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The Characterization Of Graphs With Small Bicycle Spectrum

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The Characterization of Graphs with Small Bicycle Spectrum

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Abstract

Matroids designs are defined to be matroids in which the hyperplanes all have the same size. The dual of a matroid design is a matroid with all circuits of the same size, called a dual matroid design. The connected bicircular dual matroid designs have been characterized previously. In addition, these results have been extended to connected bicircular matroids with circuits of two sizes in the case that the associated graph is a subdivision of a 3-connected graph.

In this dissertation, we will use a graph theoretic approach to discuss the characterizations of bicircular matroids with circuits of two and three sizes. We will characterize the associated graph of a bicircular matroid with circuits of two sizes. Moreover, we will provide a characterization of connected bicircular matroids with circuits of three sizes in the case that the associated graph is a subdivision of a 3-connected graph.

We will also investigate the circuit spectrum of bicircular matroids whose associated graphs have minimum degree at least $3k$, for $k \geq 1$, and show that there exists a set of bicycles with consecutive bicycle lengths.

Acknowledgements

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Contents

List of Figures

CHAPTER 1

Introduction

In 1969, U.S.R. Murty was the first to investigate matroids in which all of the hyperplanes have the same size. These matroids, called *equicardinal matroids* by Murty, were later called *matroid designs* by Young. Furthermore, Edmonds, Murty, and Young viewed such matroids in terms of their relationships to balanced incomplete block designs. The dual of a matroid with all hyperplanes having the same size is a matroid with all circuits having the same size, called a *dual matroid design*. In 1971, Murty characterized the connected binary dual matroid designs. The circuit-spectrum of a matroid M, denoted *spec(M)*, is the set of circuit sizes of a matroid. In 2010, Lemos, Reid and Wu characterized the connected binary matroids with a circuit spectrum of size two, where the largest circuit size is odd.

Also in 2010, Lewis extended the results of Murty, Lemos, Reid and Wu to the class of connected bicircular matroid designs and connected bicircular matroids where the associated graph of the matroid is a subdivision of a 3-connected graph and the matroid has circuits of two different sizes. The first result of this dissertation will focus on the class of connected bicircular matroids with circuits of at most three different sizes. The final results will describe the circuit spectrum of bicircular matroids in the case that the minimum degree of the associated graph of the matroid is $3k$ for $k \geq 2$. These matroids are, for the most part, non-binary.

In this dissertation, we will use graph theoretic techniques to investigate the circuit spectrum of bicircular matroids. In Chapter 1, we introduce the graph and matroid concepts related to the thesis results. In Chapter 2, we give some known results on matroids with small circuit-spectrum. In Chapter 3, we characterize the connected bicircular matroids where the associated graph of the matroid is a subdivision of a non-3-connected graph and the matroid has circuits of two different sizes. In Chapter 4, we characterize the connected bicircular matroids where the associated graph of the matroid is a subdivision of a 3-connected graph and the matroid has circuits of three different sizes. Finally, in Chapter 5, rather than assuming the size of the circuit spectrum of a bicircular matroid, we will suppose that the associated graph has some minimum degree and show that there exists a set of bicycles of consecutive lengths.

1. Matroid Concepts

Matroid theory is a generalization of graph theory and projective geometry. A matroid is a mathematical structure that was first introduced by Hassler Whitney in 1935 to abstractly capture the notion of dependence. Given below is the formal definition of a matroid.

DEFINITION 1.1. *A matroid M is an ordered pair $(E, \mathcal{I})$ consisting of a finite set E, called the ground set, and a collection $\mathcal{I}$ of subsets of E satisfying the following three conditions:*

(I1) $\emptyset \in \mathcal{I}$.

(I2) *If* $I \in \mathcal{I}$ *and* $I' \subseteq I$, *then* $I' \in \mathcal{I}$.

(I3) *If* I_1 *and* I_2 *are in* $\mathcal{I}$ *and* $|I_1| < |I_2|$, *then there is an element* e *of* $I_2 - I_1$ *such that* $I_1 \cup e \in \mathcal{I}$.

The elements of $\mathcal{I}$ are called the *independent sets* of the matroid M. Any subset of E that is not independent is called *dependent*. A *minimal dependent set* is a dependent set with all proper subsets being independent. A matroid M can also be defined by its minimal dependent sets, called *circuits*.

THEOREM 1.2. *A set of subsets $\mathcal{C}$ of a non-empty finite set E is the set of circuits of a matroid if and only if $\mathcal{C}$ satisfies the following three conditions.*

(C1) $\emptyset \notin \mathcal{C}$.

(C2) *If C_1 and C_2 are members of $\mathcal{C}$ and $C_1 \subseteq C_2$, then $C_1 = C_2$.*

(C3) *If C_1 and C_2 are distinct members of $\mathcal{C}$ and $e \in C_1 \cap C_2$, then there is a member C_3 of $\mathcal{C}$ such that $C_3 \subseteq (C_1 \cup C_2) - e$.* ***(Circuit Elimination Axiom)***

The collection of maximal independent sets of a matroid M is denoted $\mathcal{B}$. These also obey certain axioms.

THEOREM 1.3. *A set of subsets $\mathcal{B}$ of a non-empty finite set E is the set of bases of a matroid on E if and only if $\mathcal{B}$ satisfies the following three conditions.*

(B1) *$\mathcal{B}$ is non-empty.*

(B2) *If B_1 and B_2 are distinct members of $\mathcal{B}$ and $x \in B_1 - B_2$, then there is an element $y \in B_2 - B_1$ such that $(B_1 - x) \cup y \in \mathcal{B}$* ***(Basis Exchange Axiom)***

A maximal independent set of a matroid is called a *basis*, and all bases are equicardinal. In fact, given any set $X \subseteq E$, the maximal independent subsets of X are equicardinal, and this cardinality is called the *rank* of X, denoted $r(X)$. The rank of a matroid is $r(M) = r(E)$. The following theorem characterizes the rank function of a matroid.

Theorem 1.4. *Let E be a set. A function $r : 2^E \to Z^+ \cup \{0\}$ is the rank function of a matroid on E if and only if r satisfies the following conditions:*

(R1) *If $X \subseteq E$, then $0 \leq r(X) \leq |X|$.*

(R2) *If $X \subseteq Y \subseteq E$, then $r(X) \leq r(Y)$.*

(R3) *If X and Y are subsets of E, then $r(X \cup Y) + r(X \cap Y) \leq r(X) + r(Y)$.*

Also for any $X \subseteq E$, we say that the *closure* of X, denoted $cl(X)$, is the set of all $x \in E$ such that $r(X \cup x) = r(X)$. A set X is called a *flat* of a matroid M if $cl(X) = X$, and a flat of rank $r(M) - 1$ is called a *hyperplane*.

2. Classes of Matroids

We now discuss two common ways to possibly represent a matroid: matrices and graphs. Any matrix generates a matroid, as given in the following result [**14**].

Proposition 1.5. *Let E be the set of column labels of an m by n matrix A over a field F, and let I be the set of subsets X of E for which the multiset of columns labeled by X is linearly independent in the vector space $V(m, F)$ for some positive integers m and n. Then I satisfies axioms $(I1)$, $(I2)$, and $(I3)$ so that (E, I) is a matroid.*

The matroid M above is called the *vector matroid* of the matrix A. If M is the vector matroid of a matrix A over some field F, then M is said to be *representable over F*, or F-representable. A *binary matroid* is a matroid that is representable over $GF(2)$. Murty [**12**], as well as Lemos, Reid and Wu [**7**], studied the class of binary matroids with a circuit-spectrum of small cardinality.

In another well-known result, it is shown that any finite graph yields a matroid [**14**].

PROPOSITION 1.6. *Let E be the set of edges of a graph G and C be the set of edge sets of cycles of G. Then C is the set of circuits of a matroid on E.*

The matroid generated from G above is called the *cycle matroid* of G and is denoted by $M(G)$. A *graphic matroid* is a matroid that is the cycle matroid of some graph. The set of independent sets I of $M(G)$ is comprised of the edge sets of G that are acyclic. It has been shown that a graphic matroid is representable over every field [**14**].

We now give some special classes of matroids that will be mentioned in this dissertation.

Let r and n be non-negative integers such that $r \leq n$. Let E be an n-element set and $\mathcal{B}$ be the collection of r-element subsets of E, where $\mathcal{B}$ is the set of bases of a matroid on E. We denote this matroid by $U_{r,n}$ and call it the *uniform matroid* of rank r on an n-element set.

Let M be a matroid on E. Then the *dual matroid* of M is the matroid on E with bases $\{E - B : b \in \mathcal{B}(M)\}$. This dual matroid of M is denoted M^*. Hence ${U^*}_{r,n} \cong U_{n-r,n}$ for non-negative integers r and n with $0 \leq r \leq n$ and $n > 0$.

Let S be a finite set. Let $\mathcal{A} = (A_1, A_2, ..., A_m) = (A_j : j \in J)$, with $J = \{1, 2, ..., m\}$, be a family of subsets of S. A *system of distinct representatives* or a *transversal* of $\mathcal{A}$ is a subset $\{e_1, e_2, ..., e_m\}$ of S such that $e_i \in A_i$ for each $i \in J$. If $X \subseteq S$, then X is a *partial transversal* of $\mathcal{A}$ if for some subset K of J, X is a transveral of $\mathcal{A}$. The *transversal matroid* $M[\mathcal{A}]$ is the matroid with ground set S and independent sets being the partial transversals of $\mathcal{A}$. This class of matroids is especially important because bicircular matroids are transversal matroids, as shown by Matthews [**10**].

3. Matroid Connectivity

Let k be a positive integer. Then, for a matroid M, a partition (X, Y) of $E(M)$ is a k-separation if $\min\{|X|, |Y|\} \geq k$ and $r(X) + r(Y) - r(M) \leq k - 1$. Next let $\tau(M)$ =min$\{j$: M has a j-separation$\}$ if M has a j-separation for some $j \in \{2, 3, ...\}$, otherwise, let $\tau(M) = \infty$. For an integer $n \geq 2$, a matroid M is n-connected if and only if $\tau(M) \geq n$. The parameter $\tau(M)$ is called the *Tutte-connectivity* of M. If n is an integer exceeding one, then we say that M is n-connected if $\tau(M) \geq n$. Note that for a partition (X, Y) of $E(M)$, it can be shown that $r(X)+r(Y)-r(M) = r(X)+r(X^*)-|X| = r(X^*)+r(Y^*)-r(M^*)$. So (X, Y) is a k-separation of M if and only if it is a k-separation of M^* and $\tau(M) = \tau(M^*)$. In this dissertation, we study 2-connected matroids. A 2-connected matroid is often said to be connected; that is, a matroid is 1-connected if and only if a matroid is 2-connected. It can be shown that a matroid with at least two elements is connected if and only if each pair of distinct elements is contained in some circuit of the matroid [**14**].

4. Graph Concepts

The graph theory terminology used in this dissertation mostly follows West [**16**]. Pictures of the wheel graph W_4 with 4-spokes, the complete graph on five vertices K_5, and the complete bipartite graph $K_{3,p}$, $(p \geq 3)$, are given in Figure 1.1. We have labeled the edges of W_r, $(r \geq 3)$, by A_i and B_i for $i \in \{1, 2, ..., r\}$. The edges A_i are called the spokes of W_r and the edges B_i are called the rim edges of W_r.

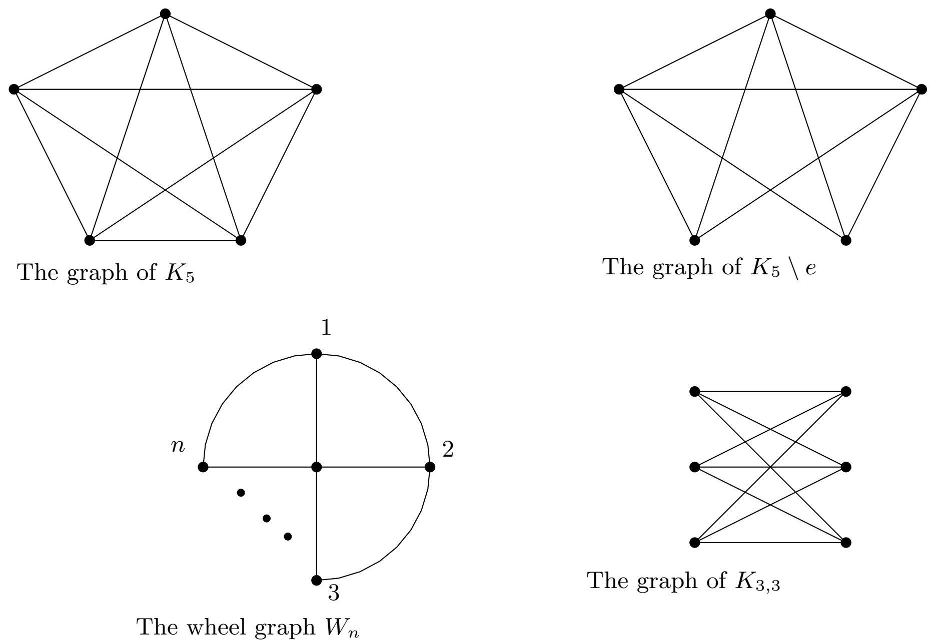


FIGURE 1.1. Some graphs without vertex-disjoint cycles

For a positive integer k, a k-subdivision of a graph is obtained by replacing each edge by a path of length k. Figure 1.2 shows a k-subdivision of the graph K_4 where $k = 3$.

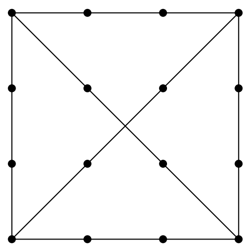


FIGURE 1.2. A 3-subdivision of the graph K_4

CHAPTER 2

Bicircular Matroids

This research focuses on the class of matroids called bicircular matroids. We discuss some definitions and basic properties of bicircular matroids in this chapter, as well as present some previous results from literature that motivate this dissertation.

1. Bicircular Matroid Concepts

We define a *bicycle* of a graph G to be a subgraph of G isomorphic to a subdivision of one of the following three graphs: (i) two loops that share a vertex, called a bowtie or a tight handcuff; (ii) two loops with distinct vertices that are joined by an edge, called a barbell or a loose handcuffs; (iii) three edges joining the same pair of vertices refered to as branch points, called a theta. Figure 2.1 shows subdividions of the three types of bicycles. For a graph G with edge set E, the bicircular matroid of G, denoted $B(G)$, has ground set E and circuits that are defined to be the edge sets of the bicycles of G. Moreover, if M is a bicircular matroid and G is a graph such that $M = B(G)$ then G is called a *representation of M.* A set of edges F is independent in the bicircular matroid $B(G)$ provided each connected component of $G[F]$, the subgraph of G induced by the edge set F, contains at most one cycle.

Here I present a result which will be useful in the subsequent chapters of this dissertation. Let G be a graph with edge set E. For a nonempty proper subset

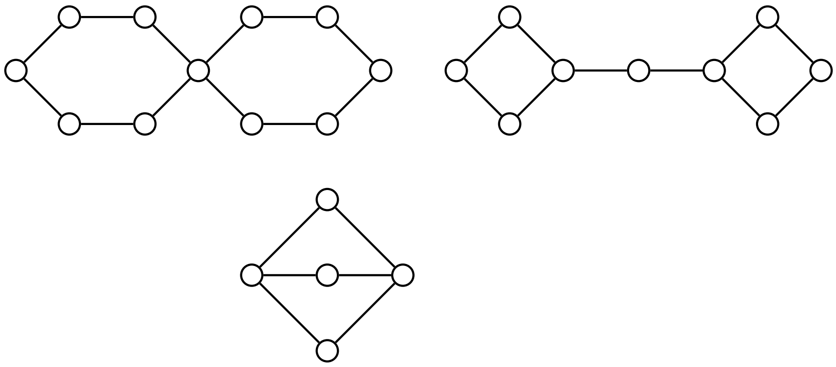


FIGURE 2.1. The types of bicycles

F of edge set E, the *vertex-boundary* of F consists of all vertices of G that are in both of the subgraphs induced by F and $E - F$. A *block* is a maximal connected subgraph without a cutvertex , and an *end-block* is a block whose vertex-boundary contains exactly one vertex. A *balloon* of a graph G is a subgraph which is a subdivision of one of the two graphs given in Figure 2.2. The vertex-boundary of a balloon is called the *tip* of the balloon. Note that in Figure 2.2, x and y are the tips of the given balloons.

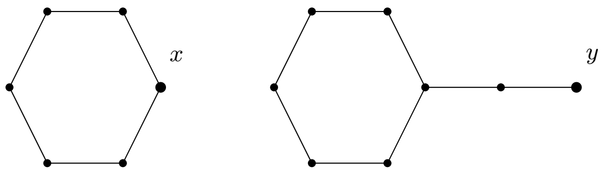


FIGURE 2.2. The types of balloons

We define a line L of a graph G to be a set of edges that forms a path in G such that all internal vertices of the path have degree two and the end-vertices have degree at least three. We require that the line not be contained in any balloon. Let x and y be the end-vertices of line L and e be the edge incident with y in L. By redefining the incidence relation of e so that e is adjacent to a vertex $u \neq y$ of

L, we obtain the graph H from G by *rolling* L away from y. Similarly, G is said to be obtained from H by *unrolling* L to y.

The following results relate rolling and unrolling to bicircular matroids.

LEMMA 2.1. [**14**] *Suppose that G and H are graphs with $B(H)$ connected and H is obtained from G by rolling a line L away from a vertex v. Then $B(G) = B(H)$ if and only if v is the tip of an end-block of G that contains L and every cycle of the end-block contains v.*

LEMMA 2.2. [**14**] *If H is a graph obtained from a graph G by replacing a balloon with another balloon on the same edge set and with the same vertex boundary, then $B(G) = B(H)$.*

2. Bicircular Matroids with Few Circuit Sizes

Here I present known results characterizing bicircular matroids with a circuit spectrum of size one. Some terminology is given first. Recall that the *circuit-spectrum* of a matroid is the set whose members are the cardinalities of its circuits. A *series class* of a matroid is a maximal subset of the ground set such that each pair of distinct elements of the subset are a cocircuit of the matroid. A *k-subdivision* of a matroid is obtained by replacing each element by a series class of size k. The notion generalizes such subdivisions from graphs to matroids.

THEOREM 2.3. [**9**] *Let M be a connected bicircular matroid. For $\eta \geq 2$, spec$(M) = \{\eta\}$ if and only if M is isomorphic to one of the following matroids:*

(i) *a k-subdivision of $U_{1,n}$ where $\eta = 2k$ and $n \geq 2$,*

(ii) *a k-subdivision of $U_{2,n}$ where $\eta = 3k$ and $n \geq 3$,*

(iii) *a k-subdivision of $U_{3,5}$ where $\eta = 4k$, or*

(iv) *a k-subdivision of* $U_{4,6}$ *where* $\eta = 5k$.

Lewis extended this result to the characterization of bicircular matroids $B(G)$ with circuits of two cardinalities given that the graph G is a subdivision of a 3-connected graph given below. Note that an (a, b)-*subdivision of a graph*, for distinct positive integers a and b, is obtained by subdividing each edge of G into a path of length a or b so that there is at least one path of each length, and that a balloon B is said to be β-subdivided if $|V(B)| = \beta$.

THEOREM 2.4. [**9**] *Let* $M = B(G)$ *be a connected bicircular matroid where* G *is a subdivision of a 3-connected graph* H*. Then* $|spec(M)| = 2$ *if and only if* H *is one of the folowing graphs.*

(1) *An* (a, b)*-subdivision of* W_3 *for distinct positive integers a and b.*

(2) *A k-subdivision of* W_4, $K_5 \setminus e$, K_5, $K_{3,3}$, $K_{4,4}$, *or the prism* P_6 *for some positive integer k. If* H *is isomorphic to* W_4, $K_5 \setminus e$, *or* K_5*, then* $spec(M) = \{5k, 6k\}$*. If* H *is isomorphic to* $K_{3,3}$, $K_{4,4}$, *or* P_6*, then* $spec(M) = \{6k, 7k\}$.

Next I present a tool to determine graphs with few bicycle sizes. Given a graph with a vertex set $V(G)$ and an edge set $E(G)$, we can define an associated graph of G such that the vertices of the associated graph of G correspond to the subdivided edges of G, and two vertices x and y in the associated graph are connected if they are the graph complement of a bicycle in G. The bicycle sizes will be assigned to corresponding edges in the associated graph. Using this assignment, we define the following:

DEFINITION 2.5. *Let* G *be a simple graph,* $\varphi : V(G) \to \mathbb{Z}^+$.

(1) *We say that* φ *is a j-vertex coloring of* G *if* $|\{\varphi(A) : A \in V(G)\}| = j$.

(2) *We say that φ is a j-edge coloring of G if* $|\{\varphi(A)+\varphi(B) : AB \in E(G)\}| = j$.

Suppose that A and B are distinct vertices of G. We refer to the number $\varphi(A)$ as the "color" of the vertex A, and to the number $\varphi(A)+\varphi(B)$ as the "color" of the edge AB. Typically, we will use $\varphi(A) = a$ for each vertex A, so that $a+b$ is the color of edge AB.

LEMMA 2.6. [**9**] *Let G be a connected graph with a two edge coloring φ.*

(1) *If U and V are vertices of G connected by an edge-monochromatic path of even length, the $\varphi(U) = \varphi(V)$.*

(2) *If a four-cycle is not edge monochromatic, then opposite edges of the four-cycle have different colors.*

(3) *If φ is a 2-vertex coloring, then one of the vertex color classes is an independent set of vertices.*

(4) *If a neighbor of a vertex V is adjacent to vertices of two different colors, then V has one of these colors.*

LEMMA 2.7. *Let $G \cong K_{m,n}$ with bipartition (U,V) and $m,n \geq 2$. If the edge-sum total coloring φ of G is an edge j-coloring with $j \in \{1,2\}$, then one of U or V is vertex monochromatic.*

CHAPTER 3

Bicircular Matroids with Circuits of Two Sizes

The first main theorem of the dissertation is given next. The following chapter characterizes the matroids whose associated graph is non-3-connected with bicycles of exactly two sizes.

1. Graph Terminology

We prove this result with a series of lemmas. Before discussing such lemmas, we will introduce some graph terminology.

Let G be a theta graph with branch points $\{u, v\}$ and branches P_1, P_2, and P_3, and let x be a vertex of P_1. Define $P_1' = P_1[u, x]$ and $P_1'' = P_1[x, v]$. If x is a branch vertex we will assume without loss of generality that $x = v$, and hence the length of $P_1{}''$ is zero. We refer to the collection of paths $\{P_1', P_1'', P_2, P_3\}$ as the subdivision of graph G *with respect to* x, denoted (G, x). An *ear attached to a graph* G is a non-trivial path that meets G exactly in its endpoints, also known as a *G-path.*

If $p_1' = p_2 = p_3 = \alpha$ and $p'' \in \{\alpha, 0\}$, we say that (G, x) is an α-subdivision, and that (G, x) is *balanced.* If $p_1' = \alpha$, $p_2 = p_3 = \beta$, and $p_1'' \in \{\beta, 0\}$, we say that (G, x) is an (α, β)-subdivision, and that (G, x) is *unbalanced.* Notice that if (G, x) is balanced and y is a vertex distinct from x, then either (G, y) is balanced such that x and y are the branch points of G or (G, y) is unbalanced. When thetas (G, x)

and (G, y) are both α-subdivisions, we say they are *equally balanced*. Note that, when the reference vertex is clear, we will say that G is balanced or unbalanced.

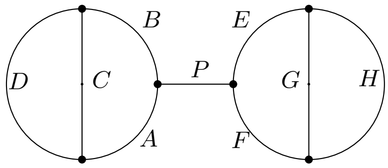


FIGURE 3.1. A theta barbell

Let G be a simple connected graph. We call G a *theta barbell* if G consists of the edge subdivisions of exactly two vertex-disjoint theta subgraphs, say H_1 and H_2, joined by a path. We label G as in Figure 3.1, and note that the paths E and B could have length zero; that is, one or both of the endpoints of the path could be on the branch point of a theta.

A graph is said to be a *bundle of thetas* if its edge set can be partitioned into disjoint thetas joined by paths that share a single common vertex, as seen in Figure 3.2.

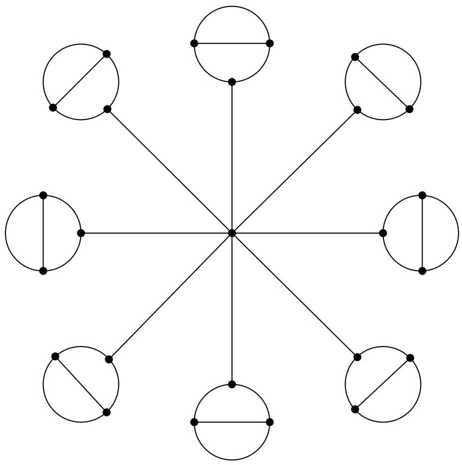


FIGURE 3.2. A bundle of thetas

2. Non-3-Connected Associated Graphs with $|spec\{B(G)\}| = 2$

In this section, we will characterize the matroid whose associated graph is non-3-connected with a circuit spectrum of cardinality two. To prove the main result of this chapter, we will study the feasible theta subgraphs of the associated graph G of a matroid. Lemmas 3.2 and 3.4 characterize the matroid whose associated graph contains either no theta subgraph or one theta subgraph.

In Lemmas 3.5-3.11, we consider the case that the associated graph contains two disjoint theta subgraphs. In Lemma 3.5, the edge-subdivisions of a theta barbell with bicycles of two sizes are given. With this knowledge, we are able to prove Lemma 3.7, which states that disjoint thetas are joined by a single path in G, with one possible exception in the case that both thetas are equally balanced. In this case, the thetas may be joined by at most two internally disjoint paths in G.

Now that we have determined that G has at most two theta subgraphs joined by at most two paths, we consider the addition of ears and balloons attached to any theta barbell in G. In Lemmas 3.8 and 3.9, we see that ears and loops may be attached to a theta barbell in G in only a handful of specific cases.

With the proof of Lemmas 3.8 and 3.9, we will have fully characterized all non-3-connected graphs with at most two disjoint theta subgraphs. To complete our characterization, we need only show that no ear can be added to a subdivision of a 3-connected graph in such a way that H loses 3-connectivity but retains exactly two bicycle sizes.

Using Lemmas 3.2-3.11, we can prove the following characterization.

THEOREM 3.1. *Let $M = B(G)$ be a connected bicircular matroid where G is not a subdivision of a 3-connected graph H. Then $|spec(M)| = 2$ if and only if H is a restricted subdivision of one of the following graphs:*

(1) *a cycle with at most three balloons,*
(2) *a theta with attached balloons,*
(3) *a bundle of thetas,*
(4) *two equally balanced thetas joined by two paths with the same endpoints,*
(5) *a theta barbell with $n \geq 0$ ears on the balanced theta,*
(6) *a theta barbell with a single balloon attached either at the conjoining subdivided edge or at the branch point of a balanced theta.*

3. Lemmas

In this section, we state and prove the lemmas discussed in the previous section of this chapter.

LEMMA 3.2. *Let G be a non-3-connected graph with bicycles of two sizes. If G has no theta subgraph, then G is an (α, β)-subdivided cycle with two balloons, an α-subdivided cycle with three balloons, or a $(2\alpha, \alpha)$-subdivided cycle with n balloons attached at distict tips.*

PROOF OF LEMMA 3.2. Suppose that C is a cycle with n balloons B_i, labeled clockwise as seen in Figure 3.3, with tips x_i for $i \in \{1, 2, ..., n\}$. Let the length of the (x_i, x_{i+1})-path be denoted c_{i+2} for $i \in \{1, 2, ..., n\}$ *mod* n.

First suppose that $C \cup (\cup_{i=1}^{n} B_i)$ is β-subdivided for some positive integer β, that is $b_i = \beta$ and $c_i = \beta$ for all i. Then the cycle and balloons have bicycle subgraphs formed by the deletion of $n - 1$ balloons or the deletion of $n - 2$ balloons and x

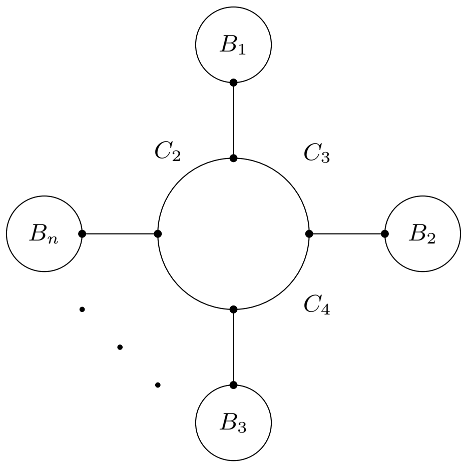


FIGURE 3.3. A cycle with n balloons

paths of length β on C where $x \in \{1, 2, ..., n-1\}$. Hence $C \cup (\cup_{i=1}^{n} B_i)$ has bicycle complements of size $\{(n-1)\beta, n\beta, (n+1)\beta, ..., (2n-3)\beta\}$. Therefore, if the cycle and balloons are β-subdivided with two bicycle sizes, then $n = 3$.

Now let $n = 2$, so that C has two balloons. Then the set of bicycle complements of $C \cup B_1 \cup B_2$ is $\{c_1, c_2, b_1, b_2\}$ and $|\{c_1, c_2, b_1, b_2\}| = 2$. Hence $C \cup B_1 \cup B_2$ is (α, β)-subdivided for some positive integers α and β.

Consider the case that $n = 3$ and the cycle subpaths and balloons have at least two lengths. Note that the set of bicycle complements of $C \cup B_1 \cup B_2 \cup B_3$ is $\{b_1 + b_2, b_2 + b_3, b_3 + b_1, b_1 + c_2 + c_3, c_1 + b_2 + c_3, c_1 + c_2 + b_3, b_1 + c_1, b_2 + c_2, b_3 + c_3\}$. Since $|\{b_1+b_2, b_2+b_3, b_3+b_1\}| \leq 2$, we have that either $b_i = \beta$ for all i or $b_1 = b_2 \neq b_3$ without loss of generality.

First suppose that $b_i = \beta$ for all i. Then $|\{\beta + c_1, \beta + c_2, \beta + c_3\}| \leq 2$. By the previous result for $n = 3$, we need only consider the case that $c_1 = c_2 \neq c_3$ without loss of generality. Then we see that $|\{\beta + c_1, \beta + c_3, \beta + c_1 + c_2 = \beta + 2c_1\}| \leq 2$. Hence $c_3 = 2c_1$, and the set of bicycle complements is $\{2\beta, \beta + 3c_1, \beta + 2c_1, \beta + c_1\}$; a contradiction.

Now suppose that $\beta = b_1 = b_2 \neq b_3 = \alpha$ without loss of generality. Then the set of bicycle complements is $\{2\beta, \alpha+\beta, \beta+c_2+c_3, \alpha+c_1+c_2, \beta+c_1, \beta+c_2, \alpha+c_3\}$. Note that $|\{\alpha+\beta, 2\beta, \beta+c_2+c_3, \beta+c_2\}| = 2$. Consider first the case that $\beta+c_2 = \alpha+\beta$ and $\beta+c_2+c_3 = 2\beta$. Then $\alpha = c_2$ and $\alpha+c_3 = \beta$. But then $|\{\alpha+\beta, 2\beta, \alpha+c_3 = \beta\}| = 3$; a contradiction. Similarly, if $\beta+c_2 = 2\beta$ and $\beta+c_2+c_3 = \alpha+\beta$, then $|\{\beta+c_1+c_3, \alpha+c_1+c_2 = \beta+c_1+c_2+c_3, \beta+c_1\}| = 3$; a contradiction. Therefore $b_1 = b_2 = b_3$.

Finally, we consider the case that $n \geq 4$. Note that any subgraph with a cycle and three balloons must be β-subdivided. Hence we see that $c_1 = c_2 = c_3 = c_4 = ... = c_n = c_1 + c_2 + ... + c_{n-2}$; a contradiction.

If a cycle C has two attached balloons B_1 and B_2 with the same tips, then $C \cup B_1 \cup B_2$ has bicycle complements $\{c, b_1, b_2\}$, and hence $C \cup B_1 \cup B_2$ is (α, β)-subdivided for distinct positive integers α and β.

If C has n attached balloons B_i all having the same tip, then by the previous argument $|\{c, b_1, b_2, ..., b_n\}| \leq 2$. Note that the cycle and balloons has bicycle complements $c + b_i$ and $b_i + b_j$ for distinct integers $i, j \in \{1, 2, ..., n\}$. Suppose that $\{c, b_1, b_2, ..., b_n\} \in \{\alpha, \beta\}$. Without loss of generality we see that at most one of $\{c, b_1, b_2, ..., b_n\}$ has length α, and hence the cycle with attached balloons has bicycle complements of size 2β and $\alpha + \beta$.

Let C be a cycle with n attached balloons B_i all having the same tip b and a balloon D attached at $v \in V(C)$, call this graph C^*. Recall that $\{c, b_i\} \in \{\alpha, \beta\}$ for all $i \in \{1, 2, ..., n\}$. Let $c = \alpha$ without loss of generality.

If $b_i = \beta$ for all i, then $|\{\alpha+d, 2\beta, d+x+\beta, d+y+\beta\}| = 2$ for $x + y = \alpha$ where x and y are the lengths of the two (b, v)-paths of C. Hence $d = \beta$, and $2\beta + x$,

$2\beta + y \in \{\alpha+\beta, 2\beta\}$. So $x = y = \frac{\alpha}{2}$ and $2x = \alpha = \beta + x$. Hence $\beta = x$ and C^* is $(2x, x)$-subdivided.

If $b_i = \beta$ for some i, say $i = 1$, and $b_j = \alpha$ for all $j \in \{2, 3, ..., n\}$, then $|\{\alpha + d, 2\alpha, \alpha+\beta, d+x+\alpha, d+y+\alpha, d+x+\beta, d+y+\beta\}| = 2$ for $x+y = \alpha$ where x and y are the lengths of the two (b, v)-paths of C. Hence $d =\in \{\alpha, \beta\}$, and $\alpha + \beta + x$, $\alpha+\beta+y \in \{\alpha+\beta, 2\alpha\}$. So $x = y = \frac{\alpha}{2}$ and $2x = \alpha = \beta + x$. Hence $\beta = x$ and C^* is $(2x, x)$-subdivided.

If C has n attached balloons B_i all having the same tip b and m attached balloons D_j having the same tip d for $i \in \{1, 2, ..., n\}$ and $j \in \{1, 2, ..., m\}$, then by the previous arguments, the (b, d)-paths on C have length α and $b_i = d_j = \alpha$ for all i and j. Then the cycle with attached $m + n$ balloons is $(2\alpha, \alpha)$-subdivided with bicycles of size 2α and 3α.

□

LEMMA 3.3. *Let G be a subdivision of a non-3-connected graph with bicycles of two sizes. If G contains one theta subgraph, but no two distinct theta subgraphs and a cycle disjoint from the theta, then G is isomorphic to one of the following, for some distinct positive integers α and β:*

- *An (α, β)-subdivision of a theta with an attached balloon,*
- *A graph consisting of $n \geq 2$, 2β-subdivided balloons attached to the branch points of a β-subdivided, 2β-subdivided, or $[2\beta, \beta]$-subdivided theta.*
- *A β-subdivision of a theta with one balloon attached to a non-branch point and at most one balloon attached at each branch point,*
- *A $(2\beta, \beta)$-subdivision of a theta with at most one balloon attached to each branch such that the ballons are 2β-subdivided and are attached either at*

the center of a 2β-subdivided edge or at a branch point of a β-subdivided theta. Moreover, there are at most two balloons attached to the theta.

PROOF. Let B be a balloon attached to a theta subgraph H of G. Then the deletion of B or the deletion of any one subdivided edge of H yields a bicycle in $H \cup B$. Hence $H \cup B$ is (α, β)-subdivided for some distinct positive integers α and β.

Suppose B_1 and B_2 are two balloons attached to the same branch point of H. Note that the deletion of a balloon and any subdivided edge of H or the deletion of H yields a bicycle in $H \cup B_1 \cup B_2$. If $H \cup B_1 \cup B_2$ is α-subdivided, then it has bicycle complements of size 2α and 3α. Suppose that $H \cup B_1 \cup B_2$ is (α, β)-subdivided. If $H \cup B_1$ is α-subdivided, then $b_2 = \beta$. Hence $H \cup B_1 \cup B_2$ has bicycle complements of size 2α, 3α and $\alpha + \beta$. Therefore $\beta = 2\alpha$.

Suppose instead that $H \cup B_i$ is (α, β)-subdivided for any $i \in \{1, 2\}$, and let $b_1 = \alpha$ without loss of generality. Then $H \cup B_1$ has bicycle complements of size $\alpha + \beta$ and $\alpha + b_2$. If $b_2 = \beta$, then H must be (α, β) subdivided or the edges of $H \cup B_i$ are equally subdivided for some $i \in \{1, 2\}$. Thus $H \cup B_1 \cup B_2$ must have bicycle complements of size $\alpha + \beta$, 2β, and 2α; hence $\alpha = \beta$; a contradiction. If $b_2 = \alpha = b_1$, then $H \cup B_1 \cup B_2$ has bicycle complements of size 2α and $\alpha + \beta$. Recall that the deletion of H yields a bicycle. Then H is a bicycle complement of size $2\alpha + \beta$, $\alpha + 2\beta$ or 3β. Note that $|\{\alpha + \beta, 2\alpha, 2\alpha + \beta\}| > 3$, so H is a bicycle complement of size $\alpha + 2\beta$ or 3β. If H has size $\alpha + 2\beta$, then $\alpha = 2\beta$ and $\{\alpha + 2\beta, \alpha + \beta, 2\alpha\} = \{4\beta, 5\beta\}$. If H has size 3β, then $\alpha = 2\beta$ and $\{3\beta, \alpha + \beta, 2\alpha\} = \{3\beta, 4\beta\}$. Hence $H \cup B_1 \cup B_2$ is $(2\beta, \beta)$-subdivided such that $b_1 = b_2 = 2\beta$ and H is either β-subdivided or $[\alpha, \beta]$-subdivided.

Let B_1, B_2,..., B_n be n balloons attached to a branch point of H. If $H' = H \cup \bigcup_{i=1}^{n} B_i$ is α subdivided, then H' has bicycle complements of size αn and $\alpha(n+1)$ from the deletion of either H and $n-2$ balloons, all n balloons, or $n-1$ balloons and a subdivided edge of H.

If H' is not α-subdivided, then by the previous result, H' is $(\beta, 2\beta)$-subdivided such that $b_i = 2\beta$ for all $i \in \{1, 2, ..., n\}$ and H is either β-subdivided or has at most one 2β-subdivided edge. If H is β-subdivided, then H' has bicycles of size 3β and 4β. If H is $(\beta, 2\beta)$-subdivided, then H' has bicycles of size 4β and 5β. Hence if there are n balloons attached to the branch point of a theta, then the balloons are equally subdivided with some α-subdivision and the theta is either equally subdivided or it is $[\alpha, \beta]$-subdivided (†).

Let B_1 and B_2 be two balloons attached to two different branch points of the theta subgraph H. Then the deletion of any two edges and/or balloons results in a bicycle, so $H \cup B_1 \cup B_2$ is $[\alpha, \beta]$-subdivided; that is, exactly one edge is α-subdivided (‡). Suppose now that there are n balloons, B_i for $i \in \{1, 2, ..., n\}$, attached to one branch point of H and exactly one balloon, B_{n+1} attached to the other branch point of H. Let $H' = H \cup \bigcup_{i=1}^{n+1} B_i$. By the previous result, we know that either $H \cup \bigcup_{i=1}^{n} B_i$ is α-subdivided, or $\bigcup_{i=1}^{n} B_i$ is 2α-subdivided and H is either α-subdivided or $[2\alpha, \alpha]$-subdivided. If $H' - B_{n-1}$ is α-subdivided, then H' has bicycles of size 2α, 3α, and $2\alpha + b_{n+1}$. Hence $b_{n+1} = \alpha$.

Suppose, on the other hand, that H' is $(2\alpha, \alpha)$-subdivided. If H is α-subdivided, then H' has bicycles of size $b_{n+1} + 2\alpha$, $b_{n+1} + 3\alpha$, 3α, and 4α. Hence $b_{n+1} = \alpha$. If H is $[2\alpha, \alpha]$-subdivided, then H' has bicycles of size $b_{n+1} + 2\alpha$, $b_{n+1} + 3\alpha$, $b_{n+1} + 4\alpha$; a contradiction.

Now add another balloon, B_{n+2}, attached at the tip of B_{n_1}. If B_i is 2α-subdivided for all $i \in \{1, 2, ..., n\}$ and all remaining edges of $H' - B_{n+2}$ are α-subdivided, then H' has bicycles of size $b_{n+1} + \alpha$, $b_{n+1} + 2\alpha$, $b_{n+1} + 3\alpha$; a contradiction. Now add $m - 1$ balloons, for some positive integer m, attached at the tip of B_{n+1}. By the previous results, all of these m balloons must be equally subdivided. Hence $b_i = \alpha$ for all $i \in \{1, 2, ..., n, n+1, n+2, ..., n+m\}$. Thus $H \cup \bigcup_{i=1}^{n+m} B_i$ has bicycles of size 2α and 3α.

We now consider the case that H has at least one balloon attached along a branch not at a branch point. If H has a balloon B_1 attached along a branch, then H has four subdivided edges, the deletion of any one of which results in a bicycle. Thus $H \cup B_1$ is (α, β)-subdivided. Let B_2 be a balloon attached along a branch of H not containing the tip of B_1.

We first consider the case that $H \cup B_1$ is β-subdivided. Then $H \cup B_2$ has edge subdivisions of length 2β, β, m, and n, where $m + n = \beta$; a contradiction. Hence, by symmetry, $m = \beta$ and $n = 0$; that is, B_2 is attached at a branch point of H. Then $H \cup B_1 \cup B_2$ has bicycle complements of size $b_1 + b_2$, $b_1 + 2\beta$, $b_1 + \beta$, 3β, and 2β. Hence $b_1 = \beta$ and $b_2 \in \{\beta, 2\beta\}$. Notice that by (†), H cannot have multiple balloons attached to the branch point at which B_2 is attached. By (‡), a balloon, B_3 may be attached to the opposite branch point of B_2 if $b_2 = b_3 = \beta$. Hence $H \cup B_1 \cup B_2 \cup B_3$ is β-subdivided and has bicycles of size 3β and 4β. Again, by (†), no additional balloons may be attached at either branch point.

Now consider the case that $H \cup B_1$ is (α, β)-subdivided, and let $\alpha > \beta$ without loss of generality. Then $H \cup B_2$ has edge subdivisions $c \in \{\alpha + \beta, 2\beta, 2\alpha\}$, m, n, and $d \in \{\alpha, \beta\}$, where $m + n \in \{\alpha, \beta\}$. Recall that $H \cup B_2$ has at most two distinct edge subdivisions, so if $m + n = \beta$, then $m, n < d < c$; a contradiction. Hence

$m + n = \alpha$. Then $m, n, \beta < \alpha$, so $m = n = \beta = \frac{\alpha}{2}$. Let $m = n = \beta = \gamma$ and $\alpha = 2\gamma$. Then $H \cup B_1$ is $(\gamma, 2\gamma)$-subdivided, $c \in \{2\gamma, 3\gamma, 4\gamma\}$, and $d \in \{2\gamma, \gamma\}$.

If $d = 2\gamma$, then $c \in \{2\gamma, 3\gamma\}$ as H is not equally subdivided with respect to B_1. Hence $c = 2\gamma$ as $H \cup B_2$ has edge subdivisions of length γ and 2γ. Hence $b_1, b_2 \in \{\gamma, 2\gamma\}$. Note that $H \cup B_1 \cup B_2$ has bicycle complements of size 2γ, 4γ, $b_i + 2\gamma$ for $i \in \{1, 2\}$. If $b_i = \gamma$, then $H \cup B_1 \cup B_2$ has bicycle complements of size 2γ, 3γ, and 4γ; a contradiction. Therefore the balloons and the branch not containing the tip of a balloon are 2γ-subdivided, and all other edges are γ-subdivided.

If $d = 2\gamma$, then $c \in \{2\gamma, 3\gamma, 4\gamma\}$. If $c = 4\gamma$, then $b_1 \in \{\gamma, 2\gamma\}$, $b_2 \in \{\gamma, 4\gamma\}$, and $H \cup B_1 \cup B_2$ has bicycle complements of size 4γ, 3γ, $b_1 + 4\gamma$, $b_2 + 2\gamma$, and $b_1 + b_2 \in \{2\gamma, 3\gamma, 5\gamma, 6\gamma\}$. Hence $b_1 = 2\gamma$ and $b_2 = \gamma$, so $b_1 + 4\gamma = 6\gamma$, $b_2 + 2\gamma = 3\gamma$; a contradiction.

If $c = 3\gamma$, then $b_1 \in \{\gamma, 2\gamma\}$, $b_2 \in \{\gamma, 3\gamma\}$, and $H \cup B_1 \cup B_2$ has bicycle complements of size 4γ, 3γ, and 2γ; a contradiction.

If $c = 2\gamma$, then $b_1 \in \{\gamma, 2\gamma\}$, $b_2 \in \{\gamma, 2\gamma\}$, and $H \cup B_1 \cup B_2$ has bicycle complements of size 3γ, 2γ, $b_1 + 2\gamma$, $b_2 + 2\gamma$, and $b_1 + b_2 \in \{2\gamma, 3\gamma, 4\gamma\}$. If $b_i = 2\gamma$ for any $i \in \{1, 2\}$, then $b_i + 2\gamma = 4\gamma$; a contradiction. Hence $b_i = \gamma$, and therefore $H \cup B_1 \cup B_2$ is γ-subdivided and has bicycle complements of size 2γ and 3γ.

Now consider adding a third balloon, B_3, to $H \cup B_1 \cup B_2$ where $H \cup B_1$ is (α, β)-subdivided. If B_1, B_2 and B_3 are attached to different branches of H, then by the previous result, either $H \cup B_1 \cup B_2$ is β-subdivided, or $H \cup B_1 \cup B_2$ is $(2\beta, \beta)$-subdivided such that B_1, B_2 and the branch containing the tip of B_3 are all 2β-subdivided. In the former case, B_3 is attached at the center of the β-subdivided branch. Then $H \cup B_1 \cup B_2 \cup B_3$ has bicycle complements of size 4β, 2β, and $4\beta + \frac{\beta}{2}$; a contradiction. In the latter case, B_3 is attached to the center of the

2β-subdivided branch. Then $H \cup B_1 \cup B_2 \cup B_3$ is $(2\beta, \beta)$-subdivided such that each of the balloons is 2β-subdivided, and all remaining edges are β-subdivided. Then $H \cup B_1 \cup B_2 \cup B_3$ has bicycles of size 5β, 6β, and 8β; a contradiction. Hence there can be balloons on at most two branches of H.

Now attach B_3 to the tip of B_i in $H \cup B_1 \cup B_2$ for some $i \in \{1, 2\}$. Note that $b_i = b_3$ by considering $H \cup B_3 \cup B_i$. By symmetry, let $i = 1$. If $H \cup B_1 \cup B_2$ is β-subdivided, we have that $H \cup B_1 \cup B_2 \cup B_3$ has bicycle complements of size 3β, 4β, and 6β; a contradiction. If $H \cup B_1 \cup B_2$ is $(2\beta, \beta)$-subdivided, we have that $H \cup B_1 \cup B_2 \cup B_3$ has bicycle complements of size 5β, 6β, and 8β; a contradiction.

By symmetry, attach B_3 to a branch point of $H \cup B_1 \cup B_2$. If $H \cup B_1 \cup B_2$ is β-subdivided, we have that $H \cup B_1 \cup B_2 \cup B_3$ has bicycle complements of size 3β, 4β, and 5β; a contradiction. If $H \cup B_1 \cup B_2$ is $(2\beta, \beta)$-subdivided, we have that $H \cup B_1 \cup B_2 \cup B_3$ has bicycle complements of size 5β, 6β, and 7β; a contradiction.

Hence balloons may be attached to at most two different branches, and in the case that there are balloons on two different branches, there is at most one balloon on each branch and no balloon on any branch points.

□

LEMMA 3.4. *Let G be a subdivision of a non-3-connected graph with bicycles of two sizes. If G contains a theta subgraph H and no cycle disjoint from H, then G is isomorphic to one of the following for distinct positive integers α and β:*

- *an (α)-subdividion of the graph Figure 3.5 (d),*
- *an (α, β)-subdivided theta with a single ear,*
- *an α-subdivision or an α, β-subdivision of the graphs Figure 3.5 (c), (e), (f), and (i),*

- *an (α, β)-subdivision of a collection of parallel edges, or*
- *an (α, β)-subdivision of a triangle with parallel edges.*

Proof of Lemma 3.4. Let G be a subdivision of a non-3-connected graph with bicycles of two sizes, and suppose that G has a theta subgraph H with an attached ear E in G. Label the subdivided edges of $H \cup E$ by A, B, C, D, E, and F, as seen in Figure 3.4, and let the corresponding lowercase letter represent the length of that subdivision. Note that the endpoint of E may be a branch point of H, so it is possible either $f = 0$, $e = 0$, or both $f = e = 0$. Then the set of bicycle complements of $H \cup E$ has cardinality $|\{a, b, c, d, e, f\}| = 2$. So $H \cup E$ is (α, β)-subdivided for some positive integers α and β.

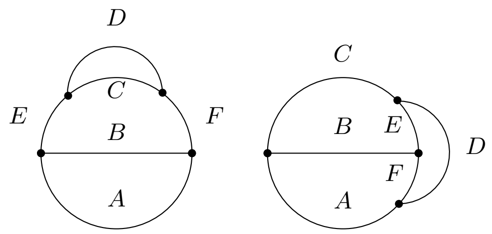


Figure 3.4. A theta graph with an attached ear

Now consider the case that a theta graph H in G has two ears. Note that the graph is isomorphic to a cycle with three chords. The possible graphs are given in Figure 3.5, and their auxiliary graphs are given in Figure 3.6.

We will consider the graphs $(a) - (f)$ first. Notice that, in each of these cases, if the auxiliary graph is 3-vertex-colored, then there are at least three edge colors; that is, there are at least three bicycle complement sizes in the orginal graph. So the auxiliary graphs are at most 2-vertex-colored, say with colors α and β. The potential edge colors of the auxiliary graphs are 2α, $\alpha + \beta$ and 2β. Therefore

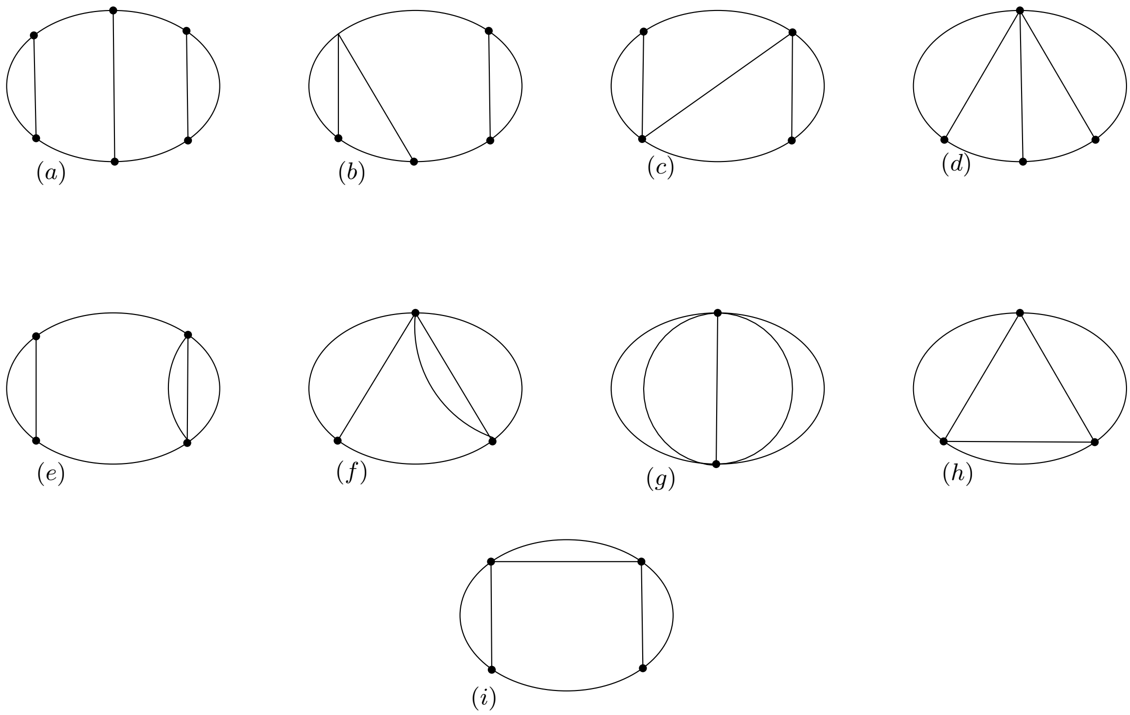


FIGURE 3.5. Cycles with three chords

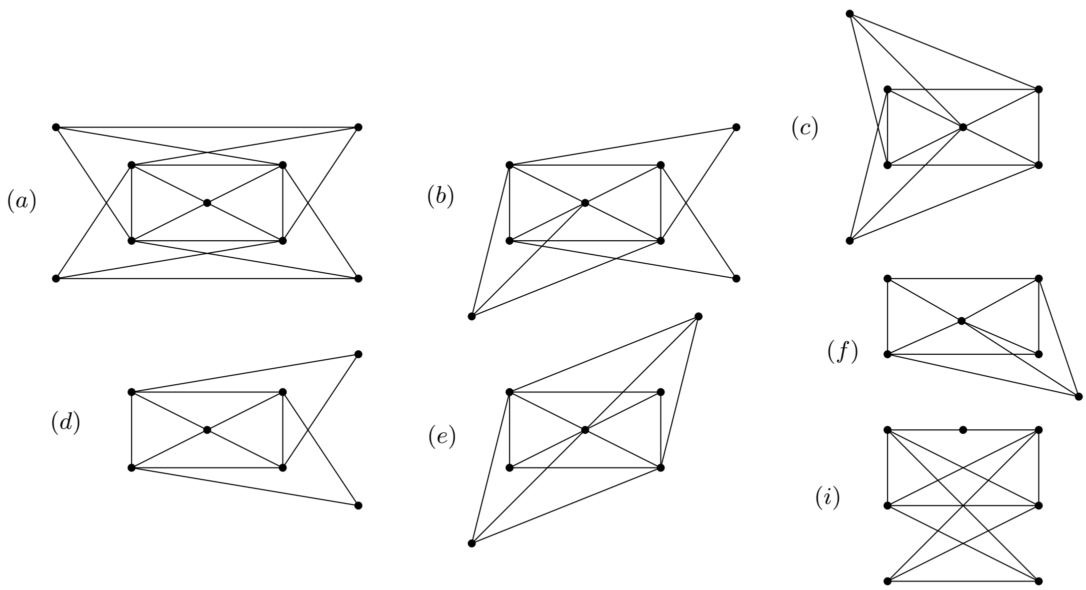


FIGURE 3.6. The auxiliary graphs of Figures 3.5 $(a) - (f)$

either $\alpha = \beta$ or all of the vertices of one color, say α, are an independent set in the auxiliary graph. In each of the graphs given in Figure 3.5 $(a) - (f)$, the maximal independent sets are exactly the sets corresponding to bicycle complements in the original graph containing more than two edges. The auxiliary graphs (a) and (b) have maximal independent sets of size three and four, the auxiliary graphs (c), (d), and (f) have maximal independent sets of size four, and the auxiliary graph (e) has maximal independent sets of size three. For the remainder of this proof, all referenced independent sets are maximal.

In graphs (a) and (b), note that if $\alpha = \beta$, then (a) and (b) have bicycle complements of size 2β, 3β, and 4β; a contradiction. Hence $\alpha \neq \beta$, and the auxiliary graphs have edge colors $\alpha + \beta$ and 2β. Note that the bicycle complements on three and four edges in the original graphs correspond to independent sets of size three and four in the auxiliary graphs of (a) and (b). Then the independent sets on three vertices that correspond to bicycle complements may be colored $\{3\alpha, 2\alpha+\beta, \alpha+2\beta, 3\beta\} \in \{\alpha+\beta, 2\beta\}$. Then $|\{\alpha+\beta, 2\beta, \alpha+2\beta\}| = 3$, so $\{3\alpha, 2\alpha+\beta, 3\beta\} \in \{\alpha+\beta, 2\beta\}$. Hence either $\alpha = 2\beta$ or $\beta = 2\alpha$. Now consider the potential sizes of the independent sets of size four in the auxiliary graphs (a) and (b). Then $\{4\alpha, 3\alpha + \beta, 2\alpha + 2\beta, \alpha + 3\beta, 4\beta\} \in \{\alpha + \beta, 2\beta\}$. Note that $|\{\alpha + \beta, 2\beta, 2\alpha + 2\beta, \alpha + 3\beta\}| \geq 3$, so $\{4\alpha, 3\alpha + \beta, 4\beta\} \in \{\alpha + \beta, 2\beta\}$. If $4\beta, 3\alpha + \beta \in \{\alpha + \beta, 2\beta\}$, then $\beta = 3\alpha$; a contradiction. Hence the bicycle complements on four edges in (a) and (b), all have size 4α. In the graph of (a), we see that all but the center chord is contained in a bicycle complement on four edges the the associated graph; hence at most one edge is β-subdivided in the graph of (a). Hence (a) has bicycle complements of size 4α, $2\alpha+\beta$, $\alpha+\beta$ and 2α; a contradiction. On the other hand, there is exactly one independent set of size four in the auxiliary graph (b); thus there are exactly four edges of (b) that are α-subdivided. Then (b) has bicycle complements of size 4α, 3β, $\alpha + \beta$, and 2β. Hence $\alpha = \beta$; a contradiction. Therefore the graphs given in Figure 3.5 (a) and (b) do not have bicycles of two sizes.

In the graph (e), if all edges are β-subdivided, then (e) has bicycle complements of size 2β and 4β. If not, then (e) has some bicycle complement on four edges with size 4α, $3\alpha+\beta$, $2\alpha+2\beta$, or $\alpha+3\beta$. Note that $|\{\alpha+\beta, 2\beta, 2\alpha+2\beta, \alpha+3\beta\}| \geq 3$. Then (e) has bicycle complements of size $\alpha+\beta$ and 2β. Hence $\{4\alpha, 3\alpha+\beta\} \in \{\alpha+\beta, 2\beta\}$. In either case, $\beta = 3\alpha$ and (e) has bicycle complements of size 4α and 6α. Therefore, if (e) has bicycle complements of two sizes, then either (e) is β-subdivided with bicycles of size 2β and 4β or (e) is (α, β)-subdivided with exactly three or four α-subdivided edges and bicycle complements of size 4α and 6α.

In the graphs (f), (c) and (d) of Figure 3.5, if all edges are β-subdivded, then the graphs have bicycle complements of size 2β and 3β. If not, then they each have some bicycle complement on three edges with size 3α, $2\alpha + \beta$, $\alpha + 2\beta$, or 3β. Notice that $|\{\alpha + \beta, 2\beta, \alpha + 2\beta\}| = 3$, so 3α, $2\alpha + \beta$, or $3\beta \in \{\alpha + \beta, 2\beta\}$. Hence either $\alpha = 2\beta$ or $\beta = 2\alpha$.

Note that in the graph (f), there is exactly one bicycle complement on three edges. Hence (f) has bicycle complements of size 3α, $\alpha + 2\beta$, or $2\alpha + \beta$. Note that $|\{\alpha + \beta, 2\beta, \alpha + 2\beta\}| > 2$, so the bicycle complement on three edges must have size 3α or $2\alpha + \beta$. Note that each of the three edges in the bicycle complement on three edges is also a bicycle complement on two edges with each of the remaining edges of (f), so none of those edges have size α. Hence (f) has exactly two or three α-subdivided edges. In either case, $\beta = 2\alpha$, and hence (f) is $(2\beta, \beta)$-subdivided with bicycle complements of size 3α and 4α.

In the graphs (c) and (i), there are exactly two bicycle complements on three edges. Note that any one edge contained in a set of three edges that form a bicycle complement in the graph is also contained a bicycle complement on two edges with any of the remaining edges of G. Hence one of the bicycle complements on three edges must have size 3β. Then we have either $|\{\alpha+\beta, 2\beta, 3\beta, \alpha+2\beta\}| = 2$ or $|\{\alpha+\beta, 2\beta, 3\beta, 3\alpha\}| = 2$. Hence $\alpha = \beta$.

Hence if the graphs of (f), (c), (d), and (i) have bicycles of two sizes, then either they are all β-subdivided with bicycle complements of size 2β and 3β, or (f) is (α, β)-subdivided with exactly two or three α-subdivided edges and bicycle complements of size 3α and 4α.

In the graph Figure 3.5 (g), if the theta has n branches, the deletion of any $n-3$ branches results in a bicycle. Hence (g) is $[\alpha, \beta]$-subdivided; that is, exactly one edge is α-subdivided.

In the graph Figure 3.5 (h), the deletion of any two edges results in a bicycle. Hence (h) is also $[\alpha, \beta]$-subdivided; that is, exactly one edge is α-subdivided. Label the vertices of Figure 3.5 (h) as x, y and z. Without loss of generality, add $n-2$ parallel xy-edges. From the subgraphs isomorphic to the graph of (h), we see that there can be at most one α-subdivided edge in the new graph.

The bicycles have size $\{3\beta, 4\beta, 3\beta+\alpha\}$. Hence $\beta = \alpha$ and the graph has bicycles of size 3β and 4β. Therefore, if any one side of the 3-cycle has more than three parallel edges, then (h) is β-subdivided.

Add $m-2$ parallel yz-edges to the graph, so that one side of the 3-cycle has n parallel edges, one side has m parallel edges, and the remaining side has 2 parallel edges for some $m, n \geq 3$. Then by the previous result, all of the edges must be β-subdivided. Then the bicycles have size $\{3\beta, 4\beta\}$. Finally, add $p-2$ multiple

xz-edges, so the each side of the 3-cycle has more than three parallel edges. By the previous result, the graph is β-subdivided, and the bicycle have size $\{3\beta, 4\beta\}$.

Let H be isomorphic to one of $(c) - (f)$ or (i). Now consider adding a third ear E_3 to the graph H, or equivalently, adding a fourth chord to the cycle. If the graph $H \cup E_3$ has bicycles of two sizes,then it contains no (a) or (b) subgraph. Note that adding any chord to the graphs (c), (e), or (i) results in some subgraph isomorphic to the graphs (a) or (b). Hence we consider adding a chord to the graphs (d) and (f) in such a way that no subgraph is isomorphic to the graphs (a) or (b).

Note that either the deletion of any chord results in a graph isomorphic to the graph (d) or the deletion of any set of parallel edges results in a subgraph isomorphic to the the graph in Figure 3.5 (a) or (i). In the former case, every edge must be β-subdivided. But the graph has bicycle complements of size 5β, 4β and 3β; a contradiction. In the latter case, every edge is β-subdivided. Hence the graphs have bicycle complements of size 5β, 4β and 3β; a contradiction.

□

Now we consider the case that the associated graph G contains at least two disjoint theta subgraphs. Note that G is connected, so the thetas form a theta barbell with some path in G. In the following lemma, we describe the possible subdivisions of a theta barbell subgraph of G given that G has bicycles of exactly two sizes.

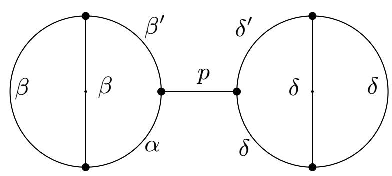


FIGURE 3.7. The edge subdivisions of an unbalanced theta barbell

Lemma 3.5. *Let G be a graph with bicycles of exactly two sizes. Suppose G contains two vertex-disjoint theta subgraphs H_1 and H_2, and some (H_1, H_2) path P with endpoints $x_1 \in V(H_1)$ and $x_2 \in V(H_2)$. Then for some $i \in \{1, 2\}$ and some positive integer δ, (H_i, x_i) is δ-subdivided. Moreover, if we assume $i = 2$, then one of the following occurs:*

- *(H_1, x_1) is δ-subdivided, and either both x_i are branch vertices or both are non-branch vertices.*
- *(H_1, x_1) is α-subdivided with $\alpha = 2\delta + \delta' + p$, with $\delta' = 0$ if x_2 is a branch vertex and $\delta' = \delta$ otherwise.*
- *(H_1, x_1) is (α, β)-subdivided with $\alpha = 2\delta + \delta' + p$, $\delta' = 0$ if x_2 is a branch vertex and $\delta' = \delta$ otherwise. Moreover, $\delta = 2\beta + \beta' + p$ where $\beta' = 0$ if x_1 is a branch point, and $\beta' = \beta$ otherwise.*

Proof of Lemma 3.5. Let $G = H_1 \cup P \cup H_2$ and label G as in Figure 3.1, where the paths E and B may have length zero.

We see that for any $X \in \{A, B, C, D\}$ and $Y \in \{E, F, G, H\}$ with $x, y > 0$, the subgraph $X \cup Y$ is the complement of a barbell in G. Let $Aux(G)$ be the graph with vertex set consisting of the non-trivial elements of $\{A, B, C, D, E, F, G, H\}$, and an edge XY for each such bicycle complement. Thus $Aux(G) \cong K_{m,n}$ for some $m, n \in \{3, 4\}$.

The edge-sum total coloring φ of $Aux(G)$ with $\varphi(V) = v$ yields an edge j-coloring of $Aux(G)$ for $j \in \{1, 2\}$. Fix $\delta = f$. From Lemma 2.7 and symmetry we may assume that $g = h = \delta$ and $e \in \{0, \delta\}$. So $[H_2, x_2]$ is δ-subdivided. From Lemma 2.5 we note that $|\{a, b, c, d\} - \{0\}| = j \leq 2$. Fix $\alpha = a$. Let Z be the set of sizes of the bicycles of G and assume $|Z| = 2$.

First suppose that $j = 1$. Then $Aux(G)$ is edge-monochromatic, $c = d = \alpha$ and $b \in \{0, \alpha\}$. So $[H_1, x_1]$ is α-subdivided. If $\alpha = \delta$, then $Z = \{3\alpha+b, 3\alpha+e, 4\alpha+b+e+p\}$, which forces $b = e$. So 3.5 is satisfied. If $\alpha \neq \delta$, we assume that $\alpha > \delta$ without loss of generality. Now $Z = \{3\alpha+b, 3\delta+e, 2\alpha+b+p+2\delta+e\}$ so $2\alpha+b+p+2\delta+e = 3\alpha+b$. Hence $\alpha = 2\delta + e + p$ and 3.5 is satisfied.

Now suppose that $j = 2$. Then $a, b, c, d \in \{\alpha, \beta\}$ for $\alpha \neq \beta$. Thus G has bicycle complements of size $\alpha + \delta$ and $\beta + \delta$ from the deletion of a subdivided edge from each theta. Note that $P \cup H_i$ is a bicycle complement of G for each $i \in \{1, 2\}$. Then $3\delta+\delta'+p \in \{\alpha+\delta, \beta+\delta\}$. Hence $2\delta+\delta'+p \in \{\alpha, \beta\}$. By symmetry, say that $\alpha = 2\delta+\delta'+p$. Then $a+b+c+d+p \in \{3\delta+\delta'+p, \beta+\delta\}$. But $\alpha \in \{a, b, c, d\}$, so $a+b+c+d+p > \alpha+\beta > \delta+\beta$. Hence $a+b+c+d = 3\delta+\delta'$; that is, the thetas of G have the same total size. Then $a+b+c+d+p-\alpha = 3\delta+\delta'+p-\alpha = \delta$. Therefore one subdivided edge of H_2 is α-subdivided and the remaining edge-subdivisions of $H_2 \cup P$ sum to δ. Hence $\delta = 2\beta + \beta' + p$.

□

LEMMA 3.6. *Let G be a connected graph with bicycles of two sizes containing a theta barbell subgraph with thetas H_i for $i \in \{1, 2\}$. Let x and y be distinct vertices of theta H_1 such that x and y are endpoints of path P_x and P_y, respectively, joining thetas H_1 and H_2. Then both of (H_1, x) and (H_1, y) are unbalanced with some (α, β)-subdivision.*

PROOF OF LEMMA 3.6. By Lemma 3.5, at least one of H_1 and H_2 must be balanced with respect to the end-vertices of P_x and P_y. Let H_2 be balanced with some γ-subdivision. Note that if (H_1, x) is unbalanced with an (α, β)-subdivided, then by Lemma [?], $\alpha \geq 4\beta + 2\beta' + 2i$, where i is the length of P_x.

We consider two cases: either (H_1, x) is balanced with some δ-subdivision or (H_1, x) is unbalanced.

Suppose that (H_1, x) is balanced with some δ-subdivision. If x and y are on different branches of H_1 and neither x nor y is a branch point of H_1, then (H_1, y) is unbalanced with some $(m, n, \delta, 2\delta)$-subdivision for $m, n \in \mathbb{Z}$ such that $m+n = \delta$; a contradiction. Hence x and y are on the same branch of H_1. Hence (H_1, y) is unbalanced with a $(\delta, \delta + n, m)$ subdivision; a contradiction.

Suppose that x is not a branch point of H_1 and y is a branch point. Then (H_1, y) is $(2\delta, \delta)$-subdivided; a contradiction by Lemma [**?**]. Now suppose that x is a branch point. If y is not a branch point, then (H_1, y) is (m, n, δ)-subdivided for some positive integers m and n such that $m+n = \delta$; a contradiction by Lemma [**?**]. Hence y is a branch point. Then since both H_1 and H_2 are balanced, both end points of P_x and P_y coincide with branch points on H_i for $i \in \{1, 2\}$. Hence $p_x = p_y = p$ by Lemma [**?**]. Then $|spec(M)| = |\{2p + 2\delta, 2p + \delta + 2\gamma, 3\delta, 3\gamma, 2\gamma + p + 2\delta\}| = 2$. So either $\delta = 2\gamma + p$ or $\gamma = 2\delta + p$. Note that $2p + \delta + 2\gamma \in \{3\delta, 3\gamma\}$. If $\delta > \gamma$, then $\delta = p + \gamma = p + 2\gamma$; a contradiction. If $\gamma > \delta$, then $\gamma = 2p + \delta = p + 2\delta$, so $p = \delta$. Then $\gamma = 3\delta$ and hence $|spec(M)| = |\{3\delta, 4\delta, 9\delta\}| > 2$; a contradiction.

Hence (H_1, y) is unbalanced with some (α, β)-subdivision such that $\alpha = 2\gamma + \gamma' + i$ and $\gamma = 2\beta + \beta' + i$. If y is on a branch of length β on (H_1, x), then (H_1, y) is (r, s, α, β)-subdivided where $r + s = \beta$; a contradiction. If y is on a branch of length α on (H_1, x), then (H_1, y) has branch lengths β, $\alpha - t$, and $\beta + t$ where $\beta + t = \alpha$, so $t = \alpha - \beta$. Thus (H_1, y) is (α, β)-subdivided.

□

LEMMA 3.7. *Let G be a connected graph with two disjoint theta subgraphs and bicycles of exactly two sizes. If the thetas are equally balanced with some δ-subdivision,* *then the thetas are joined by at most two paths of equal length* p *where* $2p = \delta$ *or* $p = 3\delta$.

Proof of Lemma 3.7. Let G be a connected graph with two distinct theta subgraphs H and K. Suppose that H and K are joined by at least two paths, say P_i for $i \geq 2$. Then let h_i be the endpoint of P_i on H and similarly, let k_i be the end point of P_i on K. Consider the paths P_1 and P_2.

Suppose that (H, h_1) is balanced with some δ-subdivision. Then h_1 and h_2 are the same vertex, call it h. By Lemma 3.6, if (K, k_1) is (α, β)-subdivided, then (K, k_2) is (α, β)-subdivided. Then by Lemma 3.5, $\delta = 2\beta + \beta' + p_1 = 2\beta + \beta' + p_2$, say that $p_i = p$ for both $i \in \{1, 2\}$. The two paths together with a cycle from K forms a bicycle of size $2p + \alpha + 2\beta$ and the two paths together with a cycle from H forms a bicycle of size $2p + \alpha - \beta + 2\delta + \delta'$. Note that $2p + \alpha + 2\beta < 2p + \alpha - \beta + 2\delta + \delta'$ as $\delta = 2\beta + \beta' + p$. Then $2p + \alpha - \beta + 2\delta + \delta' = \alpha + \beta + p + 2\delta + \delta'$ and $2p + \alpha + 2\beta = \alpha + 2\beta + \beta'$. Hence $p = 2\beta$ and $2p = \beta'$; a contradiction.

Therefore the endpoints of P_1 and P_2 must be the same, call the vertex k, and (K, k) is unbalanced with an (α, β)-subdivision. So G has bicycles $\alpha + 2\beta + 2p$, $3\beta + 2p$, and $3\delta + 2p$; a contradiction.

Hence both (K, k_1) and (H, h_1) must be balanced, say with an α-subdivision and a β-subdivision, respectively. Then by Lemma 3.5, k_1 and h_1 are the same vertex. Clearly, by Lemma 3.5, $p_1 = p_2 = p$. Hence G has bicycle sizes $2\alpha + \alpha' + 2p$, $2\beta + \beta' + 2p$, and $2\beta + \beta' + 2p + 2\alpha + \alpha'$. Note that if $\alpha > \beta$, then G has bicycles of three sizes. Suppose that $\alpha = \beta$. Then $2p = \alpha = \beta$ or $p = 3\alpha = 3\beta$.

□

Lemma 3.8. *Let G be a connected graph with circuits of two sizes and let H be a theta barbell subgraph of G with n ears, for n a positive integer. If the conjoining path of H is attached to a branch point of the balanced theta with some δ-subdivision, then each of the ears has length δ and is attached to the opposite branch point of the balanced theta of H.*

Proof of Lemma 3.8. Let H_1 and H_2 be the theta subgraphs of H. Suppose that H_2 is δ-subdivided and that E is an ear on H_2 of length e. Note that H_1 may be balanced with some β-subdivision or unbalanced with some (α, β)-subdivision. Suppose first that both endpoints of E are on the same path P_1 of length δ in H_2 and that neither endpoint coincides with a branch point of the theta. Then by Lemma 3.5, all of the paths have length δ. Then P_1 also has length δ. This implies that the endpoints of the ear must be branch points. If one of the endpoints of the ear coincides with the endpoint of the conjoining path of H_1 and H_2, then G has bicycles of size $c+2\beta+\beta'$, $3\delta+\delta'$, $2\delta+p+c+\beta+\beta'$ and $2\delta+\delta'+p+c+\beta+\beta'$ where $c \in \{\alpha, \beta\}$. Hence $\delta' = 0$. If neither of the endpoints of the ear coincide with the conjoining path, then G has bicycles of size 3δ and $3\delta+\delta'$. Hence $\delta' = 0$. Therefore, E is attached at the branch points of a balanced theta where the conjoining path attaches at a branch point. Suppose there are n such ears attached to H_2. Then H_2 has bicycle sizes 3δ and $2\delta + p + c + \beta + \beta'$ where $c \in \{\alpha, \beta\}$. Therefore H_2 has $n + 3$ branches of length δ.

Suppose now that E has endpoints on two different paths of the balanced theta H_2. Consider the bicycle formed by E and the cycle of H_2 containing both paths of H_2 to which E is attached. This bicycle has size $2\delta + \delta' + e$. Consider also the bicycle formed by E and a cycle containing at most one path to which E is attached. This bicycle has size $e+x+y+z+2\delta+\delta'$ where x and y are the lengths of subpaths that form a cycle with E and z is the length of the path joining the two cycles. Note that possibly $z = 0$. Then $2\delta+\delta'+e < e+x+y+z+2\delta+\delta'$, so $2\delta+\delta'+e = 3\delta+\delta'$. Hence $e = \delta$. But then $e+x+y+z+2\delta+\delta' = 3\delta+\delta'+x+y+z = 2\delta+\delta'+p+\text{\c{+}}\beta+\beta'$. So $\delta + x + y + z = p + c + \beta + \beta'$; a contradiction.

Now consider the case that there is an ear E of size e on an unbalanced theta, say H_2 without loss of generality. Recall that G has bicycle sizes $\alpha + 2\beta + \beta' = 3\delta + \delta' < \alpha + \beta + \beta' + p + 2\delta + \delta' = 2\alpha + \beta + \beta'$ by Lemma 3.5. Note that if the ear has both endpoints on the same path not attached at a branch point, then by Lemma 3.9, all paths of H_2 have the same length; a contradiction. Hence either at least one endpoint is attached at a branch point of H_2, or the endpoints are on two different paths.

Suppose the endpoints of E are on two different paths. Consider the bicycle formed by the ear and the cycle of H_2 containing both of the paths to which the ear is attached. Then G has a bicycle of size $c + \beta + \beta' + e$ where $c \in \{\alpha, \beta\}$. So $e \in \{\alpha, \beta\}$ as the bicycles of G have size $\alpha+\beta+\beta'+c$. Then consider the bicycles formed by the ear and a cycle of H_2 containing at most one path to which the ear is attached. Then G has a bicycle of size $c+\beta+\beta'+e+x$ where x is the length of the shortest subpath from the brachpoint to the endpoint of the ear. Then $x = 0$; a contradiction.

Now suppose that the ear has at least one endpoint on a branch point. Consider the bicycle formed by the ear and the cycle containing the path to which the ear is attached. Then G has a bicycle of size $c + \beta + \beta' + e$ where $c \in \{\alpha, \beta\}$. Hence $e \in \{\alpha, \beta\}$. Now consider the bicycle formed by the ear and a cycle containing no paths to which the ear is attached. Then G has a bicycle of size $c+\beta+\beta'+e+x$

where x is the length of the shortest subpath from the brachpoint to the endpoint of the ear. Then $x = 0$; a contradiction.

Therefore there is no ear on the unbalanced theta.

□

LEMMA 3.9. *Let G be a connected graph with bicycles of two sizes. If G contains a theta barbell subgraph H with an attached balloon, then one of the following occurs:*

- *the conjoining edge in H attaches at the branch point of a δ-subdivided balanced theta and the balloon is attached on the other branch point of the balanced theta,*
- *the thetas of H are equally balanced and the balloon is attached at the center of the subdivided conjoining edge of H.*

PROOF OF LEMMA 3.9. Let H_1 and H_2 be the theta subgraphs of G. Recall that at least one theta must be balanced, so we let H_2 be balanced with some δ-subdivision. Then say that H_1 is either β-subdivided or (α, β)-subdivided. Let $P = [x_1, x_2]$ be the conjoining path of H_1 and H_2 in G such that vertex x_i lies on theta H_i for $i \in \{1, 2\}$. Now consider some balloon B in G with tip v. Then B may be attached to H on H_1, H_2 or P.

We will first consider the case that B is attached to H_1 and H_1 is unbalanced with some (α, β)-subdivision. Then spec(M)= $\{2\alpha+\beta+\beta', \alpha+2\beta+\beta'\}$, $\alpha = 2\delta+\delta'+p$, and $\delta = 2\beta + \beta' + p$. Let P_1 be the longest v, x_1-path in H_1, and let P_2 be the smallest such path. Then $0 \leq p_2 \leq \frac{\alpha+\beta+\beta'}{2} \leq p_1 \leq \alpha + \beta + \beta'$. Hence the barbell formed by the balloon and a cycle from H_2 have size $b + p_1 + \alpha, b + p_2 + \alpha \in$ *$spec(M)$*. So $b+p_i \in \{\alpha+\beta+\beta', 2\beta+\beta'\}$ for $i \in \{1, 2\}$. If $p_1 = p_2 = \frac{\alpha+\beta+\beta'}{2}$, then $b+p_1+\alpha = b+\frac{3}{2}\alpha+\frac{\beta}{2}+\frac{\beta'}{2} \in$*$spec(M)$*. Hence $b \in \{\frac{1}{2}(\alpha+\beta+\beta'), \frac{1}{2}(-\alpha+3\beta+\beta')\}$.

Then the bowtie formed by B and the largest cycle of H_1 has size $b + \alpha + \beta + \beta' \notin$spec(M) . So $p_1 \neq p_2$. Thus $\alpha + \beta + \beta' = b + p_1 > b + p_2 = 2\beta + \beta'$. So $b = \alpha + \beta + \beta' - p_1 = 2\beta + \beta' - p_2$, and hence $p_1 = \alpha - \beta + p_2$ and $p_2 = \beta - \alpha + p_1$. Then we see that the bowtie formed by B and the largest cycle of H_1 has size $b + \alpha + \beta + \beta' \in \{2\alpha + 2\beta + 2\beta' - p_1, \alpha + 3\beta + 2\beta' - p_2\}$. So $p_1 = \beta + \beta' = p_2$; a contradiction. Hence no balloon may be attached to an unbalanced theta.

Now consider the case that B is attached to H_2. Define P_1 and P_2 as previously so that $0 \leq p_2 \leq \delta + \frac{\delta'}{2} \leq p_1 \leq 2\delta + \delta'$. Note that H_1 may be balanced or unbalanced, and hence $\delta = \beta$, $\delta = 2\beta + \beta' + p$ or $\beta = 2\delta + \delta' + p$. In either case, H contains some barbell formed by B and a cycle of H_1 of size $b + p_1 + p + 2\beta + \beta' \geq b + p_2 + p + 2\beta + \beta'$ and some bowtie formed by B and a cycle of H_2 of size $b + 2\delta + \delta'$. If $p_1 \neq p_2$, then $b + 2\delta + \delta' \in \{b + p_i + p + 2\beta + \beta'\}$ for $i \in \{1, 2\}$. So $2\delta + \delta' \in \{p_i + p + 2\beta + \beta'\}$. Hence δ and β do not satisfy one of the three required relationships. So $p_1 = p_2 = \delta + \frac{\delta'}{2}$, then $spec(M) = \{b + 2\delta + \delta', b + p_1 + p + 2\beta + \beta'\}$. We consider $\delta' \in \{0, \delta\}$. If $\delta' = \delta$, then the tip of B is on the middle of a δ-subdivided edge not incident to P. Then the barbell formed by B and a cycle not containing the tip of B has size $b + 3\delta + \frac{\delta}{2} \in \{b + 3\delta, b + \frac{3}{2}\delta + p + 2\beta + \beta'\}$. So $2\delta = p + 2\beta + \beta'$; a contradiction. Hence $\delta' = 0$ and the tip of B is on the branch point of H_1 not incident to P. Also $|\{b + 2\delta, b + \delta + p + 2\beta + \beta', 2\delta + p + 2\beta + \beta', 3\delta\}| \leq 3$, so $b = \delta$. Therefore B has size δ and is attached to a branch point of a balanced theta such that P coincides with the theta at the other branch point of the balanced theta.

Suppose that B is attached to P. Let $P_1 = [x_1, v]$ and $P_2 = [v, x_2]$ be the subpaths of P. Recall that H_1 is either β-subdivided or (α, β)-subdivided. Then G has barbells formed by B and a cycle from H_i for each $i \in \{1, 2\}$ of size $b + p_1 + c + \beta + \beta'$ where $c \in \{\alpha, \beta\}$ and $b + p_2 + 2\delta + \delta'$. If $c = \alpha$,that is, H_1 is unbalanced, then

$\delta = 2\beta + \beta' + p$, $\alpha = 2\delta + \delta' + p$ and spec(M)= $\{2\alpha + \beta + \beta', \alpha + 2\beta + \beta'\}$. Hence $b + p_1 + \alpha + \beta + \beta'$ $\in$spec(M), so $b + p_1 \in \{\alpha, \beta\}$. But $b + p_2 + 2\delta + \delta'$ $\in$spec(M), so $b \in \{2\beta + \beta' + p_2, \alpha + \beta + \beta' + p_1\}$; a contradiction. Therefore $c = \beta$; that is, H_1 is balanced.

Then $\delta \in \{2\beta + \beta' + p, \beta\}$, without loss of generality. If $\delta > \beta$, then spec(M)= $\{3\delta + \delta', 3\beta + \beta'\}$. Then $b + p_1 + 2\beta + \beta', b + p_1 + 2\delta + \delta'$ $\in$spec(M), so $b + p_1 = \beta$ and $b + p_2 = \delta$. Hence $b = \beta - p_1 = \delta - p_2 = 2\beta + \beta' + p_1$. Then $\beta = 2\beta + \beta' + 2p_1$; a contradiction. So $\delta = \beta$. Then spec(M)= $\{3\delta + \delta', 4\delta + 2\delta' + p\}$ and G has bicycles of size $b + p_i + 2\delta + \delta'$ $\in$spec(M) for $i \in \{1, 2\}$ formed be the ballon, a subpath of the subdivided edge, and a cycle from each theta. Then $b + p_i \in \{\delta, 2\delta + \delta' + p\}$. If $b + p_1 \neq b + p_2$, say that $b + p_1 = \delta$ and $b + p_2 = 2\delta + \delta' + p$, without loss of generality. Then $b = \delta - p_1 = 2\delta + \delta' + p_1$; a contradiction. Thus $p_1 = p_2$ and $b \in \{\delta - p_1, 2\delta + \delta' + p_1\}$. Therefore B is attached to the conjoining path of H in the case that the thetas are equally balanced and the tip of the balloon is centered along the path.

Hence we have that $b + p_1 \in \{\delta, 2\delta + \delta' + p_2\}$ and that $b + p_2 \in \{\delta, 2\delta + \delta' + p_1\}$. If $b + p_1 = \delta$, then $b + p_2 = \delta$. Similarly, if $b + p_1 = 2\delta + \delta' + p_2$, then $b + p_2 = 2\delta + \delta' + p_1$. Therefore $p_1 = p_2$. Hence the balloon is attached at the center of the subdivided edge of H.

□

LEMMA 3.10. *Let G be a connected graph with bicycles of exactly two sizes. The graph G has either at most two disjoint theta subgraphs or G is isomorphic to a bundle of n balloons for $n \geq 2$.*

Proof of Lemma 3.10. Notice that if G contains three disjoint theta subgraphs, then we can find a theta barbell H containing two theta subgraphs with an attached balloon using a cycle from the third theta subgraph. Therefore, given the possible edge subdivisions in Lemma 3.7, we consider a theta barbell with an attached balloon containing a subdivided chord.

By Lemma 3.7, a balloon B may be attached to the branch point of a balanced theta with some δ-subdivision or to the center of the subdivided edge P joining two theta subgraphs. In these cases, $b \in \{\delta, \delta + \frac{p}{2}, 2\delta + \delta' - \frac{p}{2}\}$. The chord of the cycle C of B partitions the cycle into edges with subdivisions of lengths b_i, for $i \in \{1, 2, 3\}$, and $b'_4 \in \{0, b_4\}$ such that $b \in \{b_1 + b_2 + b_3, b_1 + b_2 + b'_4, b_2 + b_3 + b'_4, b_1 + b_3 + b'_4\}$. Hence $b_i = \gamma$ for every $i \in \{1, 2, 3\}$ and $b'_4 \in \{0, \gamma\}$. Let Q be the edge joining C and H.

In the case that the balloon is attached to the subdivided edge joining the two thetas, we have that the thetas are equally balanced with some δ-subdivision such that $\delta' \in \{0, \delta\}$; that is, either both or neither of the endpoints of P are branch points of a theta. Then $b \in \{\delta + p', 2\delta + \delta' + p'\}$ where $p' = \frac{p}{2}$. Then $H \cup B$ has bicycles of size $3\delta + \delta'$ and $4\delta + 2\delta' + 2p'$. If $\gamma' = \delta' = 0$, then the cycle C and its chord form a bicycle of size $3\gamma \in \{3\delta, 4\delta + 2p'\}$ and the balloon B has size $2\gamma + q \in \{\delta - p', 2\delta + p'\}$. Hence $\gamma \in \{\frac{\delta - p' - q}{2}, \delta + \frac{p' - q}{2}\}$. Therefore $3\gamma \in \{\frac{3(\delta - p' - q)}{2}, 3\delta + \frac{3(p' - q)}{2}\}$. Note that $\frac{3(\delta - p' - q)}{2} \notin \{3\delta, 4\delta + 2p'\}$, so $3\delta + \frac{3(p' - q)}{2} \in \{3\delta, 4\delta + 2p'\}$. Hence $\frac{3(p' - q)}{2} = 0$, and therefore we see that $p' = q$.

Now suppose that $\gamma' = \gamma$ and $\delta' = \delta$. The cycle C and its chord form a bicycle of size $4\gamma \in \{4\delta, 6\delta + 2p'\}$ and the balloon B has size $3\gamma + q \in \{\delta - p', 3\delta + p'\}$.Hence $\gamma \in \{\frac{\delta - p' - q}{3}, \delta + \frac{p' - q}{3}\}$. Therefore $4\gamma \in \{\frac{4(\delta - p' - q)}{3}, 4\delta + \frac{4(p' - q)}{3}\}$. Note that $\frac{4(\delta - p' - q)}{3} \notin$

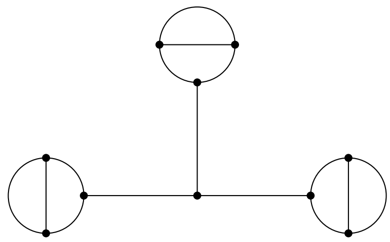


FIGURE 3.8. A bundle of thetas

$\{4\delta, 6\delta+2p'\}$, so $4\delta + \frac{4(p'-q)}{5} \in \{4\delta, 6\delta+2p'\}$. Hence $\frac{4(p'-q)}{3} = 0$, and therefore we see that $p' = q$.

Consider the case that H is composed of n equally balanced thetas with some δ-subdivision joined by subdivided edges P_i, $i \in \{1, 2, ..., n\}$, of equal length such that every edge attaches at the branch point of a theta or every edge does not, as given in Figure 3.8. Then H has bicycles of length $3\delta + \delta'$ and $4\delta + 2\delta' + 2p$ where $p_i = p$ for every i.

Now suppose that $\gamma' = 0$ and $\delta' = \delta$. The cycle C and its chord form a bicycle of size $3\gamma \in \{4\delta, 6\delta + 2p'\}$ and the balloon B has size $2\gamma + q \in \{\delta - p', 3\delta + p'\}$.Hence $\gamma \in \{\frac{\delta-p'-q}{2}, \frac{3\delta+p'-q}{2}\}$. Therefore $3\gamma \in \{\frac{3(\delta-p'-q)}{2}, \frac{9(\delta+p'-q)}{2}\} \notin \{4\delta, 6\delta + 2p'\}$; a contradiction. Similarly, if $\gamma' = \gamma$ and $\delta' = 0$. The cycle C and its chord form a bicycle of size $4\gamma \in \{3\delta, 4\delta + 2p'\}$ and the balloon B has size $3\gamma + q \in \{\delta - p', 2\delta + p'\}$.Hence $\gamma \in \{\frac{\delta-p'-q}{3}, \frac{2\delta+p'-q}{3}\}$. Therefore $4\gamma \in \{\frac{4(\delta-p'-q)}{3}, \frac{8(\delta+p'-q)}{3}\} \notin \{3\delta, 4\delta + 2p'\}$; a contradiction.

We now consider the case that a balloon is attached to the branch point of a balanced theta with some δ-subdivision. Then by Lemma 3.7, the conjoining subdivided edge attaches at the branch point of the balanced theta and $b = \delta$. Then $\delta = 2\gamma + \gamma' + q$. Hence $3\gamma + \gamma' + q = \delta + \gamma - q$ where $\gamma < \delta$. Hence

$3\gamma+\gamma'+q<2\delta$, contradicting the spectrums given in Lemma 3.5. Hence no theta is attached to the branch point of a theta barbell.

□

Lemma 3.11. *Let $M = B(G)$ be a connected bicircular matroid where G is a subdivision of a 3-connected graph H and $|spec(M)| = 2$ such that H is one of the following graphs: an (α,β)-subdivision of W_3, a k-subdivision of W_4,$K_5 \setminus e$,K_5,$K_{3,3}$,$K_{3,4}$, or P_6. Then no ear can be added to H in such a way that H loses 3-connectivity but retains $|spec(M)| = 2$.*

Proof of Lemma 3.11. Let G be a connected graph and suppose that a subgraph H of G is a theta. Then H has three paths A_i, $i \in \{1, 2, 3\}$, of length a_i. Let $P = A_4$ be an ear on H. If P has both endpoints on the branch, then $H \cup P$ has four paths with bicycles $H \cup P - A_i$ for $i \in \{1, 2, 3, 4\}$. Then $H \cup P$ has an $(a, b)-$subdivision.

If P has at least one endpoint not on the branch points, then $H \cup P$ has n branched for $n \in \{5, 6\}$, say $B_1, B_2, ..., B_n$. Therefore $H \cup P$ has bicycles $H \cup P - B_i$ for $i \in \{1, 2, ..., n\}$. Thus $H \cup P$ has an $(a, b)-$subdivision.

Consider adding k ears, each with both endpoints on the branched of H. Then $H_p = H \cup P_1 \cup P_2 \cup \cdots \cup P_k$ has $k+3$ paths, say $B_1, B_2, ..., B_{k+3}$. Then $B_l \cup B_m \cup B_n$ is a bicycle of G for distinct l, m, and n. Hence H_p has an $(a, b)-$subdivision.

Finally, consider adding $k \geq 2$ ears such that for each P_k, $k > 1$, at least one of the endpoints is not on a branchpoint of H. Then H is a subdivision of a simple 3-connected graph without two vertex-disjoint cycles. Thus (by Theorem 4.6 and Lemma 4.7, T. Lewis), H is one of the following graphs: an (a, b)-subdivision of

W_3 for distinct positive integers a and b, or a k-subdivision of W_4, $K_5 \setminus e, K_5, K_{3,3}$, or $K_{3,4}$ for some positive integer k.

Let G be a subdivision of a 3-connected graph without two vertex disjoint cycles. Then consider adding an ear P to G in such a way that G loses 3-conectivity.

First suppose that G is an (a, b)-subdivision of W_3. Then $spec\{M\} = \{5a, 4a + b\}$ Add an ear P to G in such a way that G loses 3-connectivity. Then we have a bicycle of size $p + 3a$ or $p + 2a + b$ formed by the path an one of the cycles of G. Hence $p = 2a$ or $p = a + b$. Now consider the bicycle formed by the outer cycle of G, detoured along the ear P, and two of the inner paths. Then G has a bicycle of size α such that $\alpha \leq p + 5a$ or $\alpha \leq p + 4a + b$. Then $p \leq a$; a contradiction. Thus G has no ears.

Suppose that G is a k-subdivision of W_4. Then $spec\{M\} = \{5k, 6k\}$ Add an ear P to G in such a way that G loses 3-connectivity. Then we have three bicycles. The first is formed by the ear and a cycle on four paths of G. Hence the bicycle has size $p + 4k$. The second is formed by the ear and a cycle on three paths of G, and thus the bicycle has size $p + 3k$. Thus $p = 2k$. The last bicycle is formed by the ear and five paths of G, which gives a bicycle of size $7k$; a contradiction. Thus G has no ears.

Suppose that G is a k-subdivision of $K_5 \setminus e$. Then $spec\{M\} = \{5k, 6k\}$ Add an ear P to G in such a way that G loses 3-connectivity. Then we have three bicycles. The first is formed by the ear and a cycle on four paths of G. Hence the bicycle has size $p + 4k$. The second is formed by the ear and a cycle on three paths of G, and thus the bicycle has size $p + 3k$. Thus $p = 2k$. The last bicycle is formed by the ear and five paths of G, which gives a bicycle of size $7k$; a contradiction. Thus G has no ears.

Suppose that G is a k-subdivision of K_5. Then $spec\{M\} = \{5k, 6k\}$ Add an ear P to G in such a way that G loses 3-connectivity. Then we have three bicycles. The first is formed by the ear and a cycle on four paths of G. Hence the bicycle has size $p + 4k$. The second is formed by the ear and a cycle on three paths of G, and thus the bicycle has size $p + 3k$. Thus $p = 2k$. The last bicycle is formed by the ear and five paths of G, which gives a bicycle of size $7k$; a contradiction. Thus G has no ears.

Suppose that G is a k-subdivision of $K_{3,3}$. Then $spec\{M\} = \{6k, 7k\}$ Add an ear P to G in such a way that G loses 3-connectivity. Then we consider two bicycles. The first is formed by the ear and a cycle on four paths of G. Hence the bicycle has size $p + 4k$. The second is formed by the ear and a cycle on six paths of G, and thus the bicycle has size $p + 6k$. Thus $p = 2k$ or $p = k$. Hence either G has a bicycle os size $5k$ or $7k$; a contradiction. Thus G has no ears.

Suppose that G is a k-subdivision of $K_{3,4}$. Then $spec\{M\} = \{6k, 7k\}$ Add an ear P to G in such a way that G loses 3-connectivity. Then we consider two bicycles. The first is formed by the ear and a cycle on four paths of G. Hence the bicycle has size $p + 4k$. The second is formed by the ear and a cycle on six paths of G, and thus the bicycle has size $p + 6k$. Thus $p = 2k$ or $p = k$. Hence either G has a bicycle os size $5k$ or $7k$; a contradiction. Thus G has no ears.

Suppose that G is a k-subdivision of P_6. Then $spec\{M\} = \{6k, 7k\}$ Add an ear P to G in such a way that G loses 3-connectivity. Then we consider two bicycles. The first is formed by the ear and a cycle on four paths of G. Hence the bicycle has size $p + 4k$. The second is formed by the ear and a cycle on six paths of G, and thus the bicycle has size $p + 6k$. Thus $p = 2k$ or $p = k$. Hence either G has a bicycle os size $5k$ or $7k$; a contradiction. Thus G has no ears. □

THEOREM 3.1. *Let $M = B(G)$ be a connected bicircular matroid where G is not a subdivision of a 3-connected graph H. Then $|spec(M)| = 2$ if and only if H is a restricted subdivision of one of the following graphs:*

(1) *a cycle with at $n \geq 1$ balloons,*

(2) *a theta with attached balloons,*

(3) *a bundle of thetas,*

(4) *two equally balanced thetas joined by at most two paths,*

(5) *a theta barbell with $n \geq 0$ ears on the balanced theta,*

(6) *a theta barbell with a single balloon attached either at the conjoining subdivided edge or at the branch point of a balanced theta.*

PROOF OF THEOREM 3.1. Let G be a non-3-connected graph with bicycles of exactly two sizes. Note that G contains at least two cycles that may or may not be disjoint. Suppose that G contains no theta subgraph. Then G contains at least two disjoint cycles. Since G is connected, these cycles are joined by a path in G. Hence G contains a subgraph that is a cycle with an attached balloon. By Lemma 3.2, G is a restricted subdivision of a cycle with n attached balloons for some positive integer n.

Suppose that G contains two cycles that are not disjoint. Then G contains a theta subgraph. If there exists a cycle in G disjoint from the theta subgraph, then by Lemma 3.3, G is a restricted subdivision of a theta with $n \geq 0$ attached balloons. If there is no cycle in G disjoint from the theta subgraph, then by Lemma 3.4, then G is a restricted subdivision of a theta with $n \geq 0$ ears.

Suppose that G contains two disjoint theta subgraphs. Since G is connected, the thetas are joined by a path P in G, and hence G contains a theta barbell subgraph. By Lemma 3.5, either both thetas are balanced or one theta is balanced and the other is unbalanced, with respect to the endpoints of P. By Lemma 3.7, two disjoint thetas are joined by at most two paths in G.

Now suppose that G contains a theta barbell subgraph with the restricted subdivisions given in Lemma 3.5. If there is a cycle disjoint from the theta barbell in G, then by Lemma 3.8, G is a theta barbell with a balloon attached to the branch point of a balanced theta or a balloon attached to the center of the path P. If there is no cycle disjoint from the theta barbell in G, then by Lemma 3.9, G may contain $n \geq 0$ ears attached to the branch points of the balanced theta.

Suppose that G contains more than two disjoint theta subgraphs. By Lemma 3.10, G is a bundle of thetas.

Therefore the result holds.

□

CHAPTER 4

Bicircular Matroids with Circuits of Three Sizes

The following chapter will characterize the matroid whose associated graph is 3-connected with bicycles of exactly three sizes. The characterization will be built from a series of lemmas given in subsequent sections of this chapter.

1. Graph Terminology

Dirac proved the following result in 1963.

THEOREM 4.1. [**4**] *A graph G is a subdivision of a simple 3-connected graph without two vertex-disjoint cycles if and only if G is isomorphic to a subdivision of one of the following graphs: a wheel graph, K_5, $K_5 \setminus e$, $K_{3,p}$, ${K'}_{3,p}$, ${K''}_{3,p}$, or ${K'''}_{3,p}$ for some $p \geq 3$.*

Some of the graphs mentioned in Theorem 4.1 are given in Figure 1.1. Note that the graph ${K'}_{3,p}$, ${K''}_{3,p}$, ${K'''}_{3,p}$ for some $p \geq 3$ are generated by adding one, two, and three edges, respectively, to the partite set of size three.

2. 3-connected Associated Graphs with $|spec(B(G))| = 3$

In the following sections, we will build the characterization of a bicircular matroid whose associated graph is 3-connected with bicycles of three sizes. We will use a process similar to that used in the previous chapter; that is, building the characterization through a series of lemmas. Using Dirac's result, given in Theorem 4.1, we will consider the possible edge subdivisions of some simple 3-connected graphs without vertex-disjoint cycles. We will then separately investigate the subdivisions of a simple 3-connected graph with at least two disjoint cycles.

To begin, Lemma 4.3 gives a tool, using the edge-sum total coloring on an auxiliary graph, that describes the vertex-coloring of a complete graph on four or more vertices given that the graph is 3-edge-colored. In the following result, Lemma 4.4 characterizes the subdivisions on a wheel graph, W_4, with three bicycle sizes. This lemma will be used to prove Lemma 4.6, which completes the characterization of the simple 3-connected graphs without vertex-disjoint cycles with bicycles of three sizes. Lemma 4.5 characterizes the possible subdivisions of a P_6, or a prism graph on six vertices. We will then examine the possible subgraphs of a 3-connected graph with vertex-disjoint cycles using the possible edge subdivisions of a P_6. Theorem 4.2 gives the complete characterization of a bicircular matroid whose associated graph is 3-connected with bicycles of three sizes.

Using the lemmas stated and proven in the following section, we can construct the following result for 3-connected graphs.

THEOREM 4.2. *Suppose that G is isomorphic to a subdivision of a 3-connected graph. Let $M = B(G)$. Then $|spec(M)| = 3$ if and only if G is isomorphic to one of the following graphs:*

- *an (α, β, γ)-subdivision of W_3;*
- *an β-subdivision of W_5, $K_{3,p}$, for $p \geq 4$, such that*
  - *if $G \cong W_5$, $spec(M) = \{5\beta, 6\beta, 7\beta\}$, and*
  - *if $G \cong K_{3,p}$, $spec(M) = \{6\beta, 7\beta, 8\beta\}$;*
- *a $(2\beta, \beta)$-subdivision of K_5, $K_5 \setminus e$ with $spec(M) = \{5\beta, 6\beta, 7\beta\}$;*
- *one of the following restricted $(2\beta, \beta)$-subdivisions of $K_{3,3}$:*

- *any one edge is* 2β*-subdivided with spec*$(M) = \{6\beta, 7\beta, 8\beta\}$*,*
- *a matching on three edges is* 2β*-subdivided with spec*$(M) = \{8\beta, 9\beta, 10\beta\}$*,*
- *some 4-cycle is* 2β*-subdivided with spec*$(M) = \{8\beta, 9\beta, 10\beta\}$*;*

- *one of the following resrticted* $(2\beta, \beta)$*-subdividions of* P_6*:*
  - *any one edge is* 2β*-subdivided with spec*$(M) = \{6\beta, 7\beta, 8\beta\}$*,*
  - *a* k*-edge-matching is* 2β*-subdivided for* $k \in \{2, 3\}$ *with spec*$(M) = \{k + 5\beta, k + 6\beta, k + 7\beta\}$*,*
  - *a 3-cycle is* j*-subdivided for* $j \in \{\beta, 2\beta\}$ *with spec*$(M) = \{9\beta, 11\beta, 12\beta\}$ *if* $j = \beta$ *and spec*$(M) = \{7\beta, 9\beta, 10\beta\}$ *if* $j = 2\beta$*;*
- *one of the following resrticted* $(2\beta, \beta)$*-subdividions of* W_4*. In each case, spec*$(M) = \{k + 4\beta, k + 5\beta, k + 6\beta\}$ *where* $k \in \{1, 2, 3, 4\}$ *is the number of* 2β*-subdivided edges.*
  - *the rim edges are* β*-subdivided and each of the spokes is either* β *or* 2β*-subdivided,*
  - *the edges are* $(2\beta, \beta)$*-subdivided such that exactly one rim edge is* 2β*-subdivided and both incident spokes are* β*-subdivided, or*
  - *the edges are* $(2\beta, \beta)$*-subdivided such that two opposite rim edges are* 2β*-subdivided and all of the remaining edges are* β*-subdivided.*

3. Lemmas and Theorems

LEMMA 4.3. *Let G be a complete graph on four or more vertices. The edge-sum total coloring of G is a 3-edge coloring if and only if G is 2-vertex colored with at least two vertices of each color.*

PROOF OF LEMMA 4.3. Let n be the number of vertices on a complete graph G. Suppose that $n = 4$. Clearly, if G is 1-vertex-colored, then G is 1-edge-colored.

If G is 2-vertex-colored, say with colors α and β, then either G is 3-edge colored with exactly two vertices of each color and edge colors 2α, 2β, and $\alpha + \beta$ or G is 2-edge-colored with exactly one vertex colored β and edge colors 2α and $\alpha + \beta$. If G is 3-vertex-colored, then G is 4-edge-colored. Similarly, if G is 4-vertex-colored, then G is 6-edge-colored.

Assume that the statement holds for graphs with $n-1$ vertices. Suppose that G is a complete graph on n vertices. Then for any vertex $x \in$V(G), we know that G-x is 2-vertex-colored with at least two vertices of each color, say α and β, and G-x has edge colors 2α, $\alpha + \beta$, and 2β. Then x has $n - 1$ neighbors in G with incident edges of color $x + \alpha$ or $x + \beta$. Hence $x + \alpha, x + \beta \in \{\alpha + \beta, 2\alpha, 2\beta\}$. Therefore $x \in \{\alpha, \beta\}$.

Therefore, by induction, if G is 3-edge-colored, then G is 2-vertex-colored with at least two vertices of each color. □

LEMMA 4.4. *Suppose that G is isomorphic to a subdivision of* W_4. *Let* $M = B(G)$. *Then* $|spec(M)| = 3$ *if and only if one of the following holds:*

(1) *The rim edges are* β*-subdivided and each of the spokes is either* β *or* 2β*-subdivided,*
(2) *The edges are* $(2\beta, \beta)$*-subdivided such that exactly one rim edge is* 2β*-subdivided and both incident spokes are* β*-subdivided, or*
(3) *The edges are* $(2\beta, \beta)$*-subdivided such that two opposite rim edges are* 2β*-subdivided and all of the remaining edges are* β*-subdivided.*

PROOF OF LEMMA 4.4. Consider the auxillary graph, $Aux(W_4)$, of W_4 given in Figure 4.2, and suppose that $Aux(W_4)$ is at most 3-edge-colored. Notice that the subgraph H induced by the set of vertices A_i for $i \in \{1, 2, 3, 4\}$ is a complete graph.

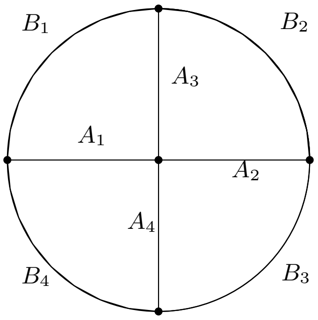


FIGURE 4.1. The graph W_4

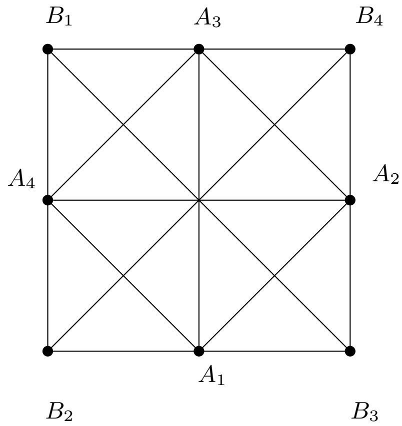


FIGURE 4.2. The auxiliary graph $Aux(W_4)$

Then H is at most 2-vertex-colored, say that $a_i \in \{\alpha, \beta\}$ for each i. Note that the bicycle complements comprised of three edges of G, $\{A_1, B_1, B_4\}$, $\{A_2, B_2, B_1\}$, $\{A_3, B_3, B_2\}$, and $\{A_4, B_4, B_3\}$, are identically the maximal independent sets of $Aux(G)$. We will use the size of these maximal independent sets to show that if G has bicycles of at most three sizes, then $Aux(G)$ is at most 2-vertex-colored.

Suppose that $Aux(G)$ is 3-vertex-colored. If H is vertex-monochromatic with color α, then the remaining two colors, say β and γ, must appear in the set of vertices B_1, B_2, B_3, and B_4. If there is some vertex of color β that is adjacent to any vertex of color γ, then G has bicycle complements of size $\alpha + \beta$, $\alpha + \gamma$, $\beta + \gamma$, and $\alpha + \beta + \gamma$; a contradiction. So no vertex of color β is adjacent to a vertex of color

γ. Then G has bicycles of size $\alpha+\beta+\gamma$, 2α, $\alpha+\beta$ and $\alpha+\gamma$. Hence $\alpha=\beta+\gamma$, and G has bicycle complements of size $2\beta+2\gamma$, $2\beta+\gamma$, and $\beta+2\gamma$. If there is an edge of $Aux(G)$ that is colored 2β, then $\beta=2\gamma$, and G has bicycle complements of size 4γ, 5γ, and 6γ. Then G has bicycle complements of size 2γ or $2\alpha+\beta=8\gamma$; a contradiction. If no edge of $Aux(G)$ is colored 2β, and by symmetry, no edge is colored 2γ, then there at most one vertex of each color β and γ in $Aux(G)$, Thus G has a bicycle complement of size $3\alpha=3\beta+3\gamma$; a contradiction. Hence we may assume that that H is 2-vertex-colored with colors α and β.

Suppose that there are exactly two vertices of each color in H; that is, G has bicycle complements of size $\alpha+\beta$, 2α and 2β. Then there is some vertex $b_i=\gamma$ for $i\in\{1,2,3,4\}$. If there is some edge colored $\alpha+\gamma$ and some edge colored $\beta+\gamma$, then $\alpha+\gamma=2\beta$ and $\beta+\gamma=2\alpha$. So $\gamma=2\beta-\alpha=2\alpha-\beta$; a contradiction. So at most one of $\alpha+\gamma$ and $\beta+\gamma$ is among the edge colors of $Aux(G)$, say $\alpha+\gamma=2\beta$. Then at most one vertex $b_i=\gamma$ for $i\in\{1,2,3,4\}$. Hence $Aux(G)$ has maximal independent sets of size $\beta+\gamma+x$ where $x\in\{\alpha,\beta\}$. If G has a bicycle complement of size $\alpha+\beta+\gamma=3\beta\in\{2\alpha,2\beta,\alpha+\beta\}$, then $\alpha+\beta+\gamma=2\alpha$, and hence $\beta+2\gamma$ and $\alpha=3\gamma$. But then $Aux(G)$ also has maximal independent sets of size $2\alpha+\beta=8\gamma$ or $2\beta+\alpha=7\gamma$; a contradiction. We reach a similar contradiction if G has a bicycle complement of size $\gamma+2\beta$.

Suppose that there is exactly one vertex of color β in H; that is G has bicycle complements of size 2α and $\alpha+\beta$. Then some vertex of $Aux(G)$ is colored γ, so $Aux(G)$ has edge colors $\alpha+\beta$, 2α and $\alpha+\gamma$. Note that if $Z(G)$ has both edge colors 2γ and 2β, then $2\gamma=\alpha+\beta$ and $2\beta=\alpha+\gamma$. But then $2\beta=\alpha+\gamma=3\gamma-\beta$; a contradiction. Hence $Z(G)$ has edges of size 2β or of size 2γ. Also note, by a similar argument, that if $Aux(G)$ has edge color $\beta+\gamma$, then $Aux(G)$ neither has edges of size 2γ nor of size 2β. Suppose $Aux(G)$ has edge color 2β, say $b_2 = \beta$, without loss of generality. Then $b_3, b_4 \neq \gamma$, so $b_1 = \gamma$. Hence $Aux(G)$ has an independent set of size $\alpha+\beta+\gamma$. But $|\{\alpha+\beta+\gamma, \alpha+\gamma, 2\alpha, \alpha+\beta, 2\beta\}| > 3$; a contradiction. If $Aux(G)$ has an edge of color 2γ, then one of B_2 or B_3 must be colored γ, and hence $Aux(G)$ also has an edge of color $\beta + \gamma$; a contradiction. So $Aux(G)$ has edges no edges of color 2β or of color 2γ. Hence $b_2, b_3 \neq \beta$. If $Aux(G)$ also has no edge of color $\beta + \gamma$, then say that $b_1 = \gamma$. Thus $b_2 = b_3 = \alpha$, and G has bicycle complements of size $\{2\alpha, \alpha + \beta, \alpha + \gamma, 2\alpha + \gamma, 3\alpha\}$. Then $\beta = 3\alpha$ and $\gamma = 2\alpha$. Hence G has bicycle complements of size 2α, 3α, and 4α. Then G has a bicycle of size $\alpha+\beta+x$ where $x \geq \alpha$. Hence $\alpha + \beta + x > 4\alpha$; a contradiction. Therefore $Aux(G)$ has an edge colored $\beta + \gamma$, and hence at least one of B_2 and B_3 is colored γ. Suppose that $b_2 = \gamma$, without loss of generality. Note that $b_4 \in \{\alpha, \beta\}$, and recall that no maximal independent set may be colored $\alpha + \beta + \gamma$. If $b_1 = \gamma$, then $b_3 = \alpha$ and $b_4 = \beta$. Hence $|\{2\alpha, \alpha+\beta, \alpha+\gamma, 2\alpha+\gamma, 2\alpha+\beta, 2\beta+\gamma, 2\gamma+\alpha\}| > 3$. Similarly, if $b_3 = \gamma$, then $b_1 = \alpha = b_4$, and hence $|\{2\alpha, \alpha+\beta, \alpha+\gamma, \beta+\gamma, 3\alpha, 2\gamma+\alpha, 2\alpha+\gamma\}| > 3$; a contradiction. So at most $b_2 = \gamma$, and hence $b_3 = \alpha$. If some independent set is colored 3α, then $3\alpha \in \{2\alpha + \gamma, \alpha + \gamma, 2\alpha\}$, and hence $\gamma = 2\alpha$. But $2\alpha = \beta + \gamma$; a contradiction. Therefore $b_4 = \beta$, and the set of bicycle complements of G has cardinality $|\{\alpha + \beta, \alpha + \gamma, \beta + \gamma, 2\alpha, 2\alpha + \gamma, 2\alpha + \beta\}| > 3$; a contradiction.

So $Aux(G)$ is 2-vertex-colored, say with colors α and β. We now show the edge subdividions if G given that G has bicycles of three sizes.

If H is vertex-monochromatic with color α, at least one of $b_i = \beta$ for $i \in \{1, 2, 3, 4\}$. If the set of vertices colored β is not an independent set, say $b_1 = b_3 = \beta$, then G has bicycle complements of size 2α, 2β, and $\alpha + \beta$. Note that G cannot have both bicycle complements of size $2\alpha + \beta$ and of size $2\beta + \alpha$. Say $b_4 = \beta$ so that

$Z(G)$ has independent sets of size $2\beta + \alpha$. Then $b_i = \beta$ for all $i \in \{1, 2, 3, 4\}$ and $\alpha = 2\beta$. If instead $b_4 = \alpha$, then $Z(G)$ has independent sets of size $2\alpha + \beta$. Then at most one set of adjacent vertices may be colored β. Hence $\beta = 2\alpha$.

If the set of vertices colored β is an independent set, then either exactly one vertex is colored β, as shown in the previous argument, or at most two vertices are colored β, say $b_1 = b_4 = \beta$. Then $b_2 = b_3 = \alpha$, and the set of bicycle complements of G has cardinality $|\{\alpha + \beta, 2\alpha, 2\beta + \alpha, 3\alpha, 2\alpha + \beta\}| > 3$; a contradiction. If H is 2-vertex-colored with exactly one vertex of color β, then either the set of vertices colored β, call it $\mathbb{B}$, is independent or not. If so, then either $b_i = \alpha$ for all $i \in \{1, 2, 3, 4\}$, $b_1 = \beta$ or $b_1 = b_4 = \beta$, without loss of generality. If $b_i = \alpha$, then $\beta = 2\alpha$, and exactly one spoke of G is 2α-subdivided. If $b_1 = \beta$, then $|\{3\alpha, \alpha + \beta, 2\alpha, 2\beta + \alpha, 2\alpha + \beta\}| > 3$; a contradiction. And if $b_1 = b_4 = \beta$, then $|\{3\alpha, 3\beta, \alpha + \beta, 2\alpha, 2\alpha + \beta\}| > 3$; a contradiction. If instead $\mathbb{B}$ is not independent, then say that $b_2 = \beta$, without loss of generality. Note that $Aux(G)$ has edge colors 2α, $\alpha + \beta$, and 2β, $Aux(G)$ cannot have independent sets of both size $2\alpha + \beta$ and $2\beta + \alpha$. If $b_1 = \beta$, then $Aux(G)$ has an independent set of size $2\beta + \alpha$. Hence all B_i are colored β, and $\alpha = 2\beta$. Thus three spokes of G are 2β-subdivided. If $b_1 = \alpha$, then $Aux(G)$ has an independent set of size $2\alpha + \beta$. Thus $b_3 = b_4 = \alpha$, and hence $\beta = 2\alpha$. Thus one spoke and one nonadjacent rim edge are 2α-subdivided.

If H is 2-vertex-colored with exactly two vertices of each color, then G has bicycle complements of size 2α, 2β, and $\alpha + \beta$. Note that G cannot have both bicycle complements of size $2\alpha + \beta$ and of size $2\beta + \alpha$. Say G has bicycle complements of size $2\alpha + \beta$, without loss of generality. If $\alpha = a_i = a_{i+1} \neq a_{i+2} = a_{i+3} = \beta$ for $i \in \{1, 2, 3, 4\}$ mod 4, then at most one $b_i = \beta$ for $i \in \{1, 2, 3, 4\}$, and G has bicycle complements of size 2α, 2β, $\alpha + \beta$, and $2\alpha + \beta$. Hence $\beta = 2\alpha$.

If $\alpha = a_i = a_{i+2} \neq a_{i+1} = a_{i+3} = \beta$ for $i \in \{1, 2, 3, 4\}$ mod 4, then G has bicycle complements of size 2α, $\alpha + \beta$, 2β, 3α, $2\alpha + \beta$. Hence $\beta = 2\alpha$.

Finally, if $\alpha = a_i = a_{i+1} = a_{i+2} \neq a_{i+3} = \beta$ for $i \in \{1, 2, 3, 4\}$ mod 4, either the set of vertices of color β, $\mathbb{B}$, is independent or not. If not, note that G has bicycle complements of size 2α, 2β, and $\alpha + \beta$, and that G cannot have both bicycle complements of size $2\alpha + \beta$ and of size $2\beta + \alpha$. If G has a bicycle complement of size $2\alpha + \beta$, then G also has a bicycle complement of size $2\beta + \alpha$; a contradiction. So G has bicycle complements of size $2\beta + \alpha$ and hence $\alpha = 2\beta$. Then $b_i = \beta$ for all $i \in \{1, 2, 3, 4\}$.

If $\mathbb{B}$ is independent, then G also has bicycle complements of size 3α and $2\alpha + \beta$. Hence $\beta = 2\alpha$. Hence G has bicycle complements of size 2α, 3α, and 4α. If $\mathbb{B}$ is maximally independent, then G has a bicycle complement of size $3\beta = 6\alpha$; a contradiction. So at most one of $b_i = \beta$ for $i \in \{1, 2, 3, 4\}$ and B_i not adjacent to A_{i+3}. □

Lemma 4.5. *Suppose that G is isomorphic to a subdivision of P_6. Let $M = B(G)$. Then $|spec(M)| = 3$ if and only if G is $(2\beta, \beta)$-subdivided and one of the following holds:*

- *Any one edge is 2β-subdivided, then $spec(M) = \{6\beta, 7\beta, 8\beta\}$;*
- *A k-edge-matching is 2β-subdivided for $k \in \{2, 3\}$, then $spec(M) = \{k + 5\beta, k + 6\beta, k + 7\beta\}$;*
- *A 3-cycle is j-subdivided for $j \in \{\beta, 2\beta\}$, then $spec(M) = \{9\beta, 11\beta, 12\beta\}$ if $j = \beta$ and $spec(M) = \{7\beta, 9\beta, 10\beta\}$ if $j = 2\beta$.*

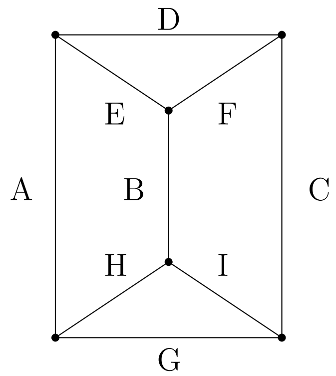


FIGURE 4.3. The graph P_6

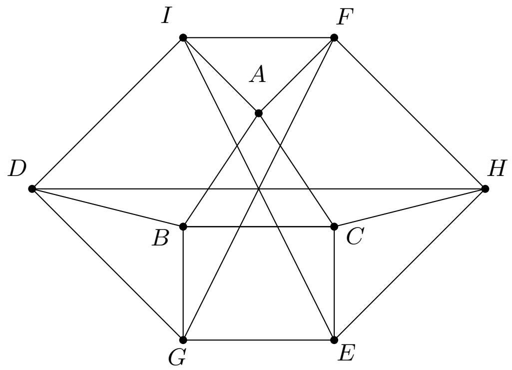


FIGURE 4.4. The auxiliary graph $Aux(P_6)$

Moreover, if G is isomorphic to a subdivision of a 3-connected graph with disjoint cycles and $|spec(B(G))| = 3$, then G is isomorphic to one of the given subdivisions of P_6.

PROOF OF LEMMA 4.5. We will first show that the auxiliary graph $Aux(G)$ of G, given in Figure 4.4, is at most two-vertex-colored given that G has bicycles of three sizes. Notice that the set of maximal independent sets of $Aux(G)$, $\{AED\}$, $\{DFC\}$, $\{CIG\}$, $\{AHG\}$, $\{BHI\}$, $\{BEF\}$, $\{DEF\}$, and $\{GHI\}$, contains all of the bicycle complements with greater than two edges in G, those being $\{AED\}$,

$\{DFC\}$, $\{CIG\}$, $\{AHG\}$, $\{BHI\}$, and $\{BEF\}$. We will use the sizes of the maximal independent sets of $Z(G)$ that are also bicycle complements in G.

We will suppose, for the sake of contradiction, that $Aux(G)$ is 3-vertex-colored with colors α, β, and γ. Suppose that each color class has some vertex adjacent to some vertex of each other color class. Hence $Aux(G)$ has edge colors $\alpha+\beta$, $\beta+\gamma$, and $\alpha+\gamma$. Note that G cannot have a bicycle complement of size $\alpha+\beta+\gamma$. So the bicycle complements consisting of three incident edges in G have at most two edge-subdivisions, that is, the corresponding vertices in $Z(G)$ are 2-vertex-colored, say that G has a bicycle complement of size $2\alpha+\beta$, without loss of generality. Hence $\gamma \in \{\alpha+\beta, 2\alpha\}$ and $\gamma > \alpha$. If $Aux(G)$ is 3-vertex-colored, then a vertex of color γ must also appear in some complement of size $\tau \in \{2\beta+\gamma, 2\alpha+\gamma, 2\gamma+\beta, 2\gamma+\alpha\}$.

If $\tau = 2\gamma + \beta \in \{\alpha+\beta, \beta+\gamma, \alpha+\gamma\}$, then either $\alpha = 2\gamma$ or $\alpha = \beta+\gamma$; a contradiction.

If $\tau = 2\gamma+\alpha \in \{\alpha+\beta, \beta+\gamma, \alpha+\gamma\}$, then $\beta \in \{2\gamma, \alpha+\gamma\}$. If $\beta = 2\gamma$, then $\gamma = 2\alpha$ and $Aux(G)$ has vertex colors α, 2α, and 4α. If, on the other hand, $\beta = \alpha+\gamma$, then $\gamma = 2\alpha$ and hence $Aux(G)$ has vertex colors α, 2α, and 3α.

If $\tau = 2\beta+\gamma \in \{\alpha+\beta, \beta+\gamma, \alpha+\gamma\}$, then either $\alpha = \beta+\gamma$, a contradiction on γ, or $\alpha = 2\beta$. Then if $\gamma = 2\alpha$, $Aux(G)$ has vertex colors α, 2α, and 4α. If $\gamma = \alpha+\beta$, $Aux(G)$ has vertex colors α, 2α, and 3α.

Finally, if $\tau = 2\alpha+\gamma \in \{\alpha+\beta, \beta+\gamma, \alpha+\gamma\}$, then either $\beta = \alpha+\gamma$, and hence $\gamma = 2\alpha$, or $\beta = 2\alpha$ and hence $\gamma = \alpha+\beta$. In either case, $Z(G)$ has vertex colors α, 2α, and 3α.

So $Aux(G)$ has vertex colores α, 2α, and either 3α or 4α. In the former case, $Aux(G)$ has edge colors 3α, 4α and 5α. If a bicycle complement consisting of three incident edges in G has a 3α-subdivided edge, then the remaining two edges must be α-subdivided. Then there is some edge in $Aux(G)$ with color 2α; a contradiction. Similarly, in the latter case, $Aux(G)$ has edge colors 3α, 5α, and 6α. If a bicycle complement consisting of three incident edges in G has a 4α-subdivided edge, then the remaining two edges must be α-subdivided. Then there is some edge in $Aux(G)$ with color 2α; a contradiction. Hence if each color class has some vertex adjacent to each other color class, then $Aux(G)$ is 2-vertex-colored.

Suppose there are two color classes, say the set of vertices colored α and β in $Aux(G)$, such that no vertex colored α is adjacent to a vertex colored β. Then $Aux(G)$ has edge colors $\alpha + \gamma$ and $\beta + \gamma$. If $Aux(G)$ has a vertex-monochromatic independent set colored α corresponding to a bicycle complement comprised of three incident edges in G, then the neighbors of those vertices colored α must be colored γ or α. Note that each of the remaining vertices of $Aux(G)$ is adjacent to at least one vertex contained in the vertex-monochromatic independent set colored α. Hence all of the remaining vertices in $Aux(G)$ must be colored γ or α, and therefore $Aux(G)$ is 2-vertex-colored.

Suppose that $Aux(G)$ has a vertex-monochromatic independent set of size two colored α. Either both vertices lie on the outer 6-cycle of $Aux(G)$, $IFHEGD$, or one lies on the outer 6-cycle and the other is A, B or C. Then the neighbors of the vertex colored α must be colored γ, and $Aux(G)$ has at most two remaining vertices. At least one of there must be colored β. Then, in either case, the set $|\{\alpha + \gamma, \beta + \gamma, 2\gamma, 2\alpha + \beta, \alpha + \beta + \gamma, 3\gamma\}| > 3$; a contradiction.

Suppose that two adjacent vertices of $Aux(G)$ are colored α, say $i = f = \alpha$, $i = a = \alpha$ or $b = c = \alpha$, without loss of generality. Then either B or C, G or C, I or F is colored β, respectively. Each case is symmetric, so choose B, G, and I, without loss of generality. Then in the first case, $|\{2\alpha, \alpha + \gamma, \beta + \gamma, 2\gamma, 3\gamma, \alpha + \beta + \gamma\}| > 3$.

In the second, $|\{2\alpha, \alpha+\gamma, \beta+\gamma, 2\gamma, 3\gamma, \alpha+2\gamma, \alpha+\beta+x\}| > 3$ for $x \in \{\beta, \gamma\}$, and in the last case $|\{2\alpha, \alpha+\gamma, \beta+\gamma, 2\gamma, 3\gamma, \alpha+\beta+x\}| > 3$ for $x \in \{\alpha, \gamma\}$.

Then $Aux(G)$ has at most one vertex colored α, say $i = \alpha$ or $a = \alpha$. Then the neighbors of the vertex colored α are colored γ. If $i = \alpha$, then some independent set containing I, say IGH, is either 2-vertex-colored with colors α and β, or 3-vertex-colored. In the former case, the set of bicycle complements of G has cardinality $|\{\alpha+\gamma, \beta+\gamma, \alpha+2\beta, 2\gamma, \beta+2\gamma, 3\gamma\}| > 3$. In the latter case, $|\{\alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma, 2\gamma, 3\gamma\}| > 3$.

If, on the other hand, $a = \alpha$, then $i = f = b = c = \gamma$. If one of the remaining vertices is colored β, then G has bicycle complements of size $\alpha+\gamma$, $\beta+\gamma$, 2γ, $2\gamma+\beta$. Then the independent set containing both I and the vertex colored β is either 2-vertex-colored such that G has some bicycle complement of size $2\beta+\alpha$ or 3-vertex-colored with some bicycle complement of size $\alpha+\beta+\gamma$. In either case, $|spec(M(G))| > 3$; a contradiction.

Hence we have shown that if G has bicycles of at most three sizes, then $Aux(G)$ is 2-vertex colored.

Using the number of vertex-monochromatic maximally independent sets that correspond to a bicycle complement in G, we will show the restricted edge subdivisions on G. Note that there are six such independent sets, each of size three and none contained entirely in the outer 6-cycle. Suppose that $Aux(G)$ is 2-vertex-colored with colors α and β. If G has some bicycle complement of size 3α, i.e. a vertex-monochromatic maximally independent set colored α in $Aux(G)$, and also some bicycle complements of size $\alpha+\beta$ and 2α, then G can have either have bicycle complements of size $2\alpha+\beta$ or of size $2\beta+\alpha$. If G also has some bicycle complement of size 2β, then G may have bicycle complements of size $2\alpha + \beta$ but not of size $2\beta + \alpha$.

If there are five or more vertex-monochromatic maximally independent sets, then $Aux(G)$ is vertex-monochromatic and G has two bicycle sizes.

If there are four vertex-monochromatic maximally independent sets colored α, then $Aux(G)$ has exactly one vertex colored β. Hence G has bicycle complements of size 2α, $\alpha + \beta$, 3α, and $2\alpha + \beta$. Hence $\beta = 2\alpha$, and G has bicycle complements of size 2α, 3α, and 4α. Hence any one edge of G may be 2α-subdivided, with all remaining edges α-subdivided.

If there are three vertex-monochromatic maximally independent sets colored α, then either no two independent sets are mutually exclusive, or exactly two independent sets are.

In the former case, say that vertex sets AED, DFC, and BEF are colored α in $Aux(G)$. Hence G has some bicycle complements of size 3α, 2α, and $\alpha + \beta$. Then at least two of the vertices G, H, and I must be colored β, or one of the remainine three maximal independent sets is also vertex-monochromatic. If two of the vertices are colored β, then G has bicycle complements of size $2\alpha + \beta$ and $2\beta + \alpha$; a contradiction. If all three are colored β, then G has bicycle complements of size $2\beta + \alpha$. Hence either $\beta = 2\alpha$ or $\alpha = 2\beta$. Thus G has some 3-cycle that is β-subdivided with the remaining edges α-subdivided such that $\beta \in \{2\alpha, \frac{\alpha}{2}\}$.

In the latter case, say that vertex sets AED, DFC, and CIG are colored α in $Z(G)$. Then vertices B and H are colored β. Hence G has bicycles of size 3α, 2α, $\alpha + \beta$, $2\alpha + \beta$ and $2\beta + \alpha$; a contradiction.

If there are two vertex-monochromatic maximally independent sets colored α, then either the sets are mutually exclusive or not. We first consider the case that they are mutually exclusive. Say that the sets AED and CIG are colored α, without loss of generality. Then G has some bicycle complements of size 3α, 2α, $\alpha+\beta$, and 2β. So the remainind maximall independent sets must have exactly two vertices colored α as G can have bicycles of size $2\alpha+\beta$ but not $2\beta+\alpha$, 3β or 3α. Hence vertices H and F are colored β, and B is colored α. Therefore $\beta = 2\alpha$, and G has two 2α-subdivided edges in two different 3-cycles and not in the same 4-cycle of G.

Now consider the case that the sets are not mutually exclusive, say that sets AED and either DFC or AHG are colored α. Then in either case, G has bicycles of size 3α, 2α, $\alpha+\beta$, and 2β. Thus the remaining independent sets are colored $2\alpha+\beta$. In the former case, C and G are colored β. In the latter case, either F and I or B and C are colored β. In any case, $\beta = 2\alpha$. Hence G has two 2α-subdivided edges such that either one edge is in some 3-cycle and the one edge is neither in any 3-cycle nor in the same 4-cycle as the other edge, or the edges are opposite edges of some 4-cycle.

If there is exactly one vertex-monochromatic maximally independent set colored α, say AED, then G has some bicycle complements of size 3α, 2α, $\alpha+\beta$, and 2β. Hence the remaining independent sets have exactly two vertices colored α. If F is colored α, then B and C are colored β, and hence G, H, and I are colored α. If F is colored β, then B and C are colored α, and hence two of G, H, and I are colored β. But then G has some bicycle complement of size 3α or of size $2\beta+\alpha$; a contradiction. So $\beta = 2\alpha$ two opposite edges of some 4-cycle of G that are both not contained in any 3-cycle are 2α-subdivided.

If there is no vertex-monochromatic maximally independent set in $Aux(G)$, then clearly G will have some bicycle complements of size $\alpha+\beta$, 2α, and 2β. Suppose without loss of generality that G has some bicycle complement of size $2\alpha + \beta$. Then G cannot also have any bicycle complements of size $2\beta + \alpha$. Suppose that $a = \beta \neq \alpha = d = e$. Then H and G are colored α. If B is colored α, then F and I are colored β. If B is colored β, then C is also colored β. If instead $a = e = \alpha \neq \beta = d$, then F and C are colored α. Hence B is colored β, and $h = i = \alpha \neq \beta = g$. In any case, $\beta = 2\alpha$. Hence G has three 2α-subdivided edges that form a matching.

Suppose now that some 3-cycle of G has a chord. Then the chord forms a theta subgraph H of G with cycles C_1 and C_2. Note that one cycle of H, say C_1, a path in H from C_1 to the vertex of H not contained in C_1 and the remaining 3-cycle of G form a P_6 subgraph of G. Then each edge of this subgraph must be β-subdivided or 2β-subdivided, including the chord of H. Hence the endpoints of the cord must be two distinct vertices of H. If H has three 2β-subdivided edges, then H has size 7β or 8β. But if a 3-cycle of G is 2β-subdivided, then all remaining edges of G must be β-subdivided. Hence G has bicycles of size 10β, 9β, or 7β. So the chord must be β-subdivided, and we can find a bicycle of size 8β containing the chord, the edge of H that shares end vertices with the chord, and the other 3-cycle of G.

If H has one 2β-subdivided edge in the outer 3-cycle, then H has size 5β or 6β. But if a 3-cycle of G has at most one 2β-subdivided edge, then by the previous result, all remaining edges of G must be β-subdivided. Hence G has bicycles of size 8β, 7β, or 6β. So the chord must be 2β-subdivided. If the chord does not share both endvertices with the other 2β-subdivided edge, then there is some P_6 subgraph with exactly two adjacent edges; a contradiction. So the chord and the 2β-subdivided edge have the same endvertices, and we can find a bicycle of size 9β

containing the chord, the other 2β-subdivided edge, two paths of G not contained in any 3-cycle, and two edges of the other 3-cycle in G.

If no edge of H is 2β-subdivided, then H has size 4β or 5β; a contradiction. Thus G has no chords.

□

THEOREM 4.6. *Suppose that G is isomorphic to a subdivision of W_r ($r \geq 3$), K_5, $K_5 \setminus e$,$K_{3,p}$,$K'_{3,p}$,$K''_{3,p}$, $K'''_{3,p}$ ($p \geq 3$). Let $M = B(G)$. Then $|spec(M)| = 3$ if and only if G is isomorphic to one of the following graphs:*

- *An (α, β, γ)-subdividion of W_3;*
- *A restricted $(2\alpha, \alpha)$-subdividion of W_4, K_5, $K_5 \setminus e$, $K_{3,3}$, $K'_{3,3}$,$K''_{3,3}$, $K'''_{3,3}$;*
- *An α-subdividion of W_5, $K_{3,4}$.*

PROOF OF THEOREM 4.6. First note that each graph represents a bicircular matroid with bicycles of three cardinalities. Suppose that $|spec(M)| = 3$. Let S denote the edge set of G.

CASE 1. *Suppose that G is a wheel graph.*

Let G be as given in Figure 1 with A_i and B_i denoting paths of G obtained by subdividing an edge of H for each $i \in \{1, 2, ..., r\}$. Hence A_i has a_i edges and B_i has b_i edges, for each i. Assume that a subdivision of $G \cong W_3$. Then $S - A_i$ and $S - B_i$ are bicycles of G for each $i \in \{1, 2, 3\}$. These bicycles are of three cardinalities so that $\{a_1, a_2, a_3, b_1, b_2, b_3\} = \{a, b, c\}$ for some distinct positive integers a, b, and c. It follows that G is obtained from W_3 by an (a, b, c)-subdivision.

We have from Lemma 4.4 the subdivisions on a W_4.

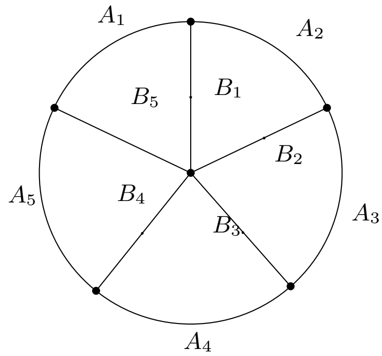


FIGURE 4.5. The graph W_5

Now suppose that H is isomorphic to W_5, as given in Figure 4.5. Note that the deletion of any spoke B_i, $i \in \{1, 2, 3, 4, 5\}$, results in a subdividion of W_4. By Lemma 4.4, either W_4 is k-subdivided with bicycles of $5k$ and $6k$ or W_4 is $(2k, k)$-subdivided with bicycles of three sizes.

Delete some edge B_i for $i \in \{1, 2, 3, 4, 5\}$. The resulting W_4 subgraph, H_i, is either k-subdivided or $(2k, k)$-subdivided. First assume that H_i is k-subdivided. Hence edges A_i and A_{i+1} for $i \in \{1, 2, 3, 4, 5\}$ mod 5 are subdivided such that $a_i + a_{i+1} = k$ and all other rim edges are k-subdivided. Now consider the W_4 subgraph H_{i+1}, resulting from the deletion of B_{i+1}. Then $a_{i+1} + a_{i+2} \in \{k, 2k\}$. But $a_{i+2} = k$, so $2k = a_{i+1} + a_{i+2} = a_{i+1} + k$. Hence $a_{i+1} = k$ and $a_i = 0$; a contradiction.

Now assume that H_i is $(2k, k)$-subdivided. Then $a_i + a_{i+1} \in \{k, 2k\}$. First suppose that $a_i + a_{i+1} = k$, then $a_{i+2} \in \{k, 2k\}$. Consider H_{i+1}. Then $a_{i+1} + a_{i+2} \in \{k, 2k\}$. If $a_{i+2} = k$, then $a_{i+1} + a_{i+2} = k + a_{i+1} > k$. So $a_{i+1} + a_{i+2} = 2k$ and $a_{i+1} = k = a_i + a_{i+1}$; a contradiction. Hence $a_{i+2} = 2k$ and $a_{i+1} + a_{i+2} = a_{i+1} + 2k > 2k$; a contradiction.

Thus we see that $a_i + a_{i+1} = 2k$. This forces $a_{i+2} = k$ as no two adjacent rim edges of a W_4 are $2k$-subdivided. Consider H_{i+1}. Then $a_{i+1} + a_{i+2} \in \{k, 2k\}$. If

$a_{i+1} + a_{i+2} = k$, then $a_{i+1} + a_{i+2} = a_{i+1} + k = k$, so $a_{i+1} = 0$; a contradiction. Hence $a_{i+1} + a_{i+2} = 2k$, and therefore $a_i = a_{i+1} = k$. By induction, we see that $a_i = k$ for all i.

Now we see that the deletion of any spoke of W_5 results in a $(2k, k)$-subdivided W_4 subgraph with exactly one $2k$-subdivided rim edge. If any spoke of W_5 is $2k$-subdivided, then that edge would be adjacent to some $2k$-subdivided rim edge of a subdivided W_4 subgraph of W_5; a contradiction. Hence all spokes of W_5 are k-subdivided.

Therefore W_5 is k-subdivided with bicycles of size $5k$, $6k$, and $7k$.

Now suppose that H is isomorphic to W_r for $r \geq 6$. Then the graph obtained from G by deleting the edge set of $r-4$ consecutive spoke paths of G is a subdivision of W_4. By the previous remarks, each subdivided path of such a W_4 with the given coloring is b-subdivided on the cycle and either b-subdivided or a-subdivided on the spokes. Remove the paths A_i for each $i \in \{1, 2, ..., r-4\}$ to obtain a subdivision of W_4 with $B_r \cup B_1 \cup B_2 \cup ... \cup B_{r-4}$ being a subdivision path. This path has b_1 edges so that $b_r + b_1 + b_2 + ... + b_{r-4} = b_1$; a contradiction.

Case 2. *Suppose that H is isomorphic to K_5.*

Note that each four-cycle is the rim of a W_4 subgraph of K_5. From the previous result, we know that either all edges of the rim are β-subdivided, exactly one edge is 2β-subdivided and the remaining three are β-subdivided, or the cycle is $(2\beta, \beta)$-subdivided such that opposite edges have equal length. We will show that there are at most two non-adjacent 2β-subdivided edges of K_5.

Suppose that two adjacent edges are 2β-subdivided in K_5. Then in some W_4 subgraph of K_5, the edges are adjacent rim edges; a contradiction. Therefore at most two non-adjacent edges of K_5 are 2β-subdivided.

If two edges of H are 2β-subdivided, then H has cycles sizes 5β, 6β, 7β, and 8β; a contradiction. If exactly one edge is 2β-subdivided, then H has cycle sizes 5β, 6β, and 7β.

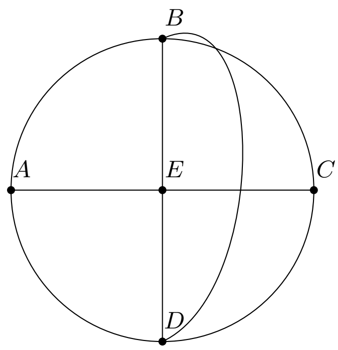


FIGURE 4.6. The graph $K_5 \setminus e$

CASE 3. *Suppose that H is isomorphic to $K_5 \setminus e$.*

Note that $K_5 \setminus e$ is a W_4 with a path P of length p connecting two opposite rim vertices, as seen in Figure 4.6. Assume that W_4 is β-subdivided with bicycle sizes 5β and 6β. Then H has bicycles containing P of size $p + 4\beta$ and $p + 5\beta$. Therefore $p = 2\beta$ and H has bicycles of size 5β, 6β, and 7β.

Consider the paths AEC, ABC, and ADC. Any two of these paths form the rim of some W_4. Clearly, any one of the edges AE, AB, AD, CE, CB, or CD may be 2β-subdivided. Suppose that any two of the edges on paths AEC, ABC, and ADC are 2β-subdivided. Note that if both edges on any one of the paths are 2β-subdivided, then there are two adjacent 2β-subdivided rim edges on some

W_4; a contradiction. Similarly, if the 2β-subdivided edges are incident to the same vertex, then there are two adjacent 2β-subdivided rim edges on some W_4; a contradiction. Hence there are no two adjacent 2β-subdivided edges on the W_4 subgraph of $K_5 \setminus e$. Therefore, at most two edges of W_4 are 2β-subdivided.

Consider the case that exactly one edge of W_4 is 2β-subdivided. Thus either the edge is adjacent to P or not. In the former case, say that AB is 2β-subdivided. In the latter case, say that AE is 2β-subdivided. In either case, $K_5 \setminus e$ has bicycle complement sizes $p + 2\beta$, $p + 3\beta$, $p + 4\beta$, 3β, 4β, and 5β. Hence $p = \beta$.

Now consider the case that two non-adjacent edges of the W_4 subgraph are 2β-subdivided. Then at least one such edge is adjacent to P. Say that AB and either DC or EC are 2β-subdivided, without loss of generality. Then in either case, $K_5 \setminus e$ has bicycle complements of size $p + 2\beta$, $p + 3\beta$, $p + 4\beta$, 3β, 4β, 5β, and 6β; a contradiction.

Therefore $K_5 \setminus e$ is $(2\beta, \beta)$-subdivided such that exactly one edge is 2β-subdivided, and $K_5 \setminus e$ has bicycles of size 5β, 6β, and 7β.

Case 4. *Suppose that H is isomorphic to $K_{3,3}$.*

Suppose that H is isomorphic to $K_{3,3}$.

Let the subdivision paths of G be as given in Figure with path X_i corresponding to edge X_i for each $i \in \{1, 2, 3\}$. Then the edge sets of $A_1 \cup A_2 \cup A_3$, $B_1 \cup B_2 \cup B_3$, $C_1 \cup C_2 \cup C_3$, $A_i \cup B_j$, $A_i \cup C_j$, and $C_i \cup B_j$ for $i, j \in \{1, 2, 3\}$ with $i \neq j$ are complements of bicycles of G. Hence the above sets are of three cardinalities. Let $Aux(G)$ be the graph with vertex set $\{A_1, A_2, A_3, B_1, B_2, B_3, C_1, C_2, C_3\}$ and edges A_iB_j, A_iC_j, and C_iB_j for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Color the vertex X by x, the number of edges in the path X of G, and color an edge XY by $x + y$. This coloring

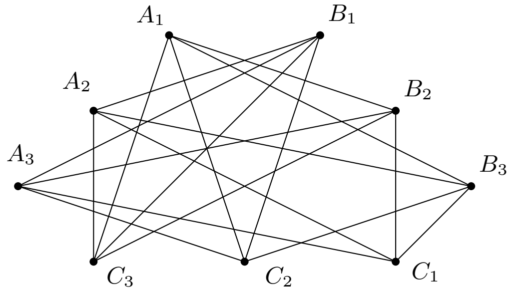


FIGURE 4.7. The auxiliary graph $Aux(K_{3,3})$

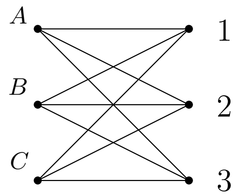


FIGURE 4.8. The graph $K_{3,3}$

yeilds a j-edge coloring of $Z(G)$ for $j \in \{1, 2, 3\}$. If $j = 1$, then $K_{3,3}$ is k-subdivided with bicycle sizes $6k$ and $7k$.

We will show that if G has bicycles of three cardinalities, then $Aux(G)$ is at most 2-vertex-colored. Hence G has subdivided paths of at most two cardinalities. Suppose that $Aux(G)$ is 3-vertex-colored with colors α, β, and γ. Note that the sets $\mathcal{A} = \{A_1, A_2, A_3\}$, $\mathcal{B} = \{B_1, B_2, B_3\}$, $\mathcal{C} = \{C_1, C_2, C_3\}$, $\mathcal{D} = \{A_1, B_1, C_1\}$, $\mathcal{E} = \{A_2, B_2, C_2\}$, and $\mathcal{F} = \{A_3, B_3, C_3\}$ are the maximal independent sets of $Z(G)$.

Note that if $Aux(G)$ has edge colors $\alpha + \beta$, $\alpha + \gamma$, $\beta + \gamma$, 2α, and 2β, then $2\alpha, 2\beta \in \{\alpha + \beta, \alpha + \gamma, \beta + \gamma\}$, then $2\alpha = \beta + \gamma$ and $2\beta = \alpha + \gamma$. Hence $2\alpha = \beta + \gamma = \beta + 2\beta - \alpha$. Therefore $\beta = \alpha$; a contradiction. Therefore if all color classes are adjacent to one another, then at least two of the color classes are independent.

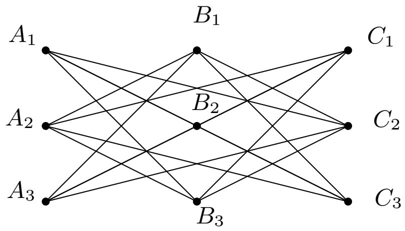


FIGURE 4.9. The auxiliary graph $Aux(K_{3,3}) \setminus \{A_iC_j\}$ for $i \neq j$

Suppose that at least two color classes are independent, say α and β. Then $Aux(G)$ has edge colors $\alpha+\beta$, $\alpha+\gamma$, and $\beta+\gamma$. If the sets of vertices colored α and β, call them $\mathscr{A}$ and $\mathscr{B}$, respectively, are both maximally independent, then the set of vertices colored γ is also maximally independent, call that set $\mathscr{C}$. Then the set of bicycle complements of G has cardinality $|\{\alpha+\beta, \alpha+\gamma, \beta+\gamma, \alpha+\beta+\gamma\}| > 3$; a contradiction. So not both $\mathscr{A}$ and $\mathscr{B}$ are maximal, say $\mathscr{B}$ is not maximally independent.

If $\mathscr{A}$ is maximally independent, G has bicycle complements of size 3α, $2\beta+\alpha$, $2\gamma+\alpha$ as a vertex of color α is adjacent to every other vertex of $Aux(G)$, exactly two vertices of $Z(G)$ are colored β and G cannot have a bicycle of size $\alpha+\beta+\gamma$. Then $|\{\alpha+\beta, \alpha+\gamma, \beta+\gamma\}| = |\{3\alpha, 2\beta+\alpha, 2\gamma+\alpha\}|$. Then $2\beta+\alpha \in \{\alpha+\gamma, \beta+\gamma\}$. In the former case, $\beta = 2\alpha$ and $\gamma = 3\alpha$. Hence $|\{\alpha+\beta, \alpha+\gamma, \beta+\gamma, 3\alpha, 2\beta+\alpha, 2\gamma+\alpha\}| > 3$. Similarly, in the latter case, $\gamma = 2\beta$ and $\beta = 2\alpha$. Then $|\{\alpha+\beta, \alpha+\gamma, \beta+\gamma, 3\alpha, 2\beta+\alpha, 2\gamma+\alpha\}| > 3$. So $\mathscr{A}$ is not maximal. Then G has bicycles of size $|\{\alpha+\beta, \alpha+\gamma, \beta+\gamma, 2\alpha+x, 2\beta+y, 2\gamma+z\}| > 3$, where $x \in \{\beta, \gamma\}$, $y \in \{\alpha, \gamma\}$, and $z \in \{\alpha, \beta, \gamma\}$.

Hence all color classes are not adjacent. Note that $Aux(G)$ is connected, so at most two color classes may be nonadjacent. Say that the set $\mathscr{A}$ has no neighbors in the set $\mathscr{C}$. Then $Aux(G)$ has edge colors $\alpha+\beta$ and $\beta+\gamma$. Also note that if $|\{\alpha+\beta, \beta+\gamma, 2\alpha, 2\beta, 2\gamma\}| \leq 3$, then $\beta+\gamma = 2\alpha$ and $\alpha+\beta = 2\gamma$. Then $2\alpha = \beta+\gamma = 2\gamma-\alpha+\gamma$, and hence $\alpha = \gamma$; a contradiction. Thus at least one color class is independent, say that the set $\mathscr{A}$ is independent.

Suppose that the set $\mathscr{A}$ is independent. If $\mathscr{A}$ is maximal, then all of the remaining vertices of $Aux(G)$ are adjacent to some vertex of color α. Hence the remaining vertices must be colored β, and $Aux(G)$ is 2-vertex-colored.

If exactly two vertices are colored α, say that $a_i = a_j = \alpha$ for $i, j \in \{1, 2, 3\}$ and $i \neq j$. Then all of $V(Aux(G)) \setminus a_k$ for $k \in \{1, 2, 3\}$ and $k \notin \{i, j\}$ are adjacent to a vertex of color α, and hence are colored β. Then a_k is colored γ. Hence the set of bicycle complements of G has cardinality $|\{\alpha+2\beta, \gamma+2\beta, 2\alpha+\gamma, 2\beta, \alpha+\beta, \beta+\gamma\}| > 3$; a contradiction.

Suppose that only one vertex of $Aux(G)$ is colored alpha, say $a_1 = \alpha$. Then $b_2 = b_3 = c_2 = c_3 = \beta$. If at least one of the remaining vertices is also colored β, then G has bicycles of size $|\{\alpha+\beta, \beta+\gamma, 2\beta, 2\beta+\gamma, 3\beta\}| > 3$. If no other vertex is colored β, then G has bicycle complements of size $|\{\alpha+\beta, \beta+\gamma, 2\beta, 2\beta+\gamma, 2\gamma+\alpha, 2\gamma\}| > 3$; a contradiction.

So $Aux(G)$ is 2-vertex-colored, and hence G has edge subdiviions of two cardinalities.

We will now consider the number of monochromatic independent sets of $Aux(G)$ to show the possible edge subdivisions of G.

If five or more independent sets of $Aux(G)$ are monochromatic, then $Aux(G)$ is vertex-monochromatic; a contradiction.

If four independent sets of $Aux(G)$ are monochromatic, then say that $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D}$ and $\mathcal{E}$ are colored α. Then vertex $c_3 = \beta$. Hence G has bicycle complements of size

$|\{3\alpha, 2\alpha+\beta, 2\alpha, \alpha+\beta\}| \leq 3$. Hence $\beta = 2\alpha$, and G has has bicycle complements of size $\{2\alpha, 3\alpha, 4\alpha\}$. Therefore, any one edge of G is 2α-subdivided and the remaining edges are α-subdivided.

If three independent sets of $Aux(G)$ are monochromatic, either the three sets are distinct or not. If they are distinct, then say that two sets are colored α and the third is colored β, without loss of generality. Then G has as bicycle complements of size $|\{3\alpha, 3\beta, 2\alpha+\beta, 2\alpha, \alpha+\beta\}| \leq 3$. Hence $\beta = 2\alpha$, and therefore G has three incident edges that are 2α-subdivided and G has bicycle complements of size 2α, 3α, 4α, and 6α; a contradiction. If the three independent sets are not distinct, say that $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are all vertex-monochromatic with color α. Then $b_3 = c_3 = \beta$. Hence G has as bicycle complements of size $|\{3\alpha, 2\alpha+\beta, 2\beta+\alpha, 2\alpha, \alpha+\beta\}| \leq 3$. Therefore $\beta = 2\alpha$, and hence G has two incident edges that are 2α-subdivided. Then G has bicycle complements of size 2α, 3α, 4α, and 5α; a contradiction

If two independent sets of $Aux(G)$ are monochromatic, then either they are distinct or not. In the former case, say one set is colored α and the other β. Then G has some bicycle complements of size 3α, 3β and $\alpha+\beta$. Note that the remaining three vertices are 2-vertex-colored. So G has bicycle complements of size 2α, 2β, $2\alpha+\beta$, without loss of generality. Then $\beta = 2\alpha$ and G has more than three bicycle complement sizes; a contradiction. If both sets are colored α, then the remaining three vertices would be a vertex-monochromatic set colored β; a contradiction.

In the latter case, $Aux(G)$ has two independent sets with a nonempty intersection. Then G has bicycle complements of size 3α, 2α and $\alpha+\beta$. If G also has a bicycle complement of size $2\alpha+\beta$, then two of the remaining vertices are colored α. So G has bicycle complements of size 2α, 3α, 2β, $2\alpha+\beta$; a contradiction. If G also has a bicycle complement of size $2\beta+\alpha$, then all of the remaining vertices are colored

β. So G has bicycle complements of size 2β, 3α, $\alpha+\beta$, and $2\beta+\alpha$. Hence $\beta = 2\alpha$. Thus some 4-cycle of G is 2α-subdivided, and G has bicycle complements of size 3α, 4α, or 5α.

If exactly one independent set of $Aux(G)$ is vertex-monochromatic with color α, then G has bicycle complements of size 2α, 3α, $\alpha+\beta$, and either $2\alpha+\beta$ or $2\beta+\alpha$. Hence $\beta = 2\alpha$ in either case. Note that both sizes cannot occur at the same time. However, G has a bicycle complement of size $2\alpha+\beta$ if and only if G has a bicycle complement of size $2\beta+\alpha$; a contradiction. So $Aux(G)$ cannot have exactly one monochormatic maximal independent set.

If no independent set of $Aux(G)$ is vertex-monochromatic, then G has bicycle complements of size $\alpha+\beta$, 2α, 2β, and $2\alpha+\beta$, without loss of generality. Then there is a matching in G if β-subdivided edges, and $\beta = 2\alpha$.

Therefore if $K_{3,3}$ has two bicycle sizes, then $K_{3,3}$ is $(2\kappa, \kappa)$-subdivided such that exacly one edge is 2κ-subdivided, or $K_{3,3}$ has a matching on two or three edges that is 2κ-subdivided, or a 4-cycle of $K_{3,3}$ is 2κ-subdivided.

Case 5. *Suppose that H is isomorphic to $K_{3,p}$ for $p \geq 4$.*

Let A_i, B_i, C_i for $i \in \{1, 2, ..., p\}$ be the edges of $K_{3,p}$ for $p \geq 4$. First consider $p = 4$. Note that every $K_{3,3}$ subgraph of $K_{3,4}$ is $(2\kappa, \kappa)$-subdivided. Suppose that at least one edge of $K_{3,4}$ is 2κ-subdivided, say $a_1 = 2\kappa$ without loss of generality. Then $K_{3,4}$ has bicycle complement sizes $m\kappa$ for $m \in \{3, 4, 5, 6, 7\}$; a contradiction.

Consider the $K_{3,3}$ subgraph K on edges A_i, B_i, and C_i for all $i \in \{1, 2, 3\}$. If K has a 2β-subdivided matching on two edges, then G has bicycles containing edges C_4 and B_4 of size $b_4 + c_4 + m\beta$ for $m \in \{4, 5, 6, 7\}$; a contradiction.

If K has a 2β-subdivided matching on three edges, then G has bicycles containing edges C_4 and B_4 of size $b_4 + c_4 + m\beta$ for $m \in \{5, 6, 7, 8\}$; a contradiction.

Similarly, if K has a 2β-subdivided 4-cycle, then G has bicycles containing edges C_4 and B_4 of size $b_4 + c_4 + m\beta$ for $m \in \{4, 5, 6, 8\}$; a contradiction.

Hence all of the $K_{3,3}$ subgraphs of $K_{3,4}$ are κ-subdivided. Therefore $K_{3,4}$ is κ-subdivided with bicycle sizes 6κ, 7κ, and 8κ. Therefore, all $K_{3,4}$ subgraphs of $K_{3,p}$ are κ-subdivided. We need only show that 8κ is the largest bicycle size of $K_{3,p}$ for $p > 4$.

Using all of the vertices from the partite set of size three and three randomly chosen vertices from the partite set of size p, we can acheive a largest cycle of size 6κ. Using a vertex not contained in the original cycle, we can get, at best, a theta subgraph of size 8κ. Thus $K_{3,p}$ is κ-subdivided with bicycles of size 6κ, 7κ, and 8κ.

Case 6. *Suppose that H is isomorphic to $K'_{3,3}$, $K''_{3,3}$, or $K'''_{3,3}$.*

Let X be the edge added to one of the partite sets. Say that X is x-subdivided. Suppose that the $K_{3,3}$ subgraph is k-subdivided. Then there are bicycles of size $4k + x$, $5k + x$, $6k + x$, $6k$ and $7k$. Hence $x = 2k$.

Suppose that the $K_{3,3}$ subgraph is $(2k, k)$-subdivided. First consider the case that any one edge is $2k$-subdivided. Then that edge is either adjacent to X or not. In the former case, we get bicycles of size $4k + x$, $5k + x$, $6k + x$, and $7k + x$; a contradiction. In the latter case, we get bicycles of size $4k + x$, $6k + x$, $7k + x$, $6k$, $7k$, and $8k$; a contradiction.

Now consider the case that two edges are $2k$-subdivided. Then we get bicycles of size $5k + x$, $6k + x$, $8k + x$, $7k$, $8k$, $9k$; a contradiction.

If three edges are $2k$-subdivided, either all three such edges are incident or they form a matching. In the former case, there are bicycle complements of size αk for $\alpha \in \{3, 4, 5, 6, 7\}$; a contradiction. In the latter case, there are bicycles of size $\beta k + x$ for $\beta \in \{6, 7, 8, 9\}$; a contradiction.

Hence $K'_{3,3}$ is $(2k, k)$-subdivided such that X is $2k$-subdivided and the remaining edges are k-subdivided.

Suppose that H is isomorphic to $K''_{3,3}$. Note that any $K'_{3,3}$ subgraph is given as above. Then let Y be the added edge to $K'_{3,3}$. Note that X and Y share at least one endpoint. Then $K''_{3,3}$ has bicycles of size $y + 4k$, $y + 6k$, $6k$, $7k$, and $8k$. Hence $y = 2k$.

Suppose that H is isomorphic to $K'''_{3,3}$. Note that any $K''_{3,3}$ subgraph is given as above. Then let Z be the added edge to $K'_{3,3}$. Then $K''_{3,3}$ has bicycles of size $z+4k$, $z + 5k$, $z + 6k$, $6k$, $7k$, and $8k$. Hence $z = 2k$.

□

CHAPTER 5

Circuit Spectrum of Bicircular Matroids

From the previous chapters, we can surmise that, for $|spec(M)| \geq 4$, the investigation of the associated graphs of bicircular matriods with circuits of few sizes becomes more tedious. Hence, for the remainder of this dissertation, rather than assuming the size of the circuit spectrum of the bicircular matroid, we will instead consider the circuit spectrum of bicircular matroids given that the associated graphs have some minimum degree condition.

1. Known Results on Cycle Lengths

Recall that bicycles in a graph are two cycles that either share edges, share a single vertex, or are distinct and joined by a path. Therefore in investigating the circuit spectrum of bicircular matroids, it would seem reasonable to consider some known results on the cycle lengths of a graph. Moreover, where existing literature on the circuit spectrum of bicircular matroids is both recent and limited, the set of cycle sizes of a graph has been a prevalent subject of investigation. In 1975, Paul Erdős raised the problem of determining the maximum number of edges in a graph in which no two cycles have the same length (see Bondy and Murty [**1**], p.247, Problem 11). In 1995, Erdős and his collaborator András Gyárfás, stated the conjecture that every graph with minimum degree 3 contains a simple cycle whose length is a power of two, which was proven in 2013 by Heckman and Krakovski [**6**] for cubic planar graphs. In 1996, Bondy and Vince answered in the affirmative

Erdős's question of whether a simple graph where every vertex has degree at least three must contain two cycles whose lengths differ by one or two [**2**].

In this section, we will present some known results on cycles in graphs with some minimum degree condition, as presented by Fan in [**5**]. In the following section, we will use these results to investigate the bicycle lengths of a graph.

Fan proved in [**5**] the following results on the cycle spectrum of graphs with some minimum degree.

THEOREM 5.1. [**5**] *If G is a 2-connected graph with minimum degree at least $3k$ for any positive integer k, and in addition, if G contains a non-separating induced odd cycle, then G contains $2k$ cycles of consecutive lengths m, $m+1$, $m+2$,..., $m+2k-1$ for some integer $m \geq k+2$.*

THEOREM 5.2. [**5**] *If G is a nonbipartite 3-connected graph with minimum degree at least $3k$ for any positive integer k, then G contains $2k$ cycles of consecutive lengths m, $m+1$, $m+2$,...,$m+2k-1$ for some integer $m \geq k+2$.*

For an edge $uv \in E(G)$, *replacing uv with a cycle* is the operation of deleting the edge uv and adding a new cycle C such that $V(C) \cap V(G) = \{u, v\}$. An (x, y)-string of k cycles is the graph obtained from an (x, y)-path by replacing k edges of the path with k cycles, one edge with one cycle. In a string, if C is the cycle replacing uv, then u and v are called the *connection vertices* of C. C is called *t-defective* if the two (u, v)-paths of C differ in length by t. A string is called *t-defective* if each of its cycles is t-defective. It is important to note that in a string of cycles, distinct cycles can only intersect only at connection vertices. Figure 5.1 gives a string of two cycles in which the first is 2-defective and the last is 3-defective.

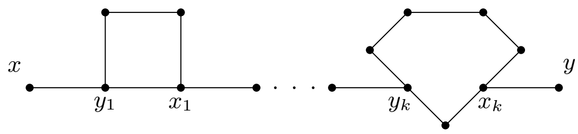


FIGURE 5.1. An (x, y)-string

An (x, y)-string S of k cycles can be represented by $S = P_0C_1P_1...C_kP_k$, where C_i is a cycle with connection vertices x_i and y_i for $1 \leq i \leq k$, and P_j is a path from x_j to y_{j+1} for $0 \leq j \leq k$. Note that $x_0 = x$ and $y_{k+1} = y$. Also, note that P_j may be trivial, that is, consisting of a single vertex $x_j = y_{j+1}$. For each i, label the two (y_i, x_i)-paths C_i' and C_i'' of C_i such that $|E(C_i'')| \geq |E(C_i')|$. The *length* of S is defined by

$$\ell(S) = \sum_{i=1}^{k} |E(C_i')| + \sum_{i=0}^{k} |E(P_i)|,$$

which is the minimum length of a path from x to y in S. For any s, $1 \leq s \leq k$, let P be a path from x_s to y in $P_sC_{s+1}...C_kP_k$. Then $P_0C_1...C_sP$ is an (x, y)-string of s cycles.

With the previous observation, Fan proved in [**5**] the following results on the existence of consecutive path lengths in an (x, y)-string.

LEMMA 5.3. [**5**] *Let S be a t defective (x, y)-string of k cycles. Then S contains (x, y)-paths of lengths m, $m + t$, $m + 2t$,..., $m + kt$, where $m = \ell(S)$.*

LEMMA 5.4. [**5**] *Let S be an (x, y)-string of k cycles in which s cycles are 1-defective and the rest $k - s$ cycles are 2-defective. If $s \geq 1$, then S contains (x, y)-paths of lengths m, $m + 1$, ..., $m + 2k - s$, where $m = \ell(S)$.*

Definition 5.5. [**5**] *Let* $S = P_0C_1P_1...C_kP_k$ *be an* (x, y)*-string of* k *cycles in a graph* G*.* S *is feasible (with respect to* k *and* G*) if all of the following three statements hold:*

(1) $\sum_{i=0}^{k} |E(P_i)| \neq 0$.
(2) C_i *is* 2*-defective for every* i*,* $1 \leq i \leq k$*, with at most one exception.*
(3) *If* C_j *is the exceptional cycle in (2), then* C_j *is* 1*-defective, and moreover, there is* $uv \in E(C_j)$ *such that* $\{u, v\} \cap \{x, y\} = \emptyset$ *and* $d_G(u) = d_G(v) = 3k$*.*

If the exceptional cycle does not exits, then S *is called a feasible 2-defective* (x, y)*-string of* k *cycles in* G*.*

Theorem 5.6. [**5**] *Let* x *and* y *be two distinct vertices in a 2-connected graph* G*. For any positive integer* k*, if every vertex other than* x *and* y *has degree at least* $3k$*, then* G *contains a feasible* (x, y)*-string of* k *cycles.*

A cycle is *non-separating* in a graph G if $G - V(C)$ is connected. In addition to the given results on consecutive path lengths and strings of cycles, we will use an induced non-separating odd cycle to construct a set of bicycles of consecutive lengths. The following results will be utilized in the proof of Theorem 5.10. In [**2**], Bondy and Vince proved that every nonbipartite 3-connected graph contains a non-separating induced odd cycle.

Lemma 5.7. [**5**] *Let* G *be a graph with minimum degree at least four. If* G *contains a nonseparating induced odd cycle, then* G *contains a non-separating induced odd cycle* C *such that either* C *is a triangle or* $e(v, C) \leq 2$ *for every* $v \in V(G) \setminus V(C)$ *which is not a cut vertex of* $G - V(C)$*.*

The following are corollaries to Theorem 5.6.

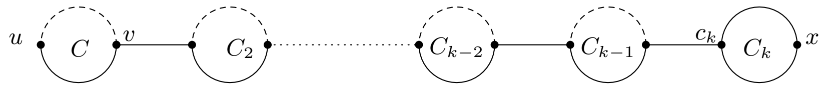


FIGURE 5.2. A (u, x)-string of cycles with $ux \in E(G)$

COROLLARY 5.8. [**5**] *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if $d_G(v) \geq 3k+1$ for every $v \in V(G) \setminus \{x, y\}$, then G contains $k+1$ (x, y)-paths of consecutive even lengths or consecutive odd lengths m, $m+2$, $m+4$, ..., $m+2k$ for some integer $m \geq k+1$.*

COROLLARY 5.9. [**5**] *Let x and y be two distinct vertices in a 2-connected graph G. For any positive integer k, if $d_G(v) \geq 3k$ for every $v \in V(G) \setminus \{x, y\}$, then G contains $k+1$ (x, y)-paths R_0, R_1, ..., R_k such that $k < |E(R_0)| < |E(R_1)| < \ < |E(R_k)|$, $|E(R_i)| - |E(R_{i-1})| = 2$, $1 \leq i \leq k-1$, and $1 \leq |E(R_k)| - |E(R_{k-1})| \leq 2$.*

2. Bicycles of Consecutive Lengths

Using the tools given in the previous section, we can show that a 2-connected graph G with minimum degree at least $3k$, $k \geq 2$, contains bicycles of consecutive lengths. In Theorem 5.10, we consider the case that G contains an induced non-separating odd cycle. The case that G is bipartite or contains no induced non-separating odd cycle is dealt with in Theorem 5.11.

THEOREM 5.10. *If G is a 2-connected graph with minimum degree at least $3(k)$ for any positive integer k, and in addition, if G contains a non-separating induced odd cycle, then G contains $2(k-1)$ bicycles of consecutive lengths.*

PROOF. The proof of Theorem 5.1 builds a feasible srting $S = P_0C_1P_1...C_kP_k$ of k cycles such that C_1 is a nonseparating induced odd cycle. Moreover, C_1 is the only 1-defective cycle of S. Therefore all of the remaining cycles C_i of S, for $i \in \{2, ..., k\}$, are 2-defective. Label the connection vertices y_1 and x_1 of C_1 such that $V(P_0) \cap V(C_1) = \{y_1\}$ and $V(P_1) \cap V(C_1) = \{x_1\}$. Then by the proof of Theorem 5.1, S is a y_1, y-string of k cycles for some $y \in V(G) \setminus V(C_1)$ and $y_1 y \in E(G)$.

We will build our bicycles using a 2-defective cycle C_i for some $2 \leq i \leq k$, say C_k, along with some (y_1, x_1)-path in C and the edge $y_1 y$. Let P_1 and P_2 be the two (y_1, x_1)-paths in C such that $|E(P_1)| > |E(P_2)|$, and let $S' = P_1 C_2 ... C_{k-1} P_{k-1}$ be a (v, c_k)-string of $k-2$ cycles where $c_k \in V(P_{k-1}) \cap V(C_k)$. By Lemma 5.3, S' has $(k-2)+1 = k-1$ (v, c_k)-paths of consecutive odd or even lengths p, $p+2$,..., $p+2(k-2)$ for $p = \ell(S')$. Then we have $k-1$ bicycles of consecutive even or odd lengths formed by C_k, P_2, the edge xu, and each of the (v, c_k)-paths in S'. Figure 5.2 shows . Say these lengths are m, $m+2$, ..., $m+2(k-1)$ for some m. Then replacing P_2 with the path P_1 in each bicycle, we increase each bicycle length by one. Hence, we have $2(k-1)$ bicycles of length m, $m+1$, $m+2$, ..., $m+2k-2$, $m+2k-1$.

□

Theorem 5.11. *Let x and y be two distinct vertices in a 2-connected graph G. If every vertex other than x and y has minimum degree at least $3k$ for $k \geq 2$, then G has $k-1$ bicycles of consecutive lengths. Moreover, if every (x, y)-string of G is a feasible 2-defective string, then G has $k-1$ bicycles of consecutive even or odd lengths.*

Proof. Let G be a 2-connected graph with minimum degree at least $3k$, $k \geq 2$, for all $v \in V(G) \setminus \{x, y\}$. Then by Lemma 5.6, G has a feasible string $S = P_0 C_1 P_1 ... C_k P_k$ of k cycles.

If $k = 2$, then $S = P_0C_1P_1C_2P_2$ and $C_1 \cup P_1 \cup C_2$ is a bicycle of G. Therefore G has $k-1(=1)$ bicycles. For the remainder of the proof, we assume that $k > 2$.

We will build barbells of consecutive cycles using C_0, C_k and a path. Let $S' = P_1C_2P_2...P_{k-1}$ of $k-2$ cycles. Recall that x_i and y_i are the connection vertices of cycle C_i such that $x_i \in V(P_{i-1}) \cap V(C_i)$ and $y_i \in V(P_i) \cap V(C_i)$. If S' contains a 1-defective cycle, then by Lemma 5.4, S' contains $(k-2)+1$ (y_1, x_k)-paths of consecutive lengths. Hence there are $k-1$ barbells of consecutive lengths built from C_0, C_k, and a (y_1, x_k)-path.

Now suppose that S' has no 1-defective cycle. Hence S' is a feasible 2-defective string. If S' does not contain a 1-defective cycle, then by Lemma 5.4, S' contains $(k-2)+1$ (y_1, x_k)-paths of consecutive even or odd lengths. Hence there are $k-1$ barbells of consecutive even or odd lengths built from C_0, C_k, and a (y_1, x_k)-path.

$\square$

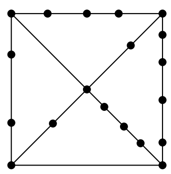
Note that in Corolloaries 5.8 and 5.9 all of the (x, y)-paths have lengths at least $k+1 \geq 2$, and hence the edge xy is not contained in any of the (x, y)-paths. We will use this fact to build thetas of consecutive even or consecutive odd lengths in the following result.

THEOREM 5.12. *Let xy be an edge in a 2-connected graph G. For any positive integer k, if $d_G(v) \geq 3k+1$ for every $v \in V(G) \setminus \{x, y\}$, then G contains k bicycles of consecutive even or consecutive odd lengths. Moreover, the k bicycles are theta subgraphs of G.*

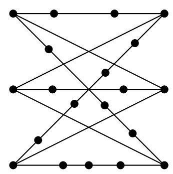
PROOF. By Corollary 5.8 and the proof of Theorem 5.6, there is an (x, y)-string $S = P_0C_1P_1...C_kP_k$ of k cycles containing $k+1$ (x, y)-paths of consecutive even or consecutive odd lengths. Label the connection vertices of C_i by y_i and x_i, and let C_i' be the shorted (y_i, x_i)-path in C_i. We will build bicycles using a cycle C_i of S, the substring $S' = P_0C_1P_1...C_i'P_i...C_kP_k$, and the edge xy.

The substring S' has $k-1$ cycles and (x, y)-paths P_j, for $j \in \{1, 2, ..., k\}$, of consecutive even or consecutive odd lengths. Then there are k theta subgraphs, $C_i \cup P_j \cup \{xy\}$ for some i and each $j \in \{1, 2, ..., k\}$, with consecutive even or consecutive odd lengths. □

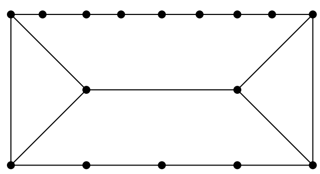
Note that in the previous results, Theorem 5.10 reports a spectrum containing only thetas, and Theorem 5.11 reports a spectrum containing only thetas and barbells. Consider the circuit spectrums for the graphs given in Figure 5.3.



$spec(B(M)) = \{14, 15, ..., 20\}$



$spec(B(M)) = \{13, 14, 15, ..., 19, 21\}$



$spec(B(M)) = \{7, 9, 10, 13, 14, 16, 17\}$

FIGURE 5.3. The circuit spectrum of several graphs

From the graphs given in Figure 5.3, we can see that the previous result still does not capture the entire circuit spectrum. We suspect that, given a minimum degree of at least $3k$, $k \geq 1$, the graphs may be pan-bicyclic. Furthermore, with more inspection, these spanning trees may yet yield a larger spectrum.

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Vita

The author, Bette Catherine Putnam, was born in Flowood, Mississippi on December 20, 1986, to Michael and Elizabeth Putnam. She graduated as the Valedictorian and Star Student of her class at Caledonia High School in 2005. Catherine went on to graduate Magna Cum Laude with her Bachelor of Arts in Mathematics from the University of Mississippi and the Sally McDonnell Barkesdale Honors College in 2009. In 2011, she graduated with her Master of Science in Mathematics from the University of Mississippi, and is currently a Ph.D. candidate in Mathematics at the University of Mississippi.