x\,N^a\,\gamma\,P^b_l\,2\,{}^3D_{[a}K^l_{b]} \\
&\quad+\int_{\mathcal{M}} \mathrm{d}^4x\,N^a\,\sigma\,P^b_l\,\tfrac{1}{2}\,\epsilon_{ij}{}^{KR}n_K\,q^L_R\,(-\epsilon^{ij}{}_m\,F_{ab}{}^M-2\sigma\gamma\epsilon^{ij}{}_m\,{}^3D_{[a}K^m_{b]} \\
&\quad+2\sigma^2\gamma^2K^i_{[a}K^j_{b]}-2\sigma\,K^i_{[a}K^j_{b]}) \\
&= -\int \mathrm{d}^4x\,N^a\sigma\,(P^b_l\,F_{ab}{}^l+(\sigma\,-\gamma^2)\epsilon^l{}_{ij}\,K^i_a\,K^j_b\,P^b_l),
\end{aligned}
\tag{2.54}
$$

where the equation (2.41) has been used to go from the 1st line to the 2nd line and (2.48) from the second to the third.

Before we start to deal with $A+C$, we want to check the following formula first,

$$
\begin{aligned}
&t^\alpha\Omega_{\alpha\beta}{}^{IJ} \\
=\;& t^\alpha(\nabla_\alpha\,\omega_\beta{}^{IJ}-\nabla_\beta\,\omega_\alpha{}^{IJ}+\omega_\alpha{}^{IP}\omega_{\beta P}{}^J-\omega_\beta{}^{IP}\omega_{\alpha P}{}^J) \\
=\;& t^\alpha\nabla_\alpha\omega_\beta{}^{IJ}+\omega_\alpha{}^{IJ}\nabla_\beta\,t^\alpha-\omega_\alpha{}^{IJ}\nabla_\beta\,t^\alpha-t^\alpha\nabla_\beta\,\omega_\alpha{}^{IJ}+t^\alpha\,\omega_\alpha{}^{IP}\omega_{\beta P}{}^J-\omega_\beta{}^{IP}\,t^\alpha\,\omega_{\alpha P}{}^J \\
=\;& \mathcal{L}_t\,\omega_\beta{}^{IJ}-\nabla_\beta(t\cdot\omega^{IJ})-\omega_\beta{}^{IP}(t\cdot\omega)_P{}^J-\omega_\beta{}^{JP}(t\cdot\omega)^I{}_P.
\end{aligned}
\tag{2.55}
$$

With this formula, we plug it in A and C,

$$\begin{aligned}
& A+C \\
= \;& \int_{\mathcal{M}} \mathrm{d}^4x \frac{\sqrt{q}}{k} e^{\delta}_{L}\, q^{\beta}_{\delta}\, q^{L}_{R}\, n_K\, [\mathcal{L}_t\, \omega_{\beta}{}^{KR} - \nabla_{\beta}(t\cdot\omega^{KR}) - \omega_{\beta}{}^{KP}(t\cdot\omega)_P{}^R - \omega_{\beta}{}^{RP}(t\cdot\omega)^K{}_P] \\
& - \int_{\mathcal{M}} \mathrm{d}^4x \frac{\sigma\sqrt{q}}{2k\gamma} e^{\delta}_{L}\, q^{\beta}_{\delta}\, \epsilon_{IJ}{}^{KL}\, n_K\, [\mathcal{L}_t\, \omega_{\beta}{}^{IJ} - \nabla_{\beta}(t\cdot\omega^{IJ}) - \omega_{\beta}{}^{IP}(t\cdot\omega)_P{}^J - \omega_{\beta}{}^{JP}(t\cdot\omega)^I{}_P] \\
= \;& \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, \mathcal{L}_t\, A^l_b \\
& + \int_{\mathcal{M}} \mathrm{d}^4x\, \gamma\, P^b_l\, q^{\beta}_b\, \nabla_{\beta}(q^l_R\, n_K\, t\cdot\omega^{RK}) \qquad (1) \\
& + \int_{\mathcal{M}} \mathrm{d}^4x\, \gamma\, P^b_l\, q^{\beta}_b\, \omega_{\beta}{}^{pK}\, n_K(t\cdot\omega_p{}^R\, q^l_R) \qquad (2) \\
& + \int_{\mathcal{M}} \mathrm{d}^4x\, \gamma\, P^b_l\, q^{\beta}_b\, q^l_R \omega_{\beta}{}^{Rp}(t\cdot\omega_p{}^K\, n_K) \qquad (3) \\
& + \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, [-q^{\beta}_b \nabla_{\beta}(-\tfrac{1}{2}\epsilon^l{}_{ij}\omega^{ij}\cdot t)] \qquad (4) \\
& + \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, \epsilon^l{}_{ij}\, q^{\beta}_b \omega_{\beta}{}^{ip}\, (t\cdot\omega_p{}^j) \qquad (5) \\
& - \int_{\mathcal{M}} \mathrm{d}^4x\, P^b_l\, \epsilon^l{}_{ij}\, q^{\beta}_b \omega_{\beta}{}^{iP}\, n_P\, (t\cdot\omega^{jQ} n_Q). \qquad (6)
\end{aligned} \tag{2.56}$$

Now we will manipulate the last 6 terms in the above formula. For each internal spatial index, since it takes values in so(3), we can use the adjoint representation of the Lie algebra,

$$\begin{aligned}
(A+C)_2 &= \int \mathrm{d}t \int_M \mathrm{d}^3x\, \gamma\, P^b_l\, q^{\beta}_b\, \omega_{\beta}{}^{pK}\, n_K\, (-\epsilon_p{}^l{}_q)(t\cdot\omega^q) \\
&= \int \mathrm{d}t \int_M \mathrm{d}^3x\, \gamma\, P^b_l\, [\epsilon^l{}_{pq}\, K^p_b (t\cdot\omega^q)],
\end{aligned} \tag{2.57}$$

where we have replaced $q^\beta_b \omega_\beta{}^{pK}\, n_K$ with K^p_b. Similarly,

$$\begin{aligned}
(A+C)_5 &= \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, \epsilon^l{}_{ij}\, q^\beta_b \omega_\beta{}^{ip}\, (t\cdot \omega_p{}^j) \\
&= \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, \epsilon^l{}_{ij}\, (-\epsilon^{ip}{}_q)(q^\beta_b \omega_\beta{}^q)(-\epsilon_p{}^j{}_m)(t\cdot \omega^m) \\
&= \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, \epsilon^l{}_{ij}(\epsilon^{ip}{}_q\, \epsilon_{mp}{}^j)\, \Gamma^q_\delta (t\cdot \omega^m) \\
&= \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, \epsilon^l{}_i{}^j\, 2\delta^{[i}_m \delta^{q]}_j\, \Gamma^q_\delta (t\cdot \omega^m) \\
&= \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, [-\epsilon^l{}_{ij}\, \Gamma^i_b (t\cdot \omega^j)].
\end{aligned} \tag{2.58}$$

Combining three of the terms above,

$$\begin{aligned}
(A+C)_{4+5+2} &= \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, [-q^\beta_b \nabla_\beta (-\tfrac{1}{2}\epsilon^l{}_{ij}\omega^{ij}\cdot t)] \\
&\quad + \int_{\mathcal{M}} \mathrm{d}^4x\, \sigma\, P^b_l\, [-\epsilon^l{}_{ij}\, \Gamma^i_\delta (t\cdot \omega^j)] \\
&\quad + \int \mathrm{d}t \int_M \mathrm{d}^3x\, \gamma\, P^b_l\, [\epsilon^l{}_{pq}\, K^p_\delta (t\cdot \omega^q)] \\
&= -\int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma\, P^b_l\, \mathcal{D}_b (t\cdot \omega^l) \\
&= \int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma (t\cdot \omega^l)\, \mathcal{D}_b\, P^b_l,
\end{aligned} \tag{2.59}$$

and also have

$$\begin{aligned}
(A+C)_3 &= \int_{\mathcal{M}} \mathrm{d}^4x\, \gamma\, P^b_l\, q^\beta_b\, q^l_R \omega_\beta{}^{Rp} (t\cdot \omega_p{}^K\, n_K) \\
&= \int_{\mathcal{M}} \mathrm{d}^4x\, \gamma\, P^b_l\, (-\epsilon^{lp}{}_q\, q^\beta_b \omega^q_\beta)(t\cdot \omega_p{}^K\, n_K) \\
&= \int_{\mathcal{M}} \mathrm{d}^4x\, \gamma\, P^b_l\, [\epsilon^l{}_{pq} \Gamma^p_\delta (t\cdot \omega^{qK} n_K)].
\end{aligned} \tag{2.60}$$

The other 3 terms left are

$$
\begin{aligned}
(A+C)_{136} &= \int_{\mathcal{M}} \mathrm{d}^4x\, \gamma\, P^b_l\, q^\beta_b\, \nabla_\beta (q^l_R\, n_K\, t\cdot\omega^{RK}) \\
&\quad + \int_{\mathcal{M}} \mathrm{d}^4x\, \gamma\, P^b_l\, [\epsilon^l{}_{pq} \Gamma^p_b (t\cdot\omega^{qK} n_K)] \\
&\quad - \int_{\mathcal{M}} \mathrm{d}^4x\, P^b_l\, \epsilon^l{}_{ij}\, q^\beta_b \omega_\beta{}^{iP}\, n_P\, (t\cdot\omega^{jQ} n_Q) \\
&= -\int \mathrm{d}t \int_M \mathrm{d}^3x\, (t\cdot\omega^{qK} n_K q^l_Q)(\gamma\, {}^3D_b P^b_l + \gamma \epsilon_{lp}{}^m \Gamma^p_b P^b_m - \epsilon_{lp}{}^m\, q^\beta_b\, \omega_\beta{}^{pI} n_I\, P^b_m),
\end{aligned}
$$

where we have carried out the integration by parts on the first term and relabelled the dummy indices on the other two terms.

Putting all the terms together we have now

$$
\begin{aligned}
S_H(e,\omega) &= \int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma\, P^b_l\, \mathcal{L}_t\, A^l_b + \int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma (t\cdot\omega^l)\, \mathcal{D}_b P^b_l \\
&\quad - \int \mathrm{d}t \int \mathrm{d}^3x (t\cdot\omega^{QK} n_K q^l_Q)(\gamma\, {}^3D_b P^b_l + \gamma \epsilon_{lp}{}^m \Gamma^p_\delta P^b_m - \epsilon_{lp}{}^m\, q^\beta_b\, \omega_\beta{}^{pI} n_I\, P^b_m) \\
&\quad - \int \mathrm{d}^4x\, N^a \sigma\, (P^b_l\, F_{ab}{}^l + (\sigma - \gamma^2) \epsilon^l{}_{ij}\, K^i_a\, K^j_b\, P^b_l) \\
&\quad - \int \mathrm{d}t \int_M \mathrm{d}^3x\, \frac{\sigma N k \gamma^2}{2\sqrt{q}}\, P^a_k\, P^b_l (\epsilon^{kl}{}_i\, F^i + (\sigma-\gamma^2) 2 K^k_{[a}\, K^l_{b]}) \\
&\quad - \int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma^2 N \frac{k\gamma}{\sqrt{q}}\, P^a_k\, P^b_l (\gamma^2 - \sigma) \epsilon^{kl}{}_i\, {}^3D_{[a} K^i_{b]}.
\end{aligned}
\tag{2.61}
$$

Since there are no time derivatives of $t\cdot\omega^l$, $t\cdot\omega^{LQ} n_Q\, q^l_L$, N^a, N, they will serve as Lagrange multipliers.

By carrying out the Poisson brackets between these constraints, it shows that there exists a second-class constraint [10, 15, 6]

$$
\Gamma^i_a - \bar{\Gamma}^i_a = 0, \tag{2.62}
$$

where ${}^3D_a\, e^i_b = {}^3\nabla_a\, e^i_b + \epsilon^i{}_{jk} \bar{\Gamma}^j_a\, e^k_b = 0$. This identifies Γ^i_a as the triad-compatible connection one-form $\bar{\Gamma}^i_a$. We should insert this solution into the action, and it then turns out that

$$G_l = \partial_b P^b_l + \epsilon_{lj}{}^k\, A^j_b\, P^b_K = {}^3D_b P^b_l - \sigma\gamma\epsilon_{lj}{}^k\, K^k_b\, P^b_k = -\sigma\gamma\epsilon_{lj}{}^k\, K^j_b\, P^b_k. \tag{2.63}$$

Consequently, we can rewrite the vector constraint,

$$\begin{aligned} C_a &= P^b_l\, F_{ab}{}^L + (\sigma - \gamma^2)\epsilon^l{}_{ij}\, K^i_a\, K^j_b\, P^b_l \\ &= P^b_l\, F_{ab}{}^l - \frac{(\sigma - \gamma^2)}{\sigma\gamma}\, K^i_a\, G_i \end{aligned} \tag{2.64}$$

and the last term in the scalar constraint

$$\begin{aligned} & \int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma^2 N \frac{k\gamma}{\sqrt{q}}\, P^a_k\, P^b_l (\gamma^2 - \sigma)\epsilon^{kl}{}_i\, {}^3D_{[a} K^i_{\delta]} \\ = & -\int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma^2 N\, k\, \gamma\, (\gamma^2 - \sigma)\, K^i_\delta\, \epsilon^{kl}{}_i\, {}^3D_a (P^a_k\, P^b_l / \sqrt{q}) \\ = & -\int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma^2 N\, k\, \gamma\, (\gamma^2 - \sigma)\, K^i_b\, \epsilon^{kl}{}_i\, P^b_l\, {}^3D_a (P^a_k / \sqrt{q}) \\ = & -\int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma\, k\, (\gamma^2 - \sigma) G^k\, {}^3D_a (P^a_k / \sqrt{q}), \end{aligned} \tag{2.65}$$

where the integration has been performed in the first line and we neglect the total derivative, and in the second step we used compatibility of the triad.

Our final form of the action is

$$S_H(e,\omega) = \int \mathrm{d}t \int_M \mathrm{d}^3x\, \sigma\, (P^a_l\, \mathcal{L}_t\, A^l_a - h(A^i_a, P^a_i, N, N^a, t\cdot\omega)), \tag{2.66}$$

where h is given by

$$h = -(t\cdot\omega) G_l + (t\cdot\omega^{lI} n_I) G_l + N^a\, C_a + N\, C \tag{2.67}$$

with

$$G_l = \mathcal{D}_b P^b_l \tag{2.68}$$

$$C_a = P^b_l \, F_{ab}{}^l - \frac{\sigma - \gamma^2}{\sigma\gamma} \, K^i_a \, G_i \tag{2.69}$$

$$C = \frac{k\gamma^2}{2\sqrt{q}} \, P^a_k \, P^b_l \, [\, \epsilon^{kl}{}_i \, F^i_{ab} + (\sigma - \gamma^2) 2 K^k_{[a} \, K^l_{b]} \,] \; + \; k\,(\gamma^2 - \sigma) G^{k\,3} D_a (P^a_k \,/\, \sqrt{q}). \tag{2.70}$$

The Hamiltonian is given by

$$H = \int \mathrm{d}t \int_M \mathrm{d}^3x \, h(N, N^a, A^i_a, P^a_i, t \cdot \omega^{ij}, t \cdot \omega^{0i}). \tag{2.71}$$

The canonical pair of variables is (A^i_a,P^a_i), where P^a_i is the densitized triad rescaled by γ. The only non-zero Poisson bracket between them is

$$[A^i_a(x), P^b_j(y)] := \delta^i_j \, \delta^b_a \, \delta(x, y). \tag{2.72}$$

Compare to the configuration variable $\omega_\mu{}^{IJ}$ in the Palatini action, the rotation part, $\Gamma^i_a = \frac{1}{2} \, q^\alpha_a \, q^i_I \, \epsilon^{IJ}{}_{KL} \, n_J \, \omega_\alpha{}^{KL}$, is now compatible with the triad, while the boost part, $K^i_a = q^i_I \, q^\alpha_a \, \omega_\alpha{}^{IJ}$, is rescaled by the Barbero-Immirzi parameter. The Lagrangian multipliers are N, N^a, $t \cdot \omega^{ij}$ and $t \cdot \omega^{0i}$.

Similar to the Palatini action, the Hamiltonian is still the sum of 3 constraints, but they are γ-dependent now. The explicit forms of the constraints seem more complicated. The Gauss constraint does not change, while the other two are γ related. Since the extra terms are all proportional to $\sigma - \gamma^2$, they will vanish if $\gamma = \pm 1$ with Riemannian signature and $\gamma = \pm i$ with the Lorentz signature.

Due to the occurrence of Γ^i_a, which is included in $K^i_a = (\sigma\gamma)^{-1}(\Gamma^i_a + A^i_a)$, the constraints seem non-polynomial again. However, it is shown by Thiemann that the Hamiltonian constraint can be made polynomial by multiplying it by a sufficiently high powers of $\det(q)$ [19]. Consequently, we arrive at a theory with real variables and the constraints are all polynomial functions of the fundamental variables.

Let's count the number of degrees of freedom now. The configuration variable A^i_a, which is a so(3) Lie algebra valued connection one-form, has $3 \times 3 = 9$ degrees of freedom. Similarly the conjugate momentum P^a_i also has 9 variables, as a result it is 18 variables in phase space. Since the second-class constraint is already solved and plugged back in the action, we can neglect them now. Each first-class constraint reduces the phase-space variables by 1 and we have 7 of them, so we are left with 11 degrees of freedom. Because of gauge transformations between the remaining variables, each first-class constraint generates one gauge transformation, then the number of gauge-invariant phase-space variables is $11-7=4$. This means that there are $4/2 = 2$ physical, gauge-invariant degrees of freedom, which is correct for General Relativity.

So far, we have done the Legendre transform of the Holst action by carrying out a 3+1 decomposition. We derived the results directly from the Holst action, rather than by a canonical transformation in the usual phase space [19]. Compare to the approach Holst originally followed with [10], both the canonical variables and constraints are more obvious to see now.

Chapter 3
DISCUSSION

After the Legendre transform of the action, the dynamical variables are obvious to see. It is straightforward to get the Hamiltonian of the theory. By deriving the Poisson brackets between constraints and canonical variables, we can see clearly the physical meaning of these constraints. In this part, I will provide a short summary about results from previous chapter.

3.1 Constraints

As shown in the previous chapter, constraints exist in all three actions. To analyze these constraints, we need to work out their Poisson brackets with the canonical variables. I will cite results from [4].

The Gauss constraint is not included in the standard Einstein-Hilbert action and it comes up once we introduce the tetrad formalism such as the Palatini action and the Holst action. It is a natural result of introducing an internal spacetime, which is Lorentz invariant. Although the original symmetry is SO(3+1)(SO(4) for Riemannian signature), once we introduce the tetrad as a field variable, which has 6 more degrees of freedom than the original metric $g_{\mu\nu}$, we are free to choose a gauge here. By choosing the time gauge, we reduce the Lorentz symmetry to an SO(3) symmetry. The Gauss constraint shows that the theory is invariant under internal SO(3) rotations.

General relativity is written in terms of tensors, hence actions used to describe it should be diffeomorphism invariant. All three actions in this thesis are of this kind. Consequently, all of them lead to vector constraints and scalar constraints. As shown by the Poisson brackets between these constraints and the canonical variables, the vector constraint is the infinitesimal generator of diffeomorphisms on the spatial surface. It leads to diffeomorphisms along N^a while the scalar constraint leads to time evolution.

3.2 Hamiltonian

As shown in the previous chapter, the Hamiltonian is always the sum of constraints. Hence, $H = 0$ on the constraints surface. This is true for any theory which is coordinate invariant [7]. Since general relativity is written in terms of tensors, it is of this kind. Coordinates do not have the same meaning as those in special relativity. Then the time parameter that we are choosing to foliate the spacetime is an arbitrary choice. It is not the true physical time, so it does not represent the real time evolution.

3.3 The Barbero-Immirzi parameter

Although the term involving γ in the Holst action vanishes when the equations of motion hold, the constraints are all γ-dependent and hence so is the Hamiltonian. So far, γ can take any real non-zero values. Different choices correspond to canonical transformations between each other [11]. It turns out that this γ dependence will stay all the way up to the quantized geometrical variables, such as area and volume. As a result, the same quantity will have different spectra depending on the choice of the parameter [17, 18]. This leads to a one-parameter ambiguity in quantum gravity [11].

At this point, it is natural to raise another question. Can we find any other similar terms, which vanish on shell, with different parameters from the Holst action? Will they also show up in the quantized spectra [15]? Is there any way to fix them?

3.4 Future work

First, in this paper we considered only the vacuum case. The next step should include both the cosmological constant and matter to see how constraints will change. Secondly, since the time gauge plays an important role to get the final Hamiltonian, we may wonder whether there are any other convenient choices and whether they would be equivalent. Thirdly, for the Barbero-Immirzi parameter, it should be a real non-zero constant to make the quantization applicable. We wonder whether it is possible to fix it or promote it as a new field [20]. We also want to find what limit the possibility of adding more similar terms to the Holst action.

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