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PORTFOLIO OPTIMIZATION METHODS: THE MEAN-VARIANCE APPROACH
AND THE BAYESIAN APPROACH

by
Hoang Nguyen

A thesis submitted to the faculty of The University of Mississippi in partial fulfillment of the
requirements of the Sally McDonnell Barksdale Honors College.

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ABSTRACT

HOANG NGUYEN: Portfolio Optimization Methods: The Mean-Variance Approach and the Bayesian Approach

(Under the direction of Dr. Andrew Lynch)

This thesis is a discussion on the mean-variance approach to portfolio optimization and an introduction of the Bayesian approach, which is designed to solve certain limitations of the classical mean-variance analysis. The primary goal of portfolio optimization is to achieve the maximum return from investment given a certain level of risk. The mean-variance approach, introduced by Harry Markowitz, sought to solve this optimization problem by analyzing the means and variances of a certain collection of stocks. However, due to its simplicity, the mean-variance approach is subject to various limitations. In this paper, we seek to solve some of these limitations by applying the Bayesian method, which is mainly based on probability theory and the Bayes' theorem. These approaches will be applied to form optimal portfolios using the data of 27 Dow Jones companies in the period of 2008-2017 for a better comparison. The topic of portfolio optimization is extremely broad, and there are many approaches that have been and are being currently researched. Yet, there is no approach that is proven to perform most efficiently. The purpose of this paper is to discuss two potential and popular approaches in forming optimal portfolios.

TABLES OF CONTENTS

Chapter 1: An Overview of Portfolio Optimization	1
1.1 Definition of risk:	1
1.2 Measure of risk:.....	1
1.3 Optimization formulations:	3
1.3.1 Maximizing portfolio expected return:.....	3
1.3.2 Minimizing portfolio volatility:.....	3
1.3.3 Maximizing portfolio's Sharpe ratio:	3
1.3.4 Maximizing risk-adjusted return:	3
Chapter 2: The Mean-Variance Approach of Portfolio Optimization	4
2.1 Return:.....	4
2.2 Variance:	5
2.3 Covariance:.....	5
2.4 Covariance matrix of a n-asset portfolio:.....	6
2.5 Variance of a portfolio:	7
2.6. Mean-variance optimal portfolio:.....	7
Chapter 3: An Overview of the Bayesian Approach for Portfolio Optimization	9
3.1 Likelihood function:	9
3.2 Bayes' theorem:.....	9
3.3 The Bayesian method in finding posterior distribution:.....	10
Chapter 4: Bayesian Application on Normal Distributions	12
Chapter 5: Mean-Variance Optimal Portfolio of the Dow Jones Industrial Average stocks	16
5.1 Risk-free rate:	16

5.2	Technique explanation and constraints:	17
5.3	Sharpe optimal portfolio of 2008-2012 period:	19
5.4	Sharpe optimal portfolio of 2009-2013 period:	21
5.5	Sharpe optimal portfolio of 2010-2014 period:	22
5.6	Sharpe optimal portfolio of 2011-2015 period:	23
5.7	Sharpe optimal portfolio of 2012-2016 period:	24
Chapter 6:	Update Sharpe Optimal Portfolio Using the Bayesian Approach	26
6.1	Technique explanation and constraints:	26
6.2	Updated optimal portfolio of 2008-2012 period given 2013 data set:	29
6.2.1	Method 1:.....	29
6.2.2	Method 2:.....	32
6.2.3	Comparisons between the portfolios:	33
6.3	Updated optimal portfolio of 2009-2013 period given 2014 data set:	36
6.4	Updated optimal portfolio of 2010-2014 period given 2015 data set:	38
6.5	Updated optimal portfolio of 2011-2015 period given 2016 data set:	40
6.6	Updated optimal portfolio of 2012-2016 period given 2017 data set:	42
Chapter 7:	Optimal Portfolios Comparisons by Real Value Investment	44
7.1	Technique explanation and constraints:	44
7.1.1	Passive mean-variance portfolio:.....	44
7.1.2	Update the optimal portfolio annually by the mean-variance approach:.....	44
7.1.3	Update the optimal portfolio annually by the Bayesian approach method 1:	44
7.1.4	Update the optimal portfolio annually by the Bayesian approach method 2:	45
7.2	Real value investments under short-sales allowed circumstance:.....	46
7.3	Real value investments under no short-sales circumstance:	47

Chapter 8: Limitations and Assumptions Using the Bayesian Approach	50
Chapter 9: Comparison Between the Mean-Variance Approach and the Bayesian Approach...	54

LIST OF TABLES

Table 1. U.S. 10-year Treasury Rate in the period of 2008-2017.....	17
Table 2. 2008-2012 Sharpe optimal portfolio with short-sales allowed.....	19
Table 3. 2008-2012 Sharpe optimal portfolio with no short-sales.....	20
Table 4. 2013 updated Sharpe optimal portfolio with short-sales allowed using method 1	30
Table 5. 2013 updated Sharpe optimal portfolio with no short-sales using method 1.....	31
Table 6. 2013 updated Sharpe optimal portfolio with short-sales allowed using method 2.....	32
Table 7. 2013 updated Sharpe optimal portfolio with no short-sales using method 2.....	33

LIST OF FIGURES

Figure 1. 2008-2012 Sharpe optimal portfolios	21
Figure 2. 2009-2013 Sharpe optimal portfolios	22
Figure 3. 2010-2014 Sharpe optimal portfolios	23
Figure 4. 2011-2015 Sharpe optimal portfolios	24
Figure 5. 2012-2016 Sharpe optimal portfolios	25
Figure 6. 2008-2012 and 2013 optimal portfolios with short-sales	34
Figure 7. 2008-2012 and 2013 optimal portfolios without short-sales	35
Figure 8. 2009-2013 and 2014 optimal portfolios with short-sales	36
Figure 9. 2009-2013 and 2014 optimal portfolios without short-sales	37
Figure 10. 2010-2014 and 2015 optimal portfolios with short-sales	38
Figure 11. 2010-2014 and 2015 optimal portfolios without short-sales	39
Figure 12. 2011-2015 and 2016 optimal portfolios with short-sales	40
Figure 13. 2011-2015 and 2016 optimal portfolios without short-sales	41
Figure 14. 2012-2016 and 2017 optimal portfolios with short-sales	42
Figure 15. 2012-2016 and 2017 optimal portfolios without short-sales	43
Figure 16. Real investment comparison under short-sales allowed circumstance.....	46
Figure 17. Real investment comparison under no short-sales circumstance	47
Figure 18. Dow Jones Industrial Average annual changes in the period of 2013-2017	48
Figure 19. Dow Jones companies monthly stock returns distribution in the period of 2008-2018	50
Figure 20. Dow Jones companies log monthly stock returns distribution in the period of 2008- 2018.....	51
Figure 21. An example of the efficient frontier	56

Chapter 1: An Overview of Portfolio Optimization

1.1 Definition of risk:

One of the major advances of investment research in the 20th century is the recognition that we cannot obtain an optimum portfolio by simply combining numerous individual securities that have desirable risk-return characteristics.¹ In fact, multiple parameters of investment choices must be considered in order to build an efficient portfolio. The primary goal of portfolio optimization is to achieve the maximum return from investment given a certain level of risk.

For simplicity, in most financial literature, risk has been understood as uncertainty of future outcomes. An alternative definition might be the probability of an adverse outcome.² Risk could be categorized into different types, including market risk, credit risk, liquidity risk or non-financial risks. However, for the purpose of this paper, risk is treated as the volatility of a stock's return. Risk premium is the amount of excess return required by the investor to compensate for this uncertainty. By excess return, the portfolio's return is compared to the risk-free rate. Risk-free rate is the rate of return that can be earned with certainty.³ In this paper, U.S. 10-year Treasury Rate is used as the risk-free rate, which will be explained in a later chapter.

1.2 Measure of risk:

The most well-known measure for risk is variance. Variance is a measure of dispersion, which is the variability around the central tendency.⁴ Some other measures of dispersion

¹ ² Reilly & Brown (2012), p. 181, 182

³ Bodie et al. (2008), p. 123

⁴ DeFusco et al. (2018), p. 430

include the range of returns, the mean absolute deviation, the coefficient variation and the Sharpe ratio. In this paper, the variance and the Sharpe ratio of a portfolio are the two most important measures of risk. The Sharpe ratio is a measure of relative dispersion and is more inclusive than the variance. It is the ratio of excess return to standard deviation of return for a portfolio, formed by William F. Sharpe:⁵

$$S_p = \frac{\mu_p - r_f}{\sigma_p}$$

where μ_p is the mean return of the portfolio, r_f is the mean return of a risk-free asset, and σ_p is the standard deviation of return on the portfolio.

In statistics, the variance is the second central moment of a random variable X around its mean μ , where the r^{th} central moment of X is:⁶

$$\mu_r = E[(X - \mu)^r]$$

Mean and variance do not adequately describe an investments' distribution of returns. We need further measures of returns in order to evaluate the distribution, such as skewness, the third central moment, or kurtosis, the fourth central moment of the random variable. However, these further measures are difficult and complicated to evaluate. Additionally, the mean and variance of a random variable already capture its most important information. Therefore, we will use the mean and variance of a portfolio in order to evaluate its distribution in this paper.⁷

⁵ DeFusco et al. (2018), p. 445

⁶ Mukhopadhyay (2000), p. 77

⁷ For the Bayesian method, we assume normal distributions for stock returns, which also have only two parameters, mean θ and variance σ^2

1.3 Optimization formulations:

Optimization problems specify the random variables that could be changed in the process and the objective to find with certain constraints. In portfolio optimization, the random variables are usually the weights of the chosen stocks, which are (w_1, w_2, \dots, w_n) . The objective of the optimization problem depends on the parameters being evaluated. And the constraints on random variables vary based on the scenario of the problem. Some common and basic optimization problems include:

1.3.1 Maximizing portfolio expected return:

$$P = \underset{\{w: \sum_1^n w_i = 1 \text{ \& other constraints}\}}{\operatorname{argmax}} \mu_p$$

1.3.2 Minimizing portfolio volatility:

$$P = \underset{\{w: \sum_1^n w_i = 1 \text{ \& other constraints}\}}{\operatorname{argmin}} \sigma_p$$

1.3.3 Maximizing portfolio's Sharpe ratio:

$$P = \underset{\{w: \sum_1^n w_i = 1 \text{ \& other constraints}\}}{\operatorname{argmax}} \frac{\mu_p - r_f}{\sigma_p}$$

1.3.4 Maximizing risk-adjusted return:

$$P = \underset{\{w: \sum_1^n w_i = 1 \text{ \& other constraints}\}}{\operatorname{argmax}} \mu_p - \tau \sigma_p^2$$

where τ is the risk aversion parameter of the investor.

Chapter 2: The Mean-Variance Approach of Portfolio Optimization

The mean-variance portfolio optimization method was one of the foundations of portfolio selection modelling recommended by Markowitz along with the concept of diversification and the efficient frontier of a portfolio.⁸ In order to understand the mean-variance approach of portfolio optimization, we need several measures and assumptions.

2.1 Return:

The existence of risk means that the investor can no longer associate a single number or payoff with investment in any asset. In fact, it must be described by a set of outcomes and each of their associated probability of occurrence, which could be called return distribution.⁹ Hence, we reflect the return of a certain stock by its expected rate of return:

$$R = \sum_{i=1}^n P_i R_i$$

where R_i is the stock's return in scenario i and P_i is the probability of scenario i happening.

We can generalize the computation of the expected rates of return for a portfolio as follows:

$$E(R_p) = \sum_{i=1}^n w_i R_i$$

⁸ Agarwal (2015), p. 20

⁹ Elton et al. (2003), p. 44

where w_i is the weight of asset i in the portfolio and R_i is the corresponding expected rate of return of that asset.

2.2 Variance:

In probability theory and statistics, the variance of a random variable is a measure of the spread of that random variable about its expected value.¹⁰ Hence, the variance of a random variable X can be written as:

$$\text{Var}(X) = E[(X - E(X))^2]$$

For a certain asset, its variance is a measure of the variation of possible rates of return R_i from the expected rate of return R where $R = E(R_i)$. The variance σ^2 of an asset is:

$$\sigma^2 = \sum_{i=1}^n P_i (R_i - E(R_i))^2 = \sum_{i=1}^n P_i (R_i - R)^2$$

where R_i is the stock's return in scenario i and P_i is the probability of scenario i happening.

2.3 Covariance:

Covariance is used to represent the relationship between two random variables. The covariance between random variables X and Y is defined by:¹¹

$$\text{Cov}(X, Y) = E[(X - E(X))(Y - E(Y))]$$

For two assets, i and j , we define the covariance of rates of return as:¹²

¹⁰ ¹¹ Finan (2018), p. 286

¹² Reilly & Brown (2012), p. 185

$$\text{Cov}_{ij} = E\{[R_i - E(R_i)][R_j - E(R_j)]\}$$

Covariance of two assets measure the extent to which their rates of return move together during a certain time period. In fact, covariance is affected by the variability of the two individual return indexes.¹³ It is difficult to compare the covariances of different assets. To standardize the measure of variables' relationship, we use another measure called correlation. The correlation ρ_{ij} of two assets i and j is measured as follows:

$$\rho_{ij} = \frac{\text{Cov}_{ij}}{\sigma_i \sigma_j}$$

where σ_i and σ_j are the standard deviations of assets i and j respectively. The standard deviation of a certain asset is the square root of its variance:

$$\sigma = \sqrt{\sigma^2} = \sqrt{\sum_{i=1}^n P_i (R_i - R)^2}$$

2.4 Covariance matrix of a n-asset portfolio:

For a n-asset portfolio, its covariance matrix Σ is a $n \times n$ matrix formulated as follows:

$$\Sigma = \begin{bmatrix} \text{Cov}_{11} & \text{Cov}_{12} & \text{Cov}_{13} & \cdots & \text{Cov}_{1n} \\ \text{Cov}_{21} & \text{Cov}_{22} & \text{Cov}_{23} & \cdots & \text{Cov}_{2n} \\ \text{Cov}_{31} & \text{Cov}_{32} & \text{Cov}_{33} & \cdots & \text{Cov}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{Cov}_{n1} & \text{Cov}_{n2} & \text{Cov}_{n3} & \cdots & \text{Cov}_{nn} \end{bmatrix}$$

where $\Sigma_{i,j} = \text{Cov}_{ij}$ is the covariance of asset i and asset j in the portfolio.

¹³ Reilly & Brown (2012), p. 188

2.5 Variance of a portfolio:

Markowitz derived the formula for the variance of a n-asset portfolio as follows:¹⁴

$$\begin{aligned}\sigma_p &= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \text{Cov}_{ij} \\ &= \sum_{i=1}^n w_i^2 \sigma_i^2 + \sum_{\substack{i=1 \\ i \neq j}}^n \sum_{j=1}^n w_i w_j \text{Cov}_{ij} \\ &= W \Sigma W^T\end{aligned}$$

where: w_i is the weight of asset i in the portfolio

σ_i^2 is the variance of asset i in the portfolio

Cov_{ij} is the covariance between asset i and asset j in the portfolio

Σ is the covariance matrix of the n-asset portfolio

$W = [w_1 \quad w_2 \quad w_3 \quad \cdots \quad w_n]$ is the $1 \times n$ weight vector of the portfolio

W^T is the transposition of W

2.6. Mean-variance optimal portfolio:

Some assumptions for the mean-variance analysis include: the investors make decisions based on the expected return and variance, and all investors have the same information; investment decisions are made for a single period.¹⁵ The mean-variance optimal portfolio is achieved when the risk is minimized for a certain goal of expected return. Hence, the optimization problem is:

¹⁴ Agarwal (2015), p. 57

¹⁵ Kim et al. (2016), p. 13

$$P = \underset{\{w: \sum_1^n w_i = 1 \ \& \ \mu_p = \mu\}}{\operatorname{argmin}} \quad W \Sigma W^T$$

where μ is the target return and $W \Sigma W^T$ is the variance of the portfolio, which is the measure of risk, as we showed in the previous section.

Chapter 3: An Overview of the Bayesian Approach for Portfolio Optimization

3.1 Likelihood function:

The method of maximum likelihood is used to estimate a certain parameter given an observed set of data with known distributions. Let X_1, X_2, \dots, X_n are independent and identically distributed with common probability mass function or probability density function $f(x, \theta)$. Let $X = (x_1, x_2, \dots, x_n)$ be an observed data set. Then the likelihood function for parameter θ is:¹⁶

$$L(\theta) = \prod_{i=1}^n f(X = x_i, \theta)$$

The likelihood function of parameter θ given observed data set X could also be used to derive the maximum likelihood estimate MLE of θ by the following formula:

$$\text{MLE}(\theta) = \underset{\theta}{\text{argmax}} L(\theta)$$

3.2 Bayes' theorem:

First, we need to understand the conditional probability of a certain event A given the knowledge that another event B has occurred. The new information about event B causes us to update the probability that event A occurs in the same sample space. The conditional probability of A given B , denoted by $P(A|B)$ is:¹⁷

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

¹⁶ Mukhopadhyay (2000), p. 345

¹⁷ Finan (2018), p. 88

Hence, we will also have the probability of B given A, $P(B|A)$ as follows:

$$P(B|A) = \frac{P(A \cap B)}{P(A)}$$

By the above formulas, we can write:

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)$$

$$P(B|A) = \frac{P(A|B)P(B)}{P(A)} \quad (1)$$

Formula (1) is known as the Bayes' formula. Bayes' formula uses the occurrence of the event to infer the probability of the scenario generating it, thus, it is sometimes called an inverse probability.¹⁸ Hence, the Bayes' formula could be written as:

Updated probability given new information

$$= \frac{\text{Probability of new information given event}}{\text{Unconditional probability of the new information}}$$

× Prior probability of event

3.3 The Bayesian method in finding posterior distribution:

Applying the Bayes' formula to continuous random variable θ with known prior distribution and a sample data set X as new information, we can obtain the posterior distribution of θ by rewriting the Bayes' theorem:¹⁹

$$f(\theta|X) = \frac{f(X|\theta)f(\theta)}{f(X)}$$

¹⁸ DeFusco et al. (2018), p. 502

¹⁹ Rachev et al. (2008), p. 19

$$= \frac{L(\theta|X)f(\theta)}{f(X)}$$

where: $f(\theta|X)$ denotes the density function for the posterior distribution of θ

$L(\theta|X) = f(X|\theta)$ is the likelihood function of θ given data set X

$f(\theta)$ is the prior distribution of the unknown parameter θ

$f(X)$ is the unconditional probability of the data set X

Since $f(X)$ does not depend on θ , we can rewrite the above formula as:

$$f(\theta|X) \propto L(\theta|X)f(\theta)$$

Given the above formula, we can find the posterior distribution of θ . For an estimator of parameter θ , the maximum likelihood estimator as discussed in section 3.1 is often used.

Chapter 4: Bayesian Application on Normal Distributions

Recall that the Bayesian formula is:

$$f(\mu|v) = \frac{f(v|\mu)f(\mu)}{f(v)}$$

Using 5 years of stock returns data, we obtained the mean and standard deviation of the prior distribution. For simplicity, the prior distribution and the updated data distribution are assumed to be normal. Let the prior mean be μ_0 and the prior standard deviation be σ_0 . The prior distribution follows $N(\mu_0, \sigma_0)$. Hence, the prior distribution's probability density function is:

$$f(\mu) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(\mu - \mu_0)^2}{2\sigma_0^2}}$$

Call the updated data set V with mean μ_1 and standard deviation σ_1 . Then the updated data distribution follows $N(\mu_1, \sigma_1)$. Since $V = (v_1, v_2, \dots, v_n)$ is a collection of known stock returns and the updated data distribution follows normal distribution with known parameters, $f(V)$ is a constant. By the Bayesian formula, we obtain the following relation:

$$f(\mu|v) \propto L(\mu|v) f(\mu)$$

where $L(\mu|v) = f(v|\mu)$ is the likelihood function of the data set V based on parameter μ and $f(\mu)$ is the prior distribution. In fact, v_1, v_2, \dots, v_n of the data set V are independent and identically distributed and follow the distribution $N(\mu_1, \sigma_1)$. Therefore, the likelihood function of V given unknown parameter μ is:

$$L(\mu|v) = f(v|\mu) = f(x_1 = v_1, x_2 = v_2, \dots, x_n = v_n)$$

$$\begin{aligned}
&= f(x_1 = v_1)f(x_2 = v_2) \dots f(x_n = v_n) \\
&= \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(v_i - \mu)^2}{2\sigma_1^2}}
\end{aligned}$$

Notice that we can write $f(x_1 = v_1, x_2 = v_2, \dots, x_n = v_n) = \prod_{i=1}^n f(x_i = v_i)$ since v_1, v_2, \dots, v_n are independent. By plugging in the above formulas for $f(\mu)$ and $f(\mu|v)$, we obtain the following relation:

$$\begin{aligned}
f(\mu|v) &\propto \left(\prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma_1} e^{-\frac{(v_i - \mu)^2}{2\sigma_1^2}} \right) \left(\frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(\mu - \mu_0)^2}{2\sigma_0^2}} \right) \\
&\propto \frac{1}{\sigma_1^n \sigma_0} \exp\left(-\frac{(\mu - \mu_0)^2}{2\sigma_0^2} - \sum_{i=1}^n \frac{(v_i - \mu)^2}{2\sigma_1^2} \right) \\
&\propto \frac{1}{\sigma_1^n \sigma_0} \exp(-g(\mu))
\end{aligned}$$

where $g(\mu) = \frac{(\mu - \mu_0)^2}{2\sigma_0^2} + \sum_{i=1}^n \frac{(v_i - \mu)^2}{2\sigma_1^2}$ is a function of a single variable μ .

$$\begin{aligned}
g(\mu) &= \frac{(\mu - \mu_0)^2}{2\sigma_0^2} + \sum_{i=1}^n \frac{(v_i - \mu)^2}{2\sigma_1^2} \\
&= \frac{\mu^2 + \mu_0^2 - 2\mu\mu_0}{2\sigma_0^2} + \sum_{i=1}^n \frac{v_i^2 + \mu^2 - 2v_i\mu}{2\sigma_1^2} \\
&= \frac{\mu^2 + \mu_0^2 - 2\mu\mu_0}{2\sigma_0^2} + \frac{\sum_{i=1}^n v_i^2 + n\mu^2 - 2\sum_{i=1}^n v_i\mu}{2\sigma_1^2} \\
&= \frac{\mu_0^2}{2\sigma_0^2} + \frac{\sum_{i=1}^n v_i^2}{2\sigma_1^2} + \frac{\mu^2 - 2\mu\mu_0}{2\sigma_0^2} + \frac{n\mu^2 - 2\sum_{i=1}^n v_i\mu}{2\sigma_1^2}
\end{aligned}$$

Since $C = \frac{\mu_0^2}{2\sigma_0^2} + \frac{\sum_{i=1}^n v_i^2}{2\sigma_1^2}$ is a constant and does not depend on μ , we write $g(\mu)$ as

follows:

$$\begin{aligned}
g(\mu) &= C + \frac{\mu^2 - 2\mu\mu_0}{2\sigma_0^2} + \frac{n\mu^2 - 2\sum_{i=1}^n v_i\mu}{2\sigma_1^2} \\
&= C + \frac{\mu^2\sigma_1^2 - 2\mu\mu_0\sigma_1^2 + n\mu^2\sigma_0^2 - 2\sum_{i=1}^n v_i\mu\sigma_0^2}{2\sigma_0^2\sigma_1^2} \\
&= C + \frac{\mu^2(\sigma_1^2 + n\sigma_0^2) - 2\mu(\mu_0\sigma_1^2 + \sum_{i=1}^n v_i\sigma_0^2)}{2\sigma_0^2\sigma_1^2}
\end{aligned}$$

The updated data set V with mean μ_1 and standard deviation σ_1 . Hence,

$$\sum_{i=1}^n v_i = n\mu_1.$$

$$\begin{aligned}
g(\mu) &= C + \frac{\mu^2(\sigma_1^2 + n\sigma_0^2) - 2\mu(\mu_0\sigma_1^2 + n\mu_1\sigma_0^2)}{2\sigma_0^2\sigma_1^2} \\
&= C + \frac{\mu^2 - 2\mu(\mu_0\sigma_1^2 + n\mu_1\sigma_0^2)/(\sigma_1^2 + n\sigma_0^2)}{2\sigma_0^2\sigma_1^2/(\sigma_1^2 + n\sigma_0^2)}
\end{aligned}$$

Since $\mu^2 - 2\mu\frac{\mu_0\sigma_1^2 + n\mu_1\sigma_0^2}{\sigma_1^2 + n\sigma_0^2}$ can be rewritten as $\left(\mu - \frac{\mu_0\sigma_1^2 + n\mu_1\sigma_0^2}{\sigma_1^2 + n\sigma_0^2}\right)^2 - C'$ where C'

is a constant and does not depend on μ . Therefore, $g(\mu)$ can be written as follows:

$$\begin{aligned}
g(\mu) &= C - \frac{C'}{2\sigma_0^2\sigma_1^2/(\sigma_1^2 + n\sigma_0^2)} + \frac{\left(\mu - \frac{\mu_0\sigma_1^2 + n\mu_1\sigma_0^2}{\sigma_1^2 + n\sigma_0^2}\right)^2}{2\sigma_0^2\sigma_1^2/(\sigma_1^2 + n\sigma_0^2)} \\
&= C - C'' + \frac{\left(\mu - \frac{\mu_0\sigma_1^2 + n\mu_1\sigma_0^2}{\sigma_1^2 + n\sigma_0^2}\right)^2}{2\sigma_0^2\sigma_1^2/(\sigma_1^2 + n\sigma_0^2)}
\end{aligned}$$

where $C'' = \frac{C'}{2\sigma_0^2\sigma_1^2/(\sigma_1^2+n\sigma_0^2)}$ is also a constant. This result shows that the posterior

distribution $f(\mu|v)$ follows a normal distribution with mean $\frac{\mu_0\sigma_1^2+n\mu_1\sigma_0^2}{\sigma_1^2+n\sigma_0^2}$ and variance

$$\frac{\sigma_0^2\sigma_1^2}{\sigma_1^2+n\sigma_0^2}.$$

Chapter 5: Mean-Variance Optimal Portfolio of the Dow Jones Industrial Average stocks

We identify the 30 constituent members of the Dow Jones Industrial Average as of December 31, 2018 and download monthly returns for each company for the prior 10 years from the Center for Research in Security Prices. Three of these companies, Visa, DowDupont, and Goldman Sachs, experience substantial mergers and/or restructuring during this period. We therefore remove them from our sample, leaving us with 27 large, liquid stocks in our sample. With a time-horizon of 10 years and the data collected on 27 Dow Jones Industrial Average component companies' returns from 2008 to 2017, we apply the mean-variance approach to obtain an optimal portfolio. Starting from 2013, we use the data of each company's monthly returns in the previous five years. Considering a data set of 60 statistics, we have sufficient amount of information to evaluate the mean and variance of each company's stock movement and evaluate the covariance matrix of 27 companies. The goal of this approach is to maximize the Sharpe ratio of the portfolio, which is:

$$S_p = \frac{\mu_p - r_f}{\sigma_p}$$

where μ_p is the portfolio's rate of return, r_f is the risk-free rate and σ_p is the portfolio's total volatility.

5.1 Risk-free rate:

Based on the time horizon, U.S. 10-year Treasury Rate is used as the risk-free rate. T-bill returns are effectively risk-free, since we know what interest rate we will earn when

buying the bills.²⁰ This rate does not change for the lifetime of the bills and the agency who issues the bills, the U.S. government, is assumed to not default, at least in the next 10 years. Furthermore, treasury bills are extremely liquid, and the other risks such as credit risk or non-financial risks are irrelevant. Therefore, this rate is the most suitable for risk-free rate in this context. U.S. 10-year Treasury Rate in the period of 2008-2017 is:

2017	2.43%
2016	2.09%
2015	1.88%
2014	2.86%
2013	1.91%
2012	1.97%
2011	3.39%
2010	3.73%
2009	2.52%
2008	3.74%

Table 1. U.S. 10-year Treasury Rate in the period of 2008-2017²¹

5.2 Technique explanation and constraints:

Most of the data calculations and evaluations of this paper are done on Microsoft Excel.

The optimal portfolios of certain years are found using the Solver tool to maximize the

Sharpe ratio with certain assumptions and constraints. The Sharpe optimal portfolio P^* is:

²⁰ Bodie et al. (2008), p. 10

²¹ U.S. Department of the Treasury (2019)

$$P^* = \underset{\{w: \sum_1^n w_i = 1\}}{\operatorname{argmax}} \frac{\mu_p - r_f}{\sigma_p}$$

where w_i 's are the weights of 27 stocks. Hence, a constraint while building the optimal portfolio is that the sum of 27 weights must be 1. For each 5-year period, two optimal portfolios are constructed: One is built without further constraint, and one is built based on the assumption that no short-sales are allowed. No short-sales constraint is assumed to consider the fact that the use of short-sales is limited and strictly regulated for certain investors.

Additionally, another notice while using Solver tool is that this tool solves for the local maxima or minima of the objective variable. Here, the objective variable is the portfolio's Sharpe ratio. Hence, we need to run Solver multiple times in order to verify that the found portfolios maximize the Sharpe ratio globally. Finally, Solver at times gives unreasonable portfolio results. For example, a portfolio with 10000% weight invested a certain stock was found. To solve this problem, we added another constraint that limits the weight of each stock to be in the interval [-10, 10]. Therefore, the Sharpe optimal portfolio P^* with short-sales allowed is:

$$P^* = \underset{\{w: \sum_1^n w_i = 1, -10 \leq w_i \leq 10\}}{\operatorname{argmax}} \frac{\mu_p - r_f}{\sigma_p}$$

and the Sharpe optimal portfolio P^* with no short-sales is:

$$P^* = \underset{\{w: \sum_1^n w_i = 1, 0 \leq w_i \leq 10\}}{\operatorname{argmax}} \frac{\mu_p - r_f}{\sigma_p}$$

5.3 Sharpe optimal portfolio of 2008-2012 period:

A notice while evaluating 2008-2012 data is the positivity of the covariance matrix. This could be due to the fact that Dow Jones stocks are all large cap companies²².

Furthermore, the financial crisis of 2007-2008 had strong impact on the U.S. economy during that period, which increased the companies' systematic risk and correlation.

Especially, the considered 27 Dow Jones companies are large, publicly owned companies based in the United States, which make them more volatile to a downturn of the economy.²³ The risk-free rate used in this period is the arithmetic mean of the 10-year Treasury Rates from 2008-2012 divided by 12 for monthly consistency.²⁴ Using solver, the Sharpe optimal portfolio with short-sales allowed for the period of 2008-2012 is:

<u>AAPL</u>	<u>AXP</u>	<u>BA</u>	<u>CAT</u>	<u>CSCO</u>	<u>CVX</u>	<u>DIS</u>	<u>HD</u>	<u>IBM</u>
0.437	0.045	-0.229	0.054	-0.520	0.135	-0.025	0.654	0.929
<u>INTC</u>	<u>JNJ</u>	<u>JPM</u>	<u>KO</u>	<u>MCD</u>	<u>MMM</u>	<u>MRK</u>	<u>MSFT</u>	<u>NKE</u>
-0.632	0.025	-0.269	-0.197	0.726	0.159	0.053	-0.224	0.222
<u>PFE</u>	<u>PG</u>	<u>TRV</u>	<u>UNH</u>	<u>UTX</u>	<u>VZ</u>	<u>WBA</u>	<u>WMT</u>	<u>XOM</u>
0.169	-0.356	0.569	-0.128	0.127	-0.205	-0.248	0.111	-0.381

Table 2. 2008-2012 Sharpe optimal portfolio with short-sales allowed

By investing in this portfolio, we obtained a Sharpe ratio of 0.709 with a monthly return of 0.046 and portfolio's total volatility of 0.061. On the other hand, Sharpe optimal portfolio with no short-sales is:

²² Chen (2019)

²³ Wikipedia (2019)

²⁴ See Table 1

<u>AAPL</u>	<u>AXP</u>	<u>BA</u>	<u>CAT</u>	<u>CSCO</u>	<u>CVX</u>	<u>DIS</u>	<u>HD</u>	<u>IBM</u>
0.086	0.000	0.000	0.000	0.000	0.000	0.000	0.283	0.160
<u>INTC</u>	<u>JNJ</u>	<u>JPM</u>	<u>KO</u>	<u>MCD</u>	<u>MMM</u>	<u>MRK</u>	<u>MSFT</u>	<u>NKE</u>
0.000	0.000	0.000	0.000	0.384	0.000	0.000	0.000	0.000
<u>PFE</u>	<u>PG</u>	<u>TRV</u>	<u>UNH</u>	<u>UTX</u>	<u>VZ</u>	<u>WBA</u>	<u>WMT</u>	<u>XOM</u>
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.086	0.000

Table 3. 2008-2012 Sharpe optimal portfolio with no short-sales

This portfolio achieves a Sharpe ratio of 0.294, with a monthly return of 0.014 and portfolio's total volatility of 0.039 specifically. Notice that the Sharpe ratio of the optimal portfolio with no short-sales is much lower than the Sharpe ratio of the optimal portfolio with short-sales. This is reasonable since we added more constraint and restricted the weight of every stock to be positive. In fact, for the portfolio with short-sales allowed, the variables are $(w_1, w_2, \dots, w_{27})$ which vary in the interval $[-10, 10]$.²⁵ In contrast, the range of the variables $(w'_1, w'_2, \dots, w'_{27})$ for the portfolio without short-sales is $[0, 10]$. Hence, with the same stocks' mean returns and covariance matrix, the Sharpe optimal portfolio with short-sales allowed is expected to achieve higher Sharpe ratio.

To make a better comparison, the following graph shows the Sharpe optimal portfolios of 27 Dow Jones stocks for the period of 2008-2012:

²⁵ Recall that the constraint $|w_i| \leq 10 \forall i$ is to help eliminate unreasonable portfolios

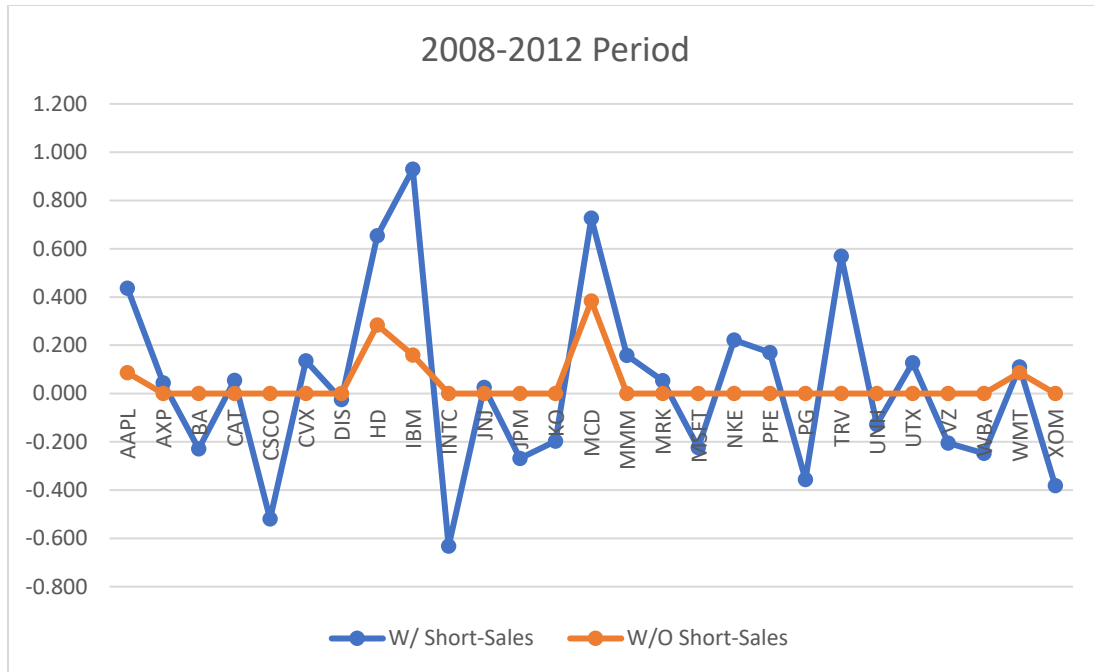


Figure 1. 2008-2012 Sharpe optimal portfolios

From the graph, we can observe that the stocks with higher positive weights in the optimal portfolios with short-sales allowed tend to have positive weights in the portfolio without short-sales. Contrarily, stocks that are shorted or have low positive weights tend to have zero weight when short-sales are not allowed.

5.4 Sharpe optimal portfolio of 2009-2013 period:

A consideration for this period is that the covariance matrix is no longer positive everywhere compared to the covariance matrix in the previous 5-year period. As explained above, the chosen 27 stocks are expected to have higher correlation when there is a big movement in the whole economy. In 2009-2013, the economy had recovered, which lowered the correlation among these large cap companies' stocks. Following the similar process, we obtain the following Sharpe optimal portfolios for the period of 2009-2013:

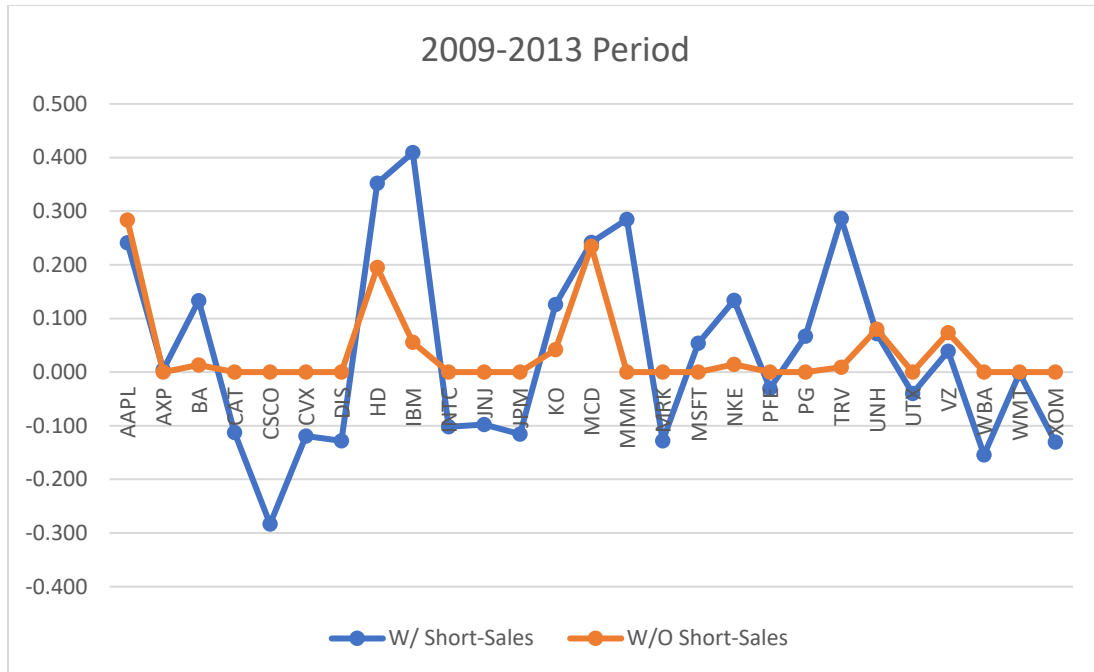


Figure 2. 2009-2013 Sharpe optimal portfolios

Here, the risk-free rate used in this period is the arithmetic mean of the 10-year Treasury Rates from 2009-2013 divided by 12. The optimal portfolio with short-sales realizes a Sharpe ratio of 0.796 with a monthly return of 0.027 and portfolio's total volatility of 0.031. On the other hand, the optimal portfolio with no short-sales acquires a Sharpe ratio of 0.543 with a monthly return of 0.022 and portfolio's total volatility of 0.037.

5.5 Sharpe optimal portfolio of 2010-2014 period:

The Sharpe optimal portfolios for the period of 2010-2014 are:

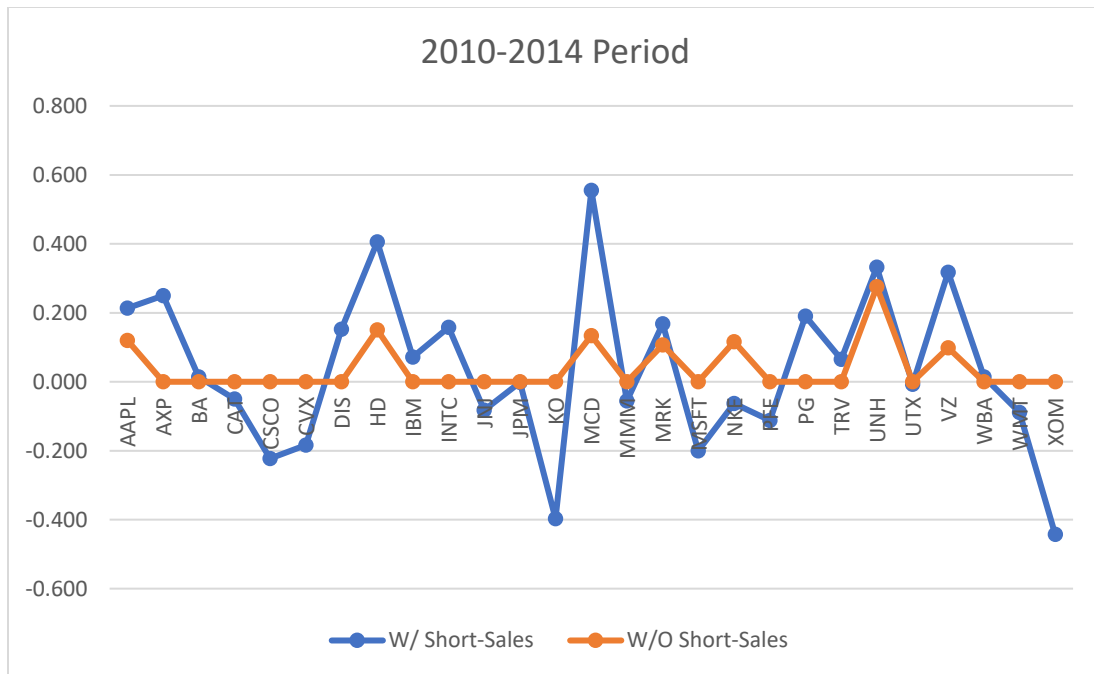


Figure 3. 2010-2014 Sharpe optimal portfolios

Here, the risk-free rate used in this period is the arithmetic mean of the 10-year Treasury Rates from 2010-2014 divided by 12. The optimal portfolio with short-sales achieves a Sharpe ratio of 0.796 with a monthly return of 0.029 and portfolio's total volatility of 0.033. On the other hand, the optimal portfolio with no short-sales obtains a Sharpe ratio of 0.579 with a monthly return of 0.019 and portfolio's total volatility of 0.029.

5.6 Sharpe optimal portfolio of 2011-2015 period:

The Sharpe optimal portfolios for the period of 2011-2015 are:

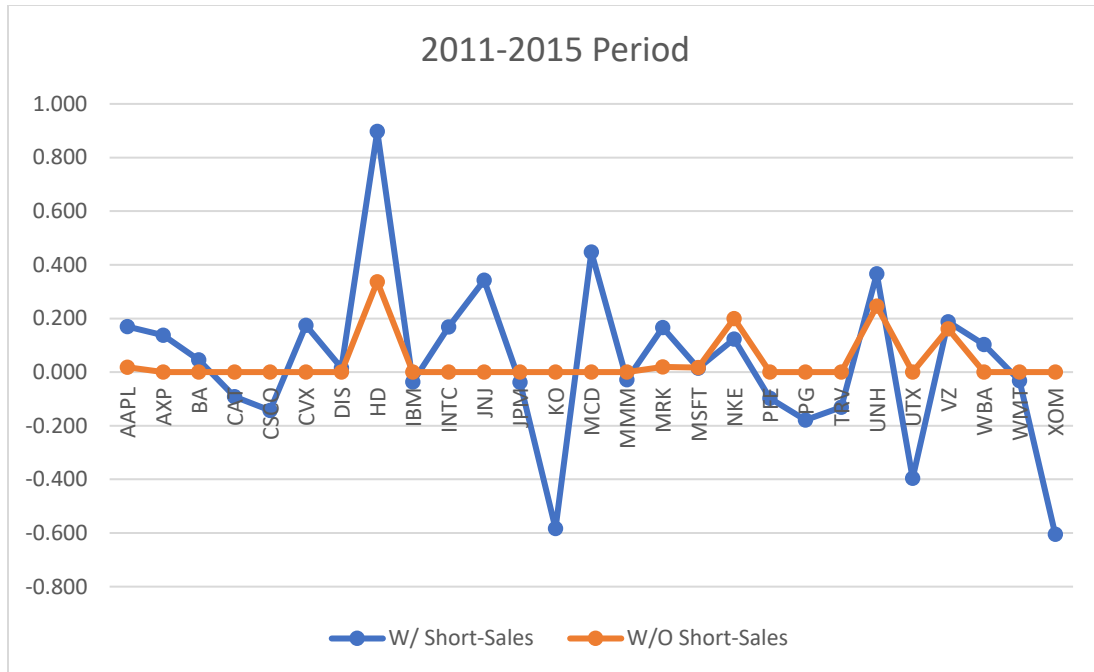


Figure 4. 2011-2015 Sharpe optimal portfolios

Here, the risk-free rate used in this period is the arithmetic mean of the 10-year Treasury Rates from 2011-2015 divided by 12. The optimal portfolio with short-sales realizes a Sharpe ratio of 0.839 with a monthly return of 0.039 and portfolio's total volatility of 0.045. On the other hand, the optimal portfolio with no short-sales acquires a Sharpe ratio of 0.622 with a monthly return of 0.021 and portfolio's total volatility of 0.030.

5.7 Sharpe optimal portfolio of 2012-2016 period:

The Sharpe optimal portfolios for the period of 2012-2016 are:

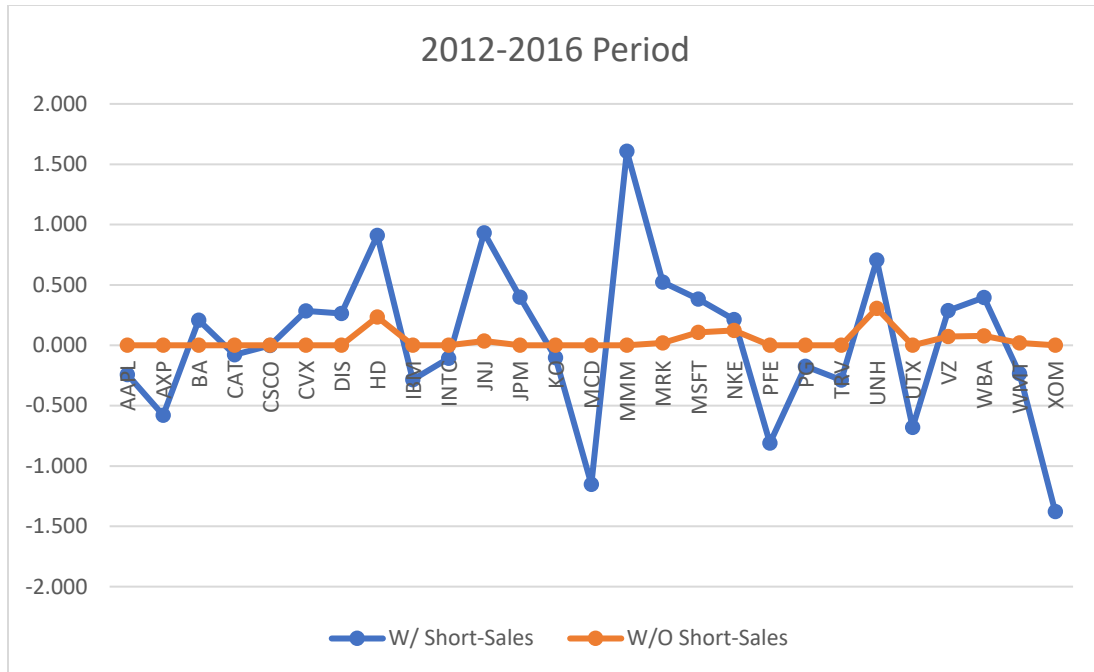


Figure 5. 2012-2016 Sharpe optimal portfolios

Here, the risk-free rate used in this period is the arithmetic mean of the 10-year Treasury Rates from 2012-2016 divided by 12. The optimal portfolio with short-sales achieves a Sharpe ratio of 0.800 with a monthly return of 0.070 and portfolio's total volatility of 0.086. On the other hand, the optimal portfolio with no short-sales obtains a Sharpe ratio of 0.578 with a monthly return of 0.019 and portfolio's total volatility of 0.030.

Chapter 6: Update Sharpe Optimal Portfolio Using the Bayesian Approach

6.1 Technique explanation and constraints:

Using the Bayesian method, we update the Sharpe optimal portfolio every year from 2013 to 2017. Let us exemplify the process by taking the 2008-2012 period portfolio and update it using the new data collected in 2013. A core assumption of this method is that stock returns follow normal distribution. Hence, for each stock of the 27 Dow Jones companies, the prior distribution is formed by 2008-2012 data with mean μ_0 and standard deviation σ_0 . The prior distribution of that stock follows $N(\mu_0, \sigma_0)$. The updated data V is the 12 monthly returns in the year of 2013. v_1, v_2, \dots, v_{12} of the data set V follow the distribution $N(\mu_1, \sigma_1)$, where μ_1 is the mean and σ_1 is the standard deviation of V .

Therefore, the posterior distribution of the stock is $N\left(\frac{\mu_0\sigma_1^2+12\mu_1\sigma_0^2}{\sigma_1^2+12\sigma_0^2}, \frac{\sigma_0^2\sigma_1^2}{\sigma_1^2+12\sigma_0^2}\right)$.²⁶

Using this posterior distribution, we have updated the mean and standard deviation of each stock in the 27-asset portfolio. By updating the portfolio using the Bayesian method instead of following the standard mean-variance approach, we expect to obtain a more accurate posterior distribution since it captures the information of the most recent data more significantly. Furthermore, if we follow the mean-variance process, we would have used five-year data during the period of 2009-2013 to form a new portfolio, which omit potentially significant data in 2008. By the Bayesian approach, we maintain the information within the last five-year period, including the data in 2008. One might argue that we could use six-year period data for the mean-variance approach to solve this

²⁶ See Chapter 4

problem. However, this approach would lower the weight of the most recent data, the year of 2013, even further, which is the most important information in forming new portfolio.

A problem that we face during the process is the resulted new variance of each stock. In fact, notice that the formula for the posterior variance is:

$$\frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 + n \sigma_0^2} = \frac{\sigma_0^2}{1 + n \frac{\sigma_0^2}{\sigma_1^2}} \ll \sigma_0^2$$

$$\frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 + n \sigma_0^2} = \frac{\sigma_1^2}{\frac{\sigma_1^2}{\sigma_0^2} + n} \ll \sigma_1^2$$

Hence, the larger n is, the closer the new variance is to zero. This is reasonable since the more data we collected for the updated distribution, the more information we have on the posterior distribution, which lower the uncertainty and thus the posterior variance. Nevertheless, the fact that the posterior variances of the stocks are all closer to zero may make the calculation of the portfolio's total volatility inaccurate. In fact, since we need the new covariance matrix to run Solver and apply the mean-variance optimization on the posterior data, we will derive each stock's variance from this updated covariance matrix. Because the posterior variance of each stock alone is insufficient to measure the uncertainty of the portfolio's variance,²⁷ it would be more consistent to use the variance derived from the new covariance matrix. The updated variances we use for 27 stocks form the horizontal line of the updated covariance matrix, which is:

²⁷ Recall that the variance of a n-asset portfolio is $\sigma_p = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \text{Cov}_{ij}$

$$\Sigma^* = \begin{bmatrix} \text{Cov}_{1,1}^* & \text{Cov}_{1,2}^* & \text{Cov}_{1,3}^* & \cdots & \text{Cov}_{1,27}^* \\ \text{Cov}_{2,1}^* & \text{Cov}_{2,2}^* & \text{Cov}_{2,3}^* & \cdots & \text{Cov}_{2,27}^* \\ \text{Cov}_{3,1}^* & \text{Cov}_{3,2}^* & \text{Cov}_{3,3}^* & \cdots & \text{Cov}_{3,27}^* \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \text{Cov}_{27,1}^* & \text{Cov}_{27,2}^* & \text{Cov}_{27,3}^* & \cdots & \text{Cov}_{27,27}^* \end{bmatrix}$$

where $\text{Cov}_{i,j}^*$ is the new covariance of stock i and stock j.

Here are the two methods we use to update the covariance matrix:

1. Using the five-year period data of 2009-2013 to form a new covariance matrix.
2. Using the correlation matrix formed by the covariance matrix in the period of 2008-2012 and using the posterior variances to derive the new covariance matrix:

The correlation matrix of the 27-asset portfolio is:

$$R = \begin{bmatrix} \rho_{1,1} & \rho_{1,2} & \rho_{1,3} & \cdots & \rho_{1,27} \\ \rho_{2,1} & \rho_{2,2} & \rho_{2,3} & \cdots & \rho_{2,27} \\ \rho_{3,1} & \rho_{3,2} & \rho_{3,3} & \cdots & \rho_{3,27} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{27,1} & \rho_{27,2} & \rho_{27,3} & \cdots & \rho_{27,27} \end{bmatrix}$$

where $\rho_{i,j} = \frac{\text{Cov}_{i,j}}{\sigma_i \sigma_j}$ is the correlation between stock i and stock j in the portfolio.

We calculate the variance of each stock by the above formula using the Bayesian approach:

$$\sigma_i^{*2} = \frac{\sigma_{i,0}^2 \sigma_{i,1}^2}{\sigma_{i,1}^2 + 12\sigma_{i,0}^2}$$

where σ_i^{*2} is the posterior variance of stock i, $\sigma_{i,0}^2$ is its prior variance and $\sigma_{i,1}^2$ is its variance on the updated data. Assuming that the correlation matrix does not change when we move forward one year, we obtain the covariance matrix using the formula:

$$\text{Cov}_{i,j}^* = \rho_{i,j} \sigma_i^* \sigma_j^*$$

where $\text{Cov}_{i,j}^*$ is the new covariance of stock i and stock j, σ_i^* and σ_j^* are the posterior standard deviations of stock i and stock j respectively.

Therefore, we achieved the posterior means and the posterior covariance matrix of the 27-asset portfolio. Applying the same approach as we showed in Chapter 5, we find the Sharpe optimal portfolios for the new data set under two circumstances: short-sales allowed and no short-sales.

6.2 Updated optimal portfolio of 2008-2012 period given 2013 data set:

6.2.1 Method 1:

Using method 1 and the updated data set of 12 monthly returns in the year of 2013, we obtain the new covariance matrix using all the data in the period of 2009-2013. The new mean return of each stock is found using the formula:

$$\mu^* = \frac{\mu_0 \sigma_1^2 + 12 \mu_1 \sigma_0^2}{\sigma_1^2 + 12 \sigma_0^2}$$

where μ^* is the new monthly return of the stock, μ_0 and σ_0^2 are the monthly return and variance of the stock in the period of 2008-2012 respectively, μ_1 and σ_1^2 are its monthly return and variance in the year of 2013.

Another notice while calculating the updated mean return for the stock is:

$$\mu^* = \frac{\mu_0 \sigma_1^2 + 12 \mu_1 \sigma_0^2}{\sigma_1^2 + 12 \sigma_0^2}$$

$$= \frac{\frac{\mu_0 \sigma_1^2}{12\sigma_0^2} + \mu_1}{\frac{\sigma_1^2}{12\sigma_0^2} + 1} \approx \mu_1$$

Therefore, the updated mean would be close to the month return of the collected data set in 2013. This is due to the fact that the collected data set contains 12 elements, which is reasonably significant information to form the posterior distribution. The more data is collected, i.e. the higher n is, the closer updated mean to the collected data set's monthly return and the lower posterior variance is.

The risk-free rate used in this period is the 10-year Treasury rate in 2013 divided by 12 since we are forming an updated portfolio within this year. Using Solver with the same constraints as shown in chapter 5, we obtain two updated Sharpe optimal portfolios under short-sales allowed circumstance and no short-sales circumstance. The Sharpe optimal portfolio with short-sales in 2013 is:

<u>AAPL</u>	<u>AXP</u>	<u>BA</u>	<u>CAT</u>	<u>CSCO</u>	<u>CVX</u>	<u>DIS</u>	<u>HD</u>	<u>IBM</u>
-0.033	-0.096	0.118	-0.479	-0.198	-0.143	-0.075	0.100	-0.083
<u>INTC</u>	<u>JNJ</u>	<u>JPM</u>	<u>KO</u>	<u>MCD</u>	<u>MMM</u>	<u>MRK</u>	<u>MSFT</u>	<u>NKE</u>
0.002	-0.360	0.059	0.070	0.070	1.045	-0.086	0.191	0.309
<u>PFE</u>	<u>PG</u>	<u>TRV</u>	<u>UNH</u>	<u>UTX</u>	<u>VZ</u>	<u>WBA</u>	<u>WMT</u>	<u>XOM</u>
-0.362	0.331	0.174	0.182	-0.070	0.020	-0.018	0.184	0.146

Table 4. 2013 updated Sharpe optimal portfolio with short-sales allowed using method 1

By investing in this portfolio, we obtained a Sharpe ratio of 1.266 with a monthly return of 0.054 and portfolio's total volatility of 0.041. Notice that if we keep the mean-

variance Sharpe optimal portfolio obtained in the period of 2008-2012, we will have a Sharpe ratio of -0.147, a monthly return of -0.009 and total volatility of 0.069, which is an extremely bad performance. This could be explained by the approach used in method 1, which eliminates the information of 2008 returns during the financial crisis in the covariance matrix. Hence, there is big difference between the covariance matrix used in the previous optimal portfolio and the updated portfolio, which causes the old portfolio to have an extremely low Sharpe ratio. Additionally, we also observe the benefit of updating the portfolio, instead of maintaining the same portfolio for a long-term period and achieve a negative Sharpe ratio.

On the other hand, the Sharpe optimal portfolio with no short-sales is:

<u>AAPL</u>	<u>AXP</u>	<u>BA</u>	<u>CAT</u>	<u>CSCO</u>	<u>CVX</u>	<u>DIS</u>	<u>HD</u>	<u>IBM</u>
0.000	0.000	0.399	0.000	0.000	0.000	0.000	0.030	0.000
<u>INTC</u>	<u>JNJ</u>	<u>JPM</u>	<u>KO</u>	<u>MCD</u>	<u>MMM</u>	<u>MRK</u>	<u>MSFT</u>	<u>NKE</u>
0.000	0.006	0.000	0.000	0.000	0.250	0.000	0.001	0.012
<u>PFE</u>	<u>PG</u>	<u>TRV</u>	<u>UNH</u>	<u>UTX</u>	<u>VZ</u>	<u>WBA</u>	<u>WMT</u>	<u>XOM</u>
0.000	0.045	0.000	0.035	0.000	0.012	0.065	0.145	0.000

Table 5. 2013 updated Sharpe optimal portfolio with no short-sales using method 1

This portfolio achieves a Sharpe ratio of 0.714 with a monthly return of 0.038 and portfolio's total volatility of 0.051. Here, we have the same observation in chapter 5 that the Sharpe ratio of the portfolio without short-sales is much lower than the portfolio with short-sales allowed due to the further constraint that all the weights must be non-negative.

Under the no short-sales constraint, the 2008-2012 portfolio would have a Sharpe ratio of 0.391 with a monthly return of 0.014 and total volatility of 0.032.

6.2.2 Method 2:

As explained in section 6.1, for method 2, we will derive the correlation matrix from the covariance matrix of the period 2008-2012. This approach's purpose is to make use of the posterior variance obtained by the Bayesian method, which is significantly lower than the prior variance of the stocks. The calculations of the posterior mean are the same as method 1. By using Solver, the Sharpe optimal portfolio with short-sales in 2013 is:

<u>AAPL</u>	<u>AXP</u>	<u>BA</u>	<u>CAT</u>	<u>CSCO</u>	<u>CVX</u>	<u>DIS</u>	<u>HD</u>	<u>IBM</u>
-0.217	0.219	0.187	-0.546	-0.169	0.218	0.042	0.227	-0.019
<u>INTC</u>	<u>JNJ</u>	<u>JPM</u>	<u>KO</u>	<u>MCD</u>	<u>MMM</u>	<u>MRK</u>	<u>MSFT</u>	<u>NKE</u>
0.264	-0.196	-0.008	0.054	0.080	0.520	0.145	0.023	0.238
<u>PFE</u>	<u>PG</u>	<u>TRV</u>	<u>UNH</u>	<u>UTX</u>	<u>VZ</u>	<u>WBA</u>	<u>WMT</u>	<u>XOM</u>
-0.305	0.056	-0.100	0.182	-0.114	-0.053	-0.016	0.216	0.071

Table 6. 2013 updated Sharpe optimal portfolio with short-sales allowed using method 2

This portfolio obtains a Sharpe ratio of 7.806 with a monthly return of 0.055 and portfolio's total volatility of 0.007. Notice that compared to method 1, this method's portfolio has a much higher Sharpe ratio. This could be explained that we used the correlations of the stocks in the year of 2008-2012, which potentially omits important information in the year of 2013. It may make some stocks overvalued or undervalued in estimation and result in the Sharpe ratio being overly optimistic. The 2008-2012 portfolio with short-sales allowed would have a Sharpe ratio of -0.793 with a monthly return of -

0.009 and total volatility of 0.013. On the other hand, the Sharpe optimal portfolio with no short-sales using method 2 is:

<u>AAPL</u>	<u>AXP</u>	<u>BA</u>	<u>CAT</u>	<u>CSCO</u>	<u>CVX</u>	<u>DIS</u>	<u>HD</u>	<u>IBM</u>
0.000	0.156	0.389	0.000	0.000	0.000	0.000	0.049	0.000
<u>INTC</u>	<u>JNJ</u>	<u>JPM</u>	<u>KO</u>	<u>MCD</u>	<u>MMM</u>	<u>MRK</u>	<u>MSFT</u>	<u>NKE</u>
0.000	0.000	0.000	0.000	0.000	0.319	0.000	0.000	0.014
<u>PFE</u>	<u>PG</u>	<u>TRV</u>	<u>UNH</u>	<u>UTX</u>	<u>VZ</u>	<u>WBA</u>	<u>WMT</u>	<u>XOM</u>
0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.073	0.000

Table 7. 2013 updated Sharpe optimal portfolio with no short-sales using method 2

This portfolio has a Sharpe ratio of 4.043 with a monthly return of 0.041 and portfolio's total volatility of 0.010. Again, compared to method 1, the Sharpe ratio obtained by this method is much higher. The 2008-2012 portfolio would achieve a Sharpe ratio of 1.665 with a monthly return of 0.014 and portfolio's total volatility of 0.008.

6.2.3 Comparisons between the portfolios:

To make a better comparison, the following graph shows the Sharpe optimal portfolios of 27 assets using method 1, method 2 and the 2008-2012 mean-variance Sharpe optimal portfolio under short-sales allowed circumstance:

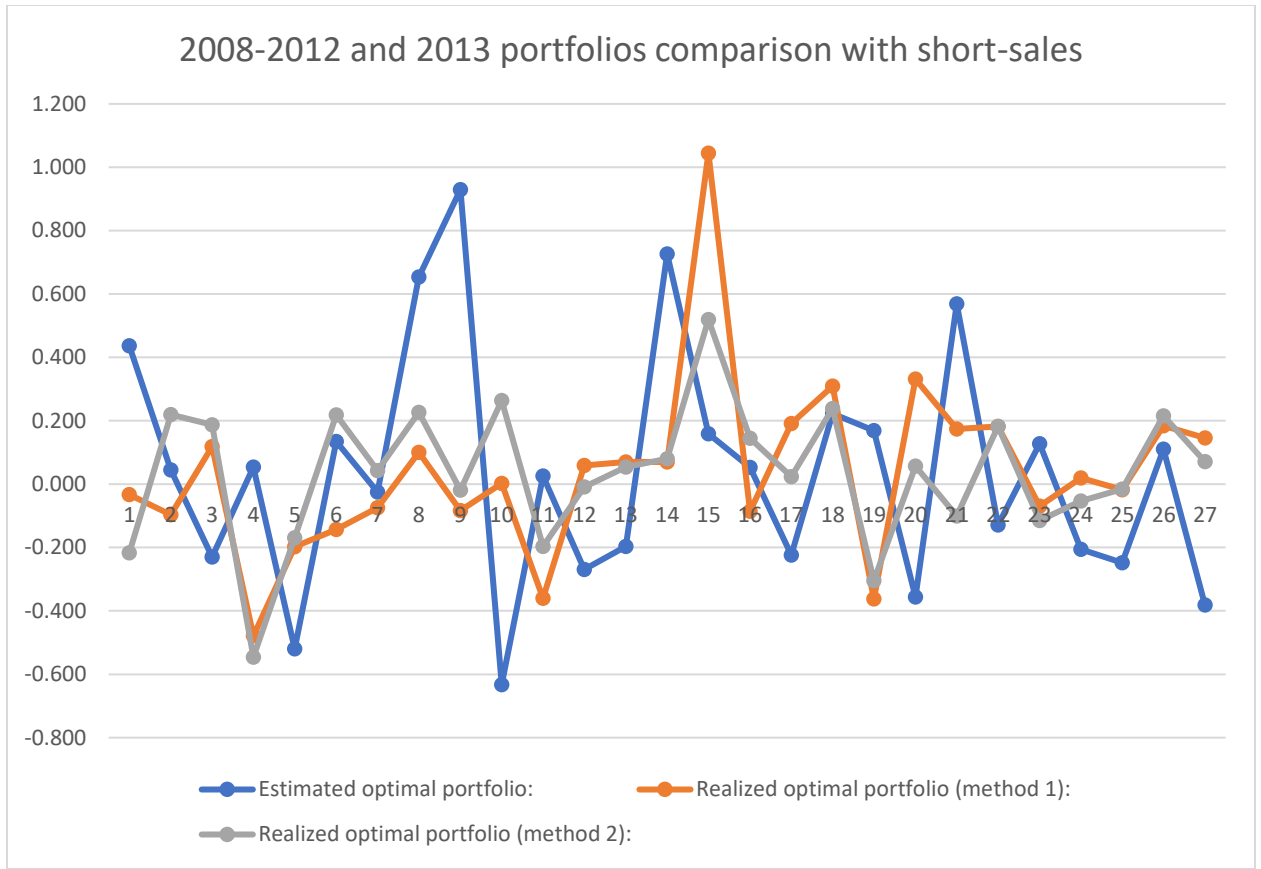


Figure 6. 2008-2012 and 2013 optimal portfolios with short-sales

Here, the estimated optimal portfolio is the mean-variance optimal portfolio in the period of 2008-2012, and the realized optimal portfolios are the Sharpe optimal portfolios obtained using the Bayesian approach. As we observe in the graph, the realized optimal portfolios of method 1 and 2 are reasonably close to each other. Under no short-sales constraint, we obtain the following graph:

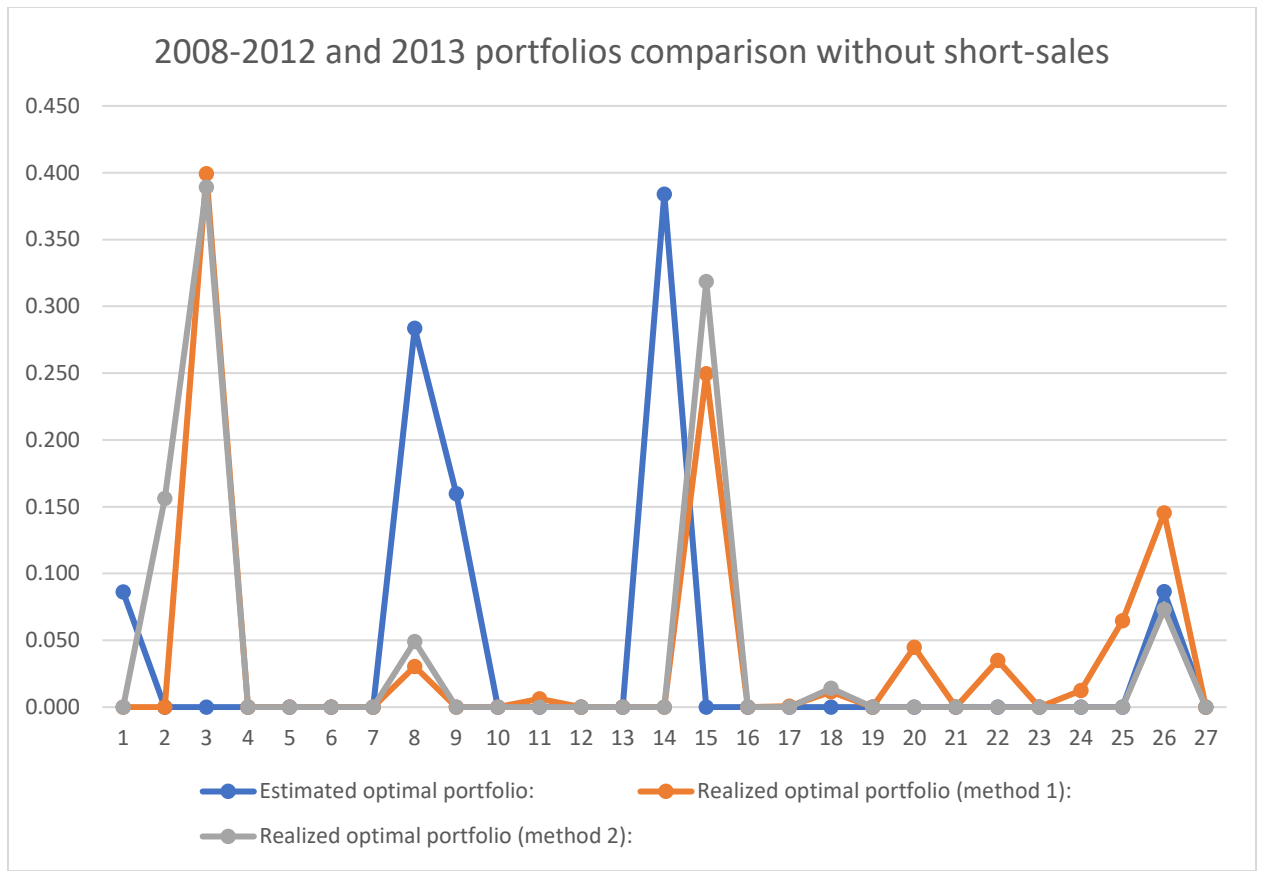


Figure 7. 2008-2012 and 2013 optimal portfolios without short-sales

Similar to the portfolios under short-sales allowed circumstance, without short-sales, the two methods of the Bayesian approach give reasonably close portfolios.

6.3 Updated optimal portfolio of 2009-2013 period given 2014 data set:

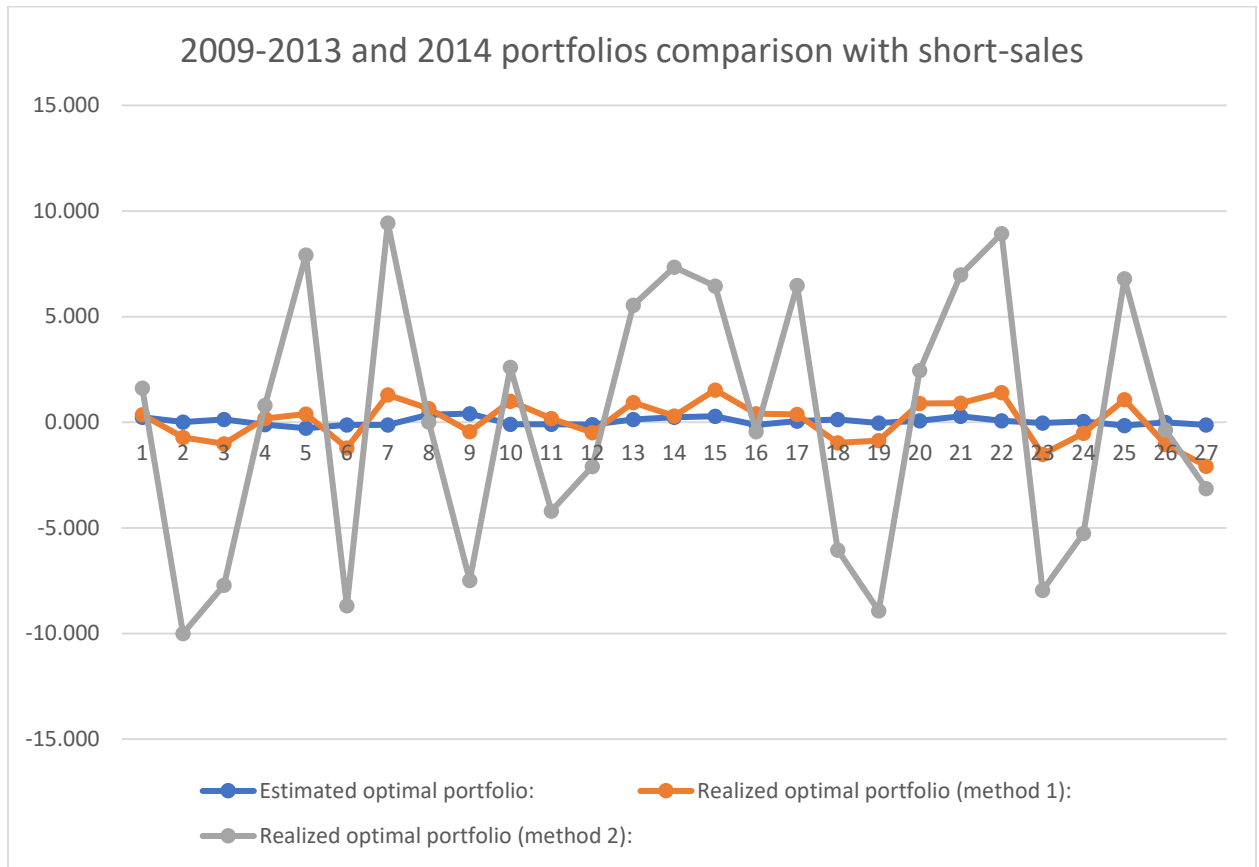


Figure 8. 2009-2013 and 2014 optimal portfolios with short-sales

The risk-free rate used in this period is the 10-year Treasury Rates in 2014 divided by 12. The optimal portfolio of method 1 realizes a Sharpe ratio of 1.466 with a monthly return of 0.200 and portfolio's total volatility of 0.135. On the other hand, the optimal portfolio of method 2 acquires a Sharpe ratio of 9.278 with a monthly return of 1.110 and portfolio's total volatility of 0.119. In this period, the Sharpe optimal portfolio of method 2 also has a much higher Sharpe ratio compared to method 1 and extremely volatile stock weights. This indicates a potential error in holding the correlation matrix constant by using Bayesian method 2.

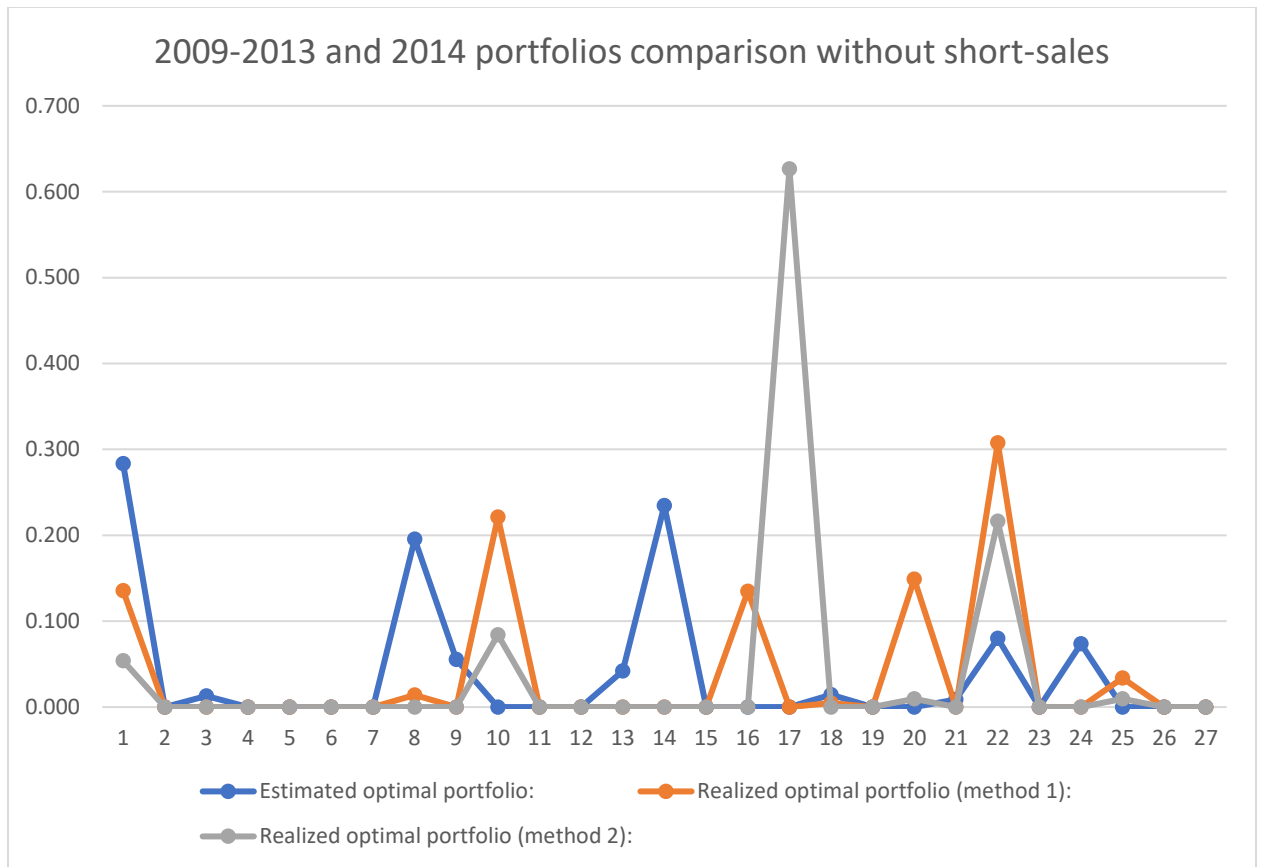


Figure 9. 2009-2013 and 2014 optimal portfolios without short-sales

In this circumstance, the optimal portfolio of method 1 obtains a Sharpe ratio of 0.687 with a monthly return of 0.025 and portfolio's total volatility of 0.032. On the other hand, the optimal portfolio of method 2 achieves a Sharpe ratio of 2.748 with a monthly return of 0.024 and portfolio's total volatility of 0.008. One more time, as we see in the figure, the two Bayesian portfolios differ significant from each other, which most likely results from the difference in the covariance matrices.

6.4 Updated optimal portfolio of 2010-2014 period given 2015 data set:

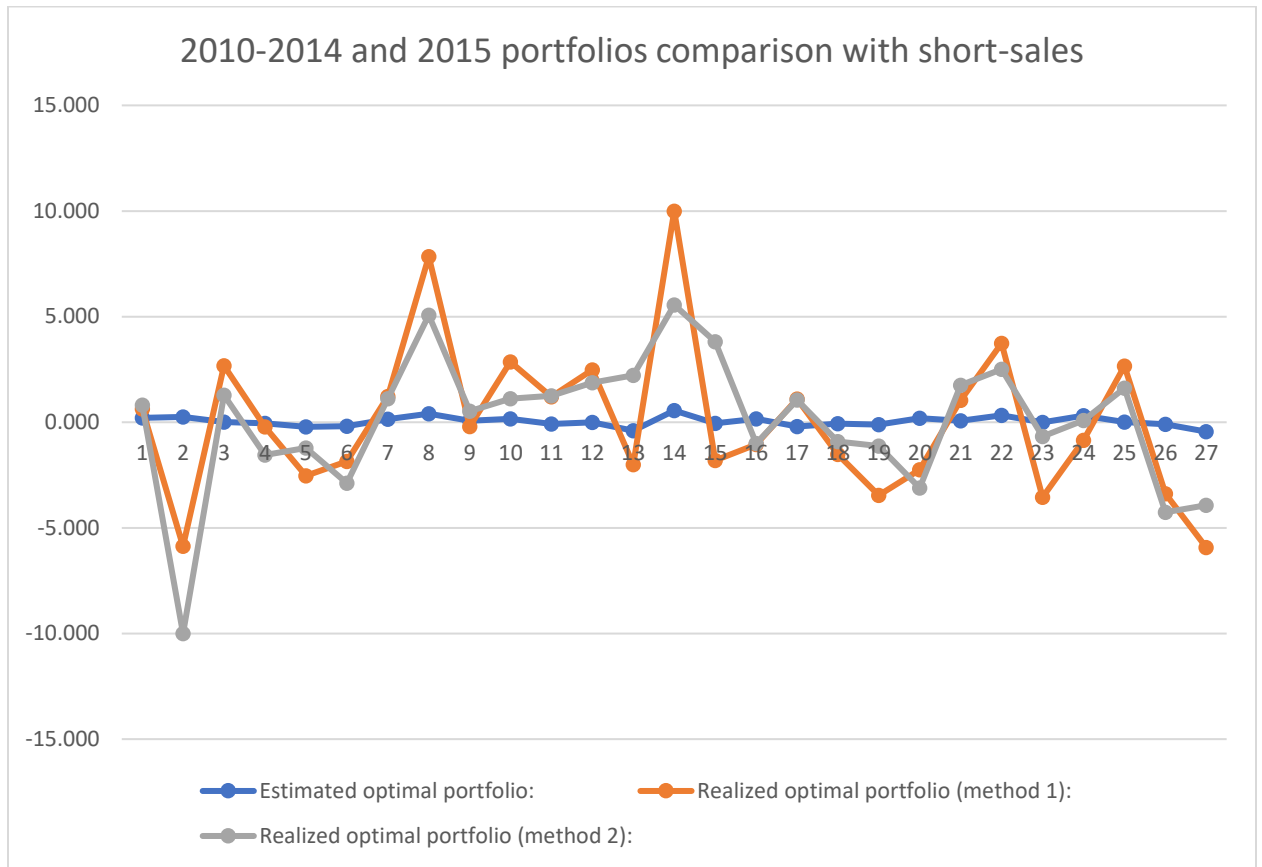


Figure 10. 2010-2014 and 2015 optimal portfolios with short-sales

The risk-free rate used in this period is the 10-year Treasury Rates in 2015 divided by 12.

The optimal portfolio of method 1 realizes a Sharpe ratio of 1.491 with a monthly return of 0.827 and portfolio's total volatility of 0.553. On the other hand, the optimal portfolio of method 2 acquires a Sharpe ratio of 6.640 with a monthly return of 0.752 and portfolio's total volatility of 0.113. Unlike the previous period, in this period, the two Bayesian portfolios are significantly close to each other. This could be explained by a stable economy, which makes the assumption that the correlation matrix does not change less significant. However, the estimated optimal portfolio obtained from the mean-variance approach diverges greatly from the two Bayesian portfolios. In fact, the mean-

variance portfolio would achieve a Sharpe ratio of 0.612 if we apply the covariance matrix of method 1 and a Sharpe ratio of 2.366 if we apply the covariance matrix of method 2.²⁸

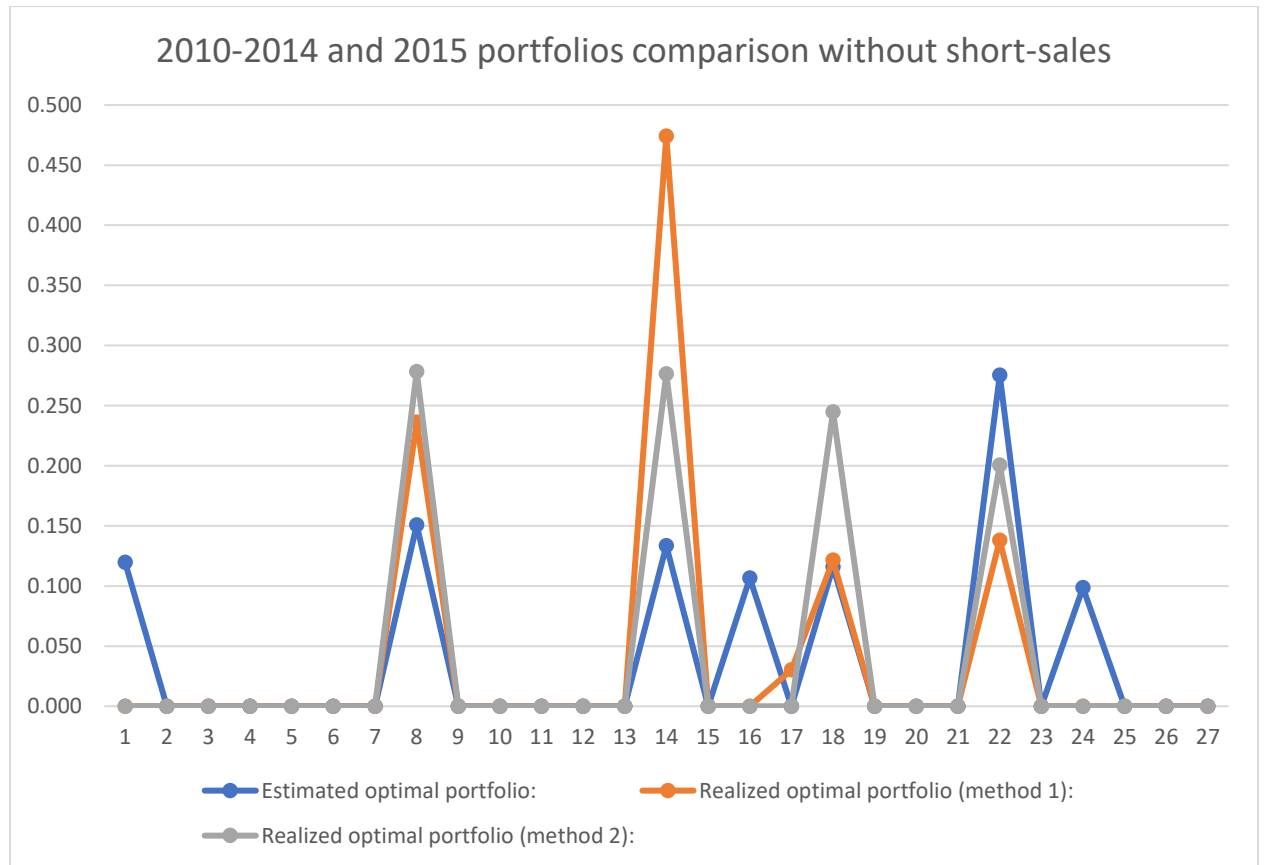


Figure 11. 2010-2014 and 2015 optimal portfolios without short-sales

In this circumstance, the optimal portfolio of method 1 obtains a Sharpe ratio of 0.659 with a monthly return of 0.021 and portfolio’s total volatility of 0.030. On the other hand, the optimal portfolio of method 2 achieves a Sharpe ratio of 2.284 with a monthly return of 0.021 and portfolio’s total volatility of 0.009. This is a reasonable result since

²⁸ Notice that the two Bayesian methods only differ in the covariance matrix. They obtain the same monthly stock returns for a given period. We consider two Bayesian methods in order to evaluate the better use of covariance matrix for 27 stocks.

the Sharpe ratios obtained without short-sales are much lower than the Sharpe ratios achieved by the portfolios with short-sales allowed.

6.5 Updated optimal portfolio of 2011-2015 period given 2016 data set:

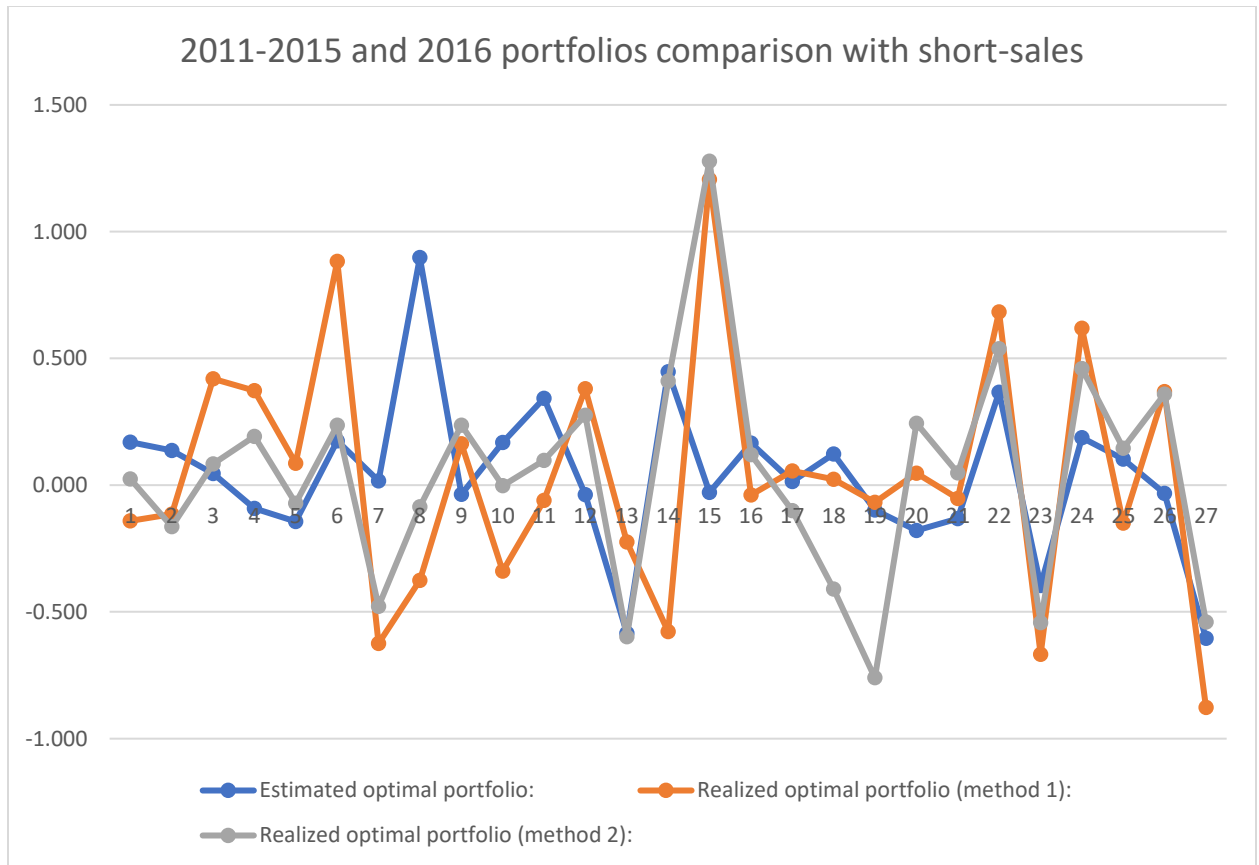


Figure 12. 2011-2015 and 2016 optimal portfolios with short-sales

The risk-free rate used in this period is the 10-year Treasury Rates in 2016 divided by 12.

The optimal portfolio of method 1 realizes a Sharpe ratio of 1.054 with a monthly return of 0.070 and portfolio's total volatility of 0.064. On the other hand, the optimal portfolio of method 2 acquires a Sharpe ratio of 4.968 with a monthly return of 0.061 and portfolio's total volatility of 0.012. This is another period that the Bayesian portfolios move closely to each other.

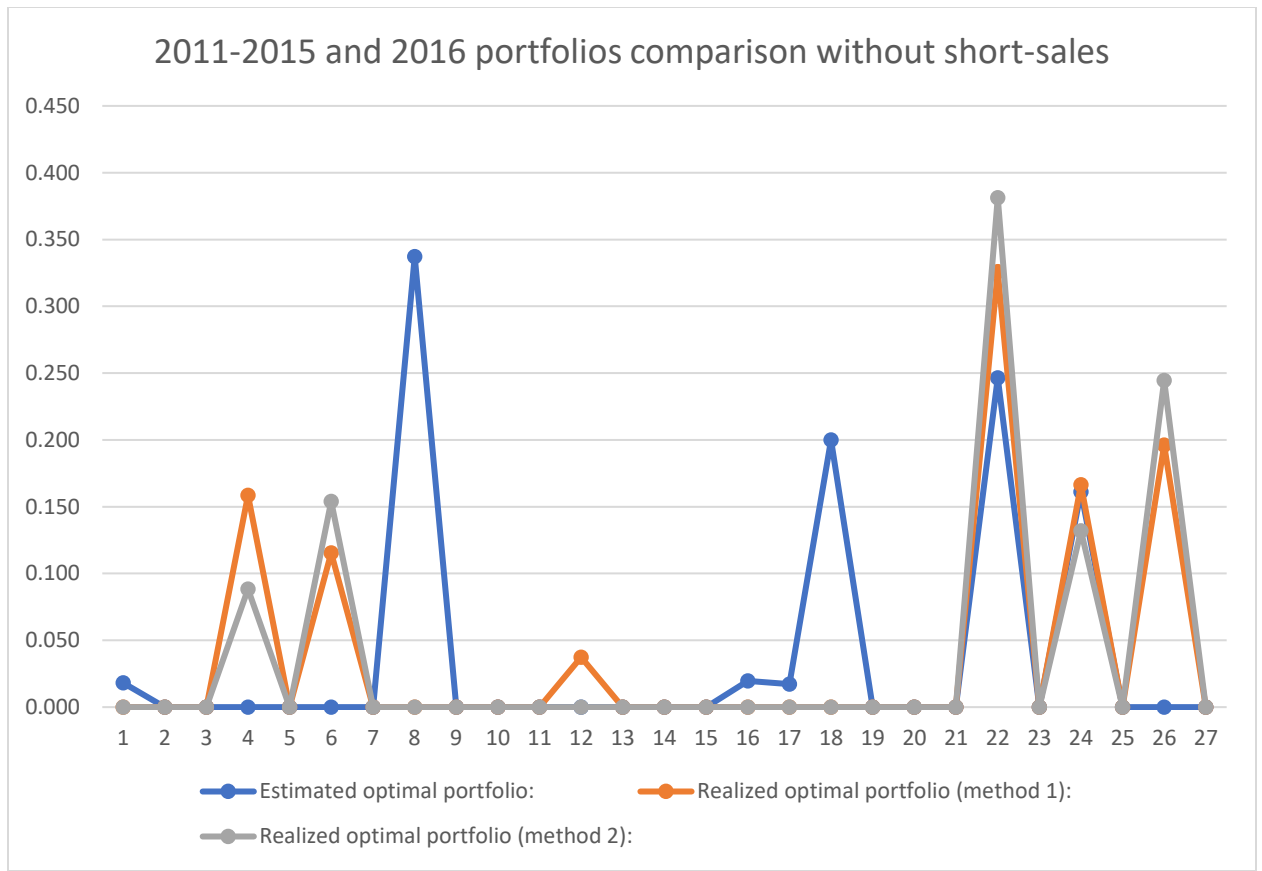


Figure 13. 2011-2015 and 2016 optimal portfolios without short-sales

In this circumstance, the optimal portfolio of method 1 obtains a Sharpe ratio of 0.729 with a monthly return of 0.023 and portfolio's total volatility of 0.029. On the other hand, the optimal portfolio of method 2 achieves a Sharpe ratio of 2.667 with a monthly return of 0.023 and portfolio's total volatility of 0.008.

6.6 Updated optimal portfolio of 2012-2016 period given 2017 data set:

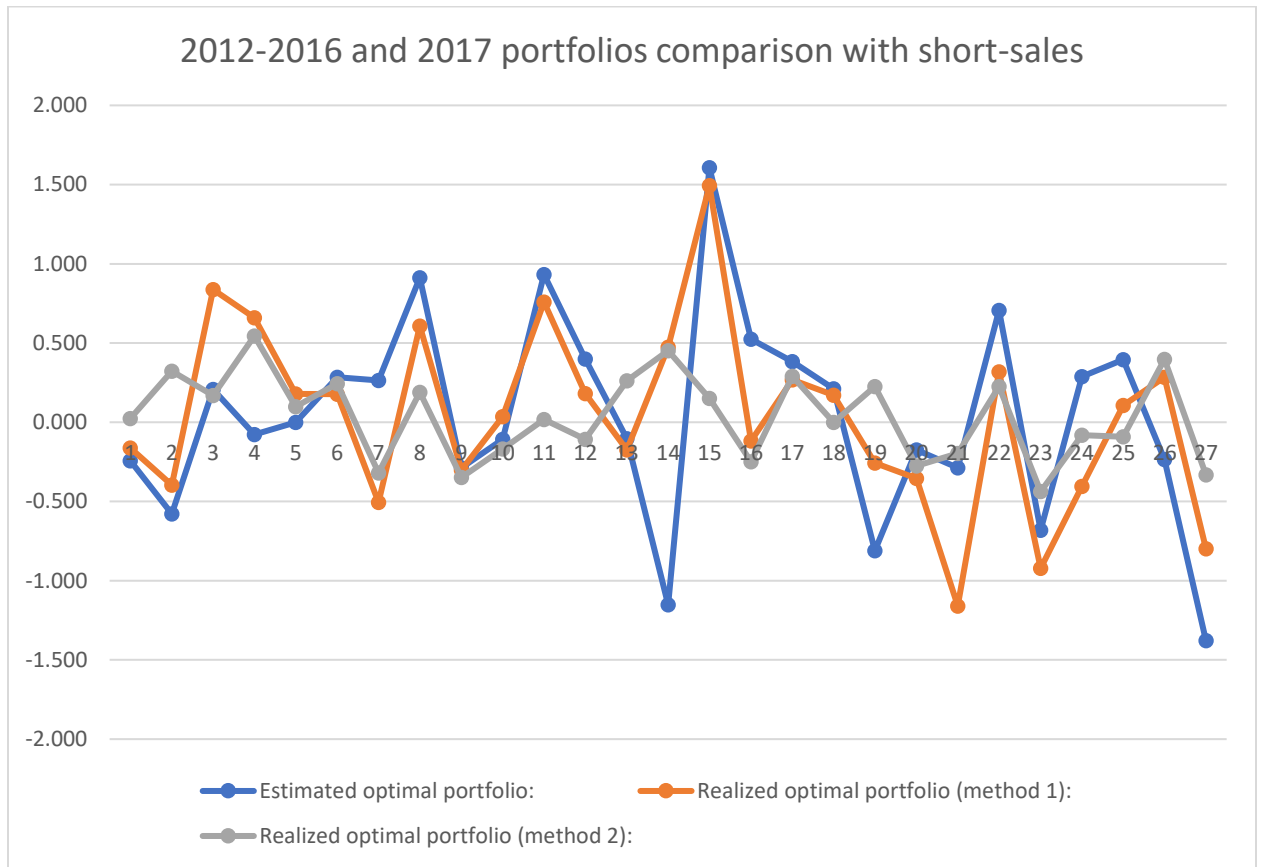


Figure 14. 2012-2016 and 2017 optimal portfolios with short-sales

The risk-free rate used in this period is the 10-year Treasury Rates in 2017 divided by 12.

The optimal portfolio of method 1 realizes a Sharpe ratio of 1.772 with a monthly return of 0.147 and portfolio's total volatility of 0.082. On the other hand, the optimal portfolio of method 2 acquires a Sharpe ratio of 8.943 with a monthly return of 0.087 and portfolio's total volatility of 0.010.

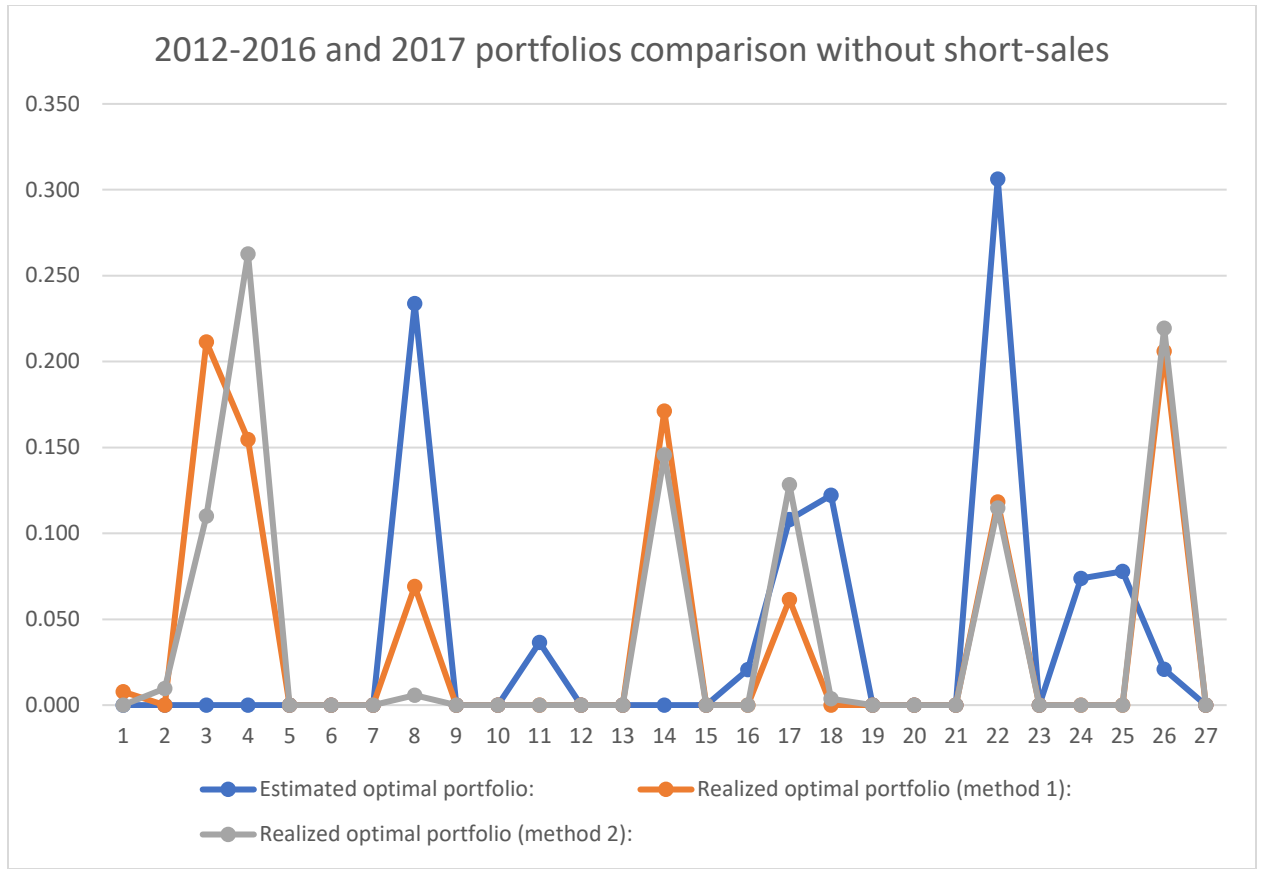


Figure 15. 2012-2016 and 2017 optimal portfolios without short-sales

In this circumstance, the optimal portfolio of method 1 obtains a Sharpe ratio of 1.175 with a monthly return of 0.038 and portfolio's total volatility of 0.031. On the other hand, the optimal portfolio of method 2 achieves a Sharpe ratio of 5.506 with a monthly return of 0.037 and portfolio's total volatility of 0.006.

Chapter 7: Optimal Portfolios Comparisons by Real Value Investment

7.1 Technique explanation and constraints:

For a better comparison among the optimal portfolios obtained by different approaches, let us assume an investment of \$1000 at the beginning of 2013 and calculate the return of approach using real returns collected in the period of 2013-2017. Under two circumstances: short-sales allowed and no short-sales, we divide each circumstance into four different scenarios using different approaches as explained in the previous chapters:

7.1.1 Passive mean-variance portfolio:

For this approach, the Sharpe optimal portfolio using the mean-variance approach with 2008-2012 data is maintained for the entire period of 2013-2017.

7.1.2 Update the optimal portfolio annually by the mean-variance approach:

For this approach, we will begin with the Sharpe optimal portfolio using the mean-variance approach with 2008-2012 data for the year of 2013. After that, for every year, the optimal portfolio is updated using the most recent five-year period data by the mean-variance approach. For example, at the beginning of 2014, the portfolio is updated given the data in the five-year period of 2009-2013.

7.1.3 Update the optimal portfolio annually by the Bayesian approach method 1:

We will still begin with the Sharpe optimal portfolio using the mean-variance approach with 2008-2012 data, since this period is the first five-year period considered for this

research, and it is only sufficient to form the prior distribution of the portfolio.²⁹ After that, for every year, we update the optimal portfolio using the Bayesian approach with method 1 as explained in chapter 6. For example, at the beginning of 2014, the portfolio is updated with the prior distribution is formed using the 2008-2012 data and the likelihood function formed by the returns collected in the year of 2013.

7.1.4 Update the optimal portfolio annually by the Bayesian approach method 2:

For this approach, the technique is similar to updating the portfolio by the Bayesian approach method 1. However, we replace method 1 by method 2.³⁰

²⁹ Recall that we need both the prior distribution of the stock returns and the likelihood function for the unknown parameters in order to form a posterior distribution using the Bayesian approach. The likelihood function is formed using the nearest year returns, which we do not have for the first five-year period of 2008-2012.

³⁰ Recall from chapter 6 that in method 1, we form a new covariance matrix annually using the most recent five-year period. In method 2, we form a new covariance matrix by using the posterior variances and holding the correlation matrix constant.

7.2 Real value investments under short-sales allowed circumstance:

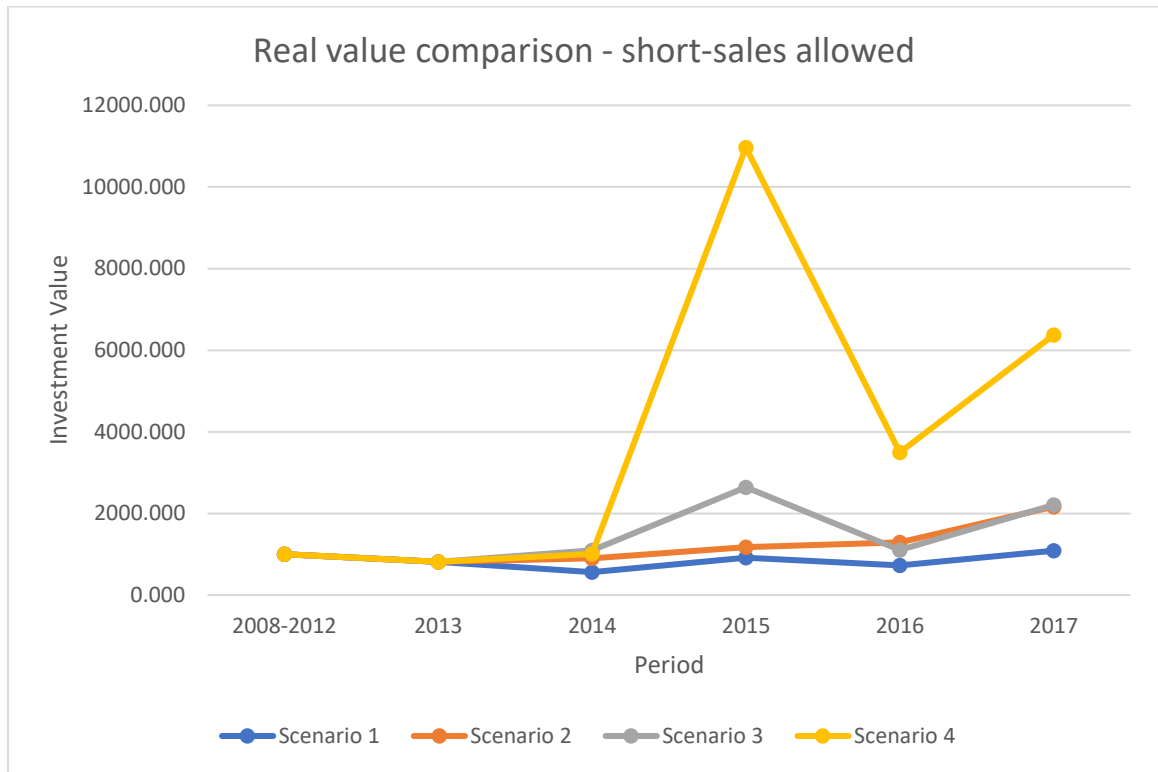


Figure 16. Real investment comparison under short-sales allowed circumstance

As explained in the above section, scenario 1 corresponds to the passive mean-variance portfolio, scenario 2 corresponds to updating the portfolio annually by the mean-variance approach, scenario 3 corresponds to updating the portfolio annually by the Bayesian approach method 1 and scenario 4 corresponds to the Bayesian approach method 2. As we observe in the figure, both the investments using the Bayesian approach method 1 and 2 did relatively better than the mean-variance approach, especially method 2. By investing \$1000 at the beginning of 2013, we will obtain \$1087 in scenario 1, \$2166 in scenario 2, \$2206 in scenario 3 and \$6376 in scenario 4. As we see in the figure, most of the methods perform poorly in the year of 2016. This could be explained by multiple unpredictable events during this year, such as Brexit, or the election of president Donald

Trump.³¹ Such events could change the covariance matrix of the 27 stocks significantly during the year, which could lead to poor performances by the portfolios. Nevertheless, the Bayesian approach performed better than the mean-variance approach since it captures and put more weight into the most recent information of the portfolio.

7.3 Real value investments under no short-sales circumstance:

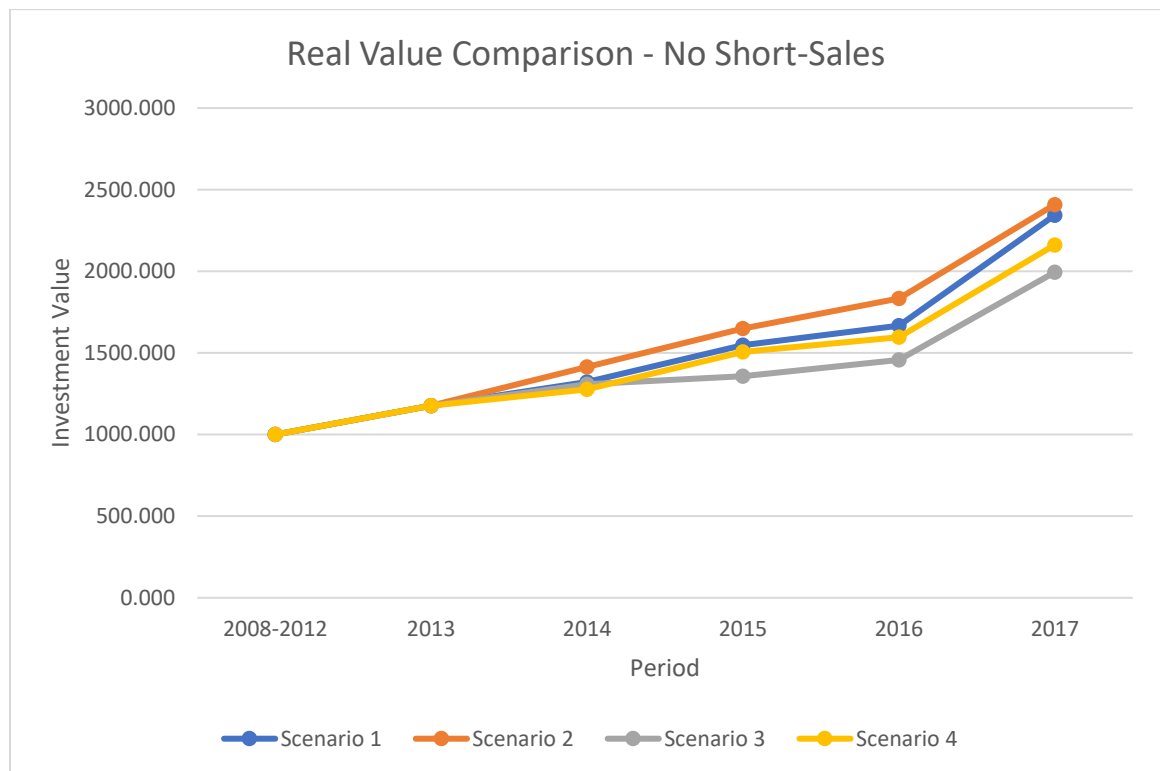


Figure 17. Real investment comparison under no short-sales circumstance

Under the constraint of no short-sales, however, both the Bayesian methods performed worse than the mean-variance approach. By investing \$1000 at the beginning of 2013, we will obtain \$2344 in scenario 1, \$2409 in scenario 2, \$1994 in scenario 3 and \$2161 in scenario 4. The constraint of no short-sales could limit the potential error of the mean-

³¹ Egan (2016)

variance approach to have short positions in variable stocks. Notice that the Dow Jones Index increases for most of the year 2013-2017, specifically, the Dow Jones Index annual changes for this period are:³²

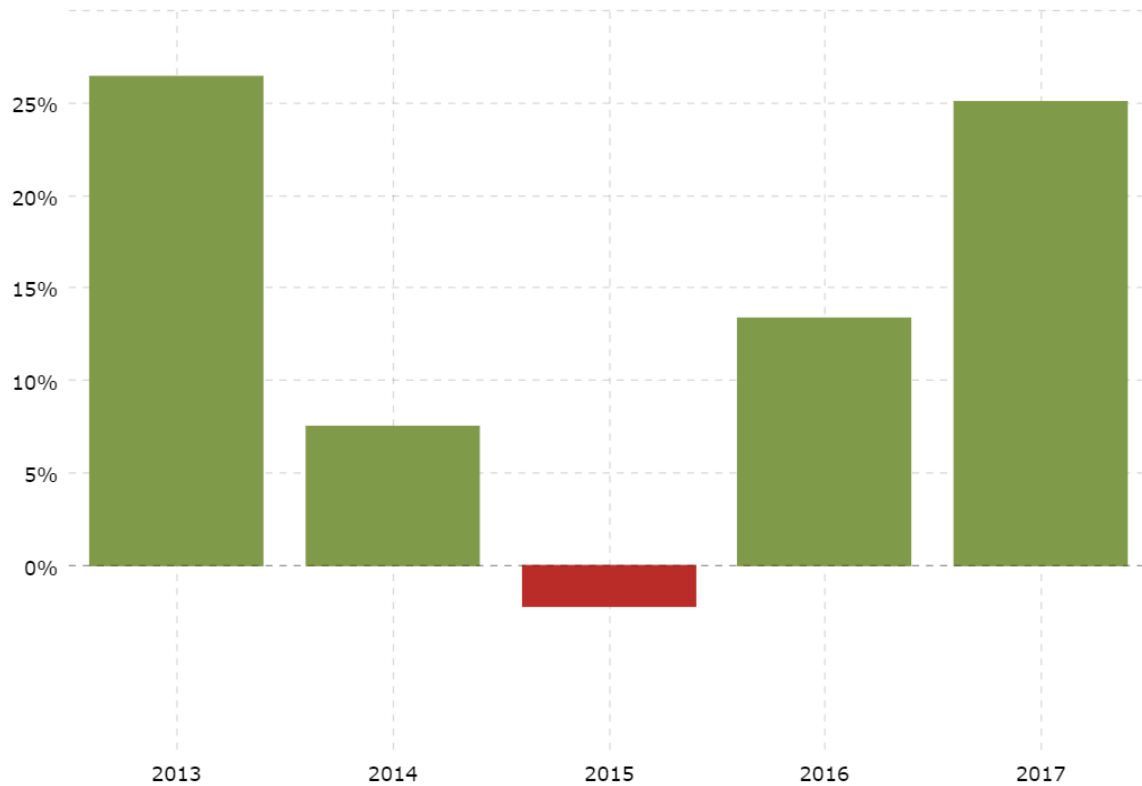


Figure 18. Dow Jones Industrial Average annual changes in the period of 2013-2017

Hence, the condition of no short-sales eliminates the probability that the mean-variance approach loses by short-selling assets. Also, we notice that 2015 is the only year that the Dow Jones Index decreases. This could explain for the performances of the portfolios in the previous circumstance, where the Bayesian portfolios performed significantly better in 2015. The Bayesian portfolios may have captured more important

³² MacroTrends (2019)

information for the 2015 portfolio and hence performed better than the mean-variance approach.

Chapter 8: Limitations and Assumptions Using the Bayesian Approach

An important assumption of the Bayesian approach application in this paper is the normalization of stock returns. In fact, the monthly stock returns distribution of 27 considered Dow Jones companies is shown in the figure below:

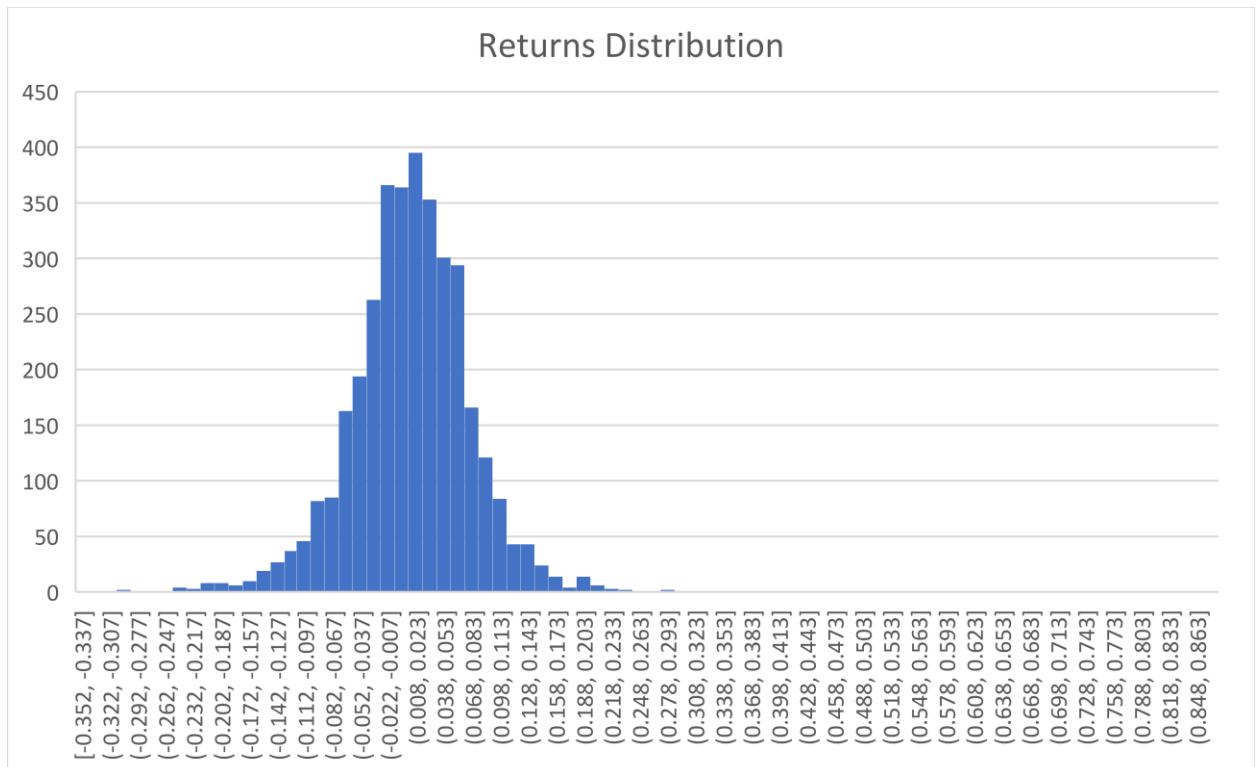


Figure 19. Dow Jones companies monthly stock returns distribution in the period of 2008-2018

Based on this figure, we can observe that the stock return distribution is relatively close to the normal distribution. There are variable statistics and normality tests for this comparison. However, this topic is not within the scope of this paper, so we will base on the visuality method for the normality of a distribution. Based on the normality

assumption, we find an alternative for the data of stock returns used in this research, which is the logarithm of the same returns data. The following figure show the distribution of the log returns of 27 Dow Jones companies in the period of 2008-2018, which is also close to normal distribution by visually:

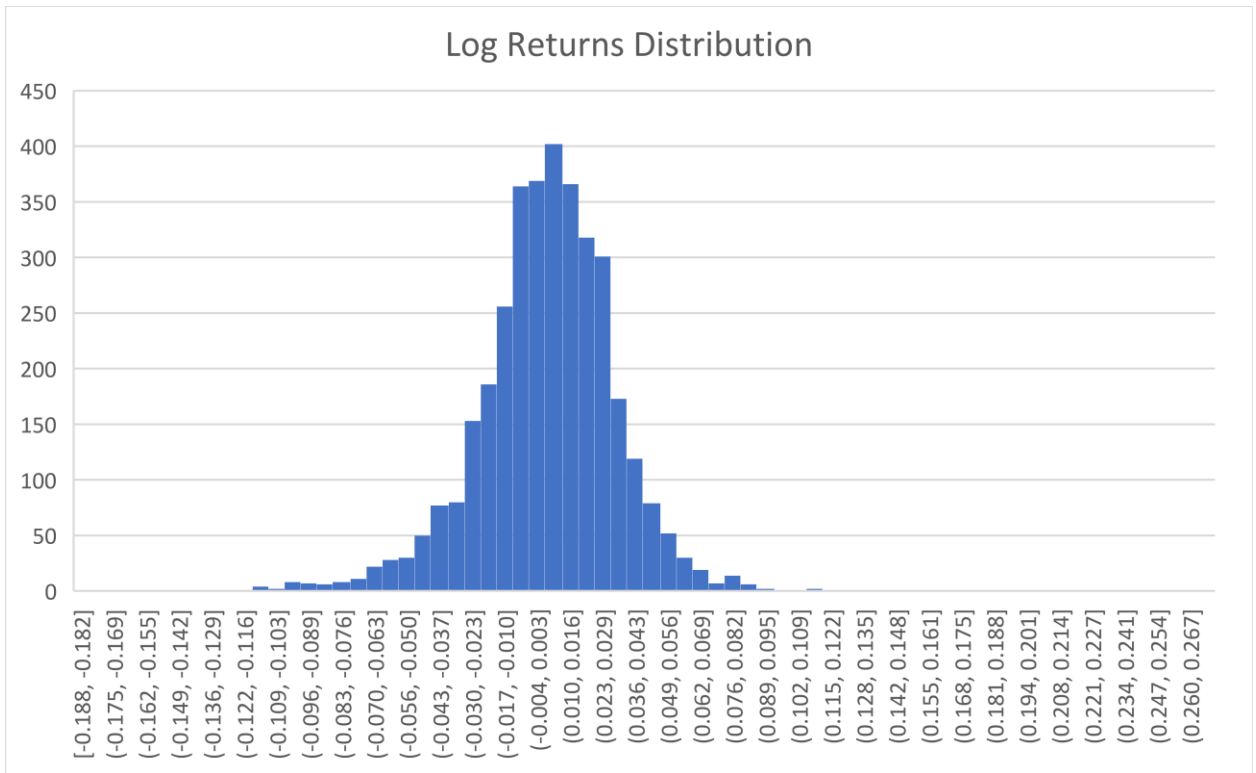


Figure 20. Dow Jones companies log monthly stock returns distribution in the period of 2008-2018

By updating the portfolio annually, we assume that the mean, variance, and the co-variance matrix of the stocks forming the portfolio do not change significantly during the year. This has been shown to be a limitation in the years when there are unpredictable major events occurring.

Additionally, we assume that there is no trading cost, or friction while trading and updating the portfolios. This assumption could lead to the annually updating portfolio methods being overestimated. However, notice that the Dow Jones companies are large, publicly owned companies in the U.S. Hence, the stocks of these companies are relatively liquid, which makes the trading cost less significant.

Furthermore, we assume that the mean and variance of a certain stock captures all the information of the stock returns' distribution, which is not the case. As explained in chapter 1, there are many measures of risk. The mean and variance are only the first and second central moments of a random variable and are not sufficient to evaluate the entire distribution of the variable. However, the mean and variance do capture the most important information. Therefore, in order to avoid complexity in calculating higher moments of the variable's distribution, the mean and variance are the only parameters considered in forming the portfolios.

In method 2 of the Bayesian approach, we assume that the correlation matrix of 27 Dow Jones stocks does not change for the entire period of 2013-2017, which may not be the case. However, correlation between two certain stocks usually depend on the industry, product characteristics, the economic policies, etc., which do not often change significantly in a five-year period. Indeed, in the previous chapter, the Bayesian approach method 2 portfolio was shown to perform much better than the other approaches under short-sales allowed circumstance. Hence, another clear assumption from the data evaluation is that under the first circumstance, all the stocks of the 27 Dow Jones companies can be shorted, which is not always the case, especially for the hedge funds with strict regulations.

A technical limitation of this research is the use of Solver tool in Excel. As mentioned in the previous chapters, the Solver tool only gives the result for the local maxima or minima of the function considered in the problem. Therefore, in each time we look for an optimal portfolio in a given problem, we need to run the Solver multiple times in order to verify the outcome. Additionally, since Solver looks for local maxima and minima, the results are at times unreasonable. Hence, the constraint that the weight of each stock must be in the interval $[-10, 10]$ was added to the optimization problems.

Chapter 9: Comparison Between the Mean-Variance Approach and the Bayesian Approach

As we have known, investors seek lower risk and higher return. Here we assume that all investors are rational and risk averse. Also, for the mean-variance approach, we assume that the expected returns, variances and covariances of all assets are known by all investors. In fact, when solving the mean-variance problem for N candidate stocks, a total of $2N + \frac{N(N-1)}{2}$ values are required as inputs:³³

$$(N \text{ mean values}) + (N \text{ variances}) + \left(\frac{N^2 - N}{2} \text{ covariances}\right)$$

On the other hand, for the Bayesian approach, we need the prior distribution of the stock returns and an updated data set. In this paper they are also derived from the returns of all assets, which are known by all investors. Nevertheless, in practice, the prior distribution could be estimated to apply the Bayesian approach. According to Rachev, Hsu, Bagasheva and Fabozzi, the priors could be categorized into two cases: informative and uninformative. Recall that the formula for the posterior variance when applying the Bayesian approach to normal distribution is:

$$\frac{\sigma_0^2 \sigma_1^2}{\sigma_1^2 + n \sigma_0^2} = \frac{1}{\frac{n}{\sigma_1^2} + \frac{1}{\sigma_0^2}}$$

The quantities $\frac{1}{\sigma_1^2}$ and $\frac{1}{\sigma_0^2}$ have they own self-explanatory names: data precision and prior precision. As we notice, in the uninformative case, i.e. not a lot of information is

³³ Kim et al. (2016), p. 17

known about the prior distribution, σ_0 will be very high because of the uncertainty. Hence the prior precision $\frac{1}{\sigma_0^2}$ is negligibly small and the posterior distribution parameters will be more concentrated on the updated data. Hence, by using the Bayesian approach, we need significantly less information than the mean-variance approach.

Another limitation of the mean-variance approach is that the Sharpe optimal portfolios change significantly each year. In other words, each time we update the portfolio after one year of data, the true efficient frontier differs greatly from the estimated efficient frontier.³⁴ Recall that the mean-variance optimal portfolio is obtained by solving the following optimization problem:

$$P = \underset{\{w: \sum_1^n w_i = 1 \ \& \ \mu_p = \mu\}}{\operatorname{argmin}} \quad W \Sigma W^T$$

By solving this problem with multiple target expected returns μ , we produce a curve with the minimum variances at each level of μ . However, because at each level of risk, or variance, we prefer the higher expected return. Therefore only the upper half of the curve is efficient, which is known as the efficient frontier, as demonstrated in an example below:³⁵

³⁴ Haugh (2016)

³⁵ Chonghui (2012)

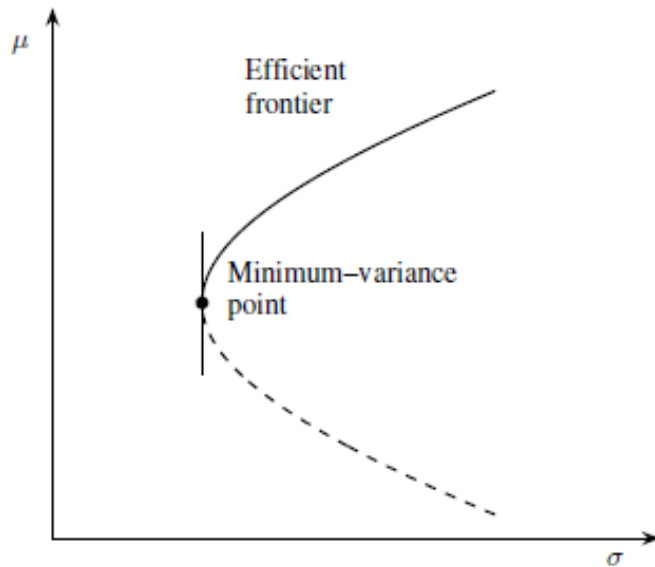


Figure 21. An example of the efficient frontier

In fact, in this paper, we solve the optimization problem on the Sharpe ratio of the portfolio, so we do not consider the efficient frontier of the mean-variance approach. Moreover, we still apply the mean-variance optimization after getting the posterior distribution using the Bayesian method. Hence, this limitation also applies to the Bayesian approach. Indeed, the purpose of the Bayesian approach is to obtain a more accurate estimation of the means and variances of the considered stocks. Afterwards, we still lack an approach to form an optimal portfolio with these estimated parameters. By using the mean-variance approach for the posterior means and variances, the portfolios are still volatile to the mean-variance limitations, such as the sensitivity of the efficient frontiers.

Another limitation applies to the mean-variance approach is the sensitivity of the portfolio weights subject to small changes in expected returns and covariance matrices.³⁶

³⁶ Haugh (2016)

This limitation could also apply to the Bayesian methods we used in this paper. However, by updating the parameters of the stocks by putting more weight to the most recent data set, the Sharpe optimal portfolios achieved by the Bayesian methods were shown to perform relatively better than the mean-variance portfolios because of more accurate estimation of the means and variances.

In conclusion, the Bayesian approach did not only help us achieve more accurate estimation for the parameters of the stocks but also lessened the amount of information we need to form the optimal portfolio. However, by applying the mean-variance approach to the posterior distribution, we are still subject to the limitations of the classical mean-variance analysis. Given the assumptions that we discussed in the previous chapter, we cannot reach the conclusion that the Bayesian portfolios will perform better than the optimal portfolios obtained by the mean-variance approach. Nevertheless, the benefit of achieving a better information regarding the stocks parameters is undebatable.

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