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FROM ACCOUNTING TO NEGATIVE NUMBERS: A SIGNAL CONTRIBUTION OF MEDIEVAL INDIA TO MATHEMATICS

Abstract: The major object of this paper is to present evidence for arguing that the highly developed Hindu accounting tradition, beginning with Kautilya's *Arthaśāstra* about 300 B.C., or even earlier, may have had a part in the more receptive attitude of medieval Indian mathematicians, compared to Europeans, in accepting negative numbers. The Hindus justified this attitude by arguing that having a debt is the inverse of possessing an asset; thus, attributing a negative number to a debt but a positive one to an asset. To advance the argument, the paper shows that the accounting aspect of debt is at least as basic as its legalistic one. Indeed, the former can be traced to the 4th millennium B.C. or earlier, while the first known legal codes go back only to the 3rd millennium B.C. However, there are other angles from which to examine the relation between accounting and negative numbers. Some accountants [e.g., Peters and Emery, 1978] believe that the long-standing hesitation of European mathematicians to accept negative numbers contributed to the accountants' debit/credit scheme, while others [e.g, Scorgie, 1989] deny this view. But this controversy concerns rather the influence of negative numbers upon accounting. It neglects to investigate the reverse possibility; namely, the influence of accounting upon the Indian mathematicians' early acceptance of negative numbers. Thus, this paper first reviews concisely, for the sake of contrast, the arguments between Peters and Emery [1978] and Scorgie [1989]; then it elaborates on the long-standing resistance of Western mathematicians to legitimizing negative numbers (which, in its entirety, did not happen before the 19th century); and, finally, it discusses the very different attitude of medieval Indian mathematicians, who were the first to accept negative magnitudes as numbers (e.g., Brahmagupta, 7th century A.D., Bhāskara, 12th century A.D.). Their interpretation of a negative number as representing "debt" as a basic accounting and legal notion may have been conditioned by the long-standing accounting tradition of India since the 3rd century B.C. or before.

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Probing more deeply into mathematical history shows that accounting aspects may have played an important role in medieval India through the earliest acceptance of negative numbers. This deserves at least as much attention as did the controversy between Peters and Emery [1978] and Scorgie [1989] as to whether or not the avoidance of negative numbers by Western mathematicians influenced the development of double-entry bookkeeping in Renaissance Europe. Peters and Emery [1978] tried to show that due to the rejection of negative numbers by Renaissance mathematicians, account balances had to be kept positive; e.g., relying on the "basic balance sheet equation" A = L + OE, instead of A - L = OE. One might counter this argument by pointing out that the balance sheet equation (A = L + OE) is more likely to have resulted from entering every transaction twice, and on opposite sides, via the trial balance because mathematicians and even accountants of this time were already sophisticated enough to know that the equation A - L = OE is an equivalent transposition of A = L + OE. But neither of these equations, nor a balance sheet, are mentioned in Pacioli's *Summa* [1494]. There one encounters merely the Profit and Loss account and the trial balance as well as the inventory, which also served as a starting basis for opening the accounts, thus approaching the notion of balance sheet. This "need for a bookkeeping system free of negative balances," in turn, was supposed to have led in commerce and in Fra Luca Pacioli's *Summa* [1494] to the notions of debits (Per) and credits (A) instead of regarding the values of assets as positive and those of all equities as negative. Scorgie [1989], quite correctly, refuted such an interpretation by pointing out the following three "critical evidential errors" contained in the argument by Peters and Emery:

(1) Omar Khayyám's (ca. 1048 - ca. 1131) rejection of negative numbers, introduced in India by Brahmagupta, b. 598, was supposed to indicate that the use of negative numbers "died out in India," if it really did at that time. Scorgie [1989, p. 317] claimed this to be invalid because a comment contained in Colebrooke [1973, p. iii], accompanying his translation of Brahmagupta together with that of Bhāskara II (b. 1115, Bhāskara hereafter), demonstrated that the work of the latter "was in the hands of both Mahammedans and Hindus between two and three centuries ago."

(2) Peters and Emery's [1978, p. 425] assertion, claimed to be based on Cajori [1919, p. 107], that "the Arabs also rejected negative numbers, in spite of knowledge of their use in India"

was shown to be invalid by Scorgie [1989, p. 317] because Cajori referred to the mathematician Abu'l-Wafa (b. 940) who authored a text that "termed the result of the subtraction of the number 10 - 5 [which is 5] from 3 a 'debt (dayn) of 2'" as quoted from Youschkevitch [1970, Vol. 1, p. 41]. Scorgie also referenced Vogel [1970, Vol. 4, p. 611], who pointed out that Leonardo Pisano (Leonardo da Pisa, also called Fibonacci, c. 1170-1250) "recognizes negative quantities and even zero as numbers."[1]

(3) Peters and Emery's [1978, p. 426] further assertion, that "there is no question that Pacioli rejected negative numbers" was called "nonsense" by Scorgie [1989, p. 318] because Pacioli [1494, ff. 114 v.-115 r.] stated 12 rules for subtraction with an example of subtracting 16 from 4 which gives a pure negative number called by Pacioli [1494, f. 114 v.] "puro meno."[2]

As the argument between Peters and Emery, on one side, and Scorgie, on the other, related accounting to negative numbers, it creates an inverse parallel to the main objective of this paper, thus offering a contrasting background as well as "counterpoint."[3] This objective lies in the search for evidence supporting the hypothesis that the highly developed Hindu accounting

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[1]But the reader should note: "Rather surprising is the fact that Al-Karkhi's algebra shows no traces whatever of Hindu indeterminate analysis. But most astonishing it is, that an arithmetic by the same author completely excludes the Hindu numerals. It is constructed wholly after Greek pattern. Abu'l-Wefa, also, in the second half of the 10th century, wrote an arithmetic in which Hindu numerals find no place. This practice is the very opposite to that of other Arabian authors" [Cajori, 1919, pp. 106-107]. The last sentence shows that, again, Peters and Emery [1978] seemed to have misread their source.

[2]Apart from my agreement with Scorgie [1989], two aspects may have to be added. First, the essence of double-entry bookkeeping goes beyond the mere interpretation of assets as positive and debts as negative; it assigns a *negative* number also to an output of an asset and, inversely, a *positive* number to a reduction of a debt or ownership claim. Second, and more importantly, a mere debit/credit scheme as, for example, employed in a "charge-and-discharge statement" [see Cooper and Ijiri, 1983, p. 95], still lacks the pivotal feature of a closed double-entry system and can hardly be regarded as such.

[3]Critics may argue that this short discussion of the papers by Peters and Emery [1978] and Scorgie [1989] is not warranted here. But just as some music fans are only interested in rhythm or a single melody, others listen no less to harmony and *counterpoint*. Similarly, I presume the readers of *AHJ* are interested not merely in one aspect but in the entire picture from which this paper evolved. After all, the above-mentioned papers dealt also with the relation between accounting and negative numbers and provided an impetus for writing this article.

tradition, beginning with Kautilya's *Arthaśāstra* about 300 B.C. or even earlier, may have had a part in the earliest acceptance or legitimization of negative numbers by mathematicians. The latter happened in India during medieval times [Brahmagupta, 7th century, Bhāskara, 12th century — see translations by Colebrooke, 1973]. But to understand the long-lasting resistance of Western mathematicians to negative numbers, it is necessary to provide in the next section an overview of this particular development. Only then, in the third section, is it possible to discuss and appreciate the Indian achievement in its relation to accounting.

THE MATHEMATICIANS' CONUNDRUM WITH NEGATIVE NUMBERS

In relating negative numbers to accounting, or vice versa, it must be noted that the status of negative numbers in mathematics from ancient times to the 19th century experienced many twists and turns in the West as well as in the Orient.[4] This development was not as straightforward as one might believe from reading Peters and Emery [1978] or even Scorgie [1989]. Despite my agreement with the latter's objections to Peters and Emery, from a more global-historical point of view, the different attitude of Indians to negative numbers as well as to accounting ought to be considered. Thus, this paper shows, among other things, that in medieval India the important connection between negative numbers in mathematics and the debtor-creditor aspects of bookkeeping point in the direction from the latter to the former rather than vice versa. If historians of mathematics found this worth remarking, then accountants should be even more interested because it confirms the wide cultural impact of accountability notions. To recognize this, two insights, formulated in the third section as auxiliary hypotheses, are necessary — (i) a debt relation is not merely a legalistic but also a basic accounting concept, and (ii) debt relations and many other basic accounting notions were conceived and described, not merely used, in India long before medieval times, thus establishing an early and relatively advanced accounting tradition.

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[4]An example of varying attitudes in Asia toward negative magnitudes is, on one side, the acceptance of negative numbers by such leading mathematicians as Brahmagupta (7th century) and Bhāskara (12th century), while other Oriental scholars (e.g., many Arabs — see footnote 2), possibly even the Persian poet and astronomer Omar Khayyám *may have* rejected negative numbers.

The earliest records of negative numbers, as Peters and Emery [1978, p. 425] mentioned, point to the Chinese, particularly to the mathematician Sun-Tsu [see *Sun-Tsu Suan-ching* or *Arithmetical Classic of Sun-Tsu*, 1st century], who not only presented different mathematical units by different positions and combinations of rods, but also distinguished positive numbers by using red rods and negative numbers by black rods [cf. Cajori, 1919, p. 72].[5] But the statement by Peters and Emery [1978, p. 425] that, "according to Cajori [1919, p. 72], the earliest reference to negative numbers is found not in mathematics, but, surprisingly, in commerce," is a puzzling misinterpretation as *Sun-Tsu Suan-ching* is undoubtedly a mathematical work. Above all, there is no pertinent reference in Cajori [1919, pp. 71-73] to commerce, merely to a possible derivation of this practice from the red and black beads of the abacus, which also is a *mathematical* device. According to Boyer [1989, p. 227]:

> The idea of negative numbers seems not to have occasioned much difficulty for the Chinese since they were accustomed to calculating with two sets of rods — a red set for positive coefficients and a black set for negatives. Nevertheless, they did not accept the notion that a negative number might be a solution of an equation.

Thus, even if the Chinese used negative numbers, the mathematical status of those numbers need not have been much higher than it was in ancient Greece. Even Cajori [1919, p. 93] agreed that the "Indians were the first to recognize the existence of absolutely negative quantities."[6] Thus, it is generally

[5]Of course, negative numbers must not be confused with the *operation sign* for subtraction; indeed, an ideogram for *minus* can already be encountered in ancient Babylonia; i.e., thousands of years before the earliest known use of a negative number as a *magnitude*. Or as Kline [1980, p. 116] pointed out, "Both Girard and Harriot used the minus sign for the operation of subtraction and for negative numbers, though separate symbols should be used because a negative number is an independent concept whereas subtraction is an operation." This reference refers to Albert Girard (1595-1632) [1629] and Thomas Harriot (1560-1621) [1631].

[6]Cajori's [1919, p. 93] expression "absolutely negative quantities" might refer to the recognition and treatment of negative quantities as genuine numbers; i.e., as those "equally important" to any other numbers presently known and in the future to be recognized. He may even have referred to the belief that reality itself possesses negative quantities, representable through negative numbers, etc.

The above qualification, "presently known and in the future to be recognized," may indicate that the legitimization of negative numbers in medieval acknowledged that the first known use and legitimization of negative numbers *in mathematics* is in Brahmagupta's *Brahma-Sphuta-Sidd'hánta* [628, partly translated and commented on, together with some work by Bhāskara, in Colebrooke, 1973].[7]

But why did negative numbers come so late to be *generally* accepted in European mathematics? In a way, our number system goes back to ancient Greece where the *natural numbers* (i.e., the positive integers, such as 1, 2, 3, . . . etc.) formed an almost sacred basis. The Pythagoreans deemed the phenomena of the universe to be reducible to those whole positive numbers or their ratios. In refining their notions, they may have come to regard numbers in a more abstract way, but for them and other ancient Greek mathematicians, a number was always something *positive*. Even when such notions as the square root of 2 or the notion of π (i.e., the non-ratios, or what we today call the *irrational* numbers) were discovered, the Greeks refused to con-

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India did not require knowledge of the entire gamut of our modern number system, from natural numbers to complex numbers or even transfinite ones. For medieval European mathematics, it would have been an immense step forward had its disciples accepted negative and irrational numbers in the same way as they accepted natural numbers and fractions.

For the reader interested in the achievements of eastern vs. western mathematicians in other areas of the number system, I refer to the internationally known text by Aleksandrov et al. [1963] which stated that "the concept of an irrational number simply did not originate among them [i.e., the Greeks]. This step was taken at a later period by the mathematicians of the East" [pp. 26-27]. "The Greeks discovered irrational magnitudes but considered them geometrically, as linear segments. . . . In this way the Greeks were already in possession of much of the material of contemporary elementary algebra but not, however, of the following essential elements: negative numbers and zero, irrational numbers abstracted entirely from geometry, and finally a well-developed system of literal symbols. It is true that Diophantus made use of literal symbols for the unknown quantity and its powers....but his algebraic equations were still written with concrete numbers" [p. 37]. Furthermore: "Omar Khayyam (about 1048-1122), and also the Azerbaijanian, Nasireddin Tsui (1201-1274), clearly showed that every ratio of magnitudes, whether commensurable or incommensurable, may be called a number; in their work we find the same general definition of number, both rational and irrational. . . . The magnitude of these achievements becomes particularly clear when we recall that complete recognition of negative and irrational numbers was attained by European mathematicians only very slowly, even after the beginning of the Renaissance of mathematics in Europe" [p. 39]. This last quote might possibly contradict what Peters and Emery [1978] assumed to be Omar Khayyám's attitude toward negative numbers.

[7]In Colebrooke [1973], Brahmagupta is spelled as "Brahmegupta"' and Bhāskara II as "Bhåscara." But here we shall adhere to what seem to be the more common notations.

sider them as numbers. The Greeks "never succeeded in uniting the notions of numbers and magnitudes, e.g., dots on a continuous line. The term 'number' was used by them in a restricted sense. What we call irrational numbers was not included under this notion. Not even rational fractions were called numbers" [Cajori, 1919, p. 22]. Since that time, every step of extending the number system, be it in the direction of the full-fledged integer system, rational numbers, and even real and complex numbers, constituted a very uneven and mixed "progression." Surprisingly enough, one of the last categories to be generally accepted by European mathematicians was that of *negative* numbers, even though from the 13th century until the second half of the 19th century, some aspects of negative magnitudes were at certain times accepted by some eminent European mathematicians.

Negative numbers became known in Europe via the Arabs and Leonardo da Pisa [e.g., his well-known *Liber Abaci,* 1202]. According to Cholerus [1944, p. 143], Leonardo da Pisa is said "to have accepted negative solutions of equations, and remarked that the solution would be meaningless if regarded as an 'asset' (Vermögen) but quite meaningful if regarded as an expression of 'debts'" (translated). Unfortunately, Cholerus did not tell us where Leonardo da Pisa made this remark. But if it was actually Leonardo's, it would confirm Scorgie's second argument against Peters and Emery [1978]. But it hardly meant a definite victory in the recognition of negative numbers in general. Most European mathematicians did not accept them as genuine numbers until the second half of the 19th century.[8] Eminent mathematicians, such as Nicholas Chuquet (1445?-1500?) and Michael Stifel (1486?-1567) called them "absurd;" Jerom Cardan (1501-1576) regarded negative roots (of equations) as mere symbols; François Vieta (also Viète, 1540-1603) abandoned negative numbers altogether; and Gottfried W. Leibniz (1646-1716) recognized them only from a *formal* point of view. On the other hand, Raphael Bombelli [1526-1572 or later] and Albert Girard (1595-1632), particularly in his *Invention novelle en algèbre* [1629], put negative and positive numbers on a par, as did Thomas Harriot (1560-1621). However, Harriot did *not* accept negative roots of equations in his posthumous work *Artis analyticae praxis* [1631]. John Wallis (1616-

[8]For details, see Kline [1980, pp. 114-116, 118-119, 153-155] and Boyer [1989, pp. 227, 245f, 256, 260, 312, 316, 321, 342f, 385, 416, 511].

1703) also accepted negative numbers as equal to positive ones. Yet, Jean d'Alembert (1717-1783) published an article in the famous *Encyclopédie,* edited by Denis Diderot and himself [1751-1759], under the title "Negative," which stated that "a problem leading to a negative solution means that some part of the hypothesis is false but assumed to be true" [quoted in Kline, 1980, p. 118]. Only Leonhard Euler (1707-1783) shared the Indians' position of vindicating negative numbers by reasserting that "we denote what a man really possesses by positive numbers, using, or understanding the sign +; whereas his debts are represented by negative numbers, or by using the sign - " [Euler, 1770, Ch. 2, item 17; p. 4 in the English reprint edition, 1972/1989].

At the end of the 18th century and the beginning of the 19th, mathematicians still continued to object to negative numbers. William Frend (1757-1841) [1796, preface] stated that a number "submits to be taken away from a number greater than itself but to attempt to take it away from a number less than itself is ridiculous;" Lazare Carnot (1753-1823) [1797/1970] affirmed that the idea of something being less than nothing is absurd; August De Morgan (1806-1871) [1831] likewise voiced his objections to negative numbers. William R. Hamilton (1805-1865) was hardly more favorably disposed toward negative numbers. Only toward the end of the 19th century was the mathematicians' conundrum with negative numbers, and rational and complex numbers in general, slowly resolved, as seen from the following quote from Kline [1980, p. 179]:

> The logic of the rational numbers was still missing. Dedekind realized this and, in *The Nature and Meaning of Numbers* [1888], he described the basic properties that one might use for an axiomatic approach to the rationals. Giuseppe Piano (1858-1932), utilizing Dedekind's ideas and some ideas in Hermann Grassmann's *Textbook on Arithmetic* [1861] succeeded in *Principles of Arithmetic* [1889] in producing a development of the rational numbers from axioms about the positive whole numbers. Thus, finally, the logical structure of the real and complex number systems was at hand.

By then, it was high time for mathematics to have caught up with humankind's perception of social and physical reality as, by the end of the 19th century, innumerable empirical applications for negative numbers had already been conceived (in fields from accounting and geography to thermodynamics and electricity).

THE HINDUS' ACCEPTANCE OF NEGATIVE NUMBERS AND THEIR INTERPRETATION AS DEBTS

The conservative European attitude toward negative numbers did not hold sway over Indian mathematicians who were not restrained by foundational considerations and proved to be more venturesome in operating with such magnitudes. Colebrooke's [1973] book and translation of two of Brahmagupta's chapters, "Gańitád'haya" and "Cuttacád'hyaya," are usually taken as evidence that Brahmagupta [628] was the first to have accepted negative numbers and operated with them.[9] Colebrooke's book also contains translations of two chapters, "Víjagańita" and "Lílávatí," by Bhāskara [1151] from which we can formulate our main hypothesis that Indian mathematicians, possibly due to a long-standing accounting tradition, seem to have been the first to give empirical meaning to *negative numbers* by interpreting them *as debts* (i.e., in terms of a basic accounting notion), while interpreting positive numbers as the *possession of assets*. The crucial evidence comes from two footnotes in Colebrooke's translation of Bhāskara's work. One of these, expressing the "rule for addition of affirmative and negative quantities," states: "For a demonstration of the rule, the [medieval] commentators, Súryadása and Crĭshń, exhibit familiar examples of the comparison of debts and assets" [Colebrooke,1973, p. 131, note 2]. The other, the "rule for the subtraction of positive and negative quantities," said: "So in respect of chattels, that, to which a man bears the relation of owner [possession], is considered as positive in regard to him: and the converse (or negative quantity) is that to which another person has the relation of owner" [Colebrooke, 1973, p. 132, note 3].[10]

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[9]See particularly item 17 and Statement of item 18 of Section I of Chapter XVIII on "Cuttacád' hyaya,"("Algebra") of Brahmagupta's book *Brahma-Sphuta-Sidd'hánta* [628], as well as items 31 and 32-33 of Section II of the same book and chapter.

[10] As to the modern usage of assigning minus signs in accounting, they are, of course, not only assigned to debt claims but also to ownership claims. But beware, the word "ownership" is often used in an ambiguous way, meaning either possession of an asset (the value of which would be expressed by a positive number) or the claim represented by an owner's equity (represented by a negative number).

It may also be noted that "debts" were not the Hindus' only interpretation

As to a better comprehension of the influence of Hindu accounting on the mathematical acceptance of negative numbers, the first section mentioned two prerequisites that might be formulated as auxiliary hypotheses. First, basic accounting notions, including asset, debt, revenues, expenses, and income, were first described in India in Kautilya's *Arthaśāstra* [ca. 300 B.C.], establishing a cultural climate that may ultimately have facilitated the association between a debt and a negative number. This claim can be verified from various presentations and translations of or commentaries on the *Arthaśāstra*, such as Shamasastry [1967], Kangle [1960, 1963, 1965], and Rangarajan [1992]. Relevant accounting interpretations and further commentaries can be found in Choudhury [1982], Bhattacharyya [1988], and Mattessich [1997, 1998b].

Kautilya's treatment of accounting was sophisticated enough to include (i) various types of income, including aspects of accounting for price and price-level changes and a possible distinction between what modern accountants call real vs. fictitious holding gains[11] and their potential relations to other accounting concepts; (ii) classifications of expenditures or costs, including possibly fixed and variable costs; and (iii) some notions of assets, debts, and capital. Thus, the description of accounting seems to have been more advanced in India than anywhere else at the time, with the possible exception of China. In consequence, the existence of cultural prerequisites for relating accounting to mathematics, particularly for attributing positive numbers to the possessions of assets but negative numbers to debts, seems to be more likely in such a sophisticated environment. This supposition is reinforced by a relative social stability and continuity in India from the 3rd century B.C. to early medieval times. Despite many terrible conflicts, it seems that during this time India did not experience anything comparable to the decline of the Roman Empire in the wake of devastating wars

of negative numbers. The note to Bhāskara's "Lílávatí" [par. 166], referring to a segment on a line or geographical direction, states: "The segment is negative, that is to say, is in the contrary direction. As the west is contrary of east; and the south the converse of north" [Colebrooke, 1973, p. 132, note 3].

[11]A fictitious holding gain merely appears to be a gain; it refers to holding a (non-monetary) commodity during an inflationary period in which, for example, the general price level increased equally or more than the specific price level pertaining to this commodity. Obviously, it is not possible to derive from mere inflation any real gain by holding a non-monetary asset (in contrast to owing a debt during such an inflationary time which, indeed, may result in a genuine holding gain).

and mass migrations. Thus, Indian insights into accounting during the 3rd century B.C., or even before, are likely to have been preserved until medieval times.

The second prerequisite or auxiliary hypothesis is that assets and debt claims are among the most basic accounting concepts. Debt claims, one of the earliest accounting notions, constitute the very pivot on which Sumerian token accounting of the 4th millennium B.C. hinged. This ancestry may be taken as further support that the accounting aspect of debt claims is at least as fundamental as its legalistic one. There exists incontrovertible archaeological evidence that the accounting notion of a debt — manifested by a kind of IOU in the form of a clay envelope (and, at times, more perishable receptacles) containing clay tokens that represented the items owed — preceded not only the codification of laws and legal regulation of debts, but even the invention of writing by at least 500 years. While archaeological findings of token accounting, i.e., clay tokens and envelopes representing debt and ownership claims, go back to the middle of the 4th millennium B.C., proto-cuneiform writing developed around 3000 B.C. [see, for example, Schmandt-Besserat, 1977, 1992; Mattessich, 1987, 1995, 1998b; Nissen et al., 1993; and Galassi, 1997]. The first known legal codes appeared about a millennium later; they are those of the kings of Isin and Shulgi (third dynasty of Ur, ca. 2000 B.C.) and King Lipit-Ishtar (2100 B.C. to 2092 B.C.) [see Ceram, 1949, p. 421], all of them precursors to the much better known code of Hammurabi, nowadays attributed to the 18th century B.C.

Even if the moral or quasi-legalistic aspect of a debt is a prerequisite to its accounting aspect, the former is so closely intertwined with the latter that in most social settings they occur conjointly.[12] What would a debt practically be without

[12]There is no evidence that five thousand years ago the Sumerians conceived of such distinct disciplines as law, accounting, and business administration. Thus, I wholly agree with one of the reviewers that historians should beware of attributing present circumstances to ancient times. But, it is quite a different matter when it comes to such basic human notions as having a "claim" on something or somebody, corresponding directly to our notions of assets and debts, liabilities and ownership. To deny that those relations existed among the Sumerians does not only run counter to the pertinent archaeological evidence, but also against the insights of anthropology and the behavioral sciences in general. Nietzsche [1887] traced even the origin of conscience to "the contractual relationship between *creditor* and debtor." Though this may be an interesting explanation, I suspect that the notion of conscience has older and deeper roots.

the two major ingredients of accounting — accountability and counting? The *recording* of a debt becomes indispensable for at least two reasons: (i) to provide for the limitations of human memory and (ii) to substantiate the existence and magnitude of the debt at due-date. This may explain why some accounting tokens go back as far as 8000 B.C., five thousand years before the invention of writing. Yet, I have no objection to one reviewer's suggestion that "the glory of negative numbers should go to 'law' as much as 'accounting'." I might even go beyond and extend the "glory" to geography as well (cf., see the second paragraph of footnote 10). However, in this venue, I deem it reasonable to concentrate on accounting aspects. The major point of this paper is unaffected; namely, that in medieval India the "existence" and use of negative numbers were justified, though not exclusively, by interpreting them as "debts," which in turn were conceived as "negative assets." Whether "debts" and "assets" have further commercial and legal connotations is here beside the point.

Perhaps there is a third prerequisite to comprehending the significance of accounting for this particular historical impact on mathematics. Only those familiar with the enduring resistance of European mathematicians to negative numbers can fully appreciate the early Indian achievement of giving the concept of negative numbers its proper place in the pantheon of mathematical concepts. Accounting seems to have played its part in this achievement. Of course, had this taken place in Europe, or had the Arabs and Leonardo da Pisa succeeded in transferring this need for a mathematical legitimization of negative numbers, Western mathematics might well have advanced more rapidly.

Admittedly, the first part of my hypothesis is supported by nothing but two short footnotes in a medieval mathematical or astronomical manuscript. Some readers might consider this fairly "slim" evidence. Accounting historians, in contrast to archaeologists, dealing with later periods are used to much more abundant evidential material and, thus, might be prone to disparage the support for the hypothesis here advanced. Yet comparing this with the diminutive evidential basis on which major advances in modern palaeontology frequently rests, one must admit that disregarding any kind of genuine evidence, be it as unobtrusive as the one supporting my hypothesis, may deprive any science of worthwhile insights.[13] As to evaluation of this evidential support, it must ultimately lie with the reader. Measurement of such support is still elusive and subjectively tainted, particularly as far as hypotheses concerning early historical or prehistoric events are concerned. Here the decisive criterion for accepting a specific hypothesis is not the "absolute" strength of evidence, but how the support compares to the evidence propping the counter-hypothesis. The latter would consist, in our case, of the two-part view that, first, "debt" is *not* a basic accounting notion and, that second, the concept of "debt" did *not* have a part in facilitating or justifying the acceptance of negative numbers by major medieval Indian mathematicians.[14]

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[13]Just as the DNA of a single human hair may constitute decisive forensic evidence in a criminal court, so a single medieval footnote or two may constitute evidence that "flips" the preference for a traditional hypothesis (e.g., the counter-hypothesis) to that for a new hypothesis. Thus, it is not so much the quantity but the quality of evidence that ultimately counts.

[14]I am reluctant to offer here any methodological recapitulation, but it seems necessary due to some misunderstanding raised during the review process of this paper. So far, neither Carnap [1950] nor anyone else has succeeded in establishing an *objective* measure of the "degree of confirmation" for measuring the strength with which a piece of specific evidence supports an hypothesis. Thus, it seems that one has to rely on Popper's [1935] assumption that a plausible hypothesis is accepted as long as no refutation is provided. As to "plausibility," it is rooted in a subjective "degree of belief" [cf., Ramsey, 1931] based on tangible evidence. The alternative of an "objective" measurement as, for example, the "degree of confirmation," first developed by Neyman and Pearson [1937] and widely used in statistical hypotheses testing, is restricted to statistical mass phenomena and, therefore, is not applicable to such historical hypotheses as advanced above. For further details see Mattessich [1978, Chs. 5 and 6, pp. 141-248].

Applying these insights to the present paper, one reaches the following twofold conclusion. First, the "link" between the evidence that relatively sophisticated accounting thoughts had existed in India since 300 B.C. and the hypothesis that it was the familiarity of medieval Indians with accounting which led them to interpret a debt as a negative asset, leading ultimately to the use of negative numbers in mathematics, cannot be established objectively but merely subjectively. Second, to invalidate this hypothesis, one has to show it impossible *that the relative accounting sophistication of early Hindu society* could have led to the pertinent influence upon medieval Indian mathematicians. Hence, this paper may well stimulate historians to continue their search for a genuine refutation of one or more of my hypotheses.

CONCLUSION

In mathematics it is not always the formal consistency alone that is decisive. In many situations the "Authority of Nature," as Kline [1980, p. 308] called it, is no less important. Although the empirical existence of a structure or relationship is not a prerequisite for its acceptance as a mathematical concept, it often happens that such existence stimulates the formulation of a concept. This seem to have happened in Sumeria and ancient Egypt when *special cases* of the "Pythagorean theorem" were formulated on the basis of experience, perhaps in large construction projects. Something similar may have happened when the Indians conceived the legitimacy of negative numbers on the basis of either debts as an inverse to the possession of assets or of opposite geographical directions (see footnote 10). Of course, one may also cite examples of reverse cases where mathematics was leading and empirical science following; e.g., the formulation of non-Euclidean geometry many decades before the discovery of the gravitational curvature of space by Einstein and Minkowski. But in the case of legitimizing negative numbers in Europe, the delay by many centuries showed its mathematicians limping much behind man's perception of reality.

The Arabs, and through them Leonardo da Pisa, might have transmitted to the West some knowledge about negative quantities; but the subsequent circumstances (greater "logical scruples" of European mathematicians and a more foundational-deductive orientation than the pragmatic one of their Indian counterparts [cf., Kline, 1980, pp. 110-112]), indicate that neither the Arabs nor Leonardo da Pisa succeeded in conveying the need for legitimizing negative numbers, though they did transmit such Indian achievements as the decimal place-order system and a symbol for zero.[15]

As demonstrated, it seems likely that the centuries-old accounting tradition of the medieval Hindus [see, e.g., Choudhury, 1982; Bhattacharyya, 1988; Mattessich, 1997, 1998a] facilitated this crucial achievement of accepting negative numbers. From an historical point of view, the fascinating details of the centuries-long struggle over the general acceptance of negative numbers and their first mathematical recognition

[15]The text by Aleksandrov et al. [1963, p. 14] pointed out that in "a rudimentary form, zero already appears in the late Babylonian cuneiform writings, but its systematic introduction was an achievement of the Indians."

by the Indians seem hardly less significant than other relations between accounting and negative numbers (e.g., those that Peters and Emery asserted and Scorgie refuted).

Many centuries after the Indians had justified the use of negative numbers to represent debts, a quite similar justification can be found in the writings of the eminent mathematician Leonhard Euler [1770]. Regrettably, this interesting cultural contribution, of which our discipline has partaken through such a basic accounting notion as that of "debt," has hitherto received scant attention from accountants, even though mathematicians have occasionally reminded us. Aleksandrov et al. [1963, p. 39], for example, observed that the "Indians invented our present system of numeration. They also introduced negative numbers, comparing the contrast between positive and negative numbers with the contrast between property and debt or between two directions on a straight line." Likewise, Kline [1980, p. 110] concluded: "The Hindus have added to the logical woes of mathematicians by introducing negative numbers to represent debts. In such uses positive numbers represent assets."

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