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Tensor Products Of Vector Seminormed Spaces

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TENSOR PRODUCTS OF VECTOR SEMINORMED SPACES

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ABSTRACT

A vector seminormed space is a triple consisting of a vector space, a Dedekind complete Riesz space, and a vector valued seminorm, called a vector seminorm, defined on the vector space and taking values in the Riesz space. The collection of vector seminormed spaces with suitably defined morphisms is shown to be a category containing finite products. A theory of vector seminorms on the tensor products of vector seminormed spaces is developed in analogy with the theory of tensor products of Banach spaces. Accordingly, a reasonable cross vector seminorm, or simply tensor seminorm, is defined such that a vector seminorm is a tensor seminorm if and only if it is in between the injective and projective vector seminorms, in analogy with the theory for normed spaces. Moreover, a theory of the complexification of a vector seminormed space is developed in analogy with the theory of compelexification of a Banach lattice, and a class of vector seminorms, called admissible vector seminorms, is defined and shown to be equivalent to the class of reasonable cross vector seminorms on the complexification. Finally, the theory is applied to Dedekind complete Riesz spaces to demonstrate uniqueness of the complexification modulus, in the process yielding multiple formulae for the modulus.

CONTENTS

I. BACKGROUND

The definitions of all terms about Riesz spaces can be found in [1] or [9]. Any category theoretic terms and concepts that are used can be found in any standard reference on category theory such as [4]. We assume familiarity with standard properties of the algebraic tensor product. Reference may be found in [7]. It is useful but not essential to have some knowledge of the concept of a reasonable cross norm on a vector space and to be familiar with the injective and projective reasonable cross norms. Reference for these concepts may be found in [7].

If E is a Riesz space, by E^+ we mean the positive cone of E. By a positive map we mean any map between Riesz spaces that maps positive elements to positive elements. By a positive bilinear map we mean a bilinear map from a product of two Riesz spaces, considered a Riesz space under the canonical product ordering, to another Riesz space such that the map is positive.

The Fremlin tensor product $(E\bar{\otimes}F, \bar{\otimes})$, where $\bar{\otimes} : E \times F \to E\bar{\otimes}F$ is positive and bilinear, of two Archimedean Riesz spaces E and F was originally defined in [2]. In addition to the positivity of $\bar{\otimes}$, we shall only make explicit use of its following universal property (see [2]): For any uniformly complete Riesz space G and any positive bilinear map $T : E \times F \to G$, there exists a unique positive linear map $T^{\bar{\otimes}} : E\bar{\otimes}F \to G$ such that $T^{\bar{\otimes}}\bar{\otimes} = T$.

The concept of a vector valued norm taking values in a vector lattice was introduced originally by L.V. Kantorovich in the 1930s along with the concept of a linear operator that is dominated by a positive operator. [1] However, aside from the core concepts and definitions, the bulk of the existing theory had not been

[1]See the foreword and preface to [3] for more background information.

developed until relatively recently, largely by the work of A.G. Kusraev and many of his students. It should be noted that in much of the existing literature on the subject, what would be called here a vector normed space is often called a lattice normed space.

Tensor products of vector normed spaces have been considered, to my knowledge, only in a paper by Shotaev ([8]). In that paper a categorical tensor product is developed using the projective vector norm where the Riesz spaces are assumed to be reflexive and Dedekind complete. Also a completeness property is assumed for the codomain of the dominated bilinear map for the universal property, and that map is assumed to have a least order continuous dominant. Moreover, Shotaev gives a definition of a vector cross norm that is equivalent to our definition, except of course that we do not assume norms throughout but seminorms. In this paper we do not require that the tensor product satisfy a universal property in the category of vector seminormed spaces. Rather, given vector seminorms $p : X \to E, q : Y \to F$ where X, Y are vector spaces and E, F Dedekind complete Riesz spaces, we focus on ways to define seminorms on the algebraic tensor product $X \otimes Y$ taking values in the Dedekind completion of the Fremlin tensor product $E\bar{\otimes}F$ such that, in addition to being reasonable vector seminorms, they satisfy an additional condition analogous to the dual space condition (See [7](p.127) or the definition of a reasonable cross norm in [6]) required for reasonable cross norms in the theory of tensor products of normed spaces. In this case we call such vector seminorms reasonable cross seminorms or tensor seminorms.

Van Neerven, in [6], defines an admissible norm on the complexification of a Banach lattice. Van Neerven defines the following two admissible norms on the complexification of a Banach lattice X.

$$\|x + iy\|_\infty := \sup\{\|x\cos\theta + y\sin\theta\| \mid \theta \in [0, 2\pi)\},$$

$$\|x+iy\|_1 := \inf\{\sum_1^n |\lambda_k|\|x_k\| \mid \sum_1^n \lambda_k \otimes x_k = x+iy, \lambda_k \in \mathbb{C}, x_k \in X\}.$$

Van Neerven goes on to prove that $\|.\|_\infty$ and $\|.\|_1$ are equal to the injective and projective, respectively, reasonable cross norms on the complexification of X. We shall arrive at analogous results for vector seminormed spaces.

If E is an Archimedean Riesz space, $\mathbb{C} \otimes E$ (or $E + iE$) is called the complexification of E if the supremum

$$|x+iy| := \sup_\theta |Re((x+iy)e^{-i\theta})| = \sup_\theta |x\cos\theta + y\sin\theta|$$

exists in E for any $x, y \in E$ (see [9]). If it exists, $|.|$ defines an extension of the absolute value on E and is called the modulus of complexification. It is shown in [9] (Theorem 13.4) that it is enough to require for E to be uniformly complete to ensure that the modulus exists. Paralleling the analysis of Van Neerven, we shall identify the above modulus with the injective tensor seminorm and in doing so discover that there are other potential formulae. Mittelmeyer and Wolff in [5], by axiomatizing a complex Riesz space, prove that the modulus is the unique extension of the absolute value on a Riesz space to a complex homogenous seminorm on the algebraic complexification (also see [6]). We show uniqueness of the modulus by viewing the complexification of a Dedekind complete Riesz space as a special case of the complexification of a vector seminormed space.

II. PRELIMINARY CONSIDERATIONS

We begin by defining our principal objects of consideration.

Definition 1. *If X is a vector space and E is a Dedekind complete Riesz space, then $p : X \to E$ is called a* **vector seminorm** *if it satisfies the following three properties:*

$$p(X) \subset E^+, \tag{S1}$$

$$\forall \alpha \in \mathbb{R}\ \forall x \in X\ [p(\alpha x) = |\alpha| p(x)], \tag{S2}$$

$$\forall x, y \in X\ [p(x+y) \leq p(x) + p(y)]. \tag{S3}$$

Definition 2. *A* **vector seminormed space** *(VSS) is a triple (X, E, p), written E_p^X, where X is a vector space, E is a Dedekind complete Riesz space, and $p : X \to E$ is a vector seminorm.*

If in addition p satisfies the property that $p^{-1}(0) = \{0\}$ then p is called a **vector norm**, and E_p^X is called a **vector normed space**. [1]

For example a seminormed vector space (X, p) can also be considered as the vector seminormed space $\mathbb{R}_p^X$. So in particular normed vector spaces are examples of vector seminormed spaces. Moreover if E is a Dedekind complete Riesz space, then since the absolute value $|.| : E \to E$ has properties (S1)-(S3), E can be considered as the VSS $E_{|.|}^E$. Hence Dedekind complete Riesz spaces constitute another ready source of examples of vector seminormed spaces.

[1] Such spaces, only assuming that the Riesz space is Archimedean, are also often called 'lattice normed spaces' (see the definition of lattice normed space in [3]). However, we avoid the term 'lattice (semi)norm', and, concomitantly 'lattice (semi)normed space' because the term 'lattice norm' is widely used to describe a real valued norm on a Riesz space which has the property of being isotonic on the positive cone (see, for example, the definition of lattice norm in [1]).

Definition 3. *Let E_p^X and F_q^Y be vector seminormed spaces. For $T : X \to Y$ a linear map and $\bar{T} : E \to F$ a positive map, $\bar{T}$* ***dominates*** *T* ***(with respect to*** *p* ***over*** *q**)** if*

$$qT \leq \bar{T}p. \tag{1}$$

Moreover, in this situation we say that T is ***dominated*** *by $\bar{T}$ or that T is a* ***dominated operator*** *and that $\bar{T}$ is a* ***dominant*** *of T (see [3]).*

Definition 4. *We define a* ***VSS morphism*** *between two vector seminormed spaces E_p^X and F_q^Y to be a pair $(T, \bar{T})$ where $T : X \to Y$ is a linear map and $\bar{T} : E \to F$ is a positive map such that $\bar{T}$ dominates T with respect to p over q.*

From the above definition of morphism, it follows that the collection of all vector seminormed spaces forms a category **VSS**. For clarity we verify that morphisms compose. Let $(T, \bar{T}) : E_p^X \longrightarrow F_q^Y$ and $(S, \bar{S}) : F_q^Y \longrightarrow G_r^Z$ be morphisms of vector seminormed spaces. Then clearly ST is linear and $\bar{S}\bar{T}$ is positive. It remains to show that ST is dominated by $\bar{S}\bar{T}$. Indeed, because S and T are both dominated by positive operators,

$$r(ST) \leq (\bar{S}q)T = \bar{S}(qT) \leq \bar{S}(\bar{T}p) = (\bar{S}\bar{T})p.$$

Hence **VSS** forms a category. We now learn how to recognize isomorphic objects in the category. To this end we have the following proposition.

Proposition 5. *Let E_p^X, F_q^Y be vector seminormed spaces. Then E_p^X is isomorphic to F_q^Y if and only if the following conditions hold: there exist* ***VSS*** *morphisms $(T, \bar{T}) : E_p^X \to F_q^Y$ and $(S, \bar{S}) : F_q^Y \to E_p^X$ such that S and T are inverses in the category of vector spaces, $\bar{S}$ and $\bar{T}$ are inverses in the category of Riesz spaces with positive morphisms, and p and q are related by $\bar{T}pS = q$.*

Proof. Suppose $E_p^X \cong F_q^Y$. Then there exist **VSS** morphisms $(T, \bar{T}) : E_p^X \to F_q^Y$ and $(S, \bar{S}) : F_q^Y \to E_p^X$ such that $(S, \bar{S})(T, \bar{T}) = (id_X, id_E)$ and $(T, \bar{T})(S, \bar{S}) =$

(id_Y, id_F). This means precisely that S and T are inverses in the category of vector spaces and $\bar{S}$ and $\bar{T}$ are inverses in the category of Riesz spaces with positive morphisms. Now since $\bar{S}$ dominates S we have $pS \leq \bar{S}q$. By applying $\bar{T}$ on the left to both sides, using that it is positive, we have

$$\bar{T}pS \leq \bar{T}\bar{S}q = id_F q = q.$$

But also $\bar{T}$ dominates T, and by applying S to the right on both sides of $qT \leq \bar{T}p$, we have

$$q = qid_Y = qTS \leq \bar{T}pS.$$

Hence $q = \bar{T}pS$.

Conversely, assuming the conditions stated in the theorem, if $q = \bar{T}pS$, then applying $\bar{S}$ to both sides on the left gives that $\bar{S}$ dominates S, and applying T on the right of both sides gives that $\bar{T}$ dominates T. So $(T, \bar{T}), (S, \bar{S})$ are in fact **VSS** morphisms. It is then clear that they are isomorphisms and inverses of each other.

■

Corollary 6. *If E_p^X is a VSS, Y is a vector space isomorphic to X under the linear isomorphism $T : X \to Y$, and F is a Riesz space isomorphic to E under the positive isomorphism $\bar{T} : E \to F$, then $\bar{T}pT^{-1}$ is a vector seminorm making $F^Y_{\bar{T}pT^{-1}} \cong E_p^X$.*

Proof. Follows immediately from the previous proposition. ■

We next show that **VSS** contains products and then develop the notion of a **VSS** bimorphism. We then propose a class of vector seminormed spaces that each behave similar to a tensor product.

Products

We show that **VSS** contains finite products. Indeed, if ${E_1}_{p_1}^{X_1}$ and ${E_2}_{p_2}^{X_2}$ are vector seminormed spaces then we demonstrate that $(E_1 \times E_2)_{p_1 \times p_2}^{X_1 \times X_2}$ is the product of ${E_1}_{p_1}^{X_1}$ and ${E_2}_{P_2}^{X_2}$ in **VSS**. First note that $p_1 \times p_2$ is easily seen to be a vector seminorm and that $E_1 \times E_2$ is Dedekind complete. Let $\pi_i : X_1 \times X_2 \to X_i$ and $\bar{\pi}_i : E_1 \times E_2 \to E_i$ be the canonical projections. Then it is clear that $\bar{\pi}_i$ dominates π_i because the diagram

$$\begin{tikzcd} X_1 \times X_2 \arrow[r, "p_1 \times p_2"] \arrow[d, "\pi_i"'] & Y_1 \times Y_2 \arrow[d, "\bar{\pi}_i"] \\ X_i \arrow[r, "p_i"'] & Y_i \end{tikzcd}$$

commutes by definition of $p_1 \times p_2$. Let G_r^Z be another VSS and $(f_i, \bar{f}_i) : G_r^Z \longrightarrow E_{p_i}^{X_i}$ be morphisms. Then because vector spaces and Dedekind complete Riesz spaces form categories containing products with respect to linear and positive maps, respectively, there exist unique maps $P : Z \to X_1 \times X_2$ and $\bar{P} : G \to E_1 \times E_2$, linear and positive, respectively, such that $\pi_i P = f_i$ and $\bar{\pi}_i \bar{P} = \bar{f}_i$. Then it only remains to show that $\bar{P}$ dominates P. However, since $\bar{f}_i$ dominates f_i, we have

$$(p_1 \times p_2)P = (p_1 \pi_1 P, p_2 \pi_2 P) = (p_1 f_1, p_2 f_2) \leq (\bar{f}_1 r, \bar{f}_2 r) = (\bar{\pi}_1 \bar{P} r, \bar{\pi}_2 \bar{P} r) = \bar{P} r,$$

which is the desired inequality. Hence $(P, \bar{P})$ is the unique VSS morphism making the diagram

$$
\begin{tikzcd}
& G_r^Z \arrow[dl, "(f_1,\bar{f}_1)"'] \arrow[d, dashed, "(P,\bar{P})"'] \arrow[dr, "(f_2,\bar{f}_2)"] & \\
{E_1}_{p_1}^{X_1} & (E_1\times E_2)_{p_1\times p_2}^{X_1\times X_2} \arrow[l, "(\pi_1,\bar{\pi}_1)"] \arrow[r, "(\pi_2,\bar{\pi}_2)"'] & {E_2}_{p_2}^{X_2}
\end{tikzcd}
$$

commute. Therefore **VSS** contains finite products, and

$$(E_1 \times E_2)_{p_1\times p_2}^{X_1\times X_2} = {E_1}_{p_1}^{X_1} \times {E_2}_{p_2}^{X_2}.$$

Bimorphisms and cross seminorms

Having introduced products of vector seminormed spaces, we introduce some new vocabulary relating to them.

Definition 7. *Let E_p^X, F_q^Y, G_r^Z be vector seminormed spaces. Then a pair $(T,\bar{T})$ is called a **VSS bimorphism** if $T : X \times Y \to Z$ is bilinear, $\bar{T} : E \times F \to G$ is positive bilinear, and $\bar{T}$ dominates T in the sense that*

$$rT \leq \bar{T}(p \times q). \tag{2}$$

Let $X \otimes Y$ be the vector space tensor product of X and Y and $E\bar{\otimes}F$ the Fremlin tensor product of E and F. Let $(E\bar{\otimes}F)^\delta$ be the Dedekind completion of $E\bar{\otimes}F$. Let $\otimes : X \times Y \to X \otimes Y$ be the canonical bilinear map and $\bar{\otimes} : E \times F \to E\bar{\otimes}F$ be the canonical positive bilinear map, and, for convenience, also let $\bar{\otimes}$ denote the composition $E \times F \to E\bar{\otimes}F \hookrightarrow (E\bar{\otimes}F)^\delta$ of the embedding of $E\bar{\otimes}F$ into its Dedekind completion following the canonical positive bilinear map.

Definition 8. *A vector seminorm t: $X \otimes Y \to (E\bar{\otimes}F)^\delta$ is called **cross** or a **cross (vector) seminorm** [2] if*

$$t\otimes = \bar{\otimes}(p \times q). \tag{C}$$

[2]An equivalent definition, assuming norms rather than seminorms, may be found in [8].

We often write $p \otimes q$ for an arbitrary cross vector seminorm. Then, using this notation and looking at the equality pointwise, (C) may be written more suggestively as follows:

$$\forall (x,y) \in X \times Y \;[\, (p \otimes q)(x \otimes y) = px \bar{\otimes} qy] \,.$$

Not quite a tensor product

Having introduced the above notions we are then led to consider $(E\bar{\otimes}F)^{\delta X\otimes Y}_{\;p\otimes q}$. We show that $\bar{\otimes}$ dominates $\otimes$ with respect to any cross vector seminorm $p \otimes q$. By definition of a cross vector seminorm, we have that the diagram

$$\begin{tikzcd} X \times Y \arrow[r, "p \times q"] \arrow[d, "\otimes"'] & E \times F \arrow[d, "\bar{\otimes}"] \\ X \otimes Y \arrow[r, "p \otimes q"'] & (E\bar{\otimes}F)^{\delta} \end{tikzcd}$$

commutes. Hence $\bar{\otimes}$ dominates $\otimes$ with respect to $p \otimes q$, as desired. We are thus led to consider $(E\bar{\otimes}F)^{\delta X\otimes Y}_{\;p\otimes q}$ as the natural codomain of the bimorphism $(\otimes, \bar{\otimes})$: $E^X_p \times F^Y_q \to (E\bar{\otimes}F)^{\delta X\otimes Y}_{\;p\otimes q}$. Now let $(T, \bar{T}) : E^X_p \times F^Y_q \to G^Z_r$ be a bimorphism. Then because T is linear and $\bar{T}$ is positive, by the definition of $X \otimes Y$ and $E\bar{\otimes}F$, respectively, there exists a unique linear map $T^{\otimes} : X \otimes Y \to Z$ and there exists a unique positive map $\bar{T}^{\bar{\otimes}} : E\bar{\otimes}F \to G$ such that the following diagrams commute:

$$\begin{tikzcd} X \times Y \arrow[r, "T"] \arrow[d, "\otimes"'] & Z \\ X \otimes Y, \arrow[ur, dashed, "T^{\otimes}"'] & \end{tikzcd} \qquad \begin{tikzcd} E \times F \arrow[r, "\bar{T}"] \arrow[d, "\bar{\otimes}"'] & G \\ E\bar{\otimes}F. \arrow[ur, dashed, "\bar{T}^{\bar{\otimes}}"'] & \end{tikzcd}$$

Now since $E \otimes F$ is a majorizing vector subspace of its Dedekind completion, we have by the Kantorovich Extension Theorem (see Theorem 2.8 in [1]) that there exists a positive extension $[\bar{T}^{\bar{\otimes}}]$ of $\bar{T}^{\bar{\otimes}}$ to all of $(E\bar{\otimes}F)^{\delta}$. Now the question arises as to whether $[\bar{T}^{\bar{\otimes}}]$ dominates $T^{\otimes}$ with respect to $p \otimes q$ over r. In general we cannot expect this to occur since we were not careful in our choice of cross seminorm $p \otimes q$, dominant $\bar{T}$ of T, or Kantorovich extension $[\bar{T}^{\bar{\otimes}}]$ of $\bar{T}^{\bar{\otimes}}$. Even making such choices may not be enough. For instance, in [8], Shotaev is able to achieve a categorical tensor product by placing further restrictions on the Riesz spaces E and F, as well as requiring the existence of a least order continuous dominant. However, even without making further such choices and assumptions, we may prove the following result: The positive operator $[\bar{T}^{\bar{\otimes}}]$ dominates $T^{\otimes}$ with respect to $p \otimes q$ over r on the set of simple tensors $\{x \otimes y \mid (x, y) \in X \times Y\} \subset X \otimes Y$, i.e.

$$\forall (x, y) \in X \times Y \; [\, rT^{\otimes}(x \otimes y) \leq [\bar{T}^{\bar{\otimes}}](p \otimes q)(x \otimes y) \,].$$

Indeed, let $(x, y) \in X \times Y$. Then

$$\begin{aligned} rT^{\otimes}(x \otimes y) &= rT^{\otimes} \otimes (x, y) = rT(x, y) \\ &\leq \bar{T}(p \times q)(x, y) = \bar{T}(px, qy) = \bar{T}^{\bar{\otimes}} \bar{\otimes}(px, qy) \\ &= \bar{T}^{\bar{\otimes}}(px \bar{\otimes} qy) = \bar{T}^{\bar{\otimes}}(p \otimes q)(x \otimes y) \\ &= [\bar{T}^{\bar{\otimes}}](p \otimes q)(x \otimes y), \end{aligned}$$

where we have used that $\bar{T}$ dominates T with respect to $p \times q$ over r, the definitions of $T^{\otimes}, \bar{T}^{\bar{\otimes}}$ and cross vector seminorm, and that $[\bar{T}^{\bar{\otimes}}]$ extends $\bar{T}^{\bar{\otimes}}$. Note that We have not yet demonstrated that $(E \bar{\otimes} F)^{\delta X \otimes Y}_{p \otimes q}$ is a VSS because we have not yet demonstrated the existence of any cross vector seminorm $p \otimes q$. Let us now turn to this problem.

III. **TENSOR SEMINORMS**

For X, Y, Z vector spaces, let $\mathscr{L}(X, Y)$ be the set of all linear mappings from X to Y and $\mathscr{B}(X \times Y, Z)$ be the set of all bilinear mappings from $X \times Y$ to Z.

Definition 9. *Let E_p^X be a VSS. The **p-unit ball**, $B(E_p^X)$, is the following subset of $\mathscr{L}(X, E)$:*

$$B(E_p^X) := \{f \in \mathscr{L}(X, E) \mid \forall x \in X\, [\, |f(x)| \leq px\,]\},$$

where $|.|$ is the absolute value on E.

That is the p-unit ball is the set of all operators from X to E dominated by the identity on E with respect to p over $|.|$. Hence it consists of all the f such that $(f, id_E) : E_p^X \to E_{|.|}^E$ is a **VSS** morphism. Also since f is linear and $p(x) = p(-x)$, f being dominated by id_E only occurs if $f \leq p$. So we also have

$$B(E_p^X) := \{f \in \mathscr{L}(X, E) \mid f \leq p\,\}.$$

Now we have the following immediate corollary of the Hahn-Banach-Kantorovich Theorem (see Theorem 2.1 in [1]).

Lemma 10. *Let E_p^X be a VSS and let $x_0 \in X$. Then $\exists \varphi \in B(E_p^X)\, [\, \varphi(x_0) = px_0\,]$. In particular we have that $px_0 = \sup\{\varphi(x_0) \mid \varphi \in B(E_p^X)\}$*

Proof. If $x_0 = 0$ then take $\varphi \equiv 0$. Otherwise suppose $x_0 \neq 0$. Then define $\varphi_0 : \mathbb{R}x_0 \to E$ by $\varphi_0(\lambda x_0) := \lambda p(x_0)$ for $\lambda \in \mathbb{R}$. Then $\varphi_0 \leq p$ on $\mathbb{R}x_0$. Since E is Dedekind complete, by the Hahn-Banach-Kantorovich Theorem

$$\exists \varphi \in \mathscr{L}(X, E)\, [\, \varphi \leq p \quad \& \quad \varphi|_{\mathbb{R}x_0} = \varphi_0\,].$$

Hence $\varphi \in B(E_p^X)$ and $\varphi(x_0) = p(x_0)$, as desired. The second statement now follows trivially. ∎

Let us now again turn our attention to the construction of particular examples of cross vector seminorms. Let E_p^X, F_q^Y be VSS. Then for $u = \sum_1^n x_k \otimes y_k \in X \otimes Y$, define

$$\varepsilon(u) := \sup\{\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\}.$$

This definition seemingly depends on choice of representation of u. We show that this is not the case and that ε takes well defined values in $(E\bar{\otimes}F)^\delta$. More generally, we show that this is an example of a cross seminorm.

Theorem 11. *If E_p^X, F_q^Y are vector seminormed spaces, then ε, as defined above, is a cross seminorm $\varepsilon : X \otimes Y \to (E\bar{\otimes}F)^\delta$.*

Proof. We organize the proof in three steps.

Step 1. *We show that ε is well defined, that is it does not depend on a chosen representation of u and that $\varepsilon(u)$ defines a specific value in $(E\bar{\otimes}F)^\delta$ for each $u \in X \otimes Y$.* With this in mind, let $\eta : X \times Y \to \mathscr{B}(\mathscr{L}(X, E) \times \mathscr{L}(Y, F), (E\bar{\otimes}F)^\delta)$ be defined by $\eta(x, y)(\varphi, \psi) := \varphi(x)\bar{\otimes}\psi(y)$ for $(x, y) \in X \times Y$ and $(\varphi, \psi) \in \mathscr{L}(X, E) \times \mathscr{L}(Y, F)$. Then because each φ, ψ is linear and $\bar{\otimes}$ is bilinear, we have that η is also bilinear as claimed. Then by the definition of the vector space tensor product, $\exists! L : X \otimes Y \to \mathscr{B}(\mathscr{L}(X, E) \times \mathscr{L}(Y, F))$ such that $L\otimes = \eta$. So since for simple tensors $x \otimes y$ we have $L(x \otimes y) = \eta(x, y)$, we must have for $u = \sum_1^n x_k \otimes y_k = \sum_1^m {x_k}' \otimes {y_k}' \in X \otimes Y$ and $(\varphi, \psi) \in \mathscr{L}(X, E) \times \mathscr{L}(Y, F)$ that

$$\begin{aligned} L(u)(\varphi, \psi) &= L(\sum_1^n x_k \otimes y_k)(\varphi, \psi) = \sum_1^n L(x_k, y_k)(\varphi, \psi) = \sum_1^n \eta(x_k, y_k)(\varphi, \psi) \\ &= \sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) = \sum_1^m \varphi(x'_k)\bar{\otimes}\psi(y'_k) \\ &= \sum_1^m \eta(x'_k, y'_k)(\varphi, \psi) = \sum_1^m L(x'_k, y'_k)(\varphi, \psi) = L(\sum_1^m x'_k \otimes y'_k)(\varphi, \psi). \end{aligned}$$

This shows that the definition of ε does not depend on the representation of u as a finite sum of simple tensors. Let $\varphi \in B(E_p^X), \psi \in B(F_q^Y)$. Then because $\bar{\otimes}$ is positive, $\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \leq \sum_1^n p(x_k)\bar{\otimes}q(y_k) \in (E\bar{\otimes}F)^\delta$. Then $\varepsilon(u)$ represents a supremum of a set that is bounded above in the Dedekind complete space $(E\bar{\otimes}F)^\delta$. So $\varepsilon(u) \in (E\bar{\otimes}F)^\delta$ as claimed. Hence $\varepsilon : X \otimes Y \to (E\bar{\otimes}F)^\delta$ is well defined.

Step 2. *We demonstrate that the vector seminorm axioms hold for ε.* Indeed, let $u = \sum_1^n x_k \otimes y_k \in X \otimes Y$. Since $\varphi \in B(E_p^X) \iff -\varphi \in B(E_p^X)$, we have that $\varepsilon(u)$ is a supremum over a set for which $\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k)$ and $-\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k)$ are both members. So $|\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k)| \leq \varepsilon(u)$, and hence $\varepsilon(u) = \sup\{|\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k)| \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\}$. Then from this it is clear that $\varepsilon(X \otimes Y) \subset ((E\bar{\otimes}F)^\delta)^+$.

Let $\alpha \in \mathbb{R}$. Then, using $\alpha(x \otimes y) = \alpha x \otimes y$ for $(x, y) \in X \otimes Y$,

$$\begin{aligned}\varepsilon(\alpha u) = \varepsilon(\sum_1^n \alpha x_k \otimes y_k) &= \sup\{|\sum_1^n \varphi(\alpha x_k)\bar{\otimes}\psi(y_k)| \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\}\\ &= \sup\{|\alpha||\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k)| \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\}\\ &= |\alpha| \sup\{|\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k)| \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\} = |\alpha|\varepsilon(u).\end{aligned}$$

Finally, if $u = \sum_1^n x_k \otimes y_k, u' = \sum_1^m x'_k \otimes y'_k$ then

$$\begin{aligned}\varepsilon(u + u') &= \sup\{\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) + \sum_1^m \varphi(x'_k)\bar{\otimes}\psi(y'_k) \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\}\\ &\leq \sup\{\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\}\\ &\quad + \sup\{\sum_1^m \varphi(x'_k)\bar{\otimes}\psi(y'_k) \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\} = \varepsilon(u) + \varepsilon(u').\end{aligned}$$

Step 3. *Lastly we show that ε is a cross seminorm, that is $\varepsilon(x \otimes y) = px\bar{\otimes}qy$ for simple tensors $x \otimes y$.* To this end, let $x \otimes y$ be a simple tensor. Clearly $\varepsilon(x \otimes y) \leq px\bar{\otimes}qy$ since $\bar{\otimes}$ is positive and by definition of the unit ball for each

VSS involved. But by Lemma 10, there exist $\varphi \in B(E_p^X)$ and $\psi \in B(F_q^Y)$ such that $\varphi(x) = p(x)$ and $\psi(y) = q(y)$. Hence $px\bar{\otimes}qy \leq \varepsilon(x \otimes y)$. So $\varepsilon(x \otimes y) = px\bar{\otimes}qy$, as desired. Therefore ε is a cross vector seminorm. ■

Definition 12. *Let E_p^X, F_q^Y be vector seminormed spaces. The **(p,q)-unit ball**, $B(E_p^X, F_q^Y)$, is the following subset of $\mathscr{B}(X \times Y, (E\bar{\otimes}F)^\delta)$:*

$$B(E_p^X, F_q^Y) := \{\psi \in \mathscr{B}(X \times Y, (E\bar{\otimes}F)^\delta) \mid \forall (x, y) \in X \times Y \; [|\psi(x, y)| \leq px\bar{\otimes}qy]\}.$$

We next relate the p-unit ball and the q-unit ball to the (p,q)-unit ball. We begin by defining a map from $\mathscr{L}(X, E) \times \mathscr{L}(Y, F)$ to $\mathscr{B}(X \times Y, (E\bar{\otimes}F)^\delta)$. Let $\varphi \in \mathscr{L}(X, E), \psi \in \mathscr{L}(Y, F)$. Then define $\varphi\bar{\otimes}\psi : X \times Y \to (E\bar{\otimes}F)^\delta$ by

$$(\varphi\bar{\otimes}\psi)(x, y) := \varphi(x)\bar{\otimes}\psi(y).$$

Then since φ, ψ are linear and $\bar{\otimes}$ is bilinear, $\varphi\bar{\otimes}\psi$ is bilinear. The following lemma then relates the p-unit ball and the q-unit ball to the (p,q)-unit ball.

Lemma 13. *Let $\varphi \in B(E_p^X), \psi \in B(F_q^Y)$. Then $\varphi\bar{\otimes}\psi \in B(E_p^X, F_q^Y)$.*

Proof. It is clear that $\varphi\bar{\otimes}\psi$ is bilinear. Moreover, since φ, ψ are in the p-unit and q-unit balls, respectively, and since $\bar{\otimes}$ is positive, we have that

$$(\varphi\bar{\otimes}\psi)(x, y) = \varphi(x)\bar{\otimes}\psi(y) \leq px\bar{\otimes}qy.$$

Hence $\varphi\bar{\otimes}\psi \in B(E_p^X, F_q^Y)$. ■

This leads us to the definition of another candidate for a cross vector seminorm on $X \otimes Y$ taking values in $(E\bar{\otimes}F)^\delta$. For $u = \sum_1^n x_k \otimes y_k$, define

$$\pi(u) := \sup\{\sum_1^n \psi(x_k, y_k) \mid \psi \in B(E_p^X, F_q^Y)\}.$$

Then, as with ε, we show that this definition is independent of representation of u as a finite sum of simple tensors and that it defines a well-defined cross vector seminorm $\pi : X \otimes Y \to (E\bar{\otimes}F)^{\delta}$.

Theorem 14. *If E_p^X, F_q^Y are vector seminormed spaces, then π, as defined above, is a cross seminorm $\pi : X \otimes Y \to (E\bar{\otimes}F)^{\delta}$.*

Proof. We organize the proof in three steps.

Step 1. *We show that π is well defined, meaning that it does not depend on a chosen representation of u and that $\pi(u)$ defines a specific value in $(E\bar{\otimes}F)^{\delta}$ for each $u \in X \otimes Y$.* With this in mind, let $\gamma : X \times Y \to \mathscr{L}(\mathscr{B}(X \times Y, (E\bar{\otimes}F)^{\delta}), (E\bar{\otimes}F)^{\delta})$ be defined by $\gamma(x, y)(\psi) := \psi(x, y)$ for $(x, y) \in X \times Y$ and $\psi \in \mathscr{B}(X \times Y, (E\bar{\otimes}F)$. Then because each ψ is bilinear, we have that γ is also bilinear as claimed. Then by the definition of the vector space tensor product, $\exists! J : X \otimes Y \to \mathscr{L}(\mathscr{B}(X \times Y, (E\bar{\otimes}F)^{\delta}), (E\bar{\otimes}F)^{\delta})$ such that $J\otimes = \gamma$. So since for simple tensors $x \otimes y$ we have $J(x \otimes y) = \gamma(x, y)$, we must have for $u = \sum_1^n x_k \otimes y_k = \sum_1^m x_k{}' \otimes y_k{}' \in X \otimes Y$ and $\psi \in \mathscr{B}(X \times Y, (E\bar{\otimes}F)^{\delta})$ that

$$\begin{aligned} J(u)(\psi) &= J(\sum_1^n x_k \otimes y_k)(\psi) = \sum_1^n J(x_k, y_k)(\psi) = \sum_1^n \gamma(x_k, y_k)(\psi) \\ &= \sum_1^n \psi(x_k, y_k) = \sum_1^m \psi(x'_k, y'_k) \\ &= \sum_1^m \gamma(x'_k, y'_k)(\psi) = \sum_1^m J(x'_k, y'_k)(\psi) = J(\sum_1^m x'_k \otimes y'_k)(\psi). \end{aligned}$$

This shows that the definition of π does not depend on the representation of u as a finite sum of simple tensors. Let $\psi \in B(E_p^X, F_q^Y)$. Then by definition of the (p,q)-unit ball, $\sum_1^n \psi(x_k, y_k) \leq \sum_1^n p(x_k)\bar{\otimes}q(y_k) \in (E\bar{\otimes}F)^{\delta}$. Then $\pi(u)$ represents a supremum of a set that is bounded above in the Dedekind complete space $(E\bar{\otimes}F)^{\delta}$. So $\pi(u) \in (E\bar{\otimes}F)^{\delta}$, as claimed. Hence $\pi : X \otimes Y \to (E\bar{\otimes}F)^{\delta}$ is well defined.

Step 2. *We establish that the vector seminorm axioms hold for π.* Let $u =$

$\sum_1^n x_k \otimes y_k \in X \otimes Y$. Since $\psi \in B(E_p^X, F_q^Y) \iff -\psi \in B(E_p^X, F_q^Y)$, we have that $\pi(u)$ is a supremum over a set for which $\sum_1^n \psi(x_k, y_k)$ and $-\sum_1^n \psi(x_k, y_k)$ are both members. So $|\sum_1^n \psi(x_k, y_k)| \leq \pi(u)$, and hence $\pi(u) = \sup\{|\sum_1^n \psi(x_k, y_k)| \mid \psi \in B(E_p^X, F_q^Y)\}$. Then from this it is clear that $\pi(X \otimes Y) \subset ((E\bar{\otimes}F)^\delta)^+$.

Let $\alpha \in \mathbb{R}$. Then, using $\alpha(x \otimes y) = \alpha x \otimes y$ for $(x, y) \in X \otimes Y$,

$$\begin{aligned}\pi(\alpha u) = \pi(\sum_1^n \alpha x_k \otimes y_k) &= \sup\{|\sum_1^n \psi(\alpha x_k, y_k)| \mid \psi \in B(E_p^X, F_q^Y)\} \\ &= \sup\{|\alpha| |\sum_1^n \psi(x_k, y_k)| \mid \psi \in B(E_p^X, F_q^Y)\} \\ &= |\alpha| \sup\{|\sum_1^n \psi(x_k, y_k)| \mid \psi \in B(E_p^X, F_q^Y)\} = |\alpha|\pi(u).\end{aligned}$$

Finally, if $u = \sum_1^n x_k \otimes y_k, u' = \sum_1^m x'_k \otimes y'_k$ then

$$\begin{aligned}\pi(u + u') &= \sup\{\sum_1^n \psi(x_k, y_k) + \sum_1^m \psi(x'_k, y'_k) \mid \psi \in B(E_p^X, F_q^Y)\} \\ &\leq \sup\{\sum_1^n \psi(x_k, y_k) \mid \psi \in B(E_p^X, F_q^Y)\} \\ &\quad + \sup\{\sum_1^m \psi(x'_k, y'_k) \mid \psi \in B(E_p^X, F_q^Y)\} = \pi(u) + \pi(u').\end{aligned}$$

Step 3. *Lastly we must show that π is a cross seminorm, that is $\pi(x \otimes y) = px\bar{\otimes}qy$ for simple tensors $x \otimes y$.* To this end, let $x \otimes y$ be a simple tensor. Clearly $\pi(x \otimes y) \leq px\bar{\otimes}qy$ by definition of the (p,q)-unit ball. But by Lemma 10, there exist $\varphi \in B(E_p^X)$ and $\psi \in B(F_q^Y)$ such that $\varphi(x) = p(x)$ and $\psi(y) = q(y)$. Then we have that

$$(\varphi\bar{\otimes}\psi)(x, y) = \varphi(x)\bar{\otimes}\psi(y) = px\bar{\otimes}qy.$$

Moreover, by the previous lemma, $\varphi\bar{\otimes}\psi \in B(E_p^X, F_q^Y)$. Hence $px\bar{\otimes}qy \leq \pi(x \otimes y)$. So $\pi(x \otimes y) = px\bar{\otimes}qy$, as desired. Therefore π is a cross vector seminorm. ∎

Let E_p^X, F_q^Y be VSS. Let $\varphi \in \mathscr{L}(X, E), \psi \in \mathscr{L}(Y, F)$. Then, as noted above,

$\varphi\bar{\otimes}\psi : X \times Y \to (E\bar{\otimes}F)^\delta$ is bilinear. Moreover, if in addition we require that $\varphi, \psi \in B(E_p^X) \times B(F_q^Y)$ then $\varphi\bar{\otimes}\psi \in B(E_p^X, F_q^Y)$. Now, in either case, since $\varphi\bar{\otimes}\psi$ is bilinear, by the definition of the vector space tensor product, there exists a unique linear map $\varphi \otimes \psi : X \otimes Y \to (E\bar{\otimes}F)^\delta$ with $(\varphi \otimes \psi)\otimes = \varphi\bar{\otimes}\psi$.

Definition 15. *A vector seminorm $t : X \otimes Y \to (E\bar{\otimes}F)^\delta$ is called **reasonable** if*

$$\forall(\varphi, \psi) \in B(E_p^X) \times B(F_q^Y)\ [\varphi \otimes \psi \in B((E\bar{\otimes}F)^{\delta X\otimes Y}_{t})]. \tag{R}$$

Definition 16. *A vector seminorm $t : X \otimes Y \to (E\bar{\otimes}F)^\delta$ is called a **tensor seminorm** or a **reasonable cross seminorm** if it is both reasonable and cross.*

So a vector seminorm $X \otimes Y \to (E\bar{\otimes}F)^\delta$ is a reasonable cross seminorm if and only if it satisfies both (R) and (C). As per definition, we shall use the terms tensor seminorm and reasonable cross seminorm interchangeably.

The reader may be aware of the notion of reasonable cross norm in the theory of tensor products of Banach spaces (see [7]). Using that $\mathbb{R}\bar{\otimes}\mathbb{R} \cong \mathbb{R}$, it is not difficult to show that when we restrict our attention to Banach spaces, the two definitions of reasonable cross norm align.

The following definitions further suggest analogy with the theory of tensor products of normed spaces.

Definition 17. *Let E_p^X and F_q^Y be vector seminormed spaces. Then the vector seminorm $\varepsilon : X \otimes Y \to (E\bar{\otimes}F)^\delta$ defined on each $u = \sum_1^n x_k \otimes y_k \in X \otimes Y$ by*

$$\varepsilon(u) := \sup\{\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\}$$

*is called the **injective vector seminorm** and $(E\bar{\otimes}F)^{\delta X\otimes Y}_{\varepsilon}$ (up to **VSS** isomorphism) is called the **injective tensor product** of E_p^X and F_q^Y.*

Definition 18. *Let E_p^X and F_q^Y be vector seminormed spaces. Then the vector seminorm $\pi : X \otimes Y \to (E\bar{\otimes}F)^\delta$ defined on each $u = \sum_1^n x_k \otimes y_k \in X \otimes Y$ by*

$$\pi(u) := \sup\{\sum_1^n \psi(x_k, y_k) \mid \psi \in B(E_p^X, F_q^Y)\}.$$

*is called the **projective vector seminorm** and $(E\bar{\otimes}F)^{\delta X \otimes Y}_{\pi}$ (up to **VSS** isomorphism) is called the **projective tensor product** of E_p^X and F_q^Y.*

Definition 19. *Let E_p^X, F_q^Y be VSS. Then if $t : X \otimes Y \to (E\bar{\otimes}F)^\delta$ is a tensor seminorm, we write $E_p^X \otimes_t F_q^Y$ for any VSS that is isomorphic (in the category of vector seminormed spaces) to $(E\bar{\otimes}F)^{\delta X \otimes Y}_{t}$ and call such a space a **t-tensor product** of E_p^X and F_q^Y.*

Moreover, as a point of notation, if G_r^Z is a t-tensor product of E_p^X and F_q^Y then we shall write $E_p^X \otimes_t F_q^Y = G_r^Z$.

We have defined two seminorms ε and π that we know to be cross. However, in the case of Banach spaces these are also reasonable cross norms. We shall find that this holds in our more general case as well, i.e., ε and π will be shown to also satisfy (R). As a consequence, the injective and projective tensor products of vector seminormed spaces E_p^X and F_1^Y, as we have already defined them, really are the ε-tensor product and π-tensor product, respectively. So our definitions will be shown to be consistent.

The following theorem, in addition to showing that ε and π are reasonable cross seminorms, further develops our analogy with the theory of tensor products of normed spaces by characterizing possible reasonable cross vector seminorms by means of ε and π.

Theorem 20. *We have that $\varepsilon \leq \pi$ and ε, π are reasonable cross seminorms. Moreover, a seminorm $t : X \otimes Y \to (E\bar{\otimes}F)^\delta$ is a reasonable cross seminorm if and only if $\varepsilon \leq t \leq \pi$.*

Proof. We organize the proof in 5 steps.

Step 1. *We show $\varepsilon \leq \pi$.* Let $u = \sum_1^n x_k \otimes y_k \in X \otimes Y$. Then consider the sets

$$\alpha_u := \{\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \mid \varphi \in B(E_p^X), \psi \in B(F_q^Y)\},$$

$$\beta_u := \{\sum_1^n \psi(x_k, y_k) \mid \psi \in B(E_p^X, F_q^Y)\}.$$

So $\varepsilon(u) = \sup \alpha_u$, and $\pi(u) = \sup \beta_u$. Let $\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \in \alpha_u$, where $(\varphi, \psi) \in B(E_p^X) \times B(F_q^Y)$. Then, by Lemma 13, we know that since $(\varphi, \psi) \in B(E_p^X) \times B(F_q^Y)$, we have that $\varphi\bar{\otimes}\psi \in B(E_p^X, F_q^Y)$. So $\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \in \beta_u$. Hence $\alpha_u \subset \beta_u$. Therefore $\varepsilon(u) \leq \pi(u)$. So $\varepsilon \leq \pi$.

Step 2. *We show that $t \leq \pi$.* Let $t : X \otimes Y \to (E\bar{\otimes}F)^\delta$ be a vector seminorm. For one implication, suppose t is a reasonable cross seminorm. Let $u = \sum_1^n x_k \otimes y_k \in X \otimes Y$. Then by Lemma 10 there exists $\varphi \in B((E\bar{\otimes}F)^{\delta X\otimes Y}_{\ t})$ such that $\varphi(u) = t(u)$. But also if $\varphi \in B((E\bar{\otimes}F)^{\delta X\otimes Y}_{\ t})$ then $\varphi(u) \leq t(u)$. Hence

$$\begin{aligned} t(u) &= \sup\{\varphi(u) \mid \varphi \in B((E\bar{\otimes}F)^{\delta X\otimes Y}_{\ t})\} \\ &= \sup\{\sum_1^n \varphi(x_k \otimes y_k) \mid \varphi \in B((E\bar{\otimes}F)^{\delta X\otimes Y}_{\ t})\} \\ &= \sup\{\sum_1^n (\varphi\otimes)(x_k, y_k) \mid \varphi \in B((E\bar{\otimes}F)^{\delta X\otimes Y}_{\ t})\}. \end{aligned}$$

Recall the definitions of α_u and β_u from earlier. Similarly, define

$$\gamma_{u,t} := \{\varphi(u) \mid \varphi \in B((E\bar{\otimes}F)^{\delta X\otimes Y}_{\ t})\}.$$

Now if $\varphi(u) = \sum_1^n (\varphi\otimes)(x_k, y_k) \in \gamma_u$ where $\varphi \in B((E\bar{\otimes}F)^{\delta X\otimes Y}_{\ t})$, then $\varphi\otimes : X \times Y \to (E\bar{\otimes}F)^\delta$ is bilinear, as a linear map composed with a bilinear map.

Moreover for $(x, y) \in X \times Y$ we have that

$$(\varphi\otimes)(x, y) = \varphi(x \otimes y) \leq t(x \otimes y) = px\bar{\otimes}qy.$$

So $\varphi\otimes \in B(E_p^X, F_q^Y)$. Hence $\gamma_{u,t} \subset \beta_u$. Then $t(u) = \sup \gamma_u \leq \pi(u) = \sup \beta_u$. Hence $t \leq \pi$. Note that we only needed that t is cross for this half of the inequality.

Step 3. *We show that ε and π are reasonable cross seminorms.* To this end, note that we have already shown that they are cross. So it suffices to show that they satisfy (R). However, this means that for $(\varphi, \psi) \in B(E_p^X) \times B(F_q^Y)$ and $u = \sum_1^n x_k \otimes y_k \in X \otimes Y$, that $(\varphi\otimes\psi) \in B((E\bar{\otimes}F)_\varepsilon^{\delta X\otimes Y})$ and $(\varphi\otimes\psi) \in B((E\bar{\otimes}F)_\pi^{\delta X\otimes Y})$. But we have shown that $\varepsilon \leq \pi$. So

$$B((E\bar{\otimes}F)_\varepsilon^{\delta X\otimes Y}) \subset B((E\bar{\otimes}F)_\pi^{\delta X\otimes Y}).$$

So it suffices only to show that $(\varphi \otimes \psi) \in B((E\bar{\otimes}F)_\varepsilon^{\delta X\otimes Y})$. That is, it suffices to show that for all $u = \sum_1^n x_k \otimes y_k \in X \otimes Y$ we have that $(\varphi \otimes \psi)(u) \leq \varepsilon(u)$. But for such a $u \in X \otimes Y$,

$$(\varphi \otimes \psi)(u) = \sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \in \alpha_u.$$

Hence $(\varphi \otimes \psi)(u) \leq \varepsilon(u)$, which is the desired inequality. Therefore ε and π are both reasonable cross seminorms.

Step 4. *We show that $\varepsilon \leq t$.* Let $\sum_1^n \varphi(x_k)\bar{\otimes}\psi(y_k) \in \alpha_u$, where $(\varphi, \psi) \in B(E_p^X) \times B(F_q^Y)$. Then since we assumed t to be a reasonable cross seminorm, we have by (R) that $\varphi \otimes \psi \in B((E\bar{\otimes}F)_t^{\delta X\otimes Y})$. Hence $(\varphi \otimes \psi)(u) \in \gamma_{u,t}$. Therefore $\varepsilon \leq t$. So

$$\varepsilon \leq t \leq \pi,$$

as claimed.

Step 5. *Lastly we show that any vector seminorm between ε and π is a reasonable cross seminorm.* Suppose that t is an arbitrary vector seminorm satisfying $\varepsilon \leq t \leq \pi$. Now since ε, π are themselves cross seminorms, they satisfy (C). Then because ε and π are equal on simple tensors and since $\varepsilon \leq t \leq \pi$, we have

$$\forall (x, y) \in X \times Y \, [\, \varepsilon(x \otimes y) = t(x \otimes y) = \pi(x \otimes y) = px \bar{\otimes} qy \,].$$

So t satisfies (C). Now let $(\varphi, \psi) \in B(E_p^X) \times B(F_q^Y)$. Then $\varphi \otimes \psi \in B((E \bar{\otimes} F)_{\varepsilon}^{\delta X \otimes Y})$ because ε is reasonable. So

$$\forall u \in X \otimes Y \, [\, (\varphi \otimes \psi)(u) \leq \varepsilon(u) \leq t(u) \,].$$

Therefore $\varphi \otimes \psi \in B((E \bar{\otimes} F)_{t}^{\delta X \otimes Y})$. So t satisfies (R) as well. Hence t is a reasonable cross seminorm, completing the proof. ■

IV. THE COMPLEXIFICATION OF A VECTOR SEMINORMED SPACE

Consider $\mathbb{C}$ as a real vector space isomorphic to $\mathbb{R}^2$. Then since $\mathbb{R}$ is Dedekind complete, $\mathbb{R}^{\mathbb{C}}_{|.|}$ is a VSS where $|.| : \mathbb{C} \to \mathbb{R}$ is the standard Euclidean norm on $\mathbb{C}$.

Let E^X_p be a VSS. It is well known that the real vector space $\mathbb{C} \otimes X$ can be considered as the algebraic complexification of X and allows us to consider $\mathbb{C} \otimes X$ as a complex vector space by defining scalar multiplication as $\alpha(\lambda \otimes x) := \alpha\lambda \otimes x$ where $\alpha, \lambda \in \mathbb{C}$ and $x \in X$. Also, since any $\sum_1^n \lambda_k \otimes x_k \in \mathbb{C} \otimes X$ can be written as $1 \otimes x + i \otimes y$ for suitable $x, y \in X$, we have $\mathbb{C} \otimes X$ is complex linear isomorphic to the formal direct sum $X + iX$ under the identification $x + iy \leftrightarrow 1 \otimes x + i \otimes y$.

As with modules, we also define the complexification of a vector seminormed space E^X_p as a tensor product with $\mathbb{C}$, where $\mathbb{C}$ is here considered as the normed vector space $\mathbb{R}^{\mathbb{C}}_{|.|}$. However, since we have a choice of tensor seminorm, there are many possible complexifications. Note that since E is Dedekind complete and $\mathbb{R}\bar{\otimes}E \cong E$, we may identify $(\mathbb{R}\bar{\otimes}E)^\delta$ with E.

Definition 21. *For E^X_p a VSS and $t : \mathbb{C} \otimes X \to E$ a tensor seminorm, the **t-complexification** of E^X_p is the t-tensor product $E^X_p \otimes_t \mathbb{R}^{\mathbb{C}}_{|.|}$.*

So in particular $E^X_p \otimes_t \mathbb{R}^{\mathbb{C}}_{|.|} = E^{\mathbb{C}\otimes X}_t$, after making the identification $(\mathbb{R}\bar{\otimes}E)^\delta \cong E$. Similarly, since every element of $\mathbb{C} \otimes X$ can be written uniquely in the form $1 \otimes x + i \otimes y$ for some $x, y \in X$, the identification $x + iy \leftrightarrow 1 \otimes x + i \otimes y$ induces a (complex) linear isomorphism $T : \mathbb{C} \otimes X \to X + iX$, and, by Corollary 6, $id_E t T^{-1} : X + iX \to E$ is a vector seminorm, and $(T, id_E) : E^{\mathbb{C}\otimes X}_t \xrightarrow{\simeq} E^{X+iX}_{id_E t T^{-1}}$ is a **VSS** isomorphism. Hence

$$t(u) = t(1 \otimes x + i \otimes y) = tT^{-1}(x + iy)$$

, where $u \in \mathbb{C} \otimes X$ and $Tu = x + iy$ for $x, y \in X$. Since we choose to identify $1 \otimes x + i \otimes y$ with $x + iy$, we shall simply identify t and $id_E tT^{-1}$. Hence $E_t^{\mathbb{C}\otimes X} \cong E_t^{X+iX}$ in **VSS**, and so we also have $E_p^X \otimes_t \mathbb{R}_{|.|}^{\mathbb{C}} = E_t^{X+iX}$.

Let E_p^X be a VSS. Consider the injective vector seminorm ε on $\mathbb{C}\otimes X \cong X+iX$ taking values in the Dedekind complete Riesz space E. Then we write

$$\varepsilon(x + iy) = \sup\{\, |\varphi(1)\psi(x) + \varphi(i)\psi(y)| \mid \varphi \in B(\mathbb{R}_{|.|}^{\mathbb{C}}), \psi \in B(E_p^X)\},$$

because we may make the identifications $x + iy \leftrightarrow 1 \otimes x + i \otimes y$ and $\alpha\bar{\otimes}x \leftrightarrow \alpha x$ for $x, y \in E, \alpha \in \mathbb{R}$, due to the isomorphisms mentioned above.

Now let us define a new candidate for a reasonable cross seminorm on the complexification. Set

$$\|x + iy\|_\infty := \sup\{p(x\cos(\theta) + y\sin(\theta)) \mid \theta \in [0, 2\pi)\}.$$

Theorem 22. *Let E_p^X be a VSS. Then $\|.\|_\infty$ is the injective vector seminorm ε on $X + iX$. That is, for $x, y \in X$, we have that*

$$\begin{aligned}\varepsilon(x + iy) &= \sup\{\, |\varphi(1)\psi(x) + \varphi(i)\psi(y)| \mid \varphi \in B(\mathbb{R}_{|.|}^{\mathbb{C}}), \psi \in B(E_p^X)\} \\ &= \sup\{p(x\cos(\theta) + y\sin(\theta)) \mid \theta \in [0, 2\pi)\} = \|x + iy\|_\infty.\end{aligned}$$

Proof. First note that

$$\begin{aligned}\varphi \in B(\mathbb{R}_{|.|}^{\mathbb{C}}) &\iff \forall\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}\,[|\varphi(\lambda_1 + i\lambda_2)| = |\varphi(1)\lambda_1 + \varphi(i)\lambda_2| \le |\lambda|] \\ &\iff \forall\lambda \in \mathbb{C}, |\lambda| \le 1\,[|\varphi(1)\lambda_1 + \varphi(i)\lambda_2| \le 1] \\ &\iff \sqrt{(\varphi(1))^2 + (\varphi(i))^2} \le 1.\end{aligned}$$

So for $x, y \in X$,

$$
\begin{aligned}
\varepsilon(x+iy) &= \sup\{\, |\varphi(1)\psi(x) + \varphi(i)\psi(y)| \mid \varphi \in B(\mathbb{R}^{\mathbb{C}}_{|.|}), \psi \in B(E_p^X)\} \\
&= \sup\{\sqrt{a^2+b^2}(\frac{a\psi(x)}{\sqrt{a^2+b^2}} + \frac{b\psi(y)}{\sqrt{a^2+b^2)}}) \mid a^2+b^2 \leq 1, \psi \in B(E_p^X)\} \\
&= \sup\{\psi(x)\cos(\theta) + \psi(y)\sin(\theta) \mid \theta \in [0, 2\pi), \psi \in B(E_p^X)\}.
\end{aligned}
$$

Let $\theta \in [0, 2\pi)$. Now since E_p^X is a VSS, by the second statement of Lemma 10,

$$
\begin{aligned}
p(x\cos(\theta) + y\sin(\theta)) &= \sup\{\psi(x\cos(\theta) + y\sin(\theta)) \mid \psi \in B(E_p^X)\} \\
&= \sup\{\psi(x)\cos(\theta) + \psi(y)\sin(\theta) \mid \psi \in B(E_p^X)\}.
\end{aligned}
$$

Therefore, putting it all together, we have

$$
\begin{aligned}
\|x+iy\|_\infty &= \sup\{p(x\cos(\theta) + y\sin(\theta)) \mid \theta \in [0, 2\pi)\} \\
&= \sup\{\sup\{\psi(x)\cos(\theta) + \psi(y)\sin(\theta) \mid \psi \in B(E_p^X)\} \mid \theta \in [0, 2\pi)\} \\
&= \sup\{|\psi(x)\cos(\theta) + \psi(y)\sin(\theta)| \mid \theta \in [0, 2\pi), \psi \in B(E_p^X)\} \\
&= \sup\{\, |\varphi(1)\psi(x) + \varphi(i)\psi(y)| \mid \varphi \in B(\mathbb{R}^{\mathbb{C}}_{|.|}), \psi \in B(E_p^X)\} = \varepsilon(x+iy),
\end{aligned}
$$

as desired. ■

Let E be a Dedekind complete Riesz space. Then, since E is Dedekind complete, the complexification $E+iE$ is well known to have a **modulus** $h : E+iE \to E$, defined, for $x, y \in E$, by the equation

$$
h(x+iy) := \sup\{|x\cos(\theta) + y\sin(\theta)| \mid \theta \in [0, 2\pi)\},
$$

and satisfying properties (S1)-(S3). So h is a vector seminorm [3] However, since $E^E_{|.|}$ is a VSS where $|.|$ is the absolute value on E, it is clear that $h = \|.\|_\infty$. So by the previous theorem h is the injective tensor seminorm on the complexification.

[3]It also satisfies $h^{-1}(0) = 0$ and is thus a vector norm on $E+iE$. See Theorem 13.5 in [9].

Therefore we have the following corollary of the previous theorem.

Corollary 23. *Let E a Dedekind complete Riesz space. We have the following formula for the complexification, where h is the complexification modulus:*

$$E_{|.|}^{E} \otimes_{\epsilon} \mathbb{R}_{|.|}^{\mathbb{C}} = E_{h}^{E+iE}.$$

Proof. Clearly $h = \|.\|_{\infty}$. Now apply the previous theorem. ■

In the theory of complexification of Banach lattices there is the concept of admissible vector seminorm (see [6]). We extend this concept to vector seminormed spaces in the following way.

Definition 24. *Let E_p^X be a VSS and $r : \mathbb{C} \otimes X \to E$ a vector seminorm. Then r is called* ***admissible***[4] *if the following two conditions hold:*

$$\forall \lambda \in \mathbb{C}\ \forall x \in X\ [r(\lambda \otimes x) = |\lambda| r(x)], \tag{A1}$$

$$\forall x, y \in X\ [p(x) \vee p(y) \leq r(x + iy) \leq p(x) + p(y)]. \tag{A2}$$

Proposition 25. *Let t be a tensor seminorm on the complexification of E_p^X. Then t is admissible.*

Proof. Let t be a tensor seminorm on the complexification of E_p^X. Clearly $px, py \leq \sup\{p(x\cos(\theta) + y\sin(\theta)) \,|\, \theta \in [0, 2\pi)\}$. Since we have already shown $\varepsilon = \|.\|_{\infty}$ and $\varepsilon \leq t$, we have that

$$px \vee py \leq \varepsilon(x + iy) \leq t(x + iy).$$

Let $\lambda \in \mathbb{C}$ and $x \in X$. Now since t is cross, $t(\lambda \otimes x) = |\lambda| p(x)$ after making the identification $(\mathbb{R}\bar{\otimes}E)^{\delta} \cong E$. In particular, $t|_X = p$. So $t(\lambda \otimes x) = |\lambda| t(x)$, which is

[4] See [6].

(A1). We also have, by the triangle inequality and because t extends p, that

$$t(x+iy) \leq t(x)+t(iy) = t(x)+t(y) = p(x)+p(y).$$

Therefore

$$px \vee py \leq t(x+iy) \leq px+py.$$

So (A2) holds for t as well. Hence t is admissible. ■

Let us define a new potential admissible seminorm on $X+iX$ as follows.

$$\|x+iy\|_1 := \inf\{\sum_1^n |\lambda_k| p(x_k) \mid \lambda_k \in \mathbb{C}, x_k \in X, x+iy = \sum_1^n \lambda_k \otimes x_k\}.$$

By taking real and imaginary parts of each representation $\sum_1^n \lambda_k \otimes x_k$ of $x+iy$, we also have

$$\|x+iy\|_1 := \inf\{\sum_1^n |(a_k+ib_k)| p(x_k) \mid x = \sum_1^n a_k x_k, y = \sum_1^n b_k x_k\},$$

where $a_k, b_k \in \mathbb{R}, x_k \in X$

Theorem 26. $\|.\|_1$ *is an admissible seminorm.*

Proof. We organize the proof in 4 steps.

Step 1. *We show $\|.\|_1$ is well defined and takes values in the positive cone of E.* Let $x, y \in X$ and $\alpha_{x,y} := \{\sum_1^n |\lambda_k| p(x_k) \mid x+iy = \sum_1^n \lambda_k \otimes x_k\}$ where $\lambda_k \in \mathbb{C}, x_k \in X$. Clearly 0 is a lower bound for $\alpha_{x.y}$ and $\alpha_{x,y} \subset E$, where E is a Dedekind complete Riesz space. So $\|x+iy\|_1 = \inf \alpha_{x,y} \in E^+$. Hence $\|.\|_1$ is well defined and takes values in the positive cone of E.

Step 2*. We show (S1)-(S3) and (A1) hold for $\|.\|_1$.* Let $x_1, x_2, y_1, y_2 \in X$. Let $x_1+iy_1 = \sum_1^n \lambda_k^1 \otimes x_k^1, x_2+iy_2 = \sum_1^m \lambda_k^2 \otimes x_k^2$, where $\lambda_i^j \in \mathbb{C}, x_i^j \in X$. Then

$(x_1+iy_1)+(x_2+iy_2)=(x_1+x_2)+i(y_1+y_2)=\sum_1^n \lambda_k^1\otimes x_k^1+\sum_1^m \lambda_k^2\otimes x_k^2$. Hence

$$\|(x_1+x_2)+i(y_1+y_2)\|_1\leq \sum_1^n \lambda_k^1\otimes x_k^1+\sum_1^m \lambda_k^2\otimes x_k^2.$$

Now since $\sum_1^n \lambda_k^1\otimes x_k^1, \sum_1^m \lambda_k^2\otimes x_k^2$ were arbitrary representations of x_1+iy_1, x_2+iy_2, respectively, we have that

$$\|(x_1+x_2)+i(y_1+y_2)\|_1\leq \|x_1+iy_1\|_1+\|x_2+iy_2\|_1.$$

Let $\gamma=\gamma_1+i\gamma_2\in\mathbb{C}$. We may assume that $\gamma\neq 0$, because clearly $\|0\|_1=0$. Then, by definition,

$$\|\gamma(x+iy)\|_1=\inf\{\sum_1^n |\lambda_k|p(x_k)\mid \gamma(x+iy)=\sum_1^n \lambda_k\otimes x_k\}.$$

But $\gamma(x+iy)=\sum_1^n \lambda_k\otimes x_k \iff x+iy=\sum_1^n \gamma^{-1}\lambda_k\otimes x_k$. So

$$\begin{aligned}\|\gamma(x+iy)\|_1&=\inf\{\sum_1^n |\lambda_k|p(x_k)\mid x+iy=\sum_1^n \gamma^{-1}\lambda_k\otimes x_k\}\\ &=\inf\{\sum_1^n |\gamma\lambda_k|p(x_k)\mid x+iy=\sum_1^n \lambda_k\otimes x_k\}\\ &=|\gamma|\inf\{\sum_1^n |\lambda_k|p(x_k)\mid x+iy=\sum_1^n \lambda_k\otimes x_k\}=|\gamma|\|x+iy\|_1.\end{aligned}$$

Therefore (S1)-(S3) and (A1) hold.

Step 3. *We show $\|.\|$ is a cross seminorm whose restriction to X is p.* Let $\lambda\in\mathbb{C}$ and $x\in X$. Then clearly $\|\lambda\otimes x\|_1\leq|\lambda|p(x)$. Now let $(\varphi,\psi)\in B(\mathbb{R}_{|.|}^{\mathbb{C}})\times B(E_p^X)$ with $\varphi(\lambda)=|\lambda|, \psi(x)=p(x)$. Let $\sum_1^n \lambda_k\otimes x_k=\lambda\otimes x$. Then

$$\begin{aligned}|\lambda|p(x)=(\varphi\otimes\psi)(\lambda\otimes x)&=(\varphi\otimes\psi)(\sum_1^n \lambda_k\otimes x_k)\\ &=\sum_1^n \varphi(\lambda_k)\psi(x_k)\leq\sum_1^n |\lambda_k|p(x_k).\end{aligned}$$

Hence, since $\sum_1^n \lambda_k \otimes x_k$ is an arbitrary representation of $\lambda \otimes x$, we have that $|\lambda|p(x) \leq \|\lambda \otimes x\|_1$. So

$$\forall \lambda \in \mathbb{C}\, \forall x \in X \quad [\, \|\lambda \otimes x\|_1 = |\lambda|p(x)\,].$$

Therefore, given the isomorphism $E \cong (\mathbb{R}\bar{\otimes}E)^\delta$, $\|.\|_1$ satisfies (C), and is therefore a cross seminorm. In particular we have that the restriction of $\|.\|_1$ to X is p because $\|x\|_1 = px$ for $x \in X$.

Step 4. *We show that* $\|.\|_1$ *is admissible.* Using similar reasoning we may show that $\|.\|_1$ satisfies (R) in addition to (C), and is therefore a reasonable cross seminorm. Indeed, let $(\varphi, \psi) \in B(\mathbb{R}^{\mathbb{C}}_{|.|}) \times B(E_p^X)$ and $x, y \in X$. Let $\sum_1^n \lambda_k \otimes x_k = x + iy$. Then

$$(\varphi \otimes \psi)(x + iy) = \sum_1^n \varphi(\lambda_k)\psi(x_k) \leq \sum_1^n |\lambda_k| p(x_k).$$

Hence, since $\sum_1^n \lambda_k \otimes x_k$ is an arbitrary representation of $x + iy$, we have that

$$(\varphi \otimes \psi)(x + iy) \leq \|x + iy\|_1,$$

which is the desired inequality. Therefore $\|.\|_1$ is a tensor seminorm. Hence it is admissible by the previous proposition. ■

Theorem 27. *Let E_p^X be a VSS and $r : X + iX \to E$ an admissible seminorm. Then r extends p and $\|.\|_\infty \leq r \leq \|.\|_1$*

Proof. By (A2), if $x \in X$ then $px = px \vee 0 = px \vee p0 \leq rx \leq px + p0 = px$. So $p = r|_X$. As for the second assertion, note that since $X + iX$ is a direct sum, there are two projections, $Re, Im : X + iX \to X$ defined by $Re : x + iy \mapsto x$ and $Im : x + iy \mapsto y$, where $x + iy \in X + iX$ is sent to its real and imaginary parts, respectively. Then since r extends p and by (A1) and (A2),

$$\begin{aligned}\|x+iy\|_\infty &= \sup\{r(x\cos(\theta)+y\sin(\theta)) \mid \theta\in[0,2\pi)\} \\ &= \sup\{r(Re(e^{-i\theta}(x+iy))) \mid \theta\in[0,2\pi)\} \le r(x+iy).\end{aligned} \tag{3}$$

Now let $x, y \in X$ and $\sum_1^n \lambda_k \otimes x_k = x + iy$. Then

$$r(x+iy) = r(\sum_1^n \lambda_k \otimes x_k) \le \sum_1^n r(\lambda_k \otimes x_k) = \sum_1^n |\lambda_k| p(x_k).$$

Hence, since $\sum_1^n \lambda_k \otimes x_k$ is an arbitrary representation of $x + iy$, we have that $r(x + iy) \le \|x + iy\|_1$. Therefore

$$\|x+iy\|_\infty \le r(x+iy) \le \|x+iy\|_1,$$

as desired. ■

Theorem 28. *A seminorm on the complexification of a VSS E_p^X is admissible if and only if it is a tensor seminorm.*

Proof. We already have shown that tensor seminorms on the complexification are admissible. So it only remains to show that admissible seminorms are also tensor seminorms. To this end let r be an admissible seminorm on the complexification of E_p^X. Then the (A1) condition coupled with the fact that $r|_X = p$ implies that r is cross. Indeed, let $\lambda \in \mathbb{C}, x \in X$. Then

$$r(\lambda \otimes x) = |\lambda| r(x) = |\lambda| p(x),$$

which shows that r is cross.

As for (R), if $x, y \in X, \varphi \in B(\mathbb{R}^{\mathbb{C}}_{|.|}), \psi \in B(E^X_p)$, then

$$\begin{aligned}(\varphi \otimes \psi)(x+iy) &= (\varphi \otimes \psi)(1 \otimes x + i \otimes y) = (\varphi \otimes \psi)(1 \otimes x) + (\varphi \otimes \psi)(i \otimes y) \\ &= \varphi(1)\psi(x) + \varphi(i)\psi(y) \leq \|x+iy\|_\infty \leq r(x+iy),\end{aligned} \tag{4}$$

which is (C). So r is a tensor seminorm, completing the proof. ■

We have already shown that $\epsilon = \|.\|_\infty$ on the complexification of a VSS. The previous theorem gives us another proof of this result as well as the additional result that $\pi = \|.\|_1$ on the complexification. We state this as a corollary.

Corollary 29. *On the complexification of a VSS E^X_p we have that $\epsilon = \|.\|_\infty$ and $\pi = \|.\|_1$.*

Proof. By the previous theorem a vector seminorm on a complexification is admissible $\iff$ it is a tensor seminorm. Moreover we know any admissible seminorm, hence tensor seminorm, r satisfies $\|.\|_\infty \leq r \leq \|.\|_1$ where $\|.\|_\infty, \|.\|_1$ are admissible, hence tensor, seminorms. So $\|.\|_\infty, \|.\|_1$ are the smallest and largest, respectively, tensor seminorms. But we have already shown that ϵ and π are the smallest and largest, respectively, tensor seminorms. The stated equalities follow immediately. ■

V. AN APPLICATION TO THE COMPLEXIFICATION OF A RIESZ SPACE

It is a remarkable fact that there is a unique tensor (admissible) seminorm on the complexification of a Dedekind complete Riesz space. Hence, if E is a Dedekind complete Riesz space, we may unambiguously call $\mathbb{R}^{\mathbb{C}}_{|.|} \otimes_h E^{E}_{|.|}$ *the complexification* of the Riesz space E where h, called the complexification modulus, is any tensor seminorm. We devote the remainder of this section towards the proof of this fact.

Let E, F be Riesz spaces. To simplify the notation of what follows, when no confusion is likely to result, we shall often simply write E even when thinking of E in its capacity as the vector seminormed space $E^{E}_{|.|}$. Similarly we will often simply write $\mathbb{C}$ when thinking of $\mathbb{C}$ in its capacity as the normed space $\mathbb{R}^{\mathbb{C}}_{|.|}$. Hence, under this convention we would write $B(E), B(E,F)$ instead of $B(E^{E}_{|.|}), B(E^{E}_{|.|}, F^{F}_{|.|})$ for respective unit balls and $\mathbb{C} \otimes_t E$ instead of $\mathbb{R}^{\mathbb{C}}_{|.|} \otimes_t E^{E}_{|.|}$ for the complexification of E with tensor seminorm t.

We shall prove that all tensor seminorms on the complexification of a Dedekind complete Riesz space are equal by demonstrating that $\|.\|_\infty = \|.\|_1$. To do this we shall need the following version of Freudenthal's Spectral Theorem for the complexification of a Riesz space, as found in Theorem 36.1 of [9].

Theorem 30. *(Zaanen) Let E be a Dedekind σ-complete Riesz space and let $e \in E^+$ and $u = x + iy$ with $x, y \in E$ satisfy $\|u\|_\infty \leq e$. Then for any real number $\epsilon > 0$ there exist disjoint components $e_1, ..., e_n$ of e and complex numbers $\lambda_1, ...\lambda_n$ with $|\lambda_k| \leq 1$ such that*

$$\|u - \sum_1^n \lambda_k \otimes e_k\|_\infty \leq \epsilon e \quad \& \quad |\|u\|_\infty - \sum_1^n |\lambda_k| e_k| \leq \epsilon e.$$

Proof. See Zaanen ([9] (p.234-235)). ∎

Corollary 31. *Let E be a Dedekind complete Riesz space and let $u = x + iy$ with $x, y \in E$. Then for any real number $\epsilon > 0$ there exist disjoint components $e_1, ..., e_n$ of $\|u\|_\infty$ and complex numbers $\lambda_1, ...\lambda_n$ with $|\lambda_k| \leq 1$ such that $\sum_1^n |\lambda_k| e_k \leq \|u\|_\infty$ and*

$$\|u - \sum_1^n \lambda_k \otimes e_k\|_\infty \leq \epsilon \|u\|_\infty \quad \& \quad \|\|u\|_\infty - \sum_1^n |\lambda_k| e_k| \leq \epsilon \|u\|_\infty.$$

Proof. Dedekind complete Riesz spaces are Dedekind σ-complete. Moreover, we may take $e = \|u\|_\infty \in E^+$. Then the hypotheses of the previous theorem are met. Hence all we need to show is that $\sum_1^n |\lambda_k| e_k \leq \|u\|_\infty$. However, we have at most that $\sum_1^n e_k = \|u\|_\infty$, and we know that $|\lambda_k| \leq 1$. Hence

$$\sum_1^n |\lambda_k| e_k \leq \sum_1^n e_k \leq \|u\|_\infty,$$

which is the desired result. ∎

Lemma 32. $\|.\|_1 \leq 2\|.\|_\infty$ *on the complexification of a Dedekind complete Riesz space E.*

Proof. Let $x, y \in E$. Then because all admissible seminorms on $\mathbb{C} \otimes E$ are equal when restricted to E, $\|x\|_1 = \|x\|_\infty \leq \|x + iy\|_\infty$ and similarly $\|y\|_1 \leq \|x + iy\|_\infty$. So

$$\|x + iy\|_1 \leq \|x\|_1 + \|y\|_1 \leq 2\|x + iy\|_\infty.$$

Hence $\|.\|_1 \leq 2\|.\|_\infty$, as desired. ∎

We are now in a position to prove that all tensor seminorms are equal on the complexification of a Dedekind complete Riesz space.

Theorem 33. *There is a unique admissible seminorm (tensor seminorm) on the complexification of a Dedekind complete Riesz space.*

Proof. Let E be a Dedekind complete Riesz space. We shall demonstrate that $\|.\|_1 = \|.\|_\infty$, from which the theorem follows. To this end let $u = x + iy$ for $x, y \in E$ and let $\epsilon \in \mathbb{R}$ with $\epsilon > 0$. Then by the Corollary 31 there exist disjoint components $e_1, ..., e_n$ of $\|u\|_\infty$ and complex numbers $\lambda_1, ...\lambda_n$ with $|\lambda_k| \leq 1$ such that $\sum_1^n |\lambda_k| e_k \leq \|u\|_\infty$ and

$$\|u - \sum_1^n \lambda_k \otimes e_k\|_\infty \leq \frac{\epsilon}{2}\|u\|_\infty \quad \& \quad |\|u\|_\infty - \sum_1^n |\lambda_k| e_k| \leq \frac{\epsilon}{2}\|u\|_\infty.$$

Then since

$$\|u\|_1 = \inf\{\sum_1^n |\lambda_k||x_k| \mid \lambda_k \in \mathbb{C}, x_k \in E, x + iy = \sum_1^n \lambda_k \otimes x_k\},$$

$\sum_1^n |\lambda_k| e_k \leq \|u\|_\infty$ implies that

$$\|\sum_1^n \lambda_k \otimes e_k\|_1 \leq \|u\|_\infty.$$

Moreover by the previous lemma

$$\|u - \sum_1^n \lambda_k \otimes e_k\|_1 \leq 2\|u - \sum_1^n \lambda_k \otimes e_k\|_\infty \leq \epsilon\|u\|_\infty.$$

Hence, putting it all together, we have that

$$\|u\|_1 \leq \|u - \sum_1^n \lambda_k \otimes e_k\|_1 + \|\sum_1^n \lambda_k \otimes e_k\|_1 \leq \epsilon\|u\|_\infty + \|u\|_\infty.$$

Therefore

$$|\|u\|_1 - \|u\|_\infty| = \|u\|_1 - \|u\|_\infty \leq \epsilon\|u\|_\infty.$$

But since $\epsilon > 0$ was arbitrary and E is Archimedean since it is Dedekind complete, we must have

$$\|u\|_1 = \|u\|_\infty.$$

Hence $\|.\|_\infty = \|.\|_1$, as desired. ■

In light of this result, if E is a Dedekind complete Riesz space, for notational simplicity, we shall simply write $|.|$ for the unique admissible seminorm on $\mathbb{C} \otimes E$ and call it the **the modulus** of the complexification of E. The previous theorem then has the following useful corollary.

Corollary 34. *Let E be a Dedekind complete Riesz space and $u = x + iy$ where $x, y \in E$. Then we have that*

$$\begin{aligned}|u| &= \sup\{x\cos(\theta) + y\sin(\theta) \mid \theta \in [0, 2\pi)\}\\ &= \inf\{\sum_1^n |\lambda_k||x_k| \mid \lambda_k \in \mathbb{C}, x_k \in E, x + iy = \sum_1^n \lambda_k \otimes x_k\}\\ &= \sup\{\psi(1, x) + \psi(i, y) \mid \psi \in B(\mathbb{C}, E)\}.\end{aligned}$$

Proof. All are examples of formulae for tensor seminorms evaluated at u on the complexification of E and are thus equal by the previous theorem. ■

We summarize our results on the complexification with the following theorem, which is essentially another corollary of Theorem 33.

Theorem 35. *Let E be a Dedekind complete Riesz space. Let t be any tensor seminorm $\mathbb{C} \otimes E \to E$. Then the complexification of E is the t-complexification, $\mathbb{C} \otimes_t E$, of E. Moreover, $\mathbb{C} \otimes_t E$ is a vector normed space.*

Proof. The modulus is well known to satisfy $|u| = 0 \iff u = 0$ for $u \in \mathbb{C} \otimes E$ (see, for instance, Theorem 13.5 in [9]). Clearly the complexification of E with modulus $|.|$ is the $|.|$-complexification, $\mathbb{C} \otimes_{|.|} E$, of E considered as a VSS. But, by Theorem 33, $t = |.|$. The result follows. ■

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