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Essays On Monopoly And Monopsony Under Imperfect Information

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ESSAYS ON MONOPOLY AND MONOPSONY UNDER IMPERFECT INFORMATION

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ABSTRACT

The first essay estimates the degree of monopsony power in the college football labor market. Previous literature suggests that the marginal revenue product of top performing college football players significantly exceeds their compensation. Such estimates overstate rents on labor because they estimate the marginal revenue product as a function of *ex post* realized labor quality, which is more information than is available *ex ante*. For 114 Football Bowl Subdivision participant schools from 2004-2011, I estimate the marginal revenue product for players grouped into three *ex post* quality tiers. Using talent ratings prospective players spanning 2002-2008, I then estimate the probability of each *ex post* quality outcome given a prospect's rating. Together these yield estimates of expected marginal revenue product as a function of *ex ante* promise, which are more appropriate for inference regarding monopsony power.

The second essay builds on a multi-market approach to second-degree price discrimination which treats each successive unit sold by the monopolist as a separate good sold in an independent market. For each unit of the good, the monopolist chooses a marginal type (lowest type served) subject to a constraint: the schedule of marginal types must be monotonically non-decreasing in the amount of the good being sold. I show that the monopolist's problem can be treated like a finite horizon multi-stage decision problem, and solved by backward induction using Bellman's equation. This approach identifies the optimal nonlinear price schedule whether or not the monotonicity constraint binds, and highlights the economic intuition behind the construction of an optimally ironed marginal type schedule an the corresponding marginal price schedule.

The third essay looks at the optimal pricing behavior of a monopolist screening consumers on a single type parameter, when the single crossing condition is violated. I show that under standard assumptions of one-dimensional screening problems, violation of the single crossing condition can allow for positive and globally incentive compatible assignments through which the monopolist extracts all equilibrium surplus. This sharply contrasts with the case of second-degree price discrimination when the single crossing condition is satisfied.

DEDICATION

I dedicate this dissertation to my wife, Katie, and my parents, Robert and Joanne, for their constant love and support.

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LIST OF ABBREVIATIONS

AQ	Automatic Qualifier
BCS	Bowl Championship Series
FBS	Football Bowl Subdivision
FE	Fixed Effects
GDP	Gross Domestic Product
IC	Incentive Compatibility
IR	Individual Rationality
LPM	Linear Probability Model
MP	Marginal Product
MRP	Marginal Revenue Product
NCAA	National Collegiate Athletic Association
NFL	National Football League
OL	Ordered Logit
US	United States
USD	United States Dollars
USDOE	United States Department of Education

CHAPTER 1

ARE YOU HIRING JOHNNY FOOTBALL OR JOHNNY DOE? THE EFFECTS OF UNCERTAIN LABOR QUALITY ON EMPLOYER MONOPSONY IN COLLEGE FOOTBALL.

Few labor markets in the United States (US) have received as much media attention and public scrutiny of late, as the market for college football players. The collegiate football industry in the US has long been offered as an example of a monopsonistic labor market,[1] primarily because National Collegiate Athletic Association (NCAA) amateurism rules limit player compensation to a full-tuition scholarship and the coverage of some additional expenses, restricting wage competition among employers. In recent years, several challenges to these rules have been mounted. For example, Gregory (2013, September 16), the cover story of *Time* magazine, advocates allowing player payment beyond the current limit. In April of 2014, football players at Northwestern University voted on the certification of a players' union, a move which the University is presently challenging [2] In August of 2014, the NCAA voted to allow member institutions of the "power five" conferences (65 schools in all), a degree of autonomy in determining the amount of expenses these schools offer to cover for athletes as part of their compensation package [see Terlep (2014, August 7)]. In January of 2015, these "power five" member schools approved adding a stipend to scholarships for 98 male athletes (85 football, 13 basketball), and a matching number of female athletes, beginning in August 2015. On average, these stipends are expected to amount to $2,500 annually

[1] See for instance Koch (1973), Becker (1983, September 30), or Fleisher et al. (1992). I use 'monopsony' to broadly refer to labor markets in which a sole employer or several employers possess some degree of market power.

[2] See Wolken (2014, April 24). The results of the vote are sealed and currently unknown while the legal challenge pends.

[see Berkowitz (2015, January 17)]. Also in August of 2014, a US District Court ruled in the O'Bannon v. NCAA class action suit, issuing an injunction against NCAA rules that prohibit schools from offering athletes a share of revenue generated from the commercial use of their images or likenesses. The NCAA intends to appeal the ruling [see Romney (2014, August 8)]. These challenges to the amateurism rules typically contend that players generate revenue for their institution which exceeds their capped compensation; and moreover, that the imposed compensation cap reflects an exercise of monopsony power by the NCAA member institutions, which allows them to systematically under-compensate college football players. Despite some evidence from Brown (1993) and Brown (2011) that elite college football players are currently under-compensated, the effect of the existing system on the average player's compensation, is unclear. In particular, the existing literature overstates the degree of monopsony power exercised over elite players, by implicitly treating the future quality of prospective labor as known at the time of hiring.

Brown (1993) is often cited as evidence of employer monopsony in college football.[3] Brown estimates that the marginal revenue product of labor (henceforth MRP) for an elite college football player is greater than $500,000 annually, substantially more than the market value of the typical compensation package.[4] Using the 2004 to 2005 season, Brown (2011) suggests that a premium player generates more than $1,000,000 annually. However, the data used to generate such estimates are from a selected sample of players who went on to be drafted in the National Football League (NFL). The majority of scholarship players participating at FBS programs do not graduate into the NFL.[5] As such, estimates based on this select sample do not reflect the value of the average player; and for the purposes of NCAA policy evaluation, are not applicable to the wider population of interest. More importantly, since the elite status of these players (as defined by a future NFL draft selection) was unknown at the time they participated in the college football labor market, the Brown (1993) MRP estimates are inappropriate for assessing the exercise of monopsony power, even as it relates to the most productive players in college football.

In order to assess the exercise of monoposony power, it is necessary to differentiate information available prior to hiring (henceforth *ex ante*), and information available after hiring (henceforth *ex post*). A monopsony rent on labor is the employer's return on a worker beyond what his return would be, given a competitive labor market. Wages and employment decisions are based only on the information available *ex ante*, and competitive wages approach *ex post* realized MRP only if future labor quality is fully known *ex ante*. Among college football players, *ex ante* promise imperfectly correlates with *ex post* realized quality. Many of the *ex ante* most promising players are not among the *ex post* highest quality, and many of the *ex post* highest quality players were not among the *ex ante* most promising. So MRP estimates from the *ex post* highest quality players overstate the expected MRP of even

[3]See for instance Boal and Ransom (1997) or Kahn (2007).

[4]Brown (1993) uses survey responses from 39 Division I-A football programs for the 1988 college football season. He correlates school football revenue with the number of players on each team that were drafted into the NFL within four years.

[5]I estimate that from 2004 to 2011, eventual NFL draft picks accounted for roughly 9% of all FBS scholarship players.

the *ex ante* most promising players. Therefore, in the college football labor market (and any other in which the future quality of labor is uncertain), a monopsony rent on labor is appropriately defined as the difference between *ex ante* expected MRP and wage. The degree of exercised monopsony power then, can be measured as the fraction of expected MRP that employers retain as rents, following Pigou (1924).

Nonetheless, there is practical justification for following Brown (1993), following Brown (2011), and others, in estimating marginal productivities using *ex post* realized quality rather than *ex ante* expected quality.[6] Given the imperfect correlation between *ex ante* promise and *ex post* quality, *ex post* quality will necessarily be a less noisy signal of a player's productive contribution. Thus, marginal product estimates will be more precise if estimated using *ex post* realized quality as opposed to *ex ante* promise. MRP estimates as a function of realized quality possess more information than employers have *ex ante*, and necessarily overstate the expected value of the most promising prospective players. Consequently, inference regarding employer monopsony power necessitates that such estimates be adjusted to reflect the *ex ante* uncertainty of labor quality.

For the empirical exercise, I collect a panel of annual institution-level revenue and winning percentage data from 114 NCAA member institutions, who competed in FBS football from 2004 to 2011. Treating on-the-field success as the output of a college football team I first estimate team winning percentage as a function of *ex post* realized labor quality, and then estimate school revenue as a function of team winning percentage. Within the institution, I proxy for differences in realized labor quality by sorting players into three separate *ex post* quality tiers. Players that go on to be drafted in the NFL constitute the *ex post* top tier.[7] An *ex post* middle tier is constructed using annual rankings of NFL draft prospects from CBS Sports.[8] Specifically, this middle tier consists of players who were ranked among

[6]This empirical approach is adopted in Brown (1994) for men's collegiate basketball, Brown and Jewell (2006) for women's collegiate basketball, and Kahane (2012) for men's collegiate hockey.

[7]NFL draft results are available at: nfl.com/draft/history/fulldraft, as well as many other sources. A total of 253 players are typically selected in each year's NFL draft.

[8]Data were retrieved from `http://www.cbssports.com/nfl/draft/prospectrankings`.

the top 500 draft prospects by CBS Sports, but not drafted. A bottom tier of all remaining scholarship players is implied. I estimate team winning percentage as function of the number of players a team employs within the *ex post* quality tiers. However, per NCAA regulations, FBS programs may have no more than 85 scholarship players annually.[9] Programs typically operate at the limit, making a team's annual counts of top, middle, and bottom tier players linearly dependent. The marginal product of labor (henceforth MP) for *ex post* top, middle and bottom tier players are therefore, not separately estimable. Instead, the estimated production equation identifies the difference in MP between top, middle, and bottom tier players. I am able to correct for this issue by establishing plausible upper and lower bound estimates for the MP of *ex post* bottom tier players. This yields interval estimates of MP, and subsequently MRP, for *ex post* top, middle, and bottom tier players.

To account for the information schools possess *ex ante*, I incorporate scouting data generated prior to player recruitment by Rivals.com.[10] Using Rivals ratings for 6,604 prospective college football players recruited from 2002 to 2008, I sort players into tiers according to *ex ante* promise. I estimate the probability that a player will belong to each of the *ex post* quality tiers, conditional on Rivals prospect ratings which serve as *ex ante* forecasts of player quality. The conditional probability and MRP interval estimates then combine to yield estimates of expected MRP as a function of *ex ante* promise. These estimates permit inference about the degree of monopsony power exercised across the college football labor market.

The results of this paper show that labor quality in college football involves considerable uncertainty *ex ante*. An initial picture of this labor quality uncertainty is found in Table 1.1. Note for instance, that of the 6604 prospects included in the 7-year sample of *ex ante* forecasts, 201 (about 3%) were given Rivals' top rating of 5-star, making this a very select group.[11] Of those top 201 prospective players, only 93 (46%) went on to be drafted,

[9] The NCAA's current Division I manual can be downloaded at: ncaapublications.com.

[10] Rivals is a scouting agency that annually rates and ranks thousands of prospective college football players based on talent. Data are retrieved from footballrecruiting.rivals.com/

[11] As a reference point 253 players are selected *annually* in the NFL draft. A complete description of the

and only 125 (62%) of the 5-star players were selected into the *ex post* groups of middle or top tier players (see Table 1.2). This *ex ante* uncertainty is quantitatively important. Thus, the MRP estimates for *ex post* top tier players, exceed the *ex ante* expected MRP estimates for 5-star prospects, by at least $251,393 and 76.56% of the expected MRP estimate.[12]

Table 1.1. Draft Outcomes by Rivals Ratings 2002-08

Rivals Rating	# with Rating	% with Rating*	# Drafted	% Drafted*
5-star	201	3.04%	93	46.27%
4-star	1839	27.84%	353	19.20%
3-star	4564	70.46%	402	8.81%
Total	6604	100%	848	12.84%

*% of 6604 player sample

Table 1.2. Ranked or Drafted by Rivals Ratings 2002-08

Rivals Rating	# with Rating	# Ranked/Drafted	% Ranked/Drafted*
5-star	201	125	62.19%
4-star	1839	596	32.41%
3-star	4654	868	19.02%
Total	6604	1589	24.06%

*% of 6604 player sample

My results also challenge any notion that the average performer in FBS college football generates revenue for his school that significantly exceeds his compensation. Over this 8-year period, *ex post* top tier players accounted for approximately the top 9% of scholarship players participating at the 114 FBS institutions sampled. Middle tier players accounted for roughly the top 9% to 18%, with bottom tier players accounting for the remaining 82%. The NCAA reports that the average out-of-state tuition scholarship at a public university carries a market value of approximately $25,000 per year. Figuring conservatively and taking that figure as the annual value of the compensation package, this paper's results do suggest that over a four year collegiate career, NFL quality players generate revenue for their institutions in excess of their compensation by an economically significant amount (consistent with the

Rivals rating scale and all other data are found in Section 1.2.

[12]These estimates are of a player's MRP over a four year collegiate career.

findings of Brown (1993), although my annual estimates are about $370,000 less). MRP estimates for *ex post* middle tier players are economically significant as well, but the difference in MRP and the compensation's market value is statistically insignificant.

Overall, my findings suggest that market-wide, NCAA member institutions do not enjoy as great a degree of monopsony power as some contend. Moreover, the current compensation may meet or even exceed the amount that a player of average ability would command in a competitive labor market. These results have interesting implications as courts and member institutions consider policy reforms. They are also notable as the potential players' union at Northwestern pends, and players nationwide weigh the merits of unionization. It appears that the current system might actually benefit a majority of FBS scholarship players, at the expense of a highly productive minority. Ironically, this result would also be consistent with the presence of a strong labor union.

1.1 The College Football Labor Market

Monopsony power in the college football labor market results primarily from NCAA amateurism rules, which limit player compensation to a full-tuition scholarship and the coverage of certain expenses.[13] These rules effectively restrict wage competition among schools.

Two different policies restrict labor mobility in the college football industry, although only one restricts mobility *ex ante*. Per NFL rules, players are not eligible to play in the NFL until 3 years after their senior year of high school. Thus collegiate football is effectively the only option for high school graduates who seek employment in the US as football players. It should be noted however that more than 1000 college football prospects receive scholarships from FBS schools each year, whereas only 253 players are selected annually in the NFL draft. Therefore, the NFL's eligibility requirement realistically restricts labor mobility only for the most promising college football prospects. The other policy limiting labor mobility is an NCAA rule that requires players to sit out one football season if they wish to transfer to another FBS program. Since this policy only applies to players *ex post*, it is not a source of monopsony power in the primary college football labor market, where prospective players are free to accept scholarship offers from any school they choose. Were the amateurism rules lifted, these transfer restrictions would prevent secondary labor markets from materializing.

As mentioned, Brown (1993, 2011) find that the realized MRP of an *ex post* NFL caliber player exceeds the market value of his compensation package significantly. Note however, that FBS programs typically award full scholarships to 85 players annually, the maximum allowed by NCAA rules.[14] Thus, within a school, wages are identical for the *ex ante* most and least promising players. Therefore, the rents implied by MRP estimates such as Brown's, overstate monopsony power in two regards. Due to the effectively uniform wage within schools, rents will necessarily be higher on players of above average *ex ante* promise.

[13] Among these expenses are room and board, textbooks, and medical insurance.

[14] NCAA regulations limit FBS schools to 85 scholarship players under normal circumstances. That limit is sometimes reduced for programs placed on probation.

Moreover, as a consequence of uncertain labor quality, MRP estimates from the *ex post* highest quality players overstate the expected MRP for even the *ex ante* most promising players. This therefore overstates rent on the *ex ante* most promising players as well.

Finally, a note on the annual provision of football scholarships by FBS schools. FBS institutions are not required to provide full scholarships to all 85 players, or even meet the 85 player limit, but they typically do.[15] My results suggest that the *ex ante* expected MRP for many FBS players is less than the cost of the scholarship. Thus, the annual provision of 85 full scholarships suggests that schools may not maximize expected profits from football. It is important to remember that football programs operate within athletic departments, which in turn operate within much larger universities as a whole. University presidents and chancellors likely have objectives beyond simply maximizing expected profits from football or even athletics. The fact that these same schools often fund scholarships for several men's sports known to operate at losses, would seem to support this notion. While decisions regarding the allocation of football scholarships to certain players are made by head coaches or members of their staff, coaches likely lack the authority to divert scholarship funds to other purposes that might yield a greater return. Thus, the provision of a scholarship to a football player does not necessarily imply that his expected MRP is greater than or equal to the cost of the scholarship.

[15] Although a player receiving a partial scholarship would count against the 85 player limit.

1.2 Data Description

The data used in this paper are collected at both the individual or player-level, as well as the university or institution-level. In this section I give a detailed explanation of these data, beginning with those collected at the player-level, and followed by those collected at the institution-level.

1.2.1 Player-Level Data

I collect Rivals.com data for 6604 college football players who meet the following criteria: they were rated 3-star or higher by Rivals, and they were recruited out of high school between 2002-2008, to one of the 114 institutions sampled. Rivals ratings are first available for the year 2002, and I collect them only up through 2008 to allow players sufficient time to matriculate to the NFL. Rivals describes a five-star prospect as “one of the nation’s top 25-30 players”, a four-star prospect as “a top 250-300 or so player”, and a three-star prospect as “a top 750 player.” I also collect the player’s position as listed by Rivals, the school that recruited him, and his recruitment year. These data are available online at: footballrecruiting.rivals.com.

All FBS scholarship players are sorted into three *ex post* quality tiers. Recall that players drafted in the NFL following their collegiate careers constitute the top tier; and players ranked among the top 500 draft prospects by CBS Sports, but not drafted, make up the middle tier. Players neither ranked top 500, nor drafted, constitute the bottom tier. Historical NFL draft records are available from numerous sources, including online at: nfl.com/draft/history/fulldraft. CBS Sports first published draft prospect rankings following the 2004 college football season, for the 2005 NFL draft. Two things should be noted about the data I collect from CBS Sports. First, in some years CBS sports ranked slightly more than 500 players. For example, prior to the 2006 NFL Draft they ranked the top 540 prospects. For consistency I use their rankings up to the same cutoff point every year, which is the top 499.[16] Thus, the middle tier actually consists of players ranked among the top 499 prospects, but not drafted. I call it the "top 500" for convenience of exposition. The other note is that players are sorted into these tiers based only on their ranking in the year they finished their collegiate careers. That is, juniors ranked among the top 500 prospects who stay in college for their senior season instead of entering the draft, are not thereby counted as middle tier players. They are instead sorted to one of the *ex post* tiers based on their ranking or draft outcome following their senior season. The CBS Sports prospect rankings are available online at: cbssports.com/nfl/draft/prospectrankings.

1.2.2 Institution-Level Data

Institution-level data are collected on 114 NCAA member schools whose football programs have participated in Division I-A/FBS since the 2004 college football season.[17] Data are collected annually from 2004-2011.

Annual football revenue figures come from the US Department of Education (USDOE). Beginning in 2003, the athletic departments of NCAA member institutions were required to submit annual financial reports to the USDOE, in compliance with the Equity in Athletics Disclosure Act of Congress. These reports are publicly available online at the USDOE's website: ope.ed.gov/athletics. The football revenue measure is fairly comprehensive. The USDOE describes it as: "revenues from appearance guarantees and options, an athletic conference, tournament or bowl games, concessions, contributions from alumni and others, institutional support, program advertising and sales, radio and television, royalties, signage and other sponsorships, sport camps, state or other government support, student activity fees, ticket and luxury box sales, and any other revenues attributable to intercollegiate athletic activities." The revenue data I collect is attributable only to the "intercollegiate athletic activity" of football. I report all revenue figures in 2005 USD, using the US GDP

[16] They only ranked 499 prospects in 2004, the smallest number in any year.

[17] There were 120 FBS programs by the 2011 college football season. The six additional programs joined the FBS during the sample period, and are excluded from the sample for that reason.

deflator. Note that revenue was not reported for the University of Maryland in years 2005-2007, leaving three missing revenue observations, which are skipped in estimation.

A school's annual winning percentage is the number of games won in a season, as a percentage of the total number of games played. Such information is available from a wide range of sources. I collect this information from the Stassen College Football Information database, which is accessible online at: football.stassen.com.[18]

[18]The Stassen Database credits: jhowell.net/cf/scores/ScoresIndex.htm, as their initial source.

1.3 Methodology

I estimate MRP by *ex post* player quality in two steps. I first estimate expected winning percentage as function of *ex post* realized player quality, which yields estimates of marginal product by *ex post* quality tier. I then estimate expected revenue as a function of winning percentage, which yields estimates of the change in expected revenue given changes in winning percentage. MRP estimates by *ex post* quality tier are imputed as products of the two sets of estimates. This two step approach is consistent with Scully (1974) and Krautmann (1999), who estimate MRP in professional baseball and football respectively. At the player level, I then estimate the probability that a player will be selected into each of the *ex post* quality tiers conditional on his *ex ante* Rivals rating. I combine the estimates of MRP by *ex post* quality tier with these estimated conditional probabilities to impute expected MRP by *ex ante* Rivals rating.

Finally, I test the sensitivity of these results to a narrower definition of the college football labor market. I restrict my sample to players and institutions that participated in Bowl Championship Series (BCS) "automatic qualifier" conferences. The BCS, an association of the four most lucrative post-season bowl games, had agreements with these conferences which guaranteed the conference champions an invitation to one of those four (and eventually five) games.[19] Over the 2004 to 2011 sample period, the BCS "automatic qualifier" conferences were analogous to the current "power five" conferences. I then estimate MRP by *ex post* player quality, and expected MRP by *ex ante* player promise, for this subsample.

[19] The "automatic qualifier" schools were members of the Atlantic Coast Conference, the Big East, the Big Ten, the Big Twelve, the Pacific Ten, the Southeastern Conference, and the University of Notre Dame.

1.3.1 Production Equation

I measure output in a given year through winning percentage.[20] $WPCT_{i,t}$ denotes winning percentage for school i in season t.[21] $TOP_{i,t}$ denotes the count of players from school i that were selected in the NFL Draft immediately following season t; that is, the number of *ex post* top tier players at school i who finish their collegiate careers in season t. $MID_{i,t}$ denotes the count of players from school i, who were ranked among the top 500 prospects by CBS Sports in year t, entered the draft immediately following season t, but were not drafted; that is, the number of *ex post* middle tier players at school i who finish their collegiate careers in year t. The estimated production equation is,

$$WPCT_{i,t} = a_i + [\mathbf{TOP}_{i,t}, \mathbf{MID}_{i,t}]\boldsymbol{\alpha} + \epsilon_{i,t}. \tag{1.1}$$

Here, $\mathbf{TOP}_{i,t}$ denotes the vector $[TOP_{i,t}, TOP_{i,t+1}, TOP_{i,t+2}, TOP_{i,t+3}]$, and $\mathbf{MID}_{i,t}$ denotes the vector $[MID_{i,t}, MID_{i,t+1}, MID_{i,t+2}, MID_{i,t+3}]$. In any year, teams consist of players who are at different stages in their college careers. I include one, two, and three year leads of $TOP_{i,t}$ and $MID_{i,t}$. This accounts for the contribution of top and middle tier players on a school's year t football team, who would complete their careers in later years. For instance, $TOP_{i,t+1}$, which is the count of players at school i drafted following season $t+1$, is effectively the number of top tier players at school i who were juniors in year t. I include institution fixed effects to control for school specific characteristics such as quality of practice facilities and coaching staffs, which correlate with both winning percentage and *ex post* player quality. Note that while these features may not be entirely time invariant, they tend to vary at a much slower frequency than winning percentage and the composite labor quality of the team.

[20] Measuring output quality in commercial sports through winning percentage is consistent with Scully (1974); Atkinson et al. (1988); and Krautmann (1999).

[21] Index t denotes the calendar year in which the college football season began, and includes all bowl games and the NFL Draft immediately following that season. For instance $t = 2004$ begins with the open of the 2004 college football season, and ends with NFL draft immediately following, which actually took place in April 2005.

1.3.2 Revenue Equation

Revenue from football for institution i in season t is denoted $REV_{i,t}$. The estimated revenue equation is,

$$REV_{i,t} = b_i + \beta_1 WPCT_{i,t} + \beta_2 WPCT_{i,t-1} + \boldsymbol{z}'_{i,t}\boldsymbol{\beta} + v_{i,t}. \tag{1.2}$$

Again, $WPCT_{i,t}$ is the winning percentage for school i in the given season, and $WPCT_{i,t-1}$ is winning percentage for i in the previous year's season. The components of revenue that likely vary the most from year to year are ticket sales, bowl appearances, and alumni contributions. All three directly correlate with the success of the team, which is accounted for by $WPCT_{i,t}$. Some ticket sales occur before the start of the season. Conceivably, most variation in pre-season ticket sales for an institution, depends on expected performance of the team among its fans. I include $WPCT_{i,t-1}$ to account for pre-season expected performance. This is consistent with Atkinson et al. (1988) who model variation in current revenue for an NFL team, as a function of the number of wins the team had in the previous season.

The vector of control variables $\boldsymbol{z}_{i,t}$ contains year dummies and the count of home games school i played in year t, against opponents ranked in the Associated Press Top 25 poll, denoted $HR_{i,t}$. This is included to control for any annual variation in a school's revenue that might be due to variation in the quality of the teams it hosts. I include school fixed effects to control for characteristics that correlate with both winning percentage and revenue. These would be features such as stadium size, regional substitutes for college football, conference membership, and especially historic success of the program. All of these features are essentially fixed, or vary at a substantially slower frequency than annual winning percentage, and are likely absorbed in school intercept estimates.

1.3.3 Selection Equation

Equations (1.1) and (1.2) allow estimation of realized MRP for *ex post* top and middle tier players. A prospective player j, is recruited to institution i in year s, and finishes his collegiate career in $s+h$. Let the latent variable $q^*_{j(i),s+h}$ denote the *ex post* realized quality of player j, recruited to school i in year s, and completing his career in year $s+h$. Let the ordinal variable $TIER_{j(i),s+h}$ indicate the *ex post* quality tier of player j, recruited to school i in year s and completing his career in year $s+h$. It takes the values

$$TIER_{j(i),s+h} = \left\{ \begin{array}{cc} 2, & \text{if top tier} \\ 1, & \text{if middle tier} \\ 0, & \text{if bottom tier} \end{array} \right\}. \tag{1.3}$$

The latent measure of *ex post* quality is modeled as the following function of information available *ex ante* (in year s),

$$q^*_{j(i),s+h} = \gamma_1 FVST_{j(i),s} + \gamma_2 FRST_{j(i),s} + \boldsymbol{z}'_{j(i),s}\boldsymbol{\gamma} + c_i + \psi_{j(i),s+h}. \tag{1.4}$$

Variation in $q^*_{j(i),s+h}$, is observed through $TIER_{j(i),s+h}$, and from equation (1.4), I estimate the conditional probability of each $TIER$ outcome, under both the logit and linear probability specifications. $FVST_{j(i),s}$ and $FRST_{j(i),s}$ are dummy variables indicating whether or not players were rated 5-star by Rivals or 4-star by Rivals, respectively. The vector of control variables $\boldsymbol{z}_{j(i),s}$, contains dummy variables indicating the position that j played when recruited, and the year in which he was recruited. The effect of the school attended on a player's *ex post* quality, c_i, likely correlates with his Rivals rating. Prospective players self-select into a school by choosing from their available scholarship offers. Players of greater *ex ante* promise have more options, and are likely able to self-select into schools with higher values of c_i.

To account for this school specific effect, c_i, I utilize two possible approaches. One approach is to assume a linear probability specification, include school fixed effects, and estimate (1.4) twice; first with a dummy dependent variable indicating whether or not the player was realized as *ex post* top or middle tier, and then with a dummy dependent variable indicating whether or not he was realized as top tier. The linear probability model has notable drawbacks, but does permit c_i to be completely controlled for through linear fixed effects estimation. A logit or probit specification implies a non-linear estimation equation, meaning c_i cannot be transformed away as in the linear model.[22] As an alternative, I estimate (1.4) under a logit specification, and partially control for c_i using the approach of Mundlak (1978). If a linear relationship exists between c_i and the explanatory variables in (1.4), then c_i can be expressed as a linear function of school averages of the explanatory variables in (1.4). That is, one can let the heterogeneous effect be

$$c_i = \theta_0 + \overline{\boldsymbol{X}}_i' \boldsymbol{\theta} + \kappa_i, \tag{1.5}$$

where the vector $\overline{\boldsymbol{X}}_i = [\overline{FVST}_i, \overline{FRST}_i, \overline{\boldsymbol{z}}_i]$ contains averages of the explanatory variables from all players recruited to school i over the full sample period. The reduced form selection equation for the Mundlak estimation approach is then

$$q^*_{j(i),s+h} = \theta_0 + \gamma_1 FVST_{j(i),s} + \gamma_2 FRST_{j(i),s} + \boldsymbol{z}'_{j(i),s}\boldsymbol{\gamma} + \overline{\boldsymbol{X}}_i' \boldsymbol{\theta} + \kappa_i + \psi_{j(i),s+h}. \tag{1.6}$$

A residual school specific effect, e_i, remains. However, since variations in school means of the explanatory variables are now removed from c_i, e_i is less likely to correlate with individual rating and position than is c_i. That is, the school means, $\overline{\boldsymbol{X}}_i$, should project onto the component of c_i that is correlated with the explanatory variables, thereby controlling for any school specific effects with which the explanatory variables would otherwise be correlated. I estimate the conditional probabilities of each $TIER$ outcome implied by (1.6), under an

[22]Moreover, estimates of c_i are inconsistent due the incidental parameters problem. See Neyman and Scott.

ordered logit specification. Conditional probability estimates from both the linear probability and ordered logit Mundlak approaches are reported in Section 1.4.4, and used in Section 1.4.5 to impute estimates of *ex ante* expected MRP.

1.4 Results

In this section, I begin by using fixed effects estimates of equation (1.2) to establish upper and lower bound estimates for the MP of a bottom tier player. These boundary estimates then imply upper and lower bound estimates for the MP of a middle tier and top tier player. I then use these MP estimates, along with estimates of fixed effects estimates of Equation (1.2), to impute upper and lower bound estimates of MRP for players of each *ex post* quality tier. These *ex post* MRP estimates are then adjusted using estimates of the probability that a player is realized within each *ex post* quality tier, conditional on his *ex ante* Rivals rating, football position, and school selection. The adjusted values are estimates of *ex ante* expected MRP. Finally, these steps are repeated for the subsample of BCS automatic qualifier schools (and players attending those schools).

1.4.1 Marginal Product Estimates

Production equation estimates are reported in Table 1.3. Column (1) reports random effects estimates of equation (1), and column (2) reports the estimates including school fixed effects. The reported "Average FE" estimate in column (2) is an average of the 114 different school-specific intercept estimates. That is,

$$\text{Average FE} = \hat{a} = \frac{1}{114}\sum_{i=1}^{114}\hat{a}_i. \tag{1.7}$$

Standard errors, clustered by school, are given in parentheses. $WPCT$ is measured in percentage points. Recall that TOP and MID are counts of *ex post* top and middle tier players, respectively. Also remember that the leads of TOP and MID are included to account for *ex post* top and middle tier players who are part of the team in year t, but complete their collegiate careers in later seasons. For instance, $TOP_{i,t+1}$ can be thought of as the count of top tier juniors at school i in year t, and so on. The column (2) estimates suggest that top tier players in their final three seasons, and middle tier players in their final two seasons, are significantly more productive than bottom tier players.

Table 1.3. Production Equation Estimates

Variable	(1) $WPCT$	(2) $WPCT$
TOP_t	3.064***	3.419***
	(0.363)	(0.531)
TOP_{t+1}	2.376***	2.711***
	(0.372)	(0.495)
TOP_{t+2}	1.177***	1.534***
	(0.371)	(0.481)
TOP_{t+3}	0.269	0.685
	(0.365)	(0.494)
MID_t	1.405***	1.468**
	(0.476)	(0.595)
MID_{t+1}	1.343***	1.452***
	(0.436)	(0.518)
MID_{t+2}	0.704	0.659
	(0.455)	(0.558)
MID_{t+3}	-0.955**	-0.961*
	(0.450)	(0.579)
Intercept/Average FE	33.378***	30.218***
	(1.877)	(3.503)
School FE	N	Y
N	570	570

***$p < .01$, **$p < .05$, *$p < .1$

The fact that the contribution of *ex post* bottom tier players is not separately identified from the school intercepts presents a complication. The coefficient on $TOP_{i,t}$ for instance does not estimate the MP of an *ex post* top tier senior, but rather the difference between the MP of an *ex post* top tier senior and the MP of an *ex post* bottom tier player. To handle this issue I infer upper and lower bound estimates for the MP of an *ex post* bottom tier player.

From these boundary estimates of bottom tier MP, I am able to construct interval estimates for the MP of *ex post* top and middle tier players.

Establishing upper and lower bounds for the MP of a bottom tier player reduces to an interpretation of the school intercept average. Note that the school specific intercepts may be interpreted as the conditional expectation

$$a_i = \mathrm{E}[WPCT_{i,t} | (\mathbf{TOP}_{i,t}, \mathbf{MID}_{i,t}) = \mathbf{0}_{1\times 6}]. \tag{1.8}$$

That is, $\hat{\alpha}_i$ estimates expected winning percentage, if all 85 players at school i in year t were *ex post* bottom tier. The average fixed effect of 30.218% reported in Table 1.3 is the industry-wide average of said estimates. The question is what portion of that estimated 30.218% is attributable to the 85 *ex post* bottom tier players, and what portion is attributable to coaches, practice facilities, and other school capital?

The natural lower bound for the annual MP of *ex post* bottom tier players is zero. This credits the 85 bottom tier players with none of the 30.218% expected $WPCT$. In this extreme, the coefficients reported in column (2) of Table 1.3 can be interpreted as marginal product estimates. Since the career MP for bottom tier players is taken to be zero, it follows that the lower bound estimate of career MP for an *ex post* top tier player is

$$\hat{\alpha_1} + \hat{\alpha_2} + \hat{\alpha_3} + \hat{\alpha_4} = 8.348\%. \tag{1.9}$$

Analogously, the upper bound treatment is to credit the 30.218% $WPCT$ entirely to the 85 bottom tier players. The upper bound estimate of annual MP for bottom tier player is then

$$\frac{\hat{\alpha}}{85} = 0.356\%, \tag{1.10}$$

and the upper bound estimate for MP over a 4-year career is simply

$$4 \cdot \left(\frac{\hat{\alpha}}{85}\right) = 1.422\%. \tag{1.11}$$

It follows then that the upper bound estimate of career MP of an *ex post* top tier player is

$$\left(4 \cdot \left(\frac{\hat{\alpha}}{85}\right)\right) + (\hat{\alpha_1} + \hat{\alpha_2} + \hat{\alpha_3} + \hat{\alpha_4}) = 9.77\%. \tag{1.12}$$

The lower and upper bound estimates of career MP for a middle tier player are calculated the same as in equations (1.9) and (1.12), but with the coefficients $(\hat{\alpha_5}, \hat{\alpha_6}, \hat{\alpha_7}, \hat{\alpha_8})$ replacing $(\hat{\alpha_1}, \hat{\alpha_2}, \hat{\alpha_3}, \hat{\alpha_4})$.

Table 1.4 reports these boundary estimates of marginal product for *ex post* top, middle, and bottom tier players, over four year careers. Standard errors for the boundary estimates are reported in parentheses, and 95% confidence intervals are reported in brackets. Recall that the dependent variable being estimated in Tables 1.3 and 1.4, is $WPCT$ from a single season (approximately 12 games). As a reference point for interpreting these coefficients, a single game is $8.\overline{333}\%$ of 12 game season. Thus, the career MP estimates for an *ex post* top tier player can be loosely interpreted as one additional win.

An initial observation from Table 1.4 is that the *ex post* top, middle, and bottom tiers appear to be appropriately defined. As expected, MP estimates increase from bottom to middle tier, and middle to top tier. Comparison of the confidence intervals on both the upper and lower bound point estimates, shows that a top tier player is significantly more productive than a middle tier player. Moreover these differences do seem substantial. At the upper bounds, *ex post* top and middle tier players are more productive than bottom players, by factors of approximately 7.25 and 2.9 respectively. At the upper and lower bound estimates, *ex post* top tier players are more productive than middle tier players by factors of roughly 2.47 and 3.23 respectively.

Table 1.4. Marginal Product Estimates: Career Contributions

Ex post Quality	Lower Bound	Upper Bound
Top Tier	8.348***	9.770***
	(1.327)	(1.226)
	[5.720, 10.977]	[7.342, 12.199]
Middle Tier	2.617*	4.039***
	(1.447)	(1.343)
	[-0.250, 5.484]	[1.379, 6.699]
Bottom Tier	——	1.422***
	——	(0.165)
	——	[1.095, 1.749]

[95% Confidence Interval]

1.4.2 Revenue Equation Estimates

Revenue equation estimates are reported in Table 1.5. Column (1) reports random effects estimates of equation (1.2), and column (2) reports the estimates including school fixed effects. The estimated marginal effects of current and lagged winning percentage are economically significant in column (2). The reported two-year effect of winning percentage is the sum of the coefficients on $WPCT_t$ and $WPCT_{t-1}$. This sum suggests that an additional win, or approximately 8.3% increase in $WPCT_t$, generates an additional $490,408 in revenue over two years.[23]

[23] All revenue figures are measured in 2005 USD.

Table 1.5. Revenue Equation Estimates

Variable	(1) *REV*	(2) *REV*
$WPCT_t$	44,873.67***	36,304.79***
	(9,809.89)	(10,987.41)
$WPCT_{t-1}$	30,898.58***	22,780.56**
	(9383.06)	(10,473.30)
2-year effect $WPCT$	**75,772.25*****	**59,085.35*****
95% C.I.	[39,676.15, 99,862.90]	[35,209.52, 82,961.18]
	(15,169.05)	(17,191.07)
HR	166,951.90	-16,381.40
	(120,121.80)	(133,014.00)
Intercept/Average FE	12,465,219.70***	13,531,982.1***
	(1,451,299.90)	(869,716.7)
Year FE	Y	Y
School FE	N	Y
N	795	795

***$p < .01$, **$p < .05$, *$p < .1$

1.4.3 MRP Estimates by *Ex Post* Quality

MRP estimates for *ex post* top, middle, bottom tier players are reported in Table 1.6. These values come from the fixed effects estimates of equations (1.1) and (1.2). Standard error estimates, clustered by school, are reported in parentheses; and 95% confidence intervals based on those standard errors are reported in brackets. These standard error estimates are based on delta method approximate distributions for the MRP estimates, which are nonlinear combinations of product and revenue equation estimates.

Table 1.6. MRP Estimates by *Ex Post* Quality: Career Contributions

Ex post Quality	Lower Bound	Upper Bound
Top Tier	$493,271.90***	$577,292.20***
	($148,444.10)	($166,488.10)
	[$202,326.80, $784,217.10]	[$250,981.60, $903,602.90]
Middle Tier	$154,635.90*	$238,656.20***
	($87,117.40)	($95,995.71)
	[-$16,329.47, $326,917.80]	[$50,508.05, $426,804.3]
Bottom Tier	—	$84,020.29***
	—	($24,048.93)
	—	[$36,885.26, $131,155.30]
Weighted Average	$74,285.82***	**$143,157.20*****
	($21,138.63)	($38,316.10)
	[$32,854.87, $115,716.80]	[$68,059.07, $218,255.40]
N	912	912

[95% Confidence Interval]

Consider the lower bound marginal product parameter for an *ex post* top tier senior, α_1. An increase in expected $WPCT_{i,t}$ of α_1 percentage points, increases expected $REV_{i,t}$ by $\alpha_1 \cdot \beta_1$ dollars. Recall that β_1 is the equation (1.2) parameter on $WPCT_{i,t}$. The increase in expected $WPCT_{i,t}$ also increases expected $REV_{i,t+1}$ by $\alpha_1 \cdot \beta_2$ dollars. The same follows for the other three seasons he plays. Thus, the lower bound estimate of MRP for an *ex post* top tier player, over a 4-year career, is

$$(\hat{\alpha_1} + \hat{\alpha_2} + \hat{\alpha_3} + \hat{\alpha_4}) \cdot (\hat{\beta}_1 + \hat{\beta}_2). \tag{1.13}$$

The upper bound MRP estimate for an *ex post* top tier player, over a 4-year career, is

$$\left[\left(\hat{a} \cdot \frac{4}{85}\right) + \hat{\alpha_1} + \hat{\alpha_2} + \hat{\alpha_3} + \hat{\alpha_4}\right] \cdot (\hat{\beta}_1 + \hat{\beta}_2). \tag{1.14}$$

The upper and lower bound MRP estimates for *ex post* middle tier players are imputed in the same way, but with the fixed effects coefficients on $\mathbf{MID}_{i,t}$ replacing the fixed effects coefficients on $\mathbf{TOP}_{i,t}$. The upper bound MRP estimate for *ex post* bottom tier players is simply

$$\left(\hat{a} \cdot \frac{4}{85}\right) \cdot (\hat{\beta}_1 + \hat{\beta}_2). \tag{1.15}$$

Note that $\left(\hat{a} \cdot \frac{1}{85}\right)$ was the upper bound estimate of the annual MP for a bottom tier player. The implied four year upper bound MP estimate for a bottom tier player is then $\left(\hat{a} \cdot \frac{4}{85}\right)$.

In Table 1.6 we see that both the upper and lower bound estimates of the four-year MRP for an *ex post* top tier player are economically significant. Note also that both point estimates are significantly different from the approximate compensation measure of \$100,000. The upper bound MRP estimates for *ex post* middle and bottom tier players are both economically significant. However, neither estimate is significantly different from \$100,000. This begins to call into question the notion that the average performer in FBS football generates revenue in excess of his compensation.

Over the 2004-2011 sample period, there were a total of 1,788 *ex post* top tier players, and 1,705 *ex post* middle tier players. Given the scholarship player limit of 85 per year, and accounting for the instances when schools faced reduced limits, there were a total of 77,436 rosters spots in my sample.[24] Assuming four year careers, and dividing that total by 4, suggests there were approximately 19,380 different scholarship players, and a total of 15,866 *ex post* bottom tier players, at the 114 institutions from 2004-2011. Based on this estimated *ex post* quality distribution, I compute the weighted average of the MRP's and report them toward the bottom of Table 1.6. Notice especially that even the upper bound weighted average estimate is not significantly different from \$100,000. These estimates fail to reject the hypotheses that the average FBS performer generates revenue in excess of his compensation.

[24]Information on scholarship limit reductions are from the NCAA's Legislative Service Database. This database records cases of major rules infractions and respective penalties, and is available online at web1.ncaa.org/LSDBi/exec/miSearch.

1.4.4 Selection Equation Estimates

Table 1.7 reports selection equation estimates under the linear probability model specification. Column (3) reports estimates from the full model with the dummy dependent variable, $I(TIER_{j(i),s+h} \geq 1)$; which indicates whether or not player j was *ex post* top or middle tier quality. Column (6) reports estimates from the full model with the dummy dependent variable, $I(TIER_{j(i),s+h} = 2)$, which indicates whether or not player j was *ex post* top tier quality. In both columns (3) and (6), the coefficients on $FVST$ and $FRST$ are statistically significant.

Table 1.8 reports selection equation estimates under the ordered logit specification. Column (3) reports estimates of the full model, which includes school averages of the explanatory variables as additional regressors. The coefficients on $FVST$ and $FRST$ are statistically significant. Under both specifications however, the parameters of interest are the conditional probabilities of each *ex post* $TIER$ outcome, which are reported in Tables 1.9, 11.10, and 1.11.

Table 1.9 reports estimated probabilities of being *ex post* top tier quality, conditional on an *ex ante* Rivals rating. Table 1.10 reports estimated probabilities of being *ex post* top or middle tier quality, given an *ex ante* Rivals rating. In both tables, linear probability model estimates are reported in columns (1) and (2), and ordered logit model estimates are reported in columns (3) and (4). Column (1) excludes school fixed effects, while column (2) includes them. Column (3) excludes the Mundlak school averages, and column (4) includes them. All columns in both tables include fixed effects for year recruited and position played. Under both specifications, controlling for heterogeneous school effects somewhat decreases the estimated probability of being *ex post* top or middle tier for 5-star and 4-star players, but somewhat increases the estimated probability of being *ex post* top or middle tier for 3-star players. This suggests that players of greater *ex ante* promise, and therefore more choices as to which school they attend, self-select into programs more capable of developing their *ex post* quality. Probability estimates of all *ex post* outcomes, conditioned on *ex ante*

Table 1.7. Selection Equation Estimates: Linear Probability Model

Variable	(1) $TIER \geq 1$	(2) $TIER \geq 1$	(3) $TIER \geq 1$	(4) $TIER = 2$	(5) $TIER = 2$	(6) $TIER = 2$
$FVST$	0.431***	0.432***	0.361***	0.374***	0.376***	0.330***
	(0.0393)	(0.0400)	(0.0392)	(0.0354)	(0.0354)	(0.0366)
$FRST$	0.134***	0.135***	0.0845***	0.103***	0.103***	0.0746***
	(0.0134)	(0.0132)	(0.0118)	(0.0117)	(0.0117)	(0.0103)
Intercept	0.203***	0.204***	0.235***	0.0987***	0.0874***	0.107***
	(0.0191)	(0.0245)	(0.0249)	(0.0129)	(0.0184)	(0.0197)
Year Recruited	Y	Y	Y	Y	Y	Y
Position	N	Y	Y	N	Y	Y
School FE	N	N	Y	N	N	Y
N	6604	6604	6604	6604	6604	6604

***$p < .01$

rating, are reported in Table 11. In Tables 1.9, 1.10, and 1.11, note the similarity between the conditional probability estimates under the linear probability specification with school fixed effects, and ordered logit specification with the Mundlak (1978) school averages. Going forward, I use the linear probability model estimates reported in the first three columns of Table 1.11, since imputed values for *ex ante* expected MRP will be essentially the same under both specifications.

Table 1.8. Selection Equation Estimates: Ordered Logit

Variable	(1) $TIER$	(2) $TIER$	(3) $TIER$
$FVST$	2.045***	2.063***	1.701***
	(0.161)	(0.165)	(0.172)
$FRST$	0.747***	0.755***	0.481***
	(0.0680)	(0.0674)	(0.0610)
$\overline{FVST}_i$	—	—	-0.008
	—	—	(2.697)
$\overline{FRST}_i$	—	—	1.870***
	—	—	(0.485)
Top Tier Threshold†	0.804***	0.805***	0.815***
	(0.028)	(0.028)	(0.028)
Year Recruited	Y	Y	Y
Position	N	Y	Y
Mundlak Averages	N	N	Y
N	6604	6604	6604

†Middle Tier threshold set to 0, ***$p < .01$

Table 1.9. Estimated Probability of being Drafted

Rivals Rating	(1)	(2)	(3)	(4)
5-star	0.4637	0.4274	0.4468	0.4178
4-star	0.1916	0.1722	0.1793	0.1670
3-star	0.0882	0.0976	0.0931	0.0945
Linear Probability	Y	Y	N	N
Ordered Logit	N	N	Y	Y
School FE	N	Y	N	N
Mundlak Averages	N	N	N	Y

Table 1.10. Estimated Probability of being Ranked or Drafted

Rivals Rating	(1)	(2)	(3)	(4)
5-star	0.6220	0.5668	0.6437	0.6171
4-star	0.3251	0.2906	0.3282	0.3104
3-star	0.1898	0.2061	0.1868	0.1899
Linear Probability	Y	Y	N	N
Ordered Logit	N	N	Y	Y
School FE	N	Y	N	N
Mundlak Averages	N	N	N	Y

Table 1.11. Estimated Conditional Probabilities of All *Ex Post* Outcomes

	Top Tier	Middle Tier	Bottom Tier	Top Tier	Middle Tier	Bottom Tier
5-star	0.4274	0.1394	0.4332	0.4178	0.1993	0.3829
4-star	0.1772	0.1134	0.7094	0.1670	0.1434	0.6896
3-star	0.0976	0.1085	0.7939	0.0945	0.0954	0.8101
Linear Probability	Y	Y	Y	N	N	N
Ordered Logit	N	N	N	Y	Y	Y
School FE	Y	Y	Y	N	N	N
Mundlak Averages	N	N	N	Y	Y	Y

1.4.5 Expected MRP Estimates

Estimates of *ex ante* expected MRP are reported in Tables 1.12 and 1.13. Table 1.12 reports estimates of *ex ante* expected MRP, rents on labor, and rent-to-MRP ratios. Rent estimates assume a four-year compensation of $100,000. Table 1.13 compares the estimates of realized MRP as a function of *ex post* quality with estimates of expected MRP as a function of *ex ante* promise. These results emphasize the effect that uncertain labor quality has on the ability of employers to exercise monopsony power, and thus, the importance of accounting for potential quality uncertainty when empirically assessing monopsony in labor markets. Looking at the upper bound estimates, MRP for an *ex post* top tier player overstates expected MRP by $260,891 or 82.46% (of expected MRP) for an *ex ante* 5-star prospect, and $388,328 or 205.5% (of expected MRP) for an *ex ante* 4-star prospect.

Table 1.12. Expected MRP and Rent Estimates by *Ex Ante* Rating: LPM with School FE

Expected MRP

Rivals Rating	Lower Bound	Upper Bound
5-star	$232,380.65	$316,400.95
4-star	$104,943.49	$188,963.78
3-star	$64,921.33	$148,941.62

Rent on Labor†

Rivals Rating	Lower Bound	Upper Bound
5-star	$132,380.65	$216,400.95
(*Rent/MRP*)	(*0.570*)	(*0.684*)
4-star	$4,943.49	$88,963.78
(*Rent/MRP*)	(*0.047*)	(*0.471*)
3-star	-$35,078.67	$48,941.62
(*Rent/MRP*)	(*-0.540*)	(*0.329*)

†Assuming $100,000 career compensation

The *ex post* top tier player's compensation falls short of the revenue he generates by approximately 80% to 83% of his realized MRP. Yet, his school retains approximately 57%

Table 1.13. Comparison: MRP by *Ex Post* Realized Quality and *Ex Ante* Promise

Ex post Quality	Lower Bound	Upper Bound
Top Tier	\$493,271.90	\$577,292.20
Middle Tier	\$154,635.90	\$238,656.20
Bottom Tier	—	\$84,020.29
Weighted Average	\$74,285.82	\$143,157.20
Rivals Rating		
5-star	\$232,380.65	\$316,400.95
4-star	\$104,943.49	\$188,963.78
3-star	\$64,921.33	\$148,941.62

to 69% of his expected MRP as a rent, if he is a 5-star prospect. The school retains anywhere from 4% to 47% of his expected MRP, if he were a 4-star prospect. Finally, if rated 3-star or lower *ex ante*, it is not clear that the school exercises any monopsony power over him at all. If he were a 3-star prospect, his school might retain as much as 33% of his expected MRP. Yet, at the other extreme, he may ultimately be compensated more than his *ex ante* expected value.

These results are limited in two ways. First, they depend on an assumed wage. I have assumed what I believe to be a conservative compensation value, in the interest of providing strong inference to the conclusion that employer monopsony power in the college football industry has been substantially overstated, not only in the degree exercised over players of average quality, but even in the degree exercised over the highest quality players.[25] The other limitation is that the sample of Rivals rated players is a selected sample, drawing only from players rated 3-star or higher. Consequently, the average player from this sample is of greater *ex ante* promise than the average player from the population. However, this does not conflict with the conclusion that employer mononopsony power in college football is limited.

[25]That is, I err on the side of possibly overstating rents.

1.5 Robustness Checks

In this section, I test the robustness of the preceding estimates to two alternative specifications. First, in light of the recent NCAA decision to allow players from "power five" conference schools to receive a stipend, I estimate expected MRP for only those players who attended a very similar set of schools to the present "power five" conference member. Then, I consider a more general specification of the revenue equation in which $REV_{i,t}$ is dependent on $REV_{i,t-1}$.

1.5.1 A Narrower Definition of the Labor Market

As previously mentioned, this paper's data come from a period in which the BCS had agreements with six major conferences guaranteeing an invitation to one of the four or five most lucrative post-season bowl games to the champion team of those conferences.[26] There were 62 BCS "automatic qualifier" (henceforth AQ) schools in 2004, and 65 thereafter.[27] Over this period, AQ schools were analogous to the present-day "power-five" conference schools. On average, the return on labor at an AQ school is likely higher than the return to a non-AQ school. For this reason, I test the robustness of the preceding results to a narrower definition of the labor market, in which AQ schools are the only employers.

Similar to Table 1.4, Table 1.14 reports lower and upper bound estimates for career MP given *ex post* quality. Columns (1) and (2) report estimates from the subsample of AQ schools only, while columns (3) and (4) report estimates from the subsample of non-AQ schools only. These values come from estimates of equation (1.1) with school fixed effects.

Table 1.15 reports lower and upper bound estimates of career MRP by *ex post* realized quality. Columns (1) and (2) report estimates the subsample of AQ schools only, and columns (3) and (4) report estimates from the subsample of non-AQ schools only. Similar to the MRP

[26]The University of Notre Dame was also included in this agreement. Although their football program is independent, it was guaranteed an invitation it finished the regular season ranked among the top eight by the BCS.

[27]The University of Cincinnati, the University of Louisville, and the University of South Florida accepted invitations to join the Big East, beginning in 2005.

Table 1.14. Marginal Product Estimates (Career Contributions): AQ vs. Non-AQ Schools

Ex post Quality	Lower Bound	Upper Bound	Lower Bound	Upper Bound
Top Tier	7.929***	9.385***	8.665**	10.083***
	(1.229)	(1.079)	(3.875)	(3.783)
	[5.521, 10.338]	[7.271, 11.499]	[1.069, 16.260]	[2.668, 17.497]
Middle Tier	1.114	2.570**	7.412*	8.830***
	(1.164)	(1.052)	(3.792)	(3.647)
	[-1.168, 3.396]	[0.508, 4.632]	[-0.019, 14.843]	[1.683, 15.977]
Bottom Tier	——	1.456***	——	1.418***
	——	(0.204)	——	(0.215)
	——	[1.057, 1.855]	——	[0.995, 1.840]
BCS AQ	Y	Y	N	N
N	517	517	395	395

[95% Confidence Interval]

estimates reported in Table 1.6, these values come from fixed effects estimates of equations (1) and (2). Between AQ and non-AQ schools, there is considerable disparity in MRP across all *ex post* quality levels, and this disparity is widest at the top tier. An *ex post* top tier player at an AQ school, appears to generate anywhere from four to five times as much revenue as a top tier player at a non-AQ school. The evidence that players generate significantly more revenue at AQ schools than at non-AQ schools is not surprising. From the 95% confidence interval, note that for AQ schools, the weighted average upper bound MRP estimate is significantly greater than $100,000. Among non-AQ school, there is no evidence that player compensation falls short of the MRP of an *ex post* top tier player, even at the upper bound.

Table 1.16 reports conditional probability estimates of all *ex post* outcomes for the subsample of prospects recruited to AQ schools only. The conditional probability estimates are not markedly different for this subsample. This is not surprising as 5,826 (or about 88%) of the 6,604 prospect sample, were recruited to one of the AQ schools.

Similar to Table 1.12, Table 1.17 reports *ex ante* expected MRP estimates for the subsample of prospects recruited AQ schools only. Rent estimates based on the assumed $100,000 compensation are reported in the bottom portion of Table 1.17. For the purpose of comparison, MRP estimates as a function of both *ex post* realized quality and *ex ante* promise, for the subsample of AQ schools only, are reported in Table 1.18. Again, the effect of uncertain labor quality is substantial. Looking at the upper bound estimates, MRP for an *ex post* top tier player overstates expected MRP by $403,878 or 84.97% (of expected MRP) for an *ex ante* 5-star prospect, and $597,029 or 211.6% (of expected MRP) for an *ex ante* 4-star prospect.

If analyses are restricted to AQ schools only, two results stand out. The first is that better football performance generates a much greater return to AQ schools than non-AQ schools. Thus, the potential "upside" on a prospective player is greater at AQ schools. From Table 1.15 we see that over a four year career, an *ex post* top tier player may be worth as much as $666,000 more to an AQ school than a non-AQ school. However, the issue of

Table 1.15. MRP Estimates by *Ex Post* Quality (Career Contributions): AQ vs. Non-AQ Schools

Ex post Quality	Lower Bound	Upper Bound	Lower Bound	Upper Bound
Top Tier	$742,789.70***	$879,174.60***	$183,546.70*	$213,586**
	($220,887.40)	($244,682.70)	($101,774.40)	($106,814.60)
	[$309,858.40, $1,175,721]	[$399,605.20, $1,358,744]	[-$15,927.47, $383,020.80]	[$4,233.16, $422,938.80]
Middle Tier	$104,360	$240,744.90**	$157,008.80*	$187,048.10*
	($114,047)	($119,770.60)	($94,581.31)	($98,077.09)
	[$-119,168, $327,888]	[$5,998.87, $475,490.90]	[-$28,367.14, $342,384.80]	[-$5,179.44, $379,275.70]
Bottom Tier	—	$136,384.90***	—	$30,039.31***
	—	($39,264.21)	—	($11,401.32)
	—	[$59,428.46, $213,341.30]	—	[$7,693.13, $52,385.49]
Weighted Average	$116,135.60***	**$252,520.50*****	$14,829.63**	**$44,868.95*****
	($36,552.06)	($64,707.06)	($6,401.41)	($15,249.28)
	[$44,494.88, $187,776.30]	[$125,697, $379,344]	[$2,283.11, $27,376.16]	[$14,980.90, $74,756.99]
BCS AQ	Y	Y	N	N
N	517	517	395	395

[95% Confidence Interval]

Table 1.16. Estimated Conditional Probabilities of All *Ex Post* Outcomes if at AQ School

	Top Tier	Middle Tier	Bottom Tier	Top Tier	Middle Tier	Bottom Tier
5-star	0.4365	0.1407	0.4228	0.4262	0.1998	0.3740
4-star	0.1787	0.1248	0.6965	0.1731	0.1475	0.6794
3-star	0.1020	0.1141	0.7839	0.1004	0.1007	0.7989
Linear Probability	Y	Y	Y	N	N	N
Ordered Logit	N	N	N	Y	Y	Y
School FE	Y	Y	Y	N	N	N
Mundlak Averages	N	N	N	Y	Y	Y
N	5826	5826	5826	5826	5826	5826

Table 1.17. Expected MRP and Rent Estimates if Attending AQ School: LPM with School FE

Expected MRP

Rivals Rating	Lower Bound	Upper Bound
5-star	\$338,911.16	\$475,296.06
4-star	\$145,760.65	\$282,145.55
3-star	\$87,672.03	\$224,056.93

Rent on Labor†

Rivals Rating	Lower Bound	Upper Bound
5-star	\$238,911.16	\$375,296.06
(Rent/MRP)	*(0.705)*	*(0.790)*
4-star	\$45,760.65	\$182,145.55
(Rent/MRP)	*(0.314)*	*(0.646)*
3-star	-\$12,327.97	\$124,056.93
(Rent/MRP)	*(-0.141)*	*(0.554)*

†Assuming \$100,000 career compensation

ex ante uncertainty regarding labor quality is no different for this subsample. The *ex ante* most promising players are still realized as *ex post* top tier less than half of the time. This uncertainty inhibits the degree to which schools are able to exercise monopsony power in the college football labor market.

1.5.2 Dynamic Revenue Function

As an alternative to the revenue equation specified by (1.2), suppose $REV_{i,t}$ depends on past values of itself. Replacing lagged winning percentage with lagged revenue, consider the dynamic revenue specification given by

$$REV_{i,t} = b_i + \rho REV_{i,t-1} + \beta_1 WPCT_{i,t} + \boldsymbol{z'_{i,t}\beta} + v_{i,t}. \tag{1.16}$$

Note that equation (1.2) restricts $\rho = 0$.

Table 1.18. Comparison: MRP by *Ex Post* Realized Quality and *Ex Ante* Promise at AQ Schools

Ex post Quality	Lower Bound	Upper Bound
Top Tier	\$742,789.70	\$879,174.60
Middle Tier	\$154,635.90	\$238,656.20
Bottom Tier	—	\$136,384.90
Weighted Average	\$116,135.60	\$252,520.50
Rivals Rating		
5-star	\$338,911.16	\$475,296.06
4-star	\$145,760.65	\$282,145.55
3-star	\$87,672.03	\$224,056.93

First differencing (1.16) removes the school-specific term b_i and gives

$$\Delta REV_{i,t} = \rho \Delta REV_{i,t-1} + \beta_1 \Delta WPCT_{i,t} + \boldsymbol{\Delta z'_{i,t}\beta} + \Delta v_{i,t}. \tag{1.17}$$

I treat all explanatory variables as potentially endogenous, and estimate equation (1.17), following Arellano and Bond (1991). This approach estimates equation (1.17) for $t \geq 3$ only. Arellano and Bond (1991) show that if $\mathrm{E}(v_{i,t}v_{i,t+h}) = 0$ for all $h \geq 2$, then level values of the dependent variable from 2 or more time periods prior are valid instruments the differenced equation (1.17). These estimates are reported in Table 1.19. The reported two-year effect of winning percentage on revenue is the estimate of $\beta_1 \cdot (1+\rho)$. The estimated two-year effect is larger than that reported in column (2) of Table 1.2. Note however that the 95% confidence interval on two-year effect reported in Table 1.19 contains the corresponding confidence interval from column (2) of Table 1.2. Thus, estimates of this two year effect are not significantly different these two alternative specifications. The reported long-run effect is the estimate of $\frac{\beta_1}{(1-\rho)}$. This estimate is not significantly different from zero at the 95% significance level.

Table 1.19. Revenue Equation: Arellano-Bond Two-Step Estimates

Variable	REV
REV_{t-1}	0.670***
	(.100)
$WPCT_t$	55,612.32**
	(25,383.16)
Two year effect $WPCT_t$	**92,872.71****
	(42,232.39)
95% C.I.	[10,098.76, 175,646.70]
Long run effect $WPCT_t$	168,523.40*
	(88,329.49)
	[-4,599.206, 341,646]
HR_t	408,548.10
	(424,920.1)
Year 3	778,208.80
	(547,550.20)
Year 4	1,651,051***
	(424,966.30)
Year 5	1,107,829**
	(485,293.40)
Year 6	1,444,776**
	(590,289.50)
Year 7	1,592,330***
	(563,515.40)
Year 8	2,391,274***
	(712,506)
N	680
Sargan Stat.	59.64384
Critical Value ($p = .05$)	76.778
(d.f.)	(58)
Rejects H_0	N

The estimates reported in Table 1.19 restrict the matrix of available instruments to lagged values of revenue only, since all explanatory variables are treated as potentially endogenous. A matrix of 67 instruments is utilized in the estimation of (1.17). Since the number of instruments is greater than the number of endogenous explanatory variables, the model is over-identified by the 67 assumptions that each instrument is uncorrelated with $\Delta v_{i,t}$ in the appropriate time periods. The lower panel of Table 1.19 reports test statistics for the Sargan (1958) test of over-identifying restrictions. Under the null hypothesis of the test, the matrix of available instruments is exogenous and the test statistic is distributed chi-square with 58 degrees of freedom.[28] The test statistic reported in Table 1.19 fails to reject the null hypothesis. Rejection of the null would suggest that at least some of the 67 instruments are endogenous, and call all estimates in Table 1.19 into question. A limitation of the Sargan test is that the test statistic can only fail to reject that the instruments are exogenous. This should not be seen as rejecting the hypothesis that an instrument is endogenous.

Arellano and Bond (1991) propose a test for autocorrelation of the first-differenced error term. Note that $\Delta v_{i,t}$ will always be autocorrelated of order one, since $\Delta v_{i,t}$ and $\Delta v_{i,t-1}$ both contain the level error $v_{i,t-1}$. However, the absence of autocorrelation of order two or higher will be consistent with the assumption that $\mathrm{E}(v_{i,t}v_{i,t+h}) = 0$ for all $h \geq 2$. Thus, autocorrelation in $\Delta v_{i,t}$ of order two or more calls the specification and estimates into question. Test statistics for zero autocorrelation of orders one, two, three, four, and five, are reported in Table 1.20. Under the null hypothesis of no autocorrelation, the test statistics are distributed standard normal.[29] As it should, the test rejects the null hypothesis of no autocorrelation of order one. The test fails to reject the null hypothesis of no autocorrelation of order two or higher. Thus, these test statistics do not call the specification into question. However, this does not reject the hypothesis of no higher order autocorrelation in $\Delta v_{i,t}$.

[28] The degrees of freedom are given by the number of identifying assumptions (67) minus the number of explanatory variables (9).

[29] A detailed explanation of the calculation of these test statistics and a proof of asymptotic normality can be found in Section 3 and the Appendix of Arellano and Bond (1991), respectively.

Table 1.20. Arellano-Bond Test of No Autocorrelation in $\Delta v_{i,t}$

Order	Test statistic	p-value	Rejects H_0
1	-4.3343	≈ 0	Y
2	-0.0979	0.9220	N
3	0.3234	0.7464	N
4	-0.3491	0.7270	N
5	0.4484	0.6539	N

Given the failure to reject the revenue function specification given in 1.16, it is worth comparing the *ex ante* expected MRP implied by the estimates reported in Table 1.19 with the initial estimates reported in Table 1.12. These alternative estimates of *ex ante* expected MRP based on the long-run effect of $WPCT$ are reported in columns (1) and (2) of Table 1.21. Recall from Table 1.19 that the long-run effect is not significantly different from zero. Thus, values reported in columns (1) and (2) should be viewed with some skepticism. The alternative estimates of *ex ante* expected MRP based on the two-year effect of $WPCT$ are reported in columns (2) and (3) of Table 1.21. The initial estimates are reported in the columns (3) and (4) for comparison.

Table 1.21. Expected MRP by *Ex Ante* Promise: Dynamic versus Static Revenue

Rivals Rating	Lower Bound	Upper Bound	Lower Bound	Upper Bound	Lower Bound	Upper Bound
5-star	$662,796.7378	$902,439.6717	$232,380.65	$316,400.95	$232,380.65	$316,400.95
4-star	$299,320.11	$538,963.02	$104,943.49	$188,963.78	$104,943.49	$188,963.78
3-star	$185,168.80	$424,811.70	$64,921.33	$148,941.62	$64,921.33	$148,941.62
Dynamic *REV*	Y	Y	Y	Y	N	N
Long-run effect	Y	Y	N	N	N	N

1.6 Concluding Remarks

My results suggest that the degree of monopsony power held by member institutions is more limited than Brown (1993, 2011) suggests. The *ex post* highest quality players do generate revenue that is well in excess of their compensation's market value. However, the employers' ability to exercise monopsony power over these elite performers is weakened by *ex ante* uncertain labor quality. Moreover, the current compensation may meet or even exceed the amount that a player of average *ex ante* promise would command in a competitive labor market. As courts and NCAA member institutions consider policy changes, and as some players propose them; these results have interesting implications. Among them, elite players would likely benefit from more comprehensive reforms, such as permitting competitive wage offers throughout the labor market. Yet, the below average performer could be made worse off by such reforms, and see at least the market value of his compensation decrease. It would seem that more modest policy changes, such as maintaining or slightly increasing the uniform wage; are more favorable to the average and below average players, than a competitive labor market alternative.

CHAPTER 2

OPTIMAL IRONING MADE EASY

In this chapter, I explore an implementability problem that arises in the construction of optimal nonlinear price schedules. I show that the one-dimensional screening problem of a monopolist practicing second-degree price discrimination, can be treated like a dynamic programming problem and solved by backward induction using Bellman's equation.[1] This approach builds on a multi-market treatment of nonlinear pricing problems developed in Goldman et al. (1984), and Wilson (1993).

The standard model of second-degree price discrimination involves a monopolist selling different amounts of a singe good to many consumers who are privately informed of their heterogeneous preferences for the good. Consumer preferences are ordered by a single parameter, their "type". The higher a consumer's type, the more she is willing to pay for any positive amount of the good. The monopolist sets a schedule, assigning prices to bundles of different amounts of the good. By offering quantity discounts, the monopolist's optimal price schedule serves as a screening mechanism, inducing higher types to buy larger bundles than lower types.

Implicit in a monopolist's price schedule are marginal prices for each unit of the good. For example, if the monopolist prices a 1-unit bundle at \$3 and a 2-unit bundle at \$5, this is equivalent to pricing the 1^{st} unit at \$3 and the 2^{nd} at \$2. The multi-market approach to the problem treats each unit that a consumer buys as though it is a different good, sold in a different market. The price of each of these "goods" is the marginal price for that unit. This approach to the search for an optimal price schedule is advantageous, as it allows a complex

[1] Dynamic programming and the Bellman equation were developed in a series of works by Richard Bellman, notably Bellman (1956) and Bellman (1957). Since then their application in economics has been widespread.

problem to be broken up into simpler subproblems of great familiarity in economics. Rather than searching for an optimal nonlinear price schedule in a single market, the monopolist is seen a searching for optimal uniform prices in many independent markets. However, an issue arises: these different markets are not in fact independent. A consumer's demand for the i^{th} unit depends on her having consumed all units up to that point. The subproblems overlap, and in some instances, the monopolist's optimal price schedule cannot be completely constructed from the optimal uniform prices treating each market in isolation. It is this issue that motivates the dynamic programming treatment developed here.

The monopolist's problem is shown to be one of choosing the set of customers he sells to in each market. The optimal marginal price which induces that set of consumers to buy, is implied. The minimum consumer type sold to in the market for a given unit will be referred to as the marginal consumer type, or "marginal type", of that market. Recall that the consumer type is just a parameter ordering consumer preference or willingness-to-pay for any given amount of the good. It follows that in a given market, the marginal type buys, as do all consumers of higher types. Thus, the monopolist's problem is modeled as one of choosing a schedule of marginal types to maximize his profits across all markets. Goldman et al. (1984) show that the monopolist is constrained in the following way: he must choose a marginal type schedule that is monotonically non-decreasing in the quantity being sold. For the monopolist to sell the i^{th} unit to a consumer type, he must induce her to buy all units leading up to the i^{th} unit. This is not achieved in any region where a marginal type schedule is decreasing in the quantity being sold.

One approach to the solution is to solve the monopolist's "relaxed problem" (that is, solve the problem ignoring the monotonicity constraint), and hope that the solution satisfies the monotonicity constraint. In the event that the relaxed solution violates the constraint it must be "ironed". That is to say, the relaxed solution must be modified so as to make it monotonically decreasing in quantity. This is typically done by way of an ironing procedure. This procedure involves choosing a monotonic modification of the relaxed solution from the set of all possible monotonic modifications, that maximizes the monopolist's profit over the region of quantities where the schedule is modified. In instances where the relaxed solution violates the monotonicity constraint, this *ad hoc* approach arrives at the monopolist's solution in a manner that is indirect and somewhat unintuitive.

Alternatively, I show that the monopolist's problem can be solved by backward induction. When searching for the monopolist's optimal marginal type schedule, the relaxed solution is sometimes insufficient because the monopolist is not actually selling in independent markets. The markets are interdependent in the following way: the set of consumers that can be sold to the market for a given unit is determined by the set sold to in the market for the previous unit, and the set sold to in the market for a given unit determines the set that can be sold to in the market for the following unit. The structure of this dependence is highly conducive to a dynamic programming solution. By incorporating Bellman equations, the problem can be in a way that reflects the interdependence of the markets. The backward induction solution method constructs the monopolist's optimal schedule over a sequence of stages. As needed, it irons the candidate schedule along the way to maintain monotonicity. Ultimately, it identifies the monopolist's optimal marginal price schedule even in instances where the solution to the relaxed problem violates the monotonicity constraint. Most importantly, this approach exhibits why ironed mechanisms like the monopolist's marginal type schedule might be optimal, and it highlights the economic intuition behind the construction of an optimally ironed schedule.

The paper proceeds as follows. In Section 2.1, I provide two very simple examples of the sort of problem being addressed. These examples highlight the relevance of the monotonicity constraint. They are also used to contrast the *ad hoc* approach to ironing (and the problem in general), with the systematic approach developed here. Section 2.2 presents a formal version of the problem, in which the monopolist sells discrete amounts of the good to a continuum of heterogeneous consumer types. Section 2.3 presents the solution by backward induction. Finally Section 2.4 presents some early results and shows that ironed schedules solved by this method satisfy a *summation condition*. This condition is analogous to an integral condition used to identify the optimal range of units over which to iron the schedule in the *ad hoc* approach. This summation condition establishes that the backward induction approach solves for the monopolist's optimal schedule, even when the solution to the relaxed problem violates the monotonicity constraint.

2.1 Two Simple Examples

Consider first a monopolist selling discrete quantities q of a single good, to two consumers. For simplicity assume he produces these units at zero marginal cost. The consumers' heterogeneous preferences for the good are private information and ordered by the type parameter $\theta \in \{\theta^L, \theta^H\}$, where $\theta^L < \theta^H$. The monopolist knows that one of the consumers is type θ^L and the other is type θ^H, but does not know which is which. Let $u(\theta, q)$ be the dollar value of the utility that type θ gets from consuming a bundle of q units. The marginal utility of the i^{th} unit to type θ, is $u(\theta, i) - u(\theta, i-1)$, denoted $u_q(\theta, i)$ by abuse of notation.

2.1.1 Example I

Suppose the two types have the utility and marginal utility schedules given in Table 1, and that this is known to the monopolist. The monopolist will set a profit maximizing price schedule which will serve as a screening mechanism by offering quantity discounts. This is just a standard second degree price discrimination problem.

Table 2.1. Utility and Marginal Utility Schedules by Type: Example I

i	$u(\theta^L, i)$	$u(\theta^H, i)$	$u_q(\theta^L, i)$	$u_q(\theta^H, i)$
1	\$4	\$7	\$4	\$7
2	\$5	\$10	\$1	\$3
3	\$5	\$11	\$0	\$1
4	\$5	\$11	\$0	\$0

Let $P(q)$ be the price for a bundle of q units. The marginal price of the i^{th} unit is $P(i) - P(i-1)$, and is denoted p_i. In this first example the monopolist will sell at most three units because neither type is willing to pay for more than a third unit. The monopolist wishes to set the price schedule $(P(1), P(2), P(3))$ that will maximize profit from sale of the good. Note that any price schedule like this, can be represented by a corresponding schedule of marginal prices $(p_1, p_2, p_3) = (P(1), P(2) - P(1), P(3) - P(2))$. Thus, one way to think about the monopolist's problem is to treat each successive unit as a separate good with its own market, where the monopolist will simply set an optimal uniform price in each of the markets, which in this case will be the marginal price. This multi-market treatment of nonlinear pricing problems is developed in Goldman et al. (1984) and Wilson (1993). To facilitate this approach, let the i^{th} market profit be the monopolist's profit from the sale of the i^{th} unit only, as a function of the marginal price for the i^{th} unit, denoted $\pi_i(p_i)$.

In this first example the monopolist's optimal price schedule will be (p_1^*, p_2^*, p_3^*), where p_i^* is the argument chosen to maximize $\pi_i(p_i)$. Recalling the marginal utility schedules in Table 2.1, the solution in this example easy. Note that since the monopolist knows each type's marginal utility value for each unit, only those marginal utility values need be considered as marginal prices. If the monopolist sets a marginal price for the 1^{st} unit of \$4, both consumers will buy the 1^{st} unit yielding a 1^{st} market profit of \$8. At a marginal price of \$7, only the consumer with type θ^H will buy the 1^{st} unit yielding a 1^{st} market profit of \$7. So the optimal 1^{st} market price is $p_1^* = \$4$. At a marginal price of \$1 for the 2^{nd} unit, he will sell to both consumers for 2^{nd} market profit of \$2. At a marginal price of \$3 he will sell only to type θ^H for a 2^{nd} market profit of \$3. Thus, the optimal 2^{nd} market price is $p_2^* = \$3$. The optimal 3^{rd} market marginal price is $p_3^* = \$1$ by the same logic.

Table 2.2. Solving for the Optimal Price Schedule: Example I

q	$u_q(\theta^L, i)$	$u_q(i, \theta^H, i)$	$\pi_i(u_q(\theta^L, i))$	$\pi_i(u_q(\theta^H, i))$	p_i^*	$\pi_q(p_i^*)$
1	\$4	\$7	**\$8**	\$7	\$4	\$8
2	\$1	\$3	\$2	**\$3**	\$3	\$3
3	\$0	\$1	\$0	**\$1**	\$1	\$1
Total						\$12

A schedule of optimal prices for each market has been identified. A monotonicity constraint must be met. The monopolists must set a price schedule such that the lowest consumer type that is willing to buy in each market, is monotonically non-decreasing in q. This is because a consumer will consider buying the 2^{nd} only if the monopolist induces her to buy the 1^{st} unit. This first example was crafted in such a way that the monopolist's optimal price schedule could be identified without consideration of the latter monotonicity constraint. In the second example, I use small changes to the problem show how this constraint can easily be violated by the *ad hoc* search for an optimal price schedule just employed.

2.1.2 Example II

Consider a similar problem with the utility schedules are amended as in Table 2.3. Table 2.4 reports the optimal price schedule solved using the same method as in the first example.

Table 2.3. Utility and Marginal Utility Schedules by Type: Example II

i	$u(\theta^L, i)$	$u(\theta^H, i)$	$u_q(\theta^L, i)$	$u_q(\theta^H, i)$
1	\$4	\$9	\$4	\$9
2	\$7	\$13	\$3	\$4
3	\$8	\$15.50	\$1	\$2.50
4	\$8	\$15.50	\$0	\$0

Table 2.4. Solving for the Optimal Price Schedule: Example II

i	$u_q(\theta^L, i)$	$u_q(\theta^H, i)$	$\pi_i(u_q(\theta^L, i))$	$\pi_i(u_q(\theta^H, i))$	p_i^*	$\pi_q(p_i^*)$
1	\$4	\$9	\$8	**\$9**	\$9	\$9
2	\$3	\$4	**\$6**	\$4	\$3	\$6
3	\$1	\$2.50	\$2	**\$2.50**	\$2.50	\$2.50
Total						\$17.50

The same approach as employed in the first example implies the optimal price schedule $(\$9, \$3, \$2.50)$ here. However it should be clear that the resulting allocation is not realistic. The monopolist intends that both consumers buy in the 2^{nd} market, when in fact, they won't. At this price schedule, θ^L prefers her outside option to any bundle offered, and buys none of the good. Type θ^H buys 3 units, and the monopolist's total profit is in fact only \$14.50, as opposed to \$17.50. This result is clearly suboptimal. One need only note that the monopolist could extract a profit of \$15.50 from the same allocation, by raising p_2 from \$3 to \$4. This suboptimal outcome results because lowest consumer type that the monopolist intends will buy in each market is not monotonically non-decreasing in q. The monopolist's intended outcome, in which a total profit of \$17.50 is generated, is not feasible.

This example serves to highlight the importance of the monotonicity constraint. Therefore, treating the minimum type sold to in each market as the monopolist's choice variable will make it easier to keep track of the monotonicity constraint.

2.1.2.1 The Marginal Consumer Type

For any implementable allocation of quantities to types, the monopolist's optimal price schedule with which to implement the allocation is immediate. Thus, an alternative way to approach this profit maximization problem is to focus, not the price schedule chosen by the monopolist, but instead on the resulting allocation. Specifically, suppose the monopolist chooses the lowest consumer type to whom he will sell each unit. The optimal marginal price in the market for each unit, is just the marginal utility of the lowest type sold to in the market for that unit. The marginal type in the i^{th} market will be the lowest type the monopolist induces to buy the i^{th} unit, denoted θ_i. Rather than considering a schedule of prices, the monopolist's decision problem can be represented as choosing a schedule of marginal types. For any schedule of marginal types, an optimal schedule of marginal prices is implied, as $p_i(\theta_i) = u_q(\theta_i, i)$. The relevant monotonicity restriction is that this schedule of marginal types must be monotonically non-decreasing in q. The monopolist can't induce a consumer to buy a 2^{nd} unit unless he makes her willing to buy the 1^{st}. Thus, the monopolist's cannot possibly implement a marginal type schedule that is non-monotonic in q.

It turns out that the schedule of marginal types, corresponding to the optimal price schedule identified in Table 2, is $(\theta^L, \theta^H, \theta^H)$. Since $\theta^L < \theta^H = \theta^H$, this schedule satisfies the monotonicity constraint that $\theta_1 \leq \theta_2 \leq \theta_3$, so this is the monopolist's optimal marginal type schedule in the first example. The schedule of marginal types corresponding to the optimal price schedule identified in Table 4, is $(\theta^H, \theta^L, \theta^H)$. This clearly violates the monotonicity constraint and must be "ironed"; that is, it must be modified in some way to be made monotonic. This schedule can be made monotonic in two possible ways. The marginal type for the second unit, θ_2, can pulled up to make the schedule $(\theta^H, \theta^H, \theta^H)$; or the marginal type for the first unit θ_1 can be pushed down, making the schedule $(\theta^L, \theta^L, \theta^H)$. Recalling that the optimal $p_i(\theta_i) = u_q(i, \theta_i)$; the marginal type schedule $(\theta^H, \theta^H, \theta^H)$ will mean a 1^{st} market profit of \$9, a 2^{nd} market profit of \$4, and a 3^{rd} market profit of \$2.50, for a total profit of \$15.50. Alternatively, The marginal type schedule $(\theta^L, \theta^L, \theta^H)$ will mean a 1^{st} market profit of \$8, a 2^{nd} market profit of \$6, and a 3^{rd} market profit of \$2.50, for a total profit of \$16.50. Thus, the optimal schedule of marginal types that can actually be implemented through a price schedule is $(\theta^L, \theta^L, \theta^H)$.

This is a simplified version what I will call an *ad hoc* approach to ironing. In a setting where the preference parameter θ is continuous, this approach amounts to finding the schedule of marginal types that satisfy the first order conditions for maxima in the different markets, ignoring the monopolist's constraint. This schedule is then checked to see if it satisfies the monotonicity constraint. If it does, the problem is solved. If not, the schedule must be ironed to be implementable. The ironed version of the schedule which maximizes the monopolist's profit across all markets, is his optimal implementable schedule.

2.1.3 An Alternative Approach

In these two examples, the monopolist's search for an optimal implementable screening mechanism, can be modeled using Bellman equations and solved by backward induction. Like the multi-market approach already adopted, this dynamic programming treatment is a useful tool for solving and better understanding the problem. The multi-market approach treats the monopolist as though he is selling different goods, in independent markets. The appeal of this approach is that it breaks up the monopolist's problem into several simple subproblems. The only issue is that the subproblems are overlapping. By incorporating Bellman equations, the problem can be modeled more realistically, as a monopolist selling different goods in markets that are interdependent. The structure of this dependence is that the demand for the i^{th} unit depends on the marginal type in the market for $(i-1)^{th}$ unit. In the above examples it is apparent in the 3^{rd} market that the set of marginal types from which the monopolist can chose (or the "“state” of the 3^{rd} market), is determined by the marginal type implemented in the 2^{nd} market; and the set of marginal types from which he chooses in the 2^{nd} market, is determined by the marginal type implemented in the 1^{st} market.

Reconsider the second example. Given the monotonicity constraint that $\theta_3 \geq \theta_2$, the set from which the monopolist can choose θ_3, depends on the marginal type chosen in the 2^{nd} market, θ_2. However, since the monopolist sells no units beyond the 3^{rd}, the 3^{rd} market is his terminal market. This means that only π_3 depends on θ_3. Therefore, his optimal choice of θ_3 conditional on θ_2 can be solved easily. Given any θ_2, his optimal choice of θ_3 is θ^H, yielding a market profit of \$2.50. This is because $\theta_3 = \theta^H$, satisfies monotonicity given either choice of θ_2. Let the maximum possible π_3 given any θ_2 be represented by the value function $V_3(\theta_2)$. Thus,

$$V_3(\theta_2) = \$2.50, \forall \theta_2 \in \{\theta^L, \theta^H\}. \tag{2.1}$$

In addition to V_3, the choice of θ_2 also determines his 2^{nd} market profit, π_2. Assuming an optimal choice of θ_3 in the resulting state of the 3^{rd} market, the monopolist's total profit as a function of θ_2 is given by the Bellman equation,

$$\pi_2(\theta_2) + V_3(\theta_2) = \left\{ \begin{array}{ll} \$4 + \$2.50 = \$6.50, & \theta_2 = \theta^H \\ \$6 + \$2.50 = \$8.50, & \theta_2 = \theta^L \end{array} \right\}. \tag{2.2}$$

From equation (2.2), the monopolist's optimal choice in the 2^{nd} market given θ_1, can be solved. If $\theta_1 = \theta^L$, so θ^L is the state of the 2^{nd} market, the optimal choice of θ_2 is θ^L (with the implication that $\theta_3 = \theta^H$), yielding a profit of \$8.50 from the 2^{nd} and 3^{rd} markets. If $\theta_1 = \theta^H$, his only implementable choice for θ_2 will be θ^H, yielding a profit of \$6.50 from the 2^{nd} and 3^{rd} markets. The maximum possible $\pi_2 + \pi_3$ conditional on the choice of the θ_1 is represented by the value function,

$$V_2(\theta_1) = \left\{ \begin{array}{ll} \$6.50, & \theta_1 = \theta^H \\ \$8.50, & \theta_1 = \theta^L \end{array} \right\}. \tag{2.3}$$

Finally, his choice of θ_1 determines π_1 in addition to V_2. Assuming that θ_2 and θ_3 are chosen optimally in the resulting states of the 2^{nd} and 3^{rd} markets, the monopolist's total profit over all units as a function of θ_1 can be expressed by the Bellman equation.

$$\pi_1(\theta_1) + V_2(\theta_1) = \left\{ \begin{array}{ll} \$9 + \$6.50 = \$15.50, & \theta_1 = \theta^H \\ \$8 + \$8.50 = \$16.50, & \theta_1 = \theta^L \end{array} \right\}. \tag{2.4}$$

It follows from (2.4) that the monopolist's optimal choice of θ_1, denoted θ_1^*, is θ^L. The optimal θ_2 and θ_1 implied by θ_1^*, are $(\theta_2^*, \theta_3^*) = (\theta^L, \theta^H)$. Altogether, the monopolist's optimal schedule is $(\theta^L, \theta^L, \theta^H)$. This is of course the same schedule identified in Section 2.1.2, when $(\theta^H, \theta^L, \theta^H)$ was "ironed". However, the *ad hoc* search for the optimal schedule from Section 2.1.2 arrives at the solution indirectly, and irons only after a complete schedule for the monopolist has been proposed.

In the following section, this problem is generalized to consider not just two consumer types, but a continuum of types $\theta \in [\underline{\theta}, \overline{\theta}]$. The backward induction approach, with the use of Bellman's equation, exhibits why an ironed marginal type schedule might be optimal.

2.2 The Monopolist's Problem

Consider a monopolist selling successive discrete quantities q of a single good, to a continuum of consumer types $\theta \in [\underline{\theta}, \overline{\theta}]$. Consumer utility depends on quantity of the good consumed and the consumer's type, $u = u(q, \theta)$. The marginal utility of the i^{th} unit for type θ, is $u(\theta, i) - u(\theta, i-1)$, denoted $u_q(\theta, i)$, by abuse of notation. Assume diminishing marginal utility (A1), and a single crossing condition (A2).

Assumption (A1). $u_q(\theta, i) \geq u_q(\theta, j) \forall i < j$.

Assumption (A2). $u_q(\theta^A, i) < u_q(\theta^B, i), \forall \theta^A < \theta^B \in [\underline{\theta}, \overline{\theta}], i = 1, ..., N$.

The distribution of θ is $f(\theta) = F'(\theta)$, where $F(\theta)$ is the CDF, so

$$F(\theta) = \int_{\underline{\theta}}^{\theta} f(t)dt, \tag{2.5}$$

and $F(\overline{\theta}) = 1$. The price a consumer pays for a bundle of i units is $P(i)$. The monopolist sets a price schedule $(P(1), P(2), ..., P(N))$, and the marginal price of the i^{th} unit, denoted by p_i, is,

$$p_i = P(i) - P(i-1). \tag{2.6}$$

The practice of second-degree price discrimination can be thought of in two different ways. The monopolist can be treated as searching for a price schedule $(P(1), P(2), ..., P(N))$, for a single good in a single market. However this price schedule will also imply N marginal prices. By treating each successive unit i, as a separate good with its own market, the monopolist can be seen as searching for N uniform prices $(p_1, p_2, ..., p_N)$, in N independent markets. I adopt the latter. This multi-market treatment developed in Goldman et al. (1984), and Wilson (1993), has the advantage of breaking the monopolist's problem into several simpler subproblems of great familiarity in economics.

2.2.1 The Marginal Consumer Type

A concept central to the multi-market approach is that of the marginal consumer type. In the market for the i^{th} unit, there will be a minimum consumer type that is willing to buy, given p_i. Let this marginal type for the i^{th} unit be denoted θ_i, where,

$$\theta_i = \min\{\theta \in [\underline{\theta}, \overline{\theta}] : u_q(i, \theta) \geq p_i\}. \tag{2.7}$$

In the i^{th} market, it will be optimal for the monopolist to extract all surplus from the marginal type θ_i. Therefore the relationship between a marginal type θ_i and the corresponding optimal p_i which implements it, is one-to-one; and any search for the optimal price schedule with which to implement a marginal type schedule, is trivial: the optimal marginal price as a function of marginal type is

$$p_i(\theta_i) = u_q(\theta_i, i). \tag{2.8}$$

Therefore, the monopolist's problem can focus solely on the choice of a marginal type schedule.

Assume the monopolist faces a constant marginal cost function in production of the good, so $c(q) = cq$. Given the single crossing condition (A2), and the implicit marginal price in (2.8), for a given marginal type θ_i, all $\theta \in [\theta_i, \overline{\theta}]$ are willing to buy in the i^{th} market. The monopolist's profit in the i^{th} market as function of θ_i is then,

$$\pi_i(\theta_i) = [u_q(\theta_i, i) - c] \cdot [1 - F(\theta_i)]. \tag{2.9}$$

The monopolist sells up the N^{th} unit, which is simply the last successive unit for which there exists some $\theta_i \in [\underline{\theta}, \overline{\theta}]$ such that $\pi_i(\theta_i)$ is non-negative. The monopolist chooses a schedule of marginal types $(\theta_1, \theta_2, ..., \theta_N)$ to maximize the sum of profits across all markets. The monopolist's objective function is then,

$$\Pi = \sum_{i=1}^{N} \pi_i(\theta_i). \tag{2.10}$$

The monopolist's problem, however, involves an important restriction regarding the set of marginal type schedules that can be implemented by way of a price schedule.

2.2.2 Monotonicity Constraint

Although the nonlinear pricing problem is being treated as a multi-market monopolist problem, it must be noted that the N markets are not independent. Only those consumers induced to buy the 1^{st} will consider buying the 2^{nd} unit, and so on. Thus, the set of consumers that the monopolist can sell to in the i^{th} market, is determined by the marginal type he chooses in the $(i-1)^{th}$ market. The monopolist must choose a schedule of marginal types that is monotonic in q. In other words, the monopolist must choose from the feasible set,

$$\{(\theta_1, \theta_2, ..., \theta_N) : \theta_1 \leq \theta_2 \leq ... \leq \theta_N\}. \tag{2.11}$$

For a consumer type to be the marginal type in the i^{th} market, she must be made willing buy up to the $(i-1)^{th}$ unit. That is, a marginal type $\theta_i < \theta_{i-1}$ is not feasible.

2.2.3 The Monopolist's Objective

The monopolist's objective is to choose the implementable schedule of marginal types that will maximize his profit over all N markets,

$$\max_{\theta_1 \leq \cdots \leq \theta_N} \sum_{i=1}^{N} \pi_i(\theta_i). \tag{2.12}$$

Finally, to ensure a global solution assume that $\pi_i(\theta_i)$ is concave in θ_i for all markets, Assumption (A3).

Assumption (A3). $\pi_i(\theta_i)$ is concave in θ_i on $[\underline{\theta}, \overline{\theta}], \forall i = 1, ..., N$.

2.3 The Monopolist's Solution

The monopolist's subproblems of maximizing profit in each market, overlap in the context of his broader problem which is the maximization of his profit across all markets. The monotonicity constraint reflects that. He can choose θ_1 from the interval $[\underline{\theta}, \overline{\theta}]$. However, subject to the monotonicity constraint, θ_2 must then be chosen from $[\theta_1, \overline{\theta}]$, and so on. The choice of marginal type in a given market therefore, not only determines the profit in that market, but also the state of the proceeding market, since θ_i is the implementable lower bound for θ_{i+1}.

The monopolist does have a terminal market. Since he only sells up to the N^{th} unit, the marginal type θ_N is not a state variable. That is, it does not determine the state of the monopolist's next market, because there no $(N+1)^{th}$ market. This means that the monopolist's problem can be solved by backward induction, beginning with a solution for the optimal choice of θ_N in any state of the N^{th} market.

2.3.1 Solution Stage N

The solution begins with what will be called Stage N. The monopolist is ultimately searching for a schedule $(\theta_1^*, ..., \theta_N^*)$ which solves the problem given in (12). Note that θ_N^* cannot be solved for without knowing θ_{N-1}^*. Thus, Stage N begins by solving for something of a placeholder, or opening conjecture as to what θ_N^* might be. Let,

$$\theta_N^*(N) = \arg\max_{\theta_N} \pi_N(\theta_N). \tag{2.13}$$

The search for the optimal schedule begins with $\theta_N^*(N)$. It is denoted $\theta_N^*(N)$ to reflect that it is the Stage N candidate for θ_N^*. As the search for the solution continues, the candidate for θ_N^* may need to be modified to maintain implementability, depending on what value θ_{N-1} takes. Let the best (profit maximizing and implementable) response of θ_N to any value of the state variable θ_{N-1} be,

$$\theta^{BR}_{N|N-1}(\theta_{N-1}) \equiv \arg \max_{\theta_N \geq \theta_{N-1}} \pi_N(\theta_N), \forall \theta_{N-1} \in [\underline{\theta}, \overline{\theta}]. \tag{2.14}$$

For any $\theta_{N-1} \leq \theta^*_N(N)$, the best response is clearly $\theta^*_N(N)$. Also, by (A3), π_N is decreasing in θ_N on the interval $[\theta^*_N(N), \overline{\theta}]$. Therefore the best response of θ_N to any value of the state variable θ_{N-1} is,

$$\theta^{BR}_{N|N-1}(\theta_{N-1}) = \max[\theta^*_N(N), \theta_{N-1}]. \tag{2.15}$$

This best response of θ_N to any choice of θ_{N-1}, allows the maximum profit in the N^{th} market, given any choice of θ_{N-1}, to be expressed as a value function,

$$V_N(\theta_{N-1}) \equiv \max_{\theta_N \geq \theta_{N-1}} \pi_N(\theta_N) = \pi_N(\theta^{BR}_{N|N-1}(\theta_{N-1})). \tag{2.16}$$

2.3.2 Stage *N-1*

As in the preceding stage, the Stage $N-1$ objective is to find an opening conjecture for θ^*_{N-1} as well as an updated candidate for θ^*_N. These will be denoted as $\theta^*_{N-1}(N-1)$ and $\theta^*_N(N-1)$, respectively. The Stage $N-1$ candidates for θ^*_{N-1} and θ^*_N will be,

$$\arg \max_{\theta_{N-1} \leq \theta_N} [\pi_{N-1}(\theta_{N-1}) + \pi_N(\theta_N)] \tag{2.17}$$

Assuming a best response of θ_N to any choice of θ_{N-1}, the initial candidate for θ^*_{N-1} is then

$$\theta^*_{N-1}(N-1) = \arg \max_{\theta_{N-1}} [\pi_{N-1}(\theta_{N-1}) + V_N(\theta_{N-1})]. \tag{2.18}$$

This in turn implies the updated candidate for θ^*_N,

$$\theta^*_N(N-1) = \theta^{BR}_{N|N-1}(\theta^*_{N-1}(N-1)) = \max[\theta^*_N(N), \theta^*_{N-1}(N-1)]. \tag{2.19}$$

The final step of (2.19) follows from (2.15).

As the search continues, the candidate for θ^*_{N-1} might need to be modified to satisfy monotonicity, depending on the value that state variable θ_{N-2} takes. Additionally, if the θ^*_{N-1} candidate is modified in response to state variable θ_{N-2}, then the candidate for θ^*_N might need to be modified in response to the θ^*_{N-1} candidate response. Assuming a best response of θ_N to θ_{N-1}, the best response of θ_{N-1} to any value of state variable θ_{N-2}, is,

$$\theta^{BR}_{N-1|N-2}(\theta_{N-2}) = \arg \max_{\theta_{N-1} \geq \theta_{N-2}} [\pi_{N-1}(\theta_{N-1}) + V_N(\theta_{N-1})]. \tag{2.20}$$

Implicit in (2.20) is a best response of θ_N to the best response of θ_{N-1} to θ_{N-2}. This composition of two best response functions, is the best response of θ_N to state variable θ_{N-2}, which is denoted by $\theta^{BR}_{N|N-2}(\theta_{N-2})$.

Assumption (A3) implies that V_N is concave in θ_{N-1} on $[\underline{\theta}, \overline{\theta}]$. Since π_{N-1} and V_N are both concave in θ_{N-1} on the same range, it follows that the sum of those two functions is concave in θ_{N-1} on that range. By the same reasoning used to express $\theta^{BR}_{N|N-1}(\theta_{N-1})$ in (2.15),

$$\theta^{BR}_{N-1|N-2}(\theta_{N-2}) = \max[\theta^*_{N-1}(N-1), \theta_{N-2}]. \tag{2.21}$$

Using (2.21), a similar expression for $\theta^{BR}_{N|N-2}(\theta_{N-2})$ can be derived as follows,

$$\theta^{BR}_{N|N-2}(\theta_{N-2}) = \theta^{BR}_{N|N-1}(\theta^{BR}_{N-1|N-2}(\theta_{N-2})) = \theta^{BR}_{N|N-1}(\max[\theta^*_{N-1}(N-1), \theta_{N-2}]). \tag{2.22}$$

Using the function $\theta^{BR}_{N|N-1}$ from (2.15) gives,

$$\theta^{BR}_{N|N-2}(\theta_{N-2}) = \max\{\theta^*_N(N), \max[\theta^*_{N-1}(N-1), \theta_{N-2}]\}. \tag{2.23}$$

Note that maxima have a distributive property such that,

$$\max\{\theta^*_N(N), \max[\theta^*_{N-1}(N-1), \theta_{N-2}]\} = \max\{\max[\theta^*_N(N), \theta^*_{N-1}(N-1)], \theta_{N-2}\}. \tag{2.24}$$

Recognizing $(\max[\theta_N^*(N), \theta_{N-1}^*(N-1)])$ as the Stage $N-1$ candidate for θ_N^*, it follows that the best response of θ_N to any value of state variable θ_{N-2} is then,

$$\theta_{N|N-2}^{BR}(\theta_{N-2}) = \max[\theta_N^*(N-1), \theta_{N-2}]. \tag{2.25}$$

Assuming best responses of θ_{N-1} and θ_N, the maximum possible $\pi_{N-1} + \pi_N$ as function of θ_{N-2} is then,

$$V_{N-1}(\theta_{N-2}) = \pi_{N-1}(\theta_{N-1|N-2}^{BR}(\theta_{N-2})) + V_N(\theta_{N-1|N-2}^{BR}(\theta_{N-2})). \tag{2.26}$$

From (2.16) note that

$$V_N(\theta_{N-1|N-2}^{BR}(\theta_{N-2})) = \pi_N[\theta_{N|N-1}^{BR}(\theta_{N-1|N-2}^{BR}(\theta_{N-2}))] = \pi_N(\theta_{N|N-2}^{BR}(\theta_{N-2})). \tag{2.27}$$

Thus the value function in (2.27) can be rewritten as

$$V_{N-1}(\theta_{N-2}) = \pi_{N-1}(\theta_{N-1|N-2}^{BR}(\theta_{N-2})) + \pi_N(\theta_{N|N-2}^{BR}(\theta_{N-2})). \tag{2.28}$$

Naturally the value function V_{N-1} sets up Stage $N-2$ and the solution by backward induction continues, with each remaining stage solved in the same manner just shown for Stage $N-1$. From here, discussion of the solution is generalized to any such stage.

2.3.3 General Stage

For any $j < N$, the Stage $j+1$ objective is to find an initial candidate θ_{j+1}^* as well as updated candidates for $(\theta_{j+2}^*, .., \theta_N^*)$. The Stage $j+1$ candidates will be,

$$\arg \max_{\theta_j \leq \cdots \leq \theta_N} \sum_{i=j+1}^{N} \pi_i(\theta_i). \tag{2.29}$$

The value function V_{j+2} is defined,

$$V_{j+2}(\theta_{j+1}) \equiv \max_{\theta_{j+1} \leq \theta_{j+2} \leq \cdots \leq \theta_N} \sum_{k=j+2}^{N} \pi_k(\theta_k) = \sum_{k=j+2}^{N} \pi_k(\theta_{k|j}^{BR}(\theta_j)). \tag{2.30}$$

Assuming best responses of $(\theta_{j+2}, ..., \theta_N)$ to any choice of θ_j, the initial candidate for θ_{j+1}^* is then,

$$\theta_{j+1}^*(j+1) = \arg\max_{\theta_{j+1}} [\pi_{j+1}(\theta_{j+1}) + V_{j+2}(\theta_{j+1})]. \tag{2.31}$$

Implicit in (2.31) are updated candidates for $(\theta_{j+2}^*, ..., \theta_N^*)$, which are,

$$\theta_k^*(j+1) = \theta_{k|j+1}^{BR}(\theta_{j+1}^*(j+1)), \forall k > j+1. \tag{2.32}$$

The initial candidate $\theta_{j+1}^*(j+1)$ may need to be modified to satisfy the monotonicity condition, depending on the value that state variable θ_j. The best response of θ_{j+1} to any value of θ_j is then,

$$\theta_{j+1|1}^{BR}(\theta_j) = \arg\max_{\theta_{j+1} \geq \theta_j} [\pi_{j+1}(\theta_{j+1}) + V_{j+2}(\theta_{j+1})]. \tag{2.33}$$

By Assumption (A3), π_{j+1} and V_{j+2} are both concave in θ_{j+1} on $[\underline{\theta}, \overline{\theta}]$, which implies that $\pi_{j+1} + V_{j+2}$ is concave in θ_{j+1} on the same range. It follows that,

$$\theta_{j+1|j}^{BR}(\theta_j) = \max[\theta_{j+1}^*(j+1), \theta_j]. \tag{2.34}$$

From (2.34), a proof by induction shows that the best response function of θ_k to state variable θ_j have a similar form, for all $k \geq j+1$.

Proposition 1. $\theta_{k|j}^{BR}(\theta_j) = \max[\theta_k^*(j+1), \theta_j], \forall k \geq j+1$.

Proof. To initialize, note that by (2.34), the proposition is true for $k = j+1$. Note also that,

$$\theta_{k|j}^{BR}(\theta_j) = \theta_{k|k-1}^{BR}(\theta_{k-1|j}^{BR}(\theta_j)). \tag{2.35}$$

Next, for any $k > j+1$, assume that the proposition is true for $k-1$. From (2.35), it then follows that,

$$\theta_{k|j}^{BR}(\theta_j) = \theta_{k|k-1}^{BR}(\max[\theta_{k-1}^*(j+1), \theta_j]) = \max\{\theta_k^*(k), \max[\theta_{k-1}^*(j+1), \theta_j]\}. \tag{2.36}$$

The first step of (2.36) follows from the inductive assumption. The second step follows from (2.34) with k replacing $j+1$. Next, maximums have a distributive property such that,

$$\theta_{k|j}^{BR}(\theta_j) = \max\{\theta_k^*(k), \max[\theta_{k-1}^*(j+1), \theta_j]\} = \max\{\max[\theta_k^*(k), \theta_{k-1}^*(j+1)], \theta_j\}. \tag{2.37}$$

Finally note that,

$$\begin{aligned} \max[\theta_k^*(k), \theta_{k-1}^*(j+1)] &= \theta_{k|k-1}^{BR}[\theta_{k-1}^*(j+1)] = \theta_{k|k-1}^{BR}[\theta_{k-1|j+1}^{BR}(\theta_{j+1}^*(j+1))] \\ &= \theta_{k|j+1}^{BR}(\theta_{j+1}^*(j+1)) = \theta_k^*(j+1). \end{aligned} \tag{2.38}$$

The first step of (2.38) is implied by (2.34) with k replacing $j+1$. The second follows from (2.32), with $k-1$ replacing k and $j+1$ replacing j. The third step follows from (2.35), and the final step follows from (2.32), with $j+1$ replacing j. Substituting (2.38) into (2.37) gives,

$$\theta_{k|j}^{BR}(\theta_j) = \max\{\theta_k^*(j+1), \theta_j\}. \tag{2.39}$$

Thus if the proposition is true for any $k-1 \geq j+1$, it is true for k. Therefore the proposition is true for all $k \geq j+1$, since it is true for $k = j+1$. **QED**

Assuming optimal responses of $(\theta_{j+1}, ..., \theta_N)$ to state variable θ_j, the value function V_{j+1} is,

$$V_{j+1}(\theta_j) = \pi_{j+1}[\theta_{j+1|j}^{BR}(\theta_j)] + V_{j+2}[\theta_{j+1|j}^{BR}(\theta_j)] = \sum_{k=j+1}^{N} \pi_k(\theta_{k|j}^{BR}(\theta_j)). \tag{2.40}$$

2.3.3.1 The Optimal Schedule

The monopolist's optimal schedule is solved in Stage 1. The initial candidate for θ_1^* is,

$$\theta_1^*(1) = \arg\max_{\theta_1}[\pi_1(\theta_1) + V_2(\theta_1)]. \tag{2.41}$$

This implies Stage 1 candidates for $(\theta_2^*, ..., \theta_N^*)$, which are,

$$\theta_k^*(1) = \theta_{k|1}^{BR}(\theta_1^*(1)), \forall k = 2, ..., N. \tag{2.42}$$

By construction $(\theta_1^*(1), ..., \theta_N^*(1))$ belongs to the feasible set given by (2.11), in other words, it satisfies the monotonicity constraint. Therefore it is the solution to the monopolist's problem (2.12),

$$(\theta_1^*(1), ..., \theta_N^*(1)) = (\theta_1^*, ..., \theta_N^*). \tag{2.43}$$

2.4 Optimal Ironing

An ironed region of a monopolist's optimal schedule is a sequence of markets over which the optimal marginal type is constant. That is, if the monopolist's optimal schedule is ironed from the j^{th} to the k^{th} market, then,

$$(\theta_j^* = \cdots = \theta_k^*). \tag{2.44}$$

This constant portion of the monopolist's schedule reflects a bunching of units j through k at a single consumer type. That is, the monopolist is making a single consumer type indifferent between a bundle of j units and bundle of k units.

Recall that the multi-market approach employed here, treats each unit as a different good that the monopolist sells in an independent market. The markets however depend on each other through the monotonicity constraint. It is this dependence that the backward induction approach builds into the solution. If the monopolist's optimal schedule is such that the monotonicity constraint does not bind, then the N markets are effectively independent or separable. That is, nothing would be lost from the *ad hoc* approach of searching for N different marginal types θ_i, to maximize N different market profit functions π_i. However, when the monopolist's optimal monotonic schedule contains ironed regions as in (2.44), this indicates that the j^{th} through to k^{th} markets should not be treated as independent. Separately solving for optimal types in these markets will lead to a violation of the monotonicity constraint. Instead, it turns out to be optimal for the monopolist to chose only one marginal type for this sequence of markets, to maximize the sum of profits from sale of the j^{th} to the k^{th} market. In this section, I establish that the backward induction approach identifies the inseparable markets and chooses a constant marginal type that is optimal over them. This optimal type, in turn, is shown to satisfy a "summation condition". If the monopolist's optimal schedule is such that $(\theta_{j-1}^* < \theta_j^* = \theta_{j+1}^* = \cdots = \theta_k^* < \theta_{k+1}^*)$, then $\theta_j^* = \cdots = \theta_k^* = \hat{\theta}$, where,

$$\sum_{i=j}^{k} \pi_i'(\hat{\theta}) = 0. \tag{2.45}$$

This summation condition is Proposition 5 below. First, I prove three propositions that will help to show the summation condition.

Proposition 2. For all $j < k \le N$, the derivative of the best response function $\theta_{k|j}^{BR}(\theta_j)$ with respect to θ_j is,

$$(\theta_{k|j}^{BR})'(\theta_j) = \left\{ \begin{array}{cc} 0, & \theta_j < \theta_k^*(j+1) \\ 1, & \theta_j > \theta_k^*(j+1) \end{array} \right\}.$$

Proof is immediate from Proposition 1.

Proposition 3. For all $j < N$, $V_{j+1}(\theta_j)$ is continuously differentiable with respect to θ_j, and this derivative is given by,

$$V_{j+1}'(\theta_j) = \left\{ \begin{array}{cc} 0, & \theta_j \le \theta_{j+1}^*(j+1) \\ \pi_{j+1}'(\theta_j) + V_{j+2}'(\theta_j), & \theta_j \ge \theta_{j+1}^*(j+1) \end{array} \right\}$$

.

Proof. First, note that by the first order condition for $\theta_{j+1}^*(j+1)$,

$$\pi_{j+1}'(\theta_{j+1}^*(j+1)) + V_{j+2}'(\theta_{j+1}^*(j+1)) = 0. \tag{2.46}$$

Next, recall that V_{j+1} can be expressed,

$$V_{j+1}(\theta_j) = \pi_{j+1}(\theta_{j+1|j}^{BR}(\theta_j)) + V_{j+2}(\theta_{j+1|j}^{BR}(\theta_j)). \tag{2.47}$$

The derivative of V_{j+1} with respect to θ_j is then,

$$V_{j+1}'(\theta_j) = [\pi_{j+1}'(\theta_{j+1|j}^{BR}(\theta_j)) + V_{j+2}'(\theta_{j+1|j}^{BR}(\theta_j))] \cdot (\theta_{j+1|j}^{BR})'(\theta_j), \forall \theta_j \ne \theta_{j+1}^*(j+1). \tag{2.48}$$

It follows from Proposition 2, that for all $\theta_j < \theta^*_{j+1}(j+1)$,

$$V'_{j+1}(\theta_j) = [\pi'_{j+1}(\theta^{BR}_{j+1|j}(\theta_j)) + V'_{j+2}(\theta^{BR}_{j+1|j}(\theta_j))] \cdot (0) = 0. \tag{2.49}$$

By Propositions 1 and 2, for all $\theta_j > \theta^*_{j+1}(j+1)$,

$$V'_{j+1}(\theta_j) = [\pi'_{j+1}(\theta_j) + V'_{j+2}(\theta_j)] \cdot (1) = \pi'_{j+1}(\theta_j) + V'_{j+2}(\theta_j). \tag{2.50}$$

To establish that this derivative is continuous at $\theta^*_{j+1}(j+1)$, note that from (2.49),

$$\lim_{\theta_j \uparrow \theta^*_{j+1}(j+1)} \; V'_{j+1}(\theta_j) = 0. \tag{2.51}$$

From (2.50) it can be seen that,

$$\lim_{\theta_j \downarrow \theta^*_{j+1}(j+1)} \; V'_{j+1}(\theta_j) = \pi'_{j+1}(\theta^*_{j+1}(j+1)) + V'_{j+2}(\theta^*_{j+1}(j+1)) = 0. \tag{2.52}$$

Therefore, for all $j < N$, V_{j+1} is continuously differentiable with respect to θ_j, and this derivative is given by,

$$V'_{j+1}(\theta_j) = \left\{ \begin{array}{cc} 0, & \theta_j \leq \theta^*_{j+1}(j+1) \\ \pi'_{j+1}(\theta_j) + V'_{j+2}(\theta_j), & \theta_j \geq \theta^*_{j+1}(j+1) \end{array} \right\}. \tag{2.53}$$

QED

Proposition 4. For all $j < N$, $V_{j+1}(\theta_j)$ is continuously differentiable with respect to θ_j, and this derivative is given by,

$$V'_{j+1}(\theta_j) = \left\{ \begin{array}{cc} 0, & \theta_j \leq \theta^*_{j+1}(j+1) \\ \pi'_{j+1}(\theta_j) + V'_{j+2}(\theta_j), & \theta_j \geq \theta^*_{j+1}(j+1) \end{array} \right\}$$

. By Proposition 3, V_{j+2} is continuously differentiable with respect to θ_j, and this derivative is given by,

$$V'_{j+2}(\theta_j) = \left\{ \begin{array}{cc} 0, & \theta_j \leq \theta^*_{j+2}(j+1) \\ \pi'_{j+2}(\theta_j) + V'_{j+3}(\theta_j), & \theta^*_{j+2}(j+1) \leq \theta_j \end{array} \right\}. \tag{2.54}$$

Substituting (2.54) into (2.53) gives,

$$V'_{j+1}(\theta_j) = \left\{ \begin{array}{cc} 0, & \theta_j \leq \theta^*_{j+1}(j+1) \\ \pi'_{j+1}(\theta_j), & \theta^*_{j+1}(j+1) \leq \theta_j \leq \theta^*_{j+2}(j+1), \\ \pi'_{j+1}(\theta_j) + \pi'_{j+2}(\theta_j) + V'_{j+3}(\theta_j), & \theta^*_{j+2}(j+1) \leq \theta_j \end{array} \right\}. \tag{2.55}$$

Thus for any $j < k \leq N$, evaluated at θ_j such that $\theta^*_k(j+1) \leq \theta_j < \theta^*_{k+1}(j+1)$, it follows from Proposition 3 that

$$V'_k(\theta_j) = \pi'_k(\theta_j) + V'_{k+1}(\theta_j) = \pi'_k(\theta_j) + 0 = \pi'_k(\theta_j). \tag{2.56}$$

Therefore, V_{j+1} is continuously differentiable with respect to θ_j, and the derivative is given by,

$$V'_{j+1}(\theta_j) = \left\{ \begin{array}{cc} 0, & \theta_j \leq \theta^*_{j+1}(j+1) \\ \sum_{i=j+1}^{k} \pi'_i(\theta_j), & \theta^*_k(j+1) \leq \theta_j < \theta^*_{k+1}(j+1) \end{array} \right\}. \tag{2.57}$$

QED

Proposition 5 (summation condition). For some $1 \leq j < k \leq N$, if the monopolist's solution is such that $\theta^*_{j-1} < \theta^*_j = \theta^*_{j+1} \cdots = \theta^*_k = \hat{\theta} < \theta^*_{k+1}$, then,

$$\sum_{i=j}^{k} \pi'_i(\hat{\theta}) = 0.$$

Proof. The monopolist's optimal schedule is given by the Stage 1 candidates from the solution by backward induction. That is, $(\theta^*_1, ..., \theta^*_N) = (\theta^*_1(1), ..., \theta^*_N(1))$. By showing that the proposition's assumption implies that $(\theta^*_j(1), ..., \theta^*_N(1)) = (\theta^*_j(j), ..., \theta^*_N(j))$, the proposition can be proved using Proposition 4, and the first order condition for $\theta_j^*(j)$.

First, recall that,

$$\theta_j^*(1) = \theta_{j|1}^{BR}(\theta_1^*(1)) = \theta_{j|j-1}^{BR}[\theta_{j-1|1}^{BR}(\theta_1^*(1))] = \theta_{j|j-1}^{BR}(\theta_{j-1}^*(1)) = \max[\theta_j^*(j), \theta_{j-1}^*(1)]. \quad (2.58)$$

Since $\theta_j^*(1) > \theta_{j-1}^*(1)$, then it must be the case that

$$\theta_j^*(1) = \max[\theta_j^*(j), \theta_{j-1}^*(1)] = \theta_j^*(j). \quad (2.59)$$

It follows that for all $k > j$,

$$\theta_k^*(1) = \theta_{k|j}^{BR}(\theta_j^*(1)) = \theta_{k|j}^{BR}(\theta_j^*(j)) = \theta_k^*(j). \quad (2.60)$$

Therefore it has been established that, $(\theta_j^*(1), ..., \theta_N^*(1)) = (\theta_j^*(j), ..., \theta_N^*(j))$.

From the assumption of the proposition, it follows that $(\theta_j^*(j) = \cdots = \theta_k^*(j) < \theta_{k+1}^*(j))$. This along with Proposition 4 imply that $V_{j+1}'(\theta_j)$ evaluated at $\theta_j^*(j)$, is given by,

$$V_{j+1}'(\theta_j^*(j)) = \sum_{i=j+1}^{k} \pi_i'(\theta_j^*(j)). \tag{2.61}$$

By the first order condition for $\theta_j^*(j)$,

$$\pi_j'(\theta_j^*(j)) + V_{j+1}'(\theta_j^*(j)) = \pi_j'(\theta_j^*(j)) + \sum_{i=j+1}^{k} \pi_i'(\theta_j^*(j)) = 0. \tag{2.62}$$

Finally, recall that $(\theta_j^*(j), ..., \theta_k^*(j)) = (\theta_j^*(1), ..., \theta_k^*(1)) = (\theta_j^*, ..., \theta_k^*)$. By assumption the monopolist's optimal schedule is such that $(\theta_j^* = \cdots = \theta_k^* = \hat{\theta})$. Therefore, (2.62) implies that

$$\sum_{i=j}^{k} \pi_i'(\theta_j^*(j)) = \sum_{i=j}^{k} \pi_i'(\theta_i^*) = \sum_{i=j}^{k} \pi_i'(\hat{\theta}) = 0. \tag{2.63}$$

QED

CHAPTER 3

MONOPOLY PRICING UNDER SINGLE CROSSING VIOLATIONS

In this chapter, I look at the optimal pricing behavior of a monopolist screening consumers on a single "type" parameter, when the single crossing (or Spence-Mirrlees) condition is violated.[1] Following Araujo et al. (2011), and Schottmüller (2015), I focus on one-time violations of this condition. I show that under standard assumptions of one-dimensional screening problems, violation of the single crossing condition can allow for positive and globally incentive compatible assignments through which the monopolist extracts all *equilibrium* surplus. That is, the monopolist can implement positive assignments such that individual rationality constraints bind for all consumer types at their respective assignments. This assignment schedule is formally introduced as $Q^x(\theta)$ in Section 3.3. In that section, Proposition 1B is proven to show that an uninformed monopolist can implement this schedule and extract all surplus from the resulting allocation.[2]

Note, that this implementable assignment schedule, which extracts all equilibrium surplus, is not necessarily the monopolist's optimal assignment schedule. I say that this schedule extracts all equilibrium surplus to emphasize that; while this assignment schedule does extract all social welfare from its resulting allocation, it does not necessarily maximize social welfare. This assignment schedule will, however, often be a component of the monopolist's optimal assignment schedule.

The existence of an assignment schedule that is implementable with binding individual rationality constraints for all consumer types, sharply contrasts with the case of

[1] See Mirrlees (1971) and Spence (1973)

[2] By referring to the monopolist as "uninformed", I mean that consumers are privately informed of their type.

second-degree price discrimination when the single crossing condition is satisfied. When the single crossing condition holds, such an assignment schedule can only be implemented when consumer type is known to the monopolist and he practices perfect first-degree price discrimination. If consumers are privately informed of their type, the individual rationality constraint can bind only for the lowest type receiving a positive assignment. All higher types will receive informational rents. Thus, relaxation of the single crossing condition can have substantial implications for nonlinear pricing.

I show that in a special scenario, the implementable and *total equilibrium surplus extracting* assignment schedule (later denoted $Q^x(\theta)$), is also the monopolist's optimal assignment schedule. In that scenario, the monopolist's marginal cost function is such that this assignment schedule assigns "first-best" quantities to all consumer types, and thus is also socially efficient. Note that when the single crossing condition holds, the monopolist's optimal assignment schedule is socially efficient only when he is informed of his consumers' types (perfect first-degree price discrimination).

Finally, I evaluate how this assignment schedule, which is implementable and extracts all equilibrium surplus, relates to the monopolist's optimal assignment schedule in more general cases. I argue that it composes part of the monopolist's optimal assignment schedule under a wide range of conditions.

3.1 Model

I model the optimal pricing behavior of a monopolist producing a single good q at a cost $c(q)$. For simplicity, I assume a constant marginal cost, $c(q) = cq$. The monopolist offers a menu of quantity bundles and corresponding prices to a set of consumers with heterogeneous preferences for the good. The total price of q units is given by $P(q) = \int_0^q p(v)dv$, where $p(q) = P'(q)$, is the marginal price of the q^{th} unit.

Consumer preferences are differentiated by a single parameter, their "type" $\theta \in [\underline{\theta}, \bar{\theta}] = \Theta$, of which they are privately informed. According to the monopolist's prior probability distribution, type θ has probability density $f(\theta)$. The cumulative distribution function, evaluated at θ, is given by

$$F(\theta) = \int_{\underline{\theta}}^{\theta} f(t)dt. \tag{3.1}$$

Consumer utility $u = u(\theta, q)$ is a function of the quantity consumed and the consumer's type. Before introducing the assumptions of the model, it will help to establish some notation.

For a given type of consumer θ, let $q_0(\theta)$ be the minimum quantity for which the marginal utility of type θ, denoted $u_q(\theta, q)$, is equal to zero. That is,

$$q_0(\theta) \equiv \inf\{q : u_q(\theta, q) = 0\}. \tag{3.2}$$

For all consumer types, assume $u_q(\theta, q)$ is decreasing in quantity over $[0, q_0(\theta)]$, and equal to zero for all $q \geq q_0(\theta)$. If $u_q(\theta, q) > 0$ for all $q > 0$, then $q_0(\theta) = \infty$.

Assumption (A1): $u_{qq}(\theta, q) < 0$, for all $q < q_0(\theta)$, and all θ in Θ.

Let $\bar{q}(\theta)$ denote the quantity at which the marginal utility for type θ is exactly equal to marginal cost. That is, $q = \bar{q}(\theta)$ solves

$$u_q(\theta, q) = c. \tag{3.3}$$

The assumptions of constant marginal cost and decreasing marginal utility imply that $\bar{q}(\theta)$ is a unique quantity.

Finally, for any $\theta^A \neq \theta^B$, let $q^x(\theta^A, \theta^B)$ denote a quantity at which the marginal utility curves of types θ^A and θ^B intersect. Note that such a quantity may not exist. The Spence-Mirrlees or single crossing condition (henceforth SCC) implies that marginal utility is increasing in type for all quantities, in which case $q^x(\theta^A, \theta^B)$ does not exist for all $\theta^A \neq \theta^B$.

$$\textbf{Single Crossing Condition (SCC):} \quad u_{q\theta}(\theta, q) > 0,\ \forall q \geq 0,\ \theta \in \Theta. \tag{3.4}$$

Thus, when the SCC is satisfied,

$$\forall \theta^A \neq \theta^B \in \Theta,\ \nexists\, q \text{ such that } u_q(\hat{\theta}, q) = u_q(\theta, q). \tag{3.5}$$

In this paper, I relax the SCC and focus on a specific violation where the marginal utility curves intersect exactly once for every pair of types in Θ. The type parameter θ, which under the SCC orders consumer preferences over all quantities of the good, will instead order consumer preferences only over initial quantities. That is, consumers with higher θ have higher marginal utilities for initial amounts of the good, but their marginal utilities also decrease at faster rates. To avoid potential confusion, I will refer to consumers with greater values of θ as "steeper" types, and consumers with lesser values of θ as "flatter" types.[3]

Assumption (A2): For all θ in Θ, $u_{q\theta}(\theta, q)|_{q=0} > 0$. For all θ in Θ and $q > 0$, $u_{\theta\theta}(\theta, q) < 0$, $u_{qq\theta}(\theta, q) < 0$, and $u_{q\theta\theta}(\theta, q) < 0$.

[3]The conventional "high" and "low" type terminology is less appropriate when the SCC is violated, since, *e.g.*, in this paper, consumers with higher θ ultimately demand less of the good than consumers with lower θ.

Assumption (A3): For all $\theta^A > \theta^B$ in Θ, there exists a quantity $0 < q^x(\theta^A, \theta^B) < \bar{q}(\theta^A)$, such that: $\forall q \in [0, q^x(\theta^A, \theta^B)]$, $u_q(\theta^A, q) \geq u_q(\theta^B, q)$, and $\forall q > q^x(\theta^A, \theta^B)$, $u_q(\theta^A, q) \leq u_q(\theta^B, q)$.

These assumptions imply a one-time violation of the SCC for every pair of types in Θ. Note also that, since $q^x(\theta^A, \theta^B)$ is assumed less than $\bar{q}(\theta^A) < q_0(\theta)$, it follows that $\bar{q}(\theta^A) < \bar{q}(\theta^B)$, and $q_0(\theta^A) < q_0(\theta^B)$, for all $\theta^A > \theta^B$ in Θ. That is, steeper types are willing to pay more for initial quantities, but willing to buy less q overall at a marginal prices of c and zero.

A pair consisting of a marginal price schedule and an assignment schedule, is *implementable* if the incentive compatibility (henceforth IC) constraint is satisfied for all types in Θ. An implementable marginal price and assignment schedule pair is *feasible* if the individual rationality (henceforth IR) constraints are satisfied for all types in Θ. The IC and IR constraints for type θ are given by $ICC(\theta)$ and $IRC(\theta)$ as follows:

$ICC(\theta)$: $u(\theta, x(\theta)) - P(x(\theta)) \geq u(\theta, x(\theta')) - P(x(\theta'))$, $\forall \theta' \in \Theta$.

$IRC(\theta)$: $u(\theta, x(\theta)) - P(x(\theta)) \geq 0$.

The monopolist chooses a feasible and implementable pair $(p(q), x(\theta))$ to maximize profit. That is, he maximizes profit subject to the $ICC(\theta)$ and $IRC(\theta)$ for all $\theta \in \Theta$. The monopolist's profit function for a pair $(p(q), x(\theta))$, is given by

$$\Pi = \int_{\underline{\theta}}^{\bar{\theta}} f(t) \left(\int_0^{x(t)} [p(v) - c] \, dv \right) dt = \int_{\underline{\theta}}^{\bar{\theta}} [(P(x(t) - c \cdot x(t)))f(t)] dt. \tag{3.6}$$

3.1.1 Consumer Surplus Under Marginal Cost Pricing

Suppose the monopolist were regulated such that $p(q) = cq$ for all $q > 0$. All $\theta \in \Theta$ will purchase $\bar{q}(\theta)$ and receive a surplus. Let $\widetilde{CS}(\theta, \bar{q}(\theta))$ represent the first-best consumer surplus that type θ receives under the hypothetical imposition of marginal cost pricing. That is,

$$\widetilde{CS}(\theta, \bar{q}(\theta)) \equiv \int_0^{\bar{q}(\theta)} [u_q(\theta, v) - c]dv = u(\theta, \bar{q}(\theta)) - c \cdot \bar{q}(\theta). \tag{3.7}$$

This hypothetical first-best consumer surplus helps to illustrate how the monopolist's optimal pricing behavior can dramatically differ when the SCC is violated.

In standard price discrimination models, where the SCC holds, higher consumer types have larger first-best quantities, and higher utilities at first-best quantities. Consequently, the monopolist's optimal price and assignment schedules exhibit well-known properties when the SCC holds. Most notably, the monopolist must pay informational rents to induce consumers to truthfully reveal their type. This is a consequence of the monopolist's imperfect information. The IR constraint binds at the assigned quantity only for the lowest type served (*i.e.* the lowest type served receives no informational rent). All other types served receive informational rents, and the rent is increasing in type. This is because the monopolist must prevent higher types from pretending they are lower types, but typically needn't worry about lower types pretending to be higher types. Finally, the highest type is the only type assigned her first-best quantity, $x(\theta) = \bar{q}(\theta)$, giving the "no distortion at the top" property.

When the SCC is violated, the monopolist's optimal marginal price and assignment schedules can produce some markedly different results. The violation of the SCC specified by assumptions (A2) and (A3), implies that $\bar{q}(\theta)$ is instead decreasing in θ. Thus, the hypothetical first-best surplus $\widetilde{CS}(\theta, \bar{q}(\theta))$ may be increasing, decreasing, or constant in consumer type. Under the SCC violation, incentive compatibility can have a variety of implications, because flatter types may be willing to pay more, less, or the same amount as steeper types, for greater assignments of the good. When the SCC is violated, the monopolist's optimal marginal price and assignment schedules can result in allocations with characteristics very different from those discussed in the preceding paragraph. In one such scenario (discussed in Section 3.2), the monopolist's optimal assignment schedule offers the first-best, $\bar{q}(\theta)$ to all θ in Θ. This is a "no distortion from top to bottom" result, if you will.

3.2 Scenario 1: Constant First-Best Surplus

Consider a scenario where $\widetilde{CS}(\theta, \bar{q}(\theta))$ is the same across all types in Θ. That is,

$$\textbf{Scenario 1:} \quad \widetilde{CS}(\theta, \bar{q}(\theta)) = \widetilde{CS}(\theta', \bar{q}(\theta')),\ \forall \theta, \theta' \in \Theta. \tag{3.8}$$

In this scenario the monopolist's optimal marginal price schedule is surprisingly straightforward. The monopolist can extract all first-best surplus from consumers with the marginal price schedule

$$p(q) = \left\{ \begin{array}{cc} u_q(\bar{\theta}, q), & q \in [0, \bar{q}(\bar{\theta})] \\ c, & q \geq \bar{q}(\bar{\theta}) \end{array} \right\}. \tag{3.9}$$

That is, the marginal price is equated to the highest type's marginal utility up to $\bar{q}(\bar{\theta})$ units, and marginal price is equated to marginal cost thereafter. See Figure 3.1.

The assignment schedule of quantities to types, implied by (3.9), is

$$x(\theta) = \bar{q}(\theta),\ \forall \theta \in \Theta. \tag{3.10}$$

I will show that the marginal price schedule and the assignment schedule given by (3.9) and (3.10) respectively, are incentive compatible and satisfy the IR constraints for all types, making (3.9) and (3.10) implementable, and feasible. I will also show that for all types, the IR constraint is binding at the type's assigned quantity $x(\theta) = \bar{q}(\theta)$. Since all types are assigned their first-best quantities it follows that the marginal price and assignment schedules (3.9) and (3.10) maximize social welfare, and since the IR constraint binds for all types, all of this social welfare is extracted by the monopolist. From this, it is immediate that (3.9) and (3.10) are the monopolist's optimal marginal price and assignment schedules under Scenario 1.

Proposition 1A. If hypothetical first-best surplus is constant in θ over Θ, the marginal price schedule given in (3.9), and the implicit assignment schedule given in (3.10), are incentive

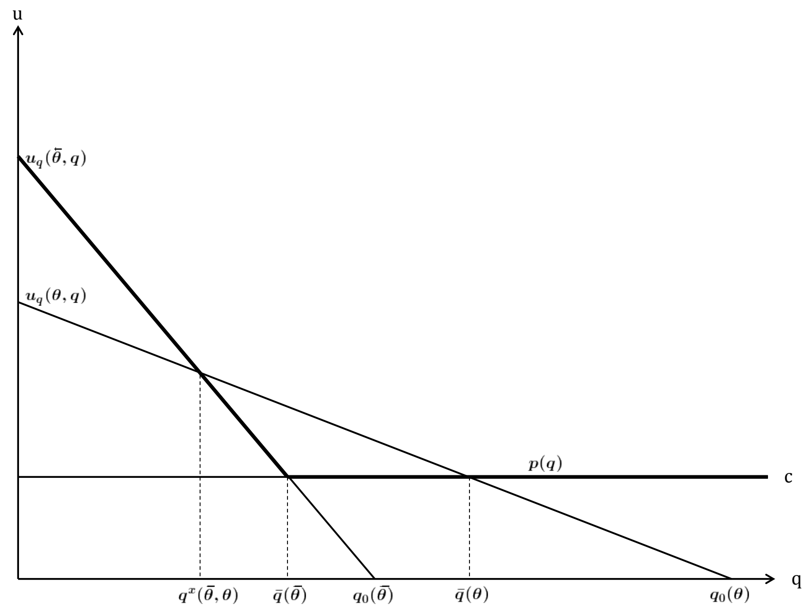

Figure 3.1. Marginal Price Scedule (constant first-best surplus)

compatible for all θ in Θ. Additionally, the IR constraint is binding at $x(\theta) = \bar{q}(\theta)$, for all θ in Θ.

Proof. First, for a given total price schedule $P(q)$, the consumer surplus that type θ actually gets from consuming q units is denoted $CS(\theta, q)$, and given by

$$CS(\theta, q) = u(\theta, q) - P(q). \tag{3.11}$$

Note that the *actual* consumer surplus given in (3.11), is a different function than the *hypothetical* first-best surplus defined in (3.7), as indicated by the removal of the tilde in (3.11). Each consumer type solves

$$\max_{q>0}[CS(\theta, q) = u(\theta, q) - P(q)], \tag{3.12}$$

and then purchases the quantity solving (3.12) if it satisfies their IR constraint.

Consider first the steepest consumer type, $\bar{\theta}$. It is immediate from (3.9) that $CS(\bar{\theta}, q)$

is equal to zero for all $q \leq \bar{q}(\bar{\theta})$. Therefore, for any $q > \bar{q}(\bar{\theta})$,

$$CS(\bar{\theta}, q) = [u(\bar{\theta}, \bar{q}(\bar{\theta})) - P(\bar{q}(\bar{\theta}))] + \int_{\bar{q}(\bar{\theta})}^{q} u_q(\bar{\theta}, v)dv - \int_{\bar{q}(\bar{\theta})}^{q} cdv = \int_{\bar{q}(\bar{\theta})}^{q} u_q(\bar{\theta}, v)dv - \int_{\bar{q}(\bar{\theta})}^{q} cdv, \quad (3.13)$$

where the second step of (3.13) makes use of $[u(\bar{\theta}, \bar{q}(\bar{\theta})) - P(\bar{q}(\bar{\theta}))] = 0$. By definition $u_q(\bar{\theta}, \bar{q}(\bar{\theta})) = c$, and $u_q(\theta, q)$ is decreasing in q over $[\bar{q}(\bar{\theta}), q_0(\bar{\theta})]$ and equal to 0 thereafter by Assumption (A1). It follows then that $u_q(\bar{\theta}, q)$ is strictly less than c for any q greater than $\bar{q}(\bar{\theta})$. Therefore, $CS(\bar{\theta}, q)$ is equal to 0 for all $q \leq x(\bar{\theta}) = \bar{q}(\bar{\theta})$, and less than 0 for all $q > x(\bar{\theta}) = \bar{q}(\bar{\theta})$. Thus, $\bar{\theta}$ weakly prefers her assignment to any other positive quantity. Note also that her IR constraint is binding at her assigned quantity, since $CS(\bar{\theta}, \bar{q}(\bar{\theta})) = 0$.

Turning now to θ less than $\bar{\theta}$, consider the derivative of $CS(\theta, q)$ with respect to q, given by

$$\frac{\partial[u(\theta, q) - P(q)]}{\partial q} = \left\{ \begin{array}{cc} u_q(\theta, q) - u_q(\bar{\theta}, q), & 0 < q \leq \bar{q}(\bar{\theta}) \\ u_q(\theta, q) - c, & q > \bar{q}(\bar{\theta}) \end{array} \right\}, \quad (3.14)$$

and shown in the lower panel of Figure 3.2. The first order condition for a local maximum is satisfied at q equal $q^x(\bar{\theta}, \theta)$, and $\bar{q}(\theta)$.

The second derivative of $CS(\theta, q)$ with respect to q is given by,[4]

$$\frac{\partial^2[u(\theta, q) - P(q)]}{\partial q^2} = \left\{ \begin{array}{cc} u_{qq}(\theta, q) - u_{qq}(\bar{\theta}, q), & 0 < q < \bar{q}(\bar{\theta}) \\ u_{qq}(\theta) - c, & q > \bar{q}(\bar{\theta}) \end{array} \right\}. \quad (3.15)$$

Recall that $q^x(\bar{\theta}, \theta)$ is less than $\bar{q}(\bar{\theta})$ by Assumption (A3), and $u_{qq}(\theta, q)$ is greater (smaller negative) than $u_{qq}(\bar{\theta}, q)$ by Assumption (A2). Thus, equation (3.15) when evaluated at $q = q^x(\bar{\theta}, \theta)$, is positive and clearly violates the second order condition for a local maximum.

Now, recall that c is a positive constant, and by Assumption (A1), $u_{qq}(\theta, q)$ is negative for all $q > 0$. Since $\bar{q}(\theta)$ is greater than $\bar{q}(\bar{\theta})$, equation (3.15) is negative when evaluated at any

[4]Note that the second derivative of $CS(\theta, q)$ with respect to q does not exist at $q = \bar{q}(\bar{\theta})$. This can seen in the lower panel of Figure 3.2 by the kink in $CS_q(\theta, q)$ at $q = \bar{q}(\bar{\theta})$.

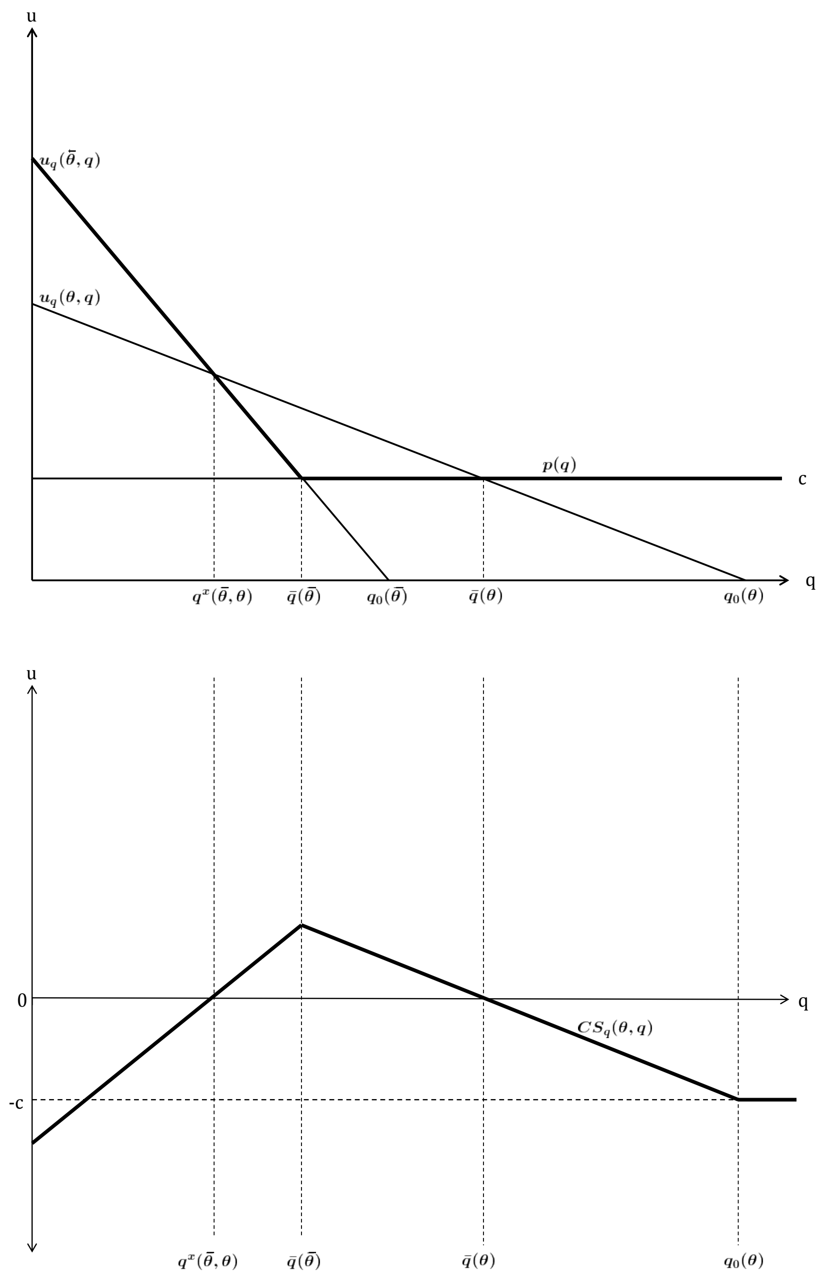

Figure 3.2. Derivative of $CS(\theta, q)$ with respect to q

$q > \bar{q}(\bar{\theta})$. Thus, $\bar{q}(\theta)$ satisfies the second order condition sufficient for a local maximum. Note however that the consumer can always choose a quantity of zero which yields a consumer surplus of zero. In the following paragraph, I verify that the IR constraint for type θ is satisfied at $\bar{q}(\theta)$, which turns out to imply that $\bar{q}(\theta)$ is a global maximum for $CS(\theta, q)$.

Turning to the IR constraints, for any θ less than $\bar{\theta}$, consumer surplus evaluated at $x(\theta) = \bar{q}(\theta)$, is

$$CS(\theta, \bar{q}(\theta)) = u(\theta, \bar{q}(\theta)) - P(\bar{q}(\theta)) = \int_{0}^{\bar{q}(\theta)} u_q(\theta, v)dv - \int_{0}^{\bar{q}(\bar{\theta})} u_q(\bar{\theta}, v)dv - \int_{\bar{q}(\bar{\theta})}^{\bar{q}(\theta)} cdv. \tag{3.16}$$

Recognizing that $\bar{q}(\bar{\theta}) < \bar{q}(\theta)$, the first integral in (3.16) can be separated into two integrals, as

$$CS(\theta, \bar{q}(\theta)) = \int_{0}^{\bar{q}(\bar{\theta})} u_q(\theta, v)dv + \int_{\bar{q}(\bar{\theta})}^{\bar{q}(\theta)} u_q(\theta, v)dv - \int_{0}^{\bar{q}(\bar{\theta})} u_q(\bar{\theta}, v)dv - \int_{\bar{q}(\bar{\theta})}^{\bar{q}(\theta)} cdv. \tag{3.17}$$

Since the second and fourth integrals in (3.17) are evaluated over the same interval, they can be combined, giving

$$CS(\theta, \bar{q}(\theta)) = \int_{0}^{\bar{q}(\bar{\theta})} u_q(\theta, v)dv + \int_{\bar{q}(\bar{\theta})}^{\bar{q}(\theta)} [u_q(\theta, v) - c]dv - \int_{0}^{\bar{q}(\bar{\theta})} u_q(\bar{\theta}, v)dv. \tag{3.18}$$

Without changing the equality, $\int_0^{\bar{q}(\bar{\theta})} cdv$ can be added and subtracted to the right-hand side of (3.18), giving

$$CS(\theta,\bar{q}(\theta)) = \int\limits_0^{\bar{q}(\bar{\theta})} u_q(\theta,v)dv + \int\limits_{\bar{q}(\bar{\theta})}^{\bar{q}(\theta)} [u_q(\theta,v) - c]dv - \int\limits_0^{\bar{q}(\bar{\theta})} u_q(\bar{\theta},v)dv + \int\limits_0^{\bar{q}(\bar{\theta})} cdv - \int\limits_0^{\bar{q}(\bar{\theta})} cdv. \quad (3.19)$$

The first, third, fourth, and fifth integrals of (3.19) are all evaluated over the same interval. Thus, the fifth integral can be combined with the first, and the fourth integral can be combined with the third, giving

$$CS(\theta,\bar{q}(\theta)) = \int\limits_0^{\bar{q}(\bar{\theta})} [u_q(\theta,v) - c]dv + \int\limits_{\bar{q}(\bar{\theta})}^{\bar{q}(\theta)} [u_q(\theta,v) - c]dv - \int\limits_0^{\bar{q}(\bar{\theta})} [u_q(\bar{\theta},v) - c]dv. \quad (3.20)$$

The first and second integrals in (3.20) have the same integrand. Since the first integral is also evaluated over $[0, \bar{q}(\bar{\theta})]$, and the second integral is evaluated over $[\bar{q}(\bar{\theta}), \bar{q}(\theta)]$, they can be combined, giving

$$CS(\theta,\bar{q}(\theta)) = \int\limits_0^{\bar{q}(\theta)} [u_q(\theta,v) - c]dv - \int\limits_0^{\bar{q}(\bar{\theta})} [u_q(\bar{\theta},v) - c]dv. \quad (3.21)$$

Finally, recall that the first integral in (3.21) is the first-best surplus for θ, and the second integral is the first-best surplus for $\bar{\theta}$. Therefore, the constant first-best surplus condition implies that

$$CS(\theta,\bar{q}(\theta)) = \widetilde{CS}(\theta,\bar{q}(\theta)) - \widetilde{CS}(\bar{\theta},\bar{q}(\bar{\theta})) = 0,\ \forall\theta < \bar{\theta}. \quad (3.22)$$

That is, the IR constraint binds at $x(\theta) = \bar{q}(\theta)$ for all θ less than $\bar{\theta}$. Recall that the IR constraint has been shown to bind for the steepest type $\bar{\theta}$ at $\bar{q}(\bar{\theta})$. Thus, the IR constraint binds at $x(\theta) = \bar{q}(\theta)$, for all θ in Θ.

Recall that for all θ, $CS(\theta, q)$ was shown to have local maxima at $q = 0$ and $q = \bar{q}(\theta)$. Consumer surplus equals zero when evaluated at $q = 0$, and since the IR constraint binds at $\bar{q}(\theta)$, consumer surplus equals zero when evaluated at $q = \bar{q}(\theta)$. Thus, $q = 0$ and $q = \bar{q}(\theta)$ are both global maxima, and all θ in Θ weakly prefer their assignment $x(\theta) = \bar{q}(\theta)$ to any other $q \geq 0$. Therefore, the marginal price schedule given in (3.9) and the assignment schedule given in (3.10), are globally incentive compatible, and the IR constraint binds at $x(\theta) = \bar{q}(\theta)$, for all θ in Θ. **QED**

Corolarry 1A. If the hypothetical first-best surplus is constant for in θ over Θ, the marginal price schedule given in (3.9), and the assignment schedule given in (3.10), are optimal for the monopolist, and the resulting allocation is socially efficient.

Proof. First-best assignments $x(\theta) = \bar{q}(\theta)$ for all θ in Θ, maximize social welfare. By Proposition 1A, the first-best assignments are globally incentive compatible and leave $CS(\theta, \bar{q}(\theta)) = 0$, for all θ in Θ. Since all consumer types receive no surplus at their assignments, it follows that all social welfare is extracted by the monopolist. Therefore, the pair $(p(q), x(\theta))$ specified by (3.9) and (3.10) must be optimal for the monopolist, and the resulting allocation is socially efficient. **QED**

3.3 Two Important Functions of Quantity and Type

In this section, I introduce two functions which map types to quantities and help to understand the monopolist's problem. Recall that for $\theta^A > \theta^B$, $q^x(\theta^A, \theta^B)$ was introduced as the quantity such that $u_q(\theta^A, q) = u_q(\theta^B, q)$, while $u_q(\theta^A, q)$ is greater than $u_q(\theta^B, q)$ for quantities less than $q^x(\theta^A, \theta^B)$, and $u_q(\theta^A, q)$ is less than or equal to $u_q(\theta^B, q)$ for quantities greater than $q^x(\theta^A, \theta^B)$. Consider a simple extension of this concept, denoted $q^x(\theta)$. For each type, $q^x(\theta) < q_0(\theta)$ is the quantity at which the cross-partial derivative of the utility function is equal to zero. That is,

$$u_{q\theta}(\theta, q^x(\theta)) = 0, \ \forall \theta \in \Theta. \tag{3.23}$$

Another way to think of $q^x(\theta)$ is as $q^x(\theta, \theta + d\theta)$, using the notation from Section 3.1. Figure 3.3 shows that $q^x(\theta)$ is a threshold which divides the (θ, q) plane into two regions. Marginal utility is constant in type along $q^x(\theta)$. In the region below $q^x(\theta)$, marginal utility is increasing in type over $[\underline{\theta}, \theta]$, and in the region above $q^x(\theta)$, marginal utility is decreasing in type over $[\theta, \bar{\theta}]$. Notice that for $q \leq q^x(\bar{\theta})$, marginal utility is increasing in type over Θ, and for $q \geq q^x(\underline{\theta})$ marginal utility is decreasing in type over Θ. That is, consumer preferences are completely ordered by θ on $[0, q^x(\bar{\theta})]$ and for $q \geq q^x(\underline{\theta})$.

The function denoted in this paper as $q^x(\theta)$, is the function denoted "$q_0(\theta)$" in Araujo et al. (2011), and denoted "$s(\theta)$" by Schottmüller (2015). Both Araujo et al. (2011) and Schottmüller (2015) focus on $q^x(\theta)$ in relation to the first-best quantity schedule $\bar{q}(\theta)$. Schottmüller (2015) deals only with cases where $q^x(\theta)$ lies entirely above $\bar{q}(\theta)$. Araujo et al. (2011) deal with cases where $q^x(\theta)$ and $\bar{q}(\theta)$ intersect once. An implication of $q^x(\theta)$ and $\bar{q}(\theta)$ crossing once is that implementable assignment schedules will not be monotonic. Following Schottmüller (2015), I focus on cases where $\bar{q}(\theta)$ lies entirely above $q^x(\theta)$, or,

$$\bar{q}(\theta) > q^x(\theta), \ \forall \theta \in \Theta. \tag{3.24}$$

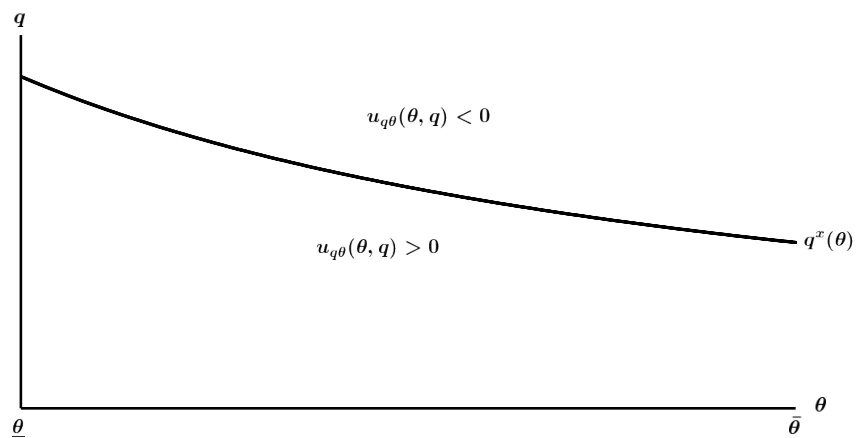

Figure 3.3. The threshold $q^x(\theta)$

This is implicit in Assumption (A3).

Lemma 1. $q^x(\theta)$ is decreasing in θ over Θ.

Proof. By definition,

$$u_{q\theta}(\theta, q^x(\theta)) = 0. \tag{3.25}$$

Implicitly differentiating (3.25) with respect to θ gives

$$u_{q\theta\theta}(\theta, q^x(\theta)) + u_{qq\theta}(\theta, q^x(\theta)) \cdot \left[(q^x)'(\theta) \right] = 0, \tag{3.26}$$

where $(q^x)'(\theta)$ denotes the derivative of $q^x(\theta)$ with respect to θ.

Rearranging (3.26) gives

$$(q^x)'(\theta) = \frac{-u_{q\theta\theta}(\theta, q^x(\theta))}{u_{qq\theta}(\theta, q^x(\theta))}. \tag{3.27}$$

By Assumption (A2), $u_{q\theta\theta}(\theta, q^x(\theta))$ and $u_{qq\theta}(\theta, q^x(\theta))$ are negative. Thus, the numerator of (3.27) becomes a positive, while the denominator remains negative. It follows that

$$(q^x)'(\theta) < 0. \tag{3.28}$$

That is, $q^x(\theta)$ is decreasing in θ over Θ. **QED**

Another important concept in these problems, is the division of $\Theta \times q$ into regions where total utility is increasing and decreasing in type. For each θ, let $Q^x(\theta)$ denote the quantity at which the partial derivative of the utility function with respect to θ is equal to zero. That is,

$$u_\theta(\theta, Q^x(\theta)) = 0, \ \forall\theta \in \Theta. \tag{3.29}$$

The function $Q^x(\theta)$ divides the (θ, q) plane into two regions. For each θ, total utility is increasing in type at any $q < Q^x(\theta)$, constant in type at $q = Q^x(\theta)$, and decreasing in type at any $q > Q^x(\theta)$. Recall that $q^x(\theta)$ can be thought of as the quantity that solves[5]

$$u_q(\theta, q) = u_q(\theta \pm d\theta, q). \tag{3.30}$$

Similarly, $Q^x(\theta)$ can be thought of as the quantity at which θ has the same total utility as those types infinitesimally close to θ. That is, $Q^x(\theta)$ is the quantity that solves

$$u(\theta, q) = u(\theta \pm d\theta, q), \tag{3.31}$$

with footnote 5 applying again to (3.31).

[5] The "$\pm$" becomes a "+", when (3.30) is evaluated at $\underline{\theta}$, and becomes a "$-$" when (3.30) is evaluated at $\bar{\theta}$.

Lemma 2. If $Q^x(\theta)$ exists for all $\theta \in \Theta$, then $Q^x(\theta)$ is decreasing in θ over Θ.

Proof. By definition,

$$u_\theta(\theta, Q^x(\theta)) = 0. \tag{3.32}$$

Implicitly differentiating (3.32) with respect to θ gives

$$u_{\theta\theta}(\theta, Q^x(\theta)) + u_{q\theta}(\theta, Q^x(\theta)) \cdot \left[(Q^x)^{'}(\theta) \right] = 0, \tag{3.33}$$

where $(Q^x)^{'}(\theta)$ denotes the derivative of $Q^x(\theta)$ with respect to θ. Rearranging (3.33) gives

$$(Q^x)^{'}(\theta) = \frac{-u_{\theta\theta}(\theta, Q^x(\theta))}{u_{q\theta}(\theta, Q^x(\theta))}. \tag{3.34}$$

By Assumption (A2), $u_{\theta\theta}(\theta, Q^x(\theta))$ is negative. If $Q^x(\theta)$ exists, it must be greater than $q^x(\theta)$. This means that $u_{q\theta}(\theta, Q^x(\theta))$ is also negative by Assumption (A2). Thus, the numerator of (3.34) becomes a positive, while the denominator remains negative. It follows that

$$(Q^x)^{'}(\theta) < 0. \tag{3.35}$$

That is, if $Q^x(\theta)$ exists for all θ in Θ, then $Q^x(\theta)$ is decreasing in θ over Θ. **QED**

Now, I turn attention to the inverse function of $Q^x(\theta)$. Over $[Q^x(\bar{\theta}), Q^x(\underline{\theta})]$, let $\theta^x(q)$ denote the inverse function of $Q^x(\theta)$. That is, $\theta^x(q) = \theta$ solves $Q^x(\theta) = q$. In general let $x^{-1}(q)$ denote the inverse function of any $x(\theta)$ that is one-to-one on Θ. I will show that for $q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta})]$, $\theta^x(q)$ is the argument in Θ which maximizes $u(\theta, q)$. Expanding this concept to all $q > 0$, let

$$\theta^\bullet(q) = \arg\max_{\theta \in \Theta} u(\theta, q), \ \forall q > 0. \tag{3.36}$$

Lemma 3. If $Q^x(\theta)$ is one-to-one on Θ, then $\arg\max_{\theta\in\Theta} u(\theta, q)$ is given by

$$\theta^\bullet(q) = \left\{ \begin{array}{cc} \bar{\theta}, & q \leq Q^x(\bar{\theta}) \\ \theta^x(q), & q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta})] \\ \underline{\theta}, & q > Q^x(\underline{\theta}) \end{array} \right\}. \tag{3.37}$$

Proof. First, for $q \leq Q^x(\bar{\theta})$, the derivative of total utility with respect to θ is such that

$$u_\theta(\theta, q) \geq 0, \ \forall\theta \in \Theta. \tag{3.38}$$

It follows that for any $q \leq Q^x(\bar{\theta})$ total utility is increasing in θ. Thus, $\theta^\bullet(q) = \bar{\theta}$ for all $q \leq Q^x(\bar{\theta})$.

Next, recall that by definition $Q^x(\theta)$ is the q that solves

$$u_\theta(\theta, q) = 0. \tag{3.39}$$

It follows that the inverse function, $\theta^x(q)$, gives

$$u_\theta(\theta^x(q), q) = 0, \tag{3.40}$$

for q in $[Q^x(\bar{\theta}), Q^x(\underline{\theta})]$. From (3.40), note that $\theta^x(q)$ satisfies the first order condition for a maximum. By Assumption (A2), $u_{\theta\theta}(\theta, q)$ is negative for all θ, which is the second order condition for a global maximum at $\theta^x(q)$. Thus,

$$\theta^\bullet(q) = \theta^x(q), \ \forall q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta})]. \tag{3.41}$$

Finally, for $q \geq Q^x(\underline{\theta})$, the derivative of total utility with respect to θ is such that

$$u_\theta(\theta, q) \leq 0, \ \forall \theta \in \Theta. \tag{3.42}$$

It follows that for any $q \geq Q^x(\underline{\theta})$, total utility is decreasing in θ. Thus $\theta^\bullet(q) = \underline{\theta}$ for all $q \geq Q^x(\underline{\theta})$. Therefore,

$$\theta^\bullet(q) = \left\{ \begin{array}{ll} \bar{\theta}, & q \leq Q^x(\bar{\theta}) \\ \theta^x(q), & q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta})] \\ \underline{\theta}, & q > Q^x(\underline{\theta}) \end{array} \right\}. \tag{3.43}$$

QED

3.4 An Envelope Theorem Result

In this section, I make use of Lemma 3 to show that an assignment schedule $x(\theta) = Q^x(\theta)$, can be implemented such that the IR constraints bind for all θ in Θ. Scenario 1, covered in Section 3.2, is seen to be a special case of this result where the first-best quantities $\bar{q}(\theta)$ are equal to $Q^x(\theta)$.

Proposition 1B.[6] If $Q^x(\theta)$ exists for all θ in Θ, the assignment schedule $x(\theta) = Q^x(\theta)$ is implementable through the marginal price schedule

$$p(q) = \left\{ \begin{array}{cc} u_q(\bar{\theta}, q), & q \in [0, Q^x(\bar{\theta})] \\ u_q(\theta^x(q), q), & q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta})] \end{array} \right\},$$

and the IR constraint binds at $x(\theta) = Q^x(\theta)$, for all θ in Θ.

Proof. Consider momentarily the following *total* price schedule,

$$P(q) = \left\{ \begin{array}{cc} u(\bar{\theta}, q), & q \in [0, Q^x(\bar{\theta})] \\ u(\theta^x(q), q), & q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta})] \end{array} \right\}. \tag{3.44}$$

I will show that under the price schedule given in (3.44), the IR constraint is binding at $x(\theta) = Q^x(\theta)$, for all θ in Θ. Then, I will show that under (3.44), each type weakly or strictly prefers their assignment $x(\theta) = Q^x(\theta)$ to any other q. Finally, I will show that the price schedule from (3.44) is also given by the marginal price schedule $p(q)$ stated in the proposition.

Recall that each consumer type solves the objective,

$$\max_{q \geq 0} [CS(\theta, q) = u(\theta, q) - P(q)]. \tag{3.45}$$

[6]This proposition is denoted 1B, since it generalizes proposition 1A from Section 3.2.

Under the price schedule given in (3.44), consumer surplus evaluated at $x(\theta) = Q^x(\theta)$, is given by

$$CS(\theta, Q^x(\theta)) = u(\theta, Q^x(\theta)) - P(Q^x(\theta)) = u\left(\theta, Q^x(\theta)\right) - u\left(\theta^x(Q^x(\theta)), Q^x(\theta)\right). \qquad (3.46)$$

Recall that $\theta^x(q)$ is the inverse function of $Q^x(\theta)$, meaning $\theta^x(Q^x(\theta)) = \theta$. Therefore, equation (3.46) reduces to

$$CS(\theta, Q^x(\theta)) = u\left(\theta, Q^x(\theta)\right) - u\left(\theta, Q^x(\theta)\right) = 0, \ \forall \theta \in \Theta. \qquad (3.47)$$

That is, under the total price schedule given in (3.44), the individual rationality constraints bind at $x(\theta) = Q^x(\theta)$, for all θ in Θ.

Next, recall that by Lemma 3, the value of θ that maximizes $u(\theta, q)$ is given by

$$\theta^\bullet(q) = \left\{ \begin{array}{ll} \bar{\theta}, & q \leq Q^x(\bar{\theta}) \\ \theta^x(q), & q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta})] \\ \underline{\theta}, & q > Q^x(\underline{\theta}) \end{array} \right\}. \qquad (3.48)$$

It follows that the total price schedule given in (3.44) is also given by,

$$P(q) = u(\theta^\bullet(q), q) = \max_{\theta \in \Theta} u(\theta, q), \ \forall q \leq Q^x(\underline{\theta}). \qquad (3.49)$$

Since the total price schedule given in (3.44) is equal to the $\max_{\theta \in \Theta} u(\tilde{\theta}, q)$ for all $q \leq Q^x(\underline{\theta})$, it follows that

$$CS(\theta, q) = u(\theta, q) - P(q) = u(\theta, q) - \max_{\tilde{\theta} \in \Theta} u(\tilde{\theta}, q) \leq 0, \ \forall q \leq Q^x(\underline{\theta}), \ \theta \in \Theta. \qquad (3.50)$$

The IR constraint has already been shown to bind for all θ in Θ. Therefore, equation (3.50) implies that all θ in Θ weakly prefer their assignment $x(\theta) = Q^x(\theta)$ to any other q in $[0, Q^x(\underline{\theta})]$. The total price schedule given by (3.44) feasibly implements $x(\theta) = Q^x(\theta)$, for all θ in Θ.

Finally, note that for $q \in [0, Q^x(\bar{\theta})]$, the marginal price schedule implied by (3.44) is given by

$$p(q) = \frac{dP(q)}{dq} = \frac{d}{dq}u(\bar{\theta}, q) = u_q(\bar{\theta}, q). \tag{3.51}$$

For $q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta}))]$, the marginal price schedule implied by (3.44) is given by

$$p(q) = \frac{dP(q)}{dq} = \frac{d}{dq}u(\theta^x(q), q) = u_q(\theta^x(q), q) + \left[u_\theta(\theta^x(q), q) \cdot (\theta^x)^{'}(q)\right]. \tag{3.52}$$

Recall that for $q \in [Q^x(\bar{\theta}), Q^x(\underline{\theta}))]$,

$$u_\theta(\theta^x(q), q) = 0, \tag{3.53}$$

by the definition of $Q^x(\theta)$. It follows that equation (3.52) reduces to

$$p(q) = u_q(\theta^x(q), q) + \left[(0) \cdot (\theta^x)^{'}(q)\right] = u_q(\theta^x(q), q), \tag{3.54}$$

where $(\theta^x)^{'}(q)$ denotes the derivative of $\theta^x(q)$ with respect to q. Thus, the total price schedule given in (3.44) implements $x(\theta) = Q^x(\theta)$ with IR constraints binding at $x(\theta)$ for all $\theta \in \Theta$, and (3.44) is also given by the marginal price schedule

$$p(q) = \left\{ \begin{array}{ll} u_q(\bar{\theta}, q), & q \in [0, Q^x(\bar{\theta})] \\ u_q(\theta^x(q), q), & q \in [Q^x(\bar{\theta}, Q^x(\underline{\theta}))] \end{array} \right\}. \tag{3.55}$$

QED

Proposition 1B shows that when the SCC is violated as it is in this paper, it is possible for the monopolist to feasibly implement an assignment schedule which extracts all equilibrium surplus, despite being uninformed of the consumer's type. Moreover, note that the constant first-best surplus scenario is in fact a special case where $Q^x(\theta) = \bar{q}(\theta)$ for all θ in Θ.

3.5 Properties of Feasible and Implementable Assignment Schedules

In this section I establish two lemmas regarding properties of implementable assignment schedules. Lemma 4 shows that if $x(\theta)$ is greater than $q^x(\theta)$ for all θ in Θ, implementability will require that flatter types receive greater assignments. Lemma 5 shows that at the monopolist's optimum, the steepest consumer type, $\bar{\theta}$, is always served.

Lemma 4. Any assignment schedule, $x(\theta)$, that is continuously differentiable and one-to-one on Θ, implementable by $P(q)$, and greater than $q^x(\theta)$ for all θ in Θ, will be strictly decreasing in θ.

Proof. The objective function for each θ is given by

$$u(\theta, q) - P(q). \tag{3.56}$$

The first order condition for a maximum of this objective function, gives the following necessary condition for $P(q)$ to implement $x(\theta)$,

$$u_q(x^{-1}(q), q) = p(q), \; q \in \{x(\theta) : \theta \in \Theta\}. \tag{3.57}$$

A second order necessary condition for implementability is given by,

$$u_{qq}(x^{-1}(q), q) - \frac{d}{dq} p(q) \leq 0, \; q \in \{x(\theta) : \theta \in \Theta\}. \tag{3.58}$$

This second order condition, evaluated at the marginal price schedule implied by the first order condition, is given by

$$u_{qq}(x^{-1}(q), q) - \frac{d}{dq} u_q(x^{-1}(q), q) \leq 0, \; q \in \{x(\theta) : \theta \in \Theta\}. \tag{3.59}$$

The derivative with respect to q of $u_q(x^{-1}(q), q)$ is given by

$$u_{qq}(x^{-1}(q), q) + u_{q\theta}(x^{-1}(q), q) \cdot \left[(x^{-1})^{'}(q) \right], \tag{3.60}$$

where $(x^{-1})^{'}(q)$ denotes the derivative of $x^{-1}(q)$ with respect to q. It follows then that the second order condition reduces to

$$- u_{q\theta}(x^{-1}(q), q) \cdot \left[(x^{-1})^{'}(q) \right] \leq 0, \; q \in \{x(\theta) : \theta \in \Theta\}. \tag{3.61}$$

Recall that $u_{q\theta}(\theta, q)$ is less than 0 at any $q > q^x(\theta)$. Since $x(\theta) > q^x(\theta)$ for all θ in Θ, it follows that $(-u_{q\theta}(x^{-1}(q), q))$ is strictly greater than zero over $\{x(\theta) : \theta \in \Theta\}$, and can therefore be divided out of (3.61) without changing the inequality. Thus, the second order condition further reduces to

$$(x^{-1})^{'}(q) \leq 0, \; q \in \{x(\theta) : \theta \in \Theta\}. \tag{3.62}$$

The inverse function theorem implies that

$$(x^{-1})^{'}(q) = \frac{1}{x'(\theta)}. \tag{3.63}$$

Therefore, (3.62) must hold as a strict inequality,[7] given by

$$\frac{1}{x'(\theta)} < 0, \; q \in \{x(\theta) : \theta \in \Theta\}. \tag{3.64}$$

This also implies that

$$x^{'}(\theta) < 0, \; q \in \{x(\theta) : \theta \in \Theta\}. \tag{3.65}$$

[7] The inequality must be strict since $\frac{1}{x'(\theta)}$ can't equal zero.

That is $x(\theta)$ is strictly decreasing in θ. **QED**

Lemma 5. If the monopolist's optimal pair $(p(q), x(\theta))$ that is both implementable and feasible, assigns a positive quantity to $\hat{\theta}$, it also assigns positive quantities to all $\theta \in [\hat{\theta}, \bar{\theta}]$. That is, *if it is optimal to sell to a given type, then it is optimal to sell to all steeper types.*

Proof. Suppose $P_\circ(q)$ is an optimal total price schedule that feasibly implements $x(\theta) = 0$ for all $\theta > \hat{\theta}$, and $x(\theta) > 0$ for all θ in $[\underline{\theta}, \hat{\theta}]$. If so, then I show that: (a) $x(\hat{\theta})$ is greater than or equal to $Q^x(\hat{\theta})$, (b) $P_\circ(q)$ is greater than or equal to $u(\hat{\theta}, q)$ for $q \geq Q^x(\hat{\theta})$, (c) the IR constraint for $\hat{\theta}$ binds at $x(\hat{\theta})$, and (d) no $\theta < \hat{\theta}$ is willing to buy a positive $q \leq Q^x(\hat{\theta})$.

To verify (a), recall that the IR constraint for $\hat{\theta}$ requires $P_\circ(x(\hat{\theta}))$ to be less than or equal to $u(\hat{\theta}, x(\hat{\theta}))$. Note also that $u_\theta(\hat{\theta}, q)$ is greater than zero for all $q < Q^x(\hat{\theta})$. This means that for all $q < Q^x(\hat{\theta})$, $u(\hat{\theta}, q)$ is less than $u(\theta, q)$ for some $\theta > \hat{\theta}$. Thus, at any $q < Q^x(\hat{\theta})$, if $P_\circ(q) \leq u(\hat{\theta}, q)$, then $P_\circ(q) < u(\theta, q)$ for some $\theta > \hat{\theta}$. Therefore, the monopolist cannot feasibly implement a positive $x(\hat{\theta}) < Q^x(\hat{\theta})$ as well as zero assignments for all $\theta > \hat{\theta}$. The assignment $x(\hat{\theta})$ must therefore be greater than or equal to $Q^x(\hat{\theta})$, proving (a).

Next, note that if $P_\circ(q) < u(\hat{\theta}, q)$ at $q \geq Q^x(\hat{\theta})$, then $P(q) > u(\theta + \epsilon, q)$ for $\epsilon > 0$ small, by the continuity of $u(\theta, q)$ in θ. Therefore, $P_\circ(q)$ must be greater than or equal to $u(\hat{\theta}, q)$ for all $q \geq Q^x(\hat{\theta})$, proving (b). It follows that the IR constraint for $\hat{\theta}$ binds at $x(\hat{\theta})$, or $P_\circ(x(\hat{\theta})) = u(\hat{\theta}, x(\hat{\theta}))$, proving (c).

Finally, note that for $q < Q^x(\hat{\theta})$, $P_\circ(q)$ must be greater than $u(\theta^\bullet(q))$ to prevent $\theta > \hat{\theta}$ from buying. By Lemma 3 $\theta^\bullet(q) > \hat{\theta}$ for $q < Q^x(\hat{\theta})$. It follows that no θ in Θ is willing to buy $q < Q^x(\hat{\theta})$, proving (d).

Now, consider an alternative total price schedule, denoted $P_A(q)$ and given by

$$P_A(q) = \left\{ \begin{array}{ll} u(\hat{\theta}, q), & q < Q^x(\hat{\theta}) \\ P_\circ(q), & q \geq Q^x(\hat{\theta}) \end{array} \right\}. \tag{3.66}$$

Since $P_A(q)$ is equal to $u(\hat{\theta}, q)$ for all $q < Q^x(\hat{\theta})$, type $\hat{\theta}$ still weakly prefers $x(\hat{\theta})$, to any other quantity. Recall that $u(\hat{\theta}, q) > u(\theta, q)$ for all $\theta < \hat{\theta}$ and $q < Q^x(\hat{\theta})$. Since $P_A(q)$ equals

$u(\hat{\theta}, q)$ for $q < Q^x(\theta)$, and $P_A(q)$ equals $P_\circ(q)$ at $q = Q^x(\hat{\theta})$, all $\theta < \hat{\theta}$ are still unwilling to buy a positive $q < Q^x(\hat{\theta})$. Finally, since $P_A(q)$ is the same as the initial total price schedule for $q \geq Q^x(\hat{\theta})$, it follows that all $\theta \leq \hat{\theta}$ buy their initial assignments and pay the initial total price. Notice however that all $\theta > \hat{\theta}$ now strictly prefer a positive $q < Q^x(\theta)$ to $q = 0$.

All $\theta \leq \hat{\theta}$ still buy the same amount under $P_A(q)$ as they do under $P_\circ(q)$, and they pay the same total price. Thus, the monopolist's profit from $\theta \leq \hat{\theta}$ remains the same under $P_A(q)$. Additionally, all $\theta > \hat{\theta}$ now buy a positive $q < Q^x(\hat{\theta})$ and pay a positive total price. It follows that the monopolist's profit is greater under $P_A(q)$ than under $P_\circ(q)$, and $P_\circ(q)$ therefore cannot be optimal. If the monopolist's optimal assignment schedule gives $x(\hat{\theta}) > 0$, then $x(\theta) > 0$ for all $\theta \geq \hat{\theta}$. **QED**

3.6 The Monopolist's Optimum: A preliminary characterization

The monopolist's objective is to chose a pair $(p(q), x(\theta))$ that is implementable, feasible, and maximizes profit given by

$$\Pi = \int_{\underline{\theta}}^{\bar{\theta}} f(t) \left(\int_{0}^{x(t)} [p(v) - c] \, dv \right) dt. \tag{3.67}$$

The monopolist's optimal schedule that is both implementable and feasible is denoted $x^*(\theta)$.

Recall that if $\bar{q}(\theta) = Q^x(\theta)$ for all θ in Θ, then $x^*(\theta) = Q^x(\theta)$. In any scenario where $\bar{q}(\theta)$ may or may not equal $Q^x(\theta)$, the monopolist's maximum profit is less than or equal to first-best profit. If $\bar{q}(\theta) \geq Q^x(\theta)$ for all θ in Θ, then $x^*(\theta)$ should be greater than or equal $Q^x(\theta)$ for all θ in Θ. To see this, recall that for all θ in Θ, total utility is strictly increasing in q on $[0, q_0(\theta)]$. By proposition 1B, the monopolist can feasibly implement $x(\theta) = Q^x(\theta)$ for all θ in Θ and extract all surplus from said assignment. Suppose the monopolist were to decrease $x(\theta)$ below $Q^x(\theta)$ for a given θ. The most the monopolist can possibly extract from θ at that decreased assignment, is less than he can certainly extract from θ at the larger $Q^x(\theta)$ assignment. Thus, the direct effect on profit of assigning a quantity less than $Q^x(\theta)$ must be negative. It follows that assigning a quantity less than $Q^x(\theta)$ can only be optimal if it allows the monopolist to increase the assignment of other types and extract enough additional surplus to dominate the direct effect.

For a given $\hat{\theta} < \bar{\theta}$, the monopolist can implement $x(\theta) > Q^x(\theta)$ for all $\theta \leq \hat{\theta}$ through the marginal price schedule,

$$p(q) = \left\{ \begin{array}{ll} u_q(\bar{\theta}, q), & q \in [0, Q^x(\bar{\theta})] \\ u_q(\theta^x, q), & q \in [Q^x(\bar{\theta}), Q^x(\hat{\theta})] \\ u_q(\hat{\theta}, q), & q \in [Q^x(\hat{\theta}), x(\hat{\theta})] \\ u_q(x^{-1}(q), q), & q \in [x(\hat{\theta}), x(\underline{\theta})] \end{array} \right\}. \tag{3.68}$$

Note that all $\theta > \hat{\theta}$ still prefer $Q^x(\theta)$ to any other quantity, and the monopolist still extracts all surplus from said assignments. For $\theta \leq \hat{\theta}$, the total price $P(q)$ is greater than $u(\theta, q)$ for all $q < Q^x(\hat{\theta})$. Note also that $P(q)$ is equal to $u(\hat{\theta}, q)$ over $[Q^x(\hat{\theta}), x(\theta)]$. Moreover, $CS(\theta, x(\theta))$ will be positive at $x(\theta)$ and decreasing in quantity thereafter, for all $\theta < \hat{\theta}$. Thus, the marginal price schedule given in (62) feasibly implements $x(\theta) > Q^x(\theta)$ for all $\theta \leq \hat{\theta}$, while still feasibly implementing $x(\theta) = Q^x(\theta)$ with binding IR constraints, for all $\theta > \hat{\theta}$. It follows that the monopolist can increase assignments above $Q^x(\theta)$ for flatter types without decreasing assignments below $Q^x(\theta)$ for steeper types.

Conversely, the monopolist can feasibly implement $x(\theta) > Q^x(\theta)$ for all $\theta \geq \hat{\theta}$ through the marginal price schedule

$$p(q) = \left\{ \begin{array}{cc} u_q(\bar{\theta}, q), & q \in [0, x(\bar{\theta})] \\ u_q(x^{-1}(q), q), & q \in [x(\bar{\theta}), x(\hat{\theta})] \end{array} \right\}. \tag{3.69}$$

The monopolist needn't decrease the assignments of flatter types below $Q^x(\theta)$ to do so. It follows that when $\bar{q}(\theta) \geq Q^x(\theta)$, assigning $x(\theta) < Q^x(\theta)$ for any θ always has a negative direct effect, and never has a positive direct effect. Therefore, assignments strictly less than $Q^x(\theta)$ can never be optimal in these circumstances.

BIBLIOGRAPHY

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Araujo, A., H. Moreira, and S. Vieira (2011), The marginal tariff approach without single-crossing, Working Paper.

Arellano, M., and S. Bond (1991), Some tests of specification for panel data: Monte carlo evidence and an application to employment equations, *The Review of Economic Studies*, 58(2), 277–297.

Atkinson, S. E., L. R. Stanley, and J. Tschirhart (1988), Revenue sharing as an incentive in an agency problem: An example from the National Football League, *The RAND Journal of Economics*, 9(1), 27–43.

Becker, G. (1983, September 30), College athletes should get paid what they are worth, *Business Week*.

Bellman, R. (1956), Dynamic programming and Lagrange multipliers, *Proceedings of the National Academy of Sciences of the United States of America*, 42(10), 767.

Bellman, R. (1957), *Dynamic Programming*, Princeton University Press.

Berkowitz, S. (2015, January 17), NCAA increases value of scholarships in historic vote, *USA Today*, retrieved from `http://www.usatoday.com/story/sports/college/2015/01/17/ncaa-convention-cost-of-attendance-student-athletes-scholarships/21921073/`.

Boal, W. M., and M. R. Ransom (1997), Monopsony in the labor market, *Journal of Economic Literature*, 35(1), 86–112.

Brown, R. W. (1993), An estimate of the rent generated by a premium college football player, *Economic Inquiry*, 31(4), 671–684.

Brown, R. W. (1994), Measuring cartel rents in the college basketball player recruitment market, *Applied Economics*, 26(1), 27–34.

Brown, R. W. (2011), Research note: Estimates of college football player rents, *Journal of Sports Economics*, 12(2), 200–212.

Brown, R. W., and R. Jewell (2006), The marginal revenue product of a women's college basketball player, *Industrial Relations: A Journal of Economy and Society*, 45(1), 96–101.

Fleisher, A. A., B. L. Goff, and R. D. Tollison (1992), *The National Collegiate Athletic Association: A study in cartel behavior*, The University of Chicago Press.

Goldman, M. B., H. E. Leland, and D. S. Sibley (1984), Optimal nonuniform prices, *The Review of Economic Studies*, 51(2), 305–319.

Greene, W. H. (2012), *Econometric analysis, 7th. Ed.*, Prentice Hall.

Gregory, S. (2013, September 16), It's time to pay college athletes, *Time*, retreived from `http://time.com/568/its-time-to-pay-college-athletes/`.

Howell, J. (2015), [division i-a historical scores], data retrieved from `http://jhowell.net/cf/scores/ScoresIndex.htm`.

Kahane, L. H. (2012), The estimated rents of a top-flight men's college hockey player, *International Journal of Sport Finance*, 7(1), 19–29.

Kahn, L. M. (2007), Markets: Cartel behavior and amateurism in college sports, *The Journal of Economic Perspectives*, 21(1), 209–226.

Koch, J. V. (1973), A troubled cartel: the NCAA, *Law and Contemporary Problems*, 38, 135.

Krautmann, A. C. (1999), What's wrong with Scully-estimates of a player's marginal revenue product, *Economic Inquiry*, 37(2), 369–381.

Lerner, A. P. (1934), The concept of monopoly and the measurement of monopoly power, *The Review of Economic Studies*, 1(3), 157–175.

Maskin, E., and J. Riley (1984), Monopoly with incomplete information, *The RAND Journal of Economics*, 15(2), 171–196.

Mirrlees, J. A. (1971), An exploration in the theory of optimum income taxation, *The review of economic studies*, 38(2), 175–208.

Mundlak, Y. (1978), On the pooling of time series and cross section data, *Econometrica: Journal of the Econometric Society*, 46(1), 69–85.

Mussa, M., and S. Rosen (1978), Monopoly and product quality, *Journal of Economic Theory*, 18(2), 301–317.

Myerson, R. B. (1981), Optimal auction design, *Mathematics of Operations Research*, 6(1), 58–73.

Neyman, J., and E. L. Scott (), Consistent estimates based on partially consistent observations, *Econometrica: Journal of the Econometric Society*, 16.

Oi, W. Y. (1971), A Disneyland dilemma: Two-part tariffs for a Mickey Mouse monopoly, *The Quarterly Journal of Economics*, 85(1), 77–96.

Pigou, A. C. (1924), *The economics of welfare*, Transaction Publishers.

Robinson, J. (1933), *Economics of imperfect competition*, Macmillan.

Romney, L. (2014, August 8), Judge rules against NCAA in Ed O'Bannon antitrust lawsuit, *The Los Angeles Times*, retrieved from http://www.latimes.com/local/lanow/la-me-ln-ncaa-obannon-ruling-20140808-story.html.

Sargan, J. D. (1958), The estimation of economic relationships using instrumental variables, *Econometrica: Journal of the Econometric Society*, 26(3), 393–415.

Schottmüller, C. (2015), Adverse selection without single crossing: Monotone solutions, *Journal of Economic Theory*, 158, 127–164.

Scully, G. W. (1974), Pay and performance in Major League Baseball, *The American Economic Review*, 64(6), 915–930.

Spence, M. (1973), Job market signaling, *The Quarterly Journal of Economics*, 87(3), 355–374.

Wolken, D. (2014, April 24), Northwestern vote another milestone on path to NCAA reform, *The USA Today*, retrieved from www.usatoday.com/story/sports/ncaaf/2014/04/24/northwestern-labor-union-players-vote-nlrb/8113263/.

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