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BASES IN SPACES OF REGULAR MULTILINEAR OPERATORS AND
HOMOGENEOUS POLYNOMIALS ON BANACH LATTICES

A Dissertation
presented in partial fulfillment of requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
The University of Mississippi

by

KHAZHAK VARAZDAT NAVOYAN

May 2018

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ABSTRACT

For Banach lattices E_1, \dots, E_m and F with 1-unconditional bases, we show that the monomial sequence forms a 1-unconditional basis of $\mathcal{L}^r(E_1, \dots, E_m; F)$, the Banach lattice of all regular m -linear operators from $E_1 \times \dots \times E_m$ to F , if and only if each basis of E_1, \dots, E_m is shrinking and every positive m -linear operator from $E_1 \times \dots \times E_m$ to F is weakly sequentially continuous. As a consequence, we obtain necessary and sufficient conditions for which the m -fold Fremlin projective tensor product $E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$ (resp. the m -fold positive injective tensor product $E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m$) has a shrinking basis or a boundedly complete basis.

For Banach lattices E and F with 1-unconditional bases, we show that the monomial sequence forms a 1-unconditional basis of $\mathcal{P}^r({}^m E; F)$, the Banach lattice of all regular m -homogeneous polynomials from E to F , if and only if E has a shrinking basis and every positive m -homogeneous polynomial from E to F is weakly sequentially continuous. As a consequence, we obtain necessary and sufficient conditions for which the m -fold symmetric positive projective tensor product $\hat{\otimes}_{m,s,|\pi|} E$ (resp. the m -fold symmetric positive injective tensor product $\check{\otimes}_{m,s,|\epsilon|} E$) has a shrinking basis or a boundedly complete basis.

For a vector lattice E and $n \in \mathbb{N}$, let $\bar{\otimes}_{n,s} E$ denote the n -fold Fremlin vector lattice symmetric tensor product of E . For $m, n \in \mathbb{N}$ with $m > n$, we prove that (i) if $\bar{\otimes}_{m,s} E$ is uniformly complete then $\bar{\otimes}_{n,s} E$ is positively isomorphic to a complemented subspace of $\bar{\otimes}_{m,s} E$, and (ii) if there exists $\phi \in E_+^\sim$ such that $\ker(\phi)$ is a projection band in E then $\bar{\otimes}_{n,s} E$ is lattice isomorphic to a projection band of $\bar{\otimes}_{m,s} E$. We also obtain analogous results for the n -fold Fremlin Banach lattice symmetric tensor product $\hat{\otimes}_{n,s,|\pi|} E$ of E where E is a Banach lattice.

DEDICATION

To the memory and 100th anniversary of my Grandfather, Khazhak Navoyan, who was a well-known scientist in Soviet Armenia.

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1 INTRODUCTION

Grothendieck, in his “Résumé” [30], exhibited the importance of the use of tensor products in the theory of Banach spaces. Tensor products had appeared in functional analysis since the late thirties, but it was Grothendieck, who realized the local nature of many properties of tensor products, and this allowed him to establish a very useful theory of duality. Another paper of Grothendieck, which had a considerable influence on the development of Banach space theory, was “Produits tensoriels topologiques et espaces nucléaires” [31]. These two papers contained a general theory of tensor norms on tensor products of Banach spaces, introduced the duality theory of these tensor products, and studied the linearization of multilinear operators through the tensor products. Grothendieck’s Résumé and Memoir were a source of many new investigations in functional analysis.

In 1980, Ryan in [42] introduced symmetric tensor products of Banach spaces, as a tool for the study of polynomials on Banach spaces and holomorphic mappings, and then characterized the linearization of homogeneous polynomials through the symmetric tensor products. In his survey [22], Floret presented the algebraic basics of symmetric tensor products, together with a thorough account of fundamental metric results for the two extreme tensor norms: the symmetric projective tensor norm $\|\cdot\|_{\pi,s}$, and the symmetric injective tensor norm $\|\cdot\|_{\epsilon,s}$. Although the theory of symmetric tensor products steadily evolved in the last decades, there are still some open questions on general symmetric tensor norms.

Dineen’s book [21], and Mujica’s book [38] give an idea of the impact of the symmetric tensor products and homogeneous polynomials on the theory of holomorphic functions.

Recently, Banach lattice theory has influenced the theory of holomorphic functions on infinite dimensional spaces. In [5], Benyamini, Lassalle and Llavona, generalizing the results of Sundaresan [46], showed that Banach space valued orthogonally additive polynomials on a Banach lattice can be linearized using a concavification of the Banach lattice. They proved that the space of n -homogeneous orthogonally additive polynomials on a Banach lattice with values in a Banach space, is isometrically isomorphic with the Banach space of all bounded linear maps on the n -concavification of the Banach space. A homogeneous polynomial P on a vector lattice, is called orthogonally additive if $P(x + y) = P(x) + P(y)$ for disjoint x and y .

In [44], Greu and Ryan studied multilinear forms on Banach lattices with an unconditional basis, and the linearization of positive multilinear operators on Banach lattices was studied by Fremlin in [23, 24]. Later, Schep [45], Grobler and Labuschagne [29], developed further the theory of positive tensor products and positive multilinear operators on Banach lattices.

In [11], Bu, Buskes and Li studied regular multilinear operators and orthogonally additive and regular polynomials on Banach and vector lattices, using Fremlin tensor product of vector lattices [23] and Fremlin's projective tensor product of Banach lattices [24]. In 2012, Bu and Buskes [9] studied n -homogeneous orthogonally additive polynomials via the quotient of the positive tensor products of Banach lattices. They extended Sundaresan's results in [46] in 1989, and Benyamini, Lassalle and Llavona's results in [5] in 2006.

The existence of bases in the space of multilinear operators and the space of homogeneous polynomials on Banach spaces was studied in recent years (see, e.g. [1, 14, 20, 19, 25, 40]). For instance, let $\mathcal{L}(E_1, \dots, E_m; F)$ denote the space of all continuous m -linear operators from $E_1 \times \dots \times E_m$ to F , where E_1, \dots, E_m , and F are Banach spaces with bases. By using the square order in \mathbb{N}^m , the monomial sequence is well defined and forms a basis in $\mathcal{L}(E_1, \dots, E_m; F)$ under some conditions (see, e.g., [20, 19, 25, 42]). However, this basis may not be an unconditional basis in $\mathcal{L}(E_1, \dots, E_m; F)$ even though each basis of E_1, \dots, E_m ,

and F is an unconditional basis (see, e.g.,[17]). If each basis of E_1, \dots, E_m , and F is a 1-unconditional basis, then they are Banach lattices with the order defined coordinatewise. From the positivity perspective, we have positive multilinear operators from $E_1 \times \dots \times E_m$ to F (which take positive elements to positive elements) and regular multilinear operators from $E_1 \times \dots \times E_m$ to F (which are differences of two positive multilinear operators)(see, e.g., [44]). Let $\mathcal{L}^r(E_1, \dots, E_m; F)$ denote the space of all regular m -linear operators from $E_1 \times \dots \times E_m$ to F . Then it is a Banach lattice with its lattice norm $\|T\|_r = \||T|\|$ for each $T \in \mathcal{L}^r(E_1, \dots, E_m; F)$ (see, e.g.,[9, 11]). In Chapter 3, we study the existence of an unconditional basis in $\mathcal{L}^r(E_1, \dots, E_m; F)$.

In particular, in Chapter 3 we show that the monomial sequence is disjoint and thus forms an unconditional basic sequence in $\mathcal{L}^r(E_1, \dots, E_m; F)$. By characterizing regular multilinear operators that are weakly sequentially continuous, we obtain necessary and sufficient conditions for which the monomial sequence forms an unconditional basis of $\mathcal{L}^r(E_1, \dots, E_m; F)$. Moreover, if E_1, \dots, E_m , and F are reflexive, then we show in Section 3.5 that $\mathcal{L}^r(E_1, \dots, E_m; F)$ is reflexive if and only if the monomial sequence forms a basis of $\mathcal{L}^r(E_1, \dots, E_m; F)$.

The space of regular multilinear operators on Banach lattices is closely related to the positive projective and injective tensor products of Banach lattices. For Banach lattices E_1, \dots, E_m , let $E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$ denote the m -fold positive projective tensor product and $E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m$ denote the m -fold positive injective tensor product of E_1, \dots, E_m . If E_1, \dots, E_m have unconditional bases, then the monomial sequence forms an unconditional basis in both $E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$ and $E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m$ (see, e.g., [12, 32]). In Section 3.4, we obtain necessary and sufficient conditions for which the monomial sequence is a shrinking basis or a boundedly complete basis in both $E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$ and $E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m$.

For a positive integer n and a vector space X , Blasco [6] proved that the n -fold symmetric tensor product $\otimes_{n,s} X$ is complemented in the $(n+1)$ -fold symmetric tensor product $\otimes_{n+1,s} X$. From the positivity perspective, for a vector lattice E , $\otimes_{n,s} E$ is an ordered vector

space (not necessarily a vector lattice). Fremlin [23, 24] (also see [32, 45]) constructed the n -fold vector lattice symmetric tensor product $\bar{\otimes}_{n,s}E$ which is a vector lattice and contains $\otimes_{n,s}E$ as a linear subspace. In Chapter 4, using the mappings given by Blasco in [6], we prove that the image of $\bar{\otimes}_{n,s}E$ is a complemented subspace in $\bar{\otimes}_{n+1,s}E$ if $\bar{\otimes}_{n+1,s}E$ is uniformly complete. Moreover, we consider the image of $\bar{\otimes}_{n,s}E$ being a sublattice or a band in $\bar{\otimes}_{n+1,s}E$. We prove that if there is a positive linear functional ϕ on E such that $\ker(\phi)$ is a projection band in E , then the image of $\bar{\otimes}_{n,s}E$ is also a sublattice and a band in $\bar{\otimes}_{n+1,s}E$. As a consequence, we obtain the complementation of $\mathcal{P}^r({}^nE; F)$ in $\mathcal{P}^r({}^{n+1}E; F)$, the space of regular $(n+1)$ -homogeneous polynomials from E to F where F is a Dedekind complete vector lattice. If E is a Banach lattice, Fremlin [23, 24] (also see [9, 45]) constructed the n -fold projective symmetric tensor product $\hat{\otimes}_{n,s,|\pi|}E$ which is a Banach lattice and contains $\bar{\otimes}_{n,s}E$ as a dense sublattice. We also obtain the complementation of $\hat{\otimes}_{n,s,|\pi|}E$ in $\hat{\otimes}_{n+1,s,|\pi|}E$ in different cases.

For Banach lattices E and F with bases let $\mathcal{P}({}^mE; F)$ denote the space of all continuous m -homogeneous polynomials from E to F . With the square order in \mathbb{N}^m , the monomial sequence is well defined and forms a basis in $\mathcal{P}({}^mE; \mathbb{R})$ if E has a shrinking basis and every continuous m -homogeneous polynomial from E to \mathbb{R} is weakly sequentially continuous [40]. If E and F have 1-unconditional bases, then they are Banach lattices with coordinatewise order. $\mathcal{P}^r({}^mE; F)$, the space of all regular m -homogeneous polynomials from E to F , is a Banach lattice with its lattice norm $\|P\|_r = \| \|P\| \|$ for each regular m -homogeneous polynomial P from E to F . In Chapter 5 we study the existence of an unconditional basis in $\mathcal{P}^r({}^mE; F)$.

For the Banach lattice E , let $\hat{\otimes}_{m,s,|\pi|}E$ denote the m -fold positive symmetric projective tensor product and $\check{\otimes}_{m,s,|\epsilon|}E$ denote the m -fold positive symmetric injective tensor product of E . It is known that the completion of the space of n -symmetric tensors endowed with the (positive) projective topology, $\hat{\otimes}_{m,s,\pi}E$ ($\hat{\otimes}_{m,s,|\pi|}E$), is a predual for the space

$\mathcal{P}({}^m E; F)$ ($\mathcal{P}^r({}^m E; F)$). Considering this and the results of Chapter 3, we see that the monomial sequence is disjoint and thus forms an unconditional basic sequence in $\mathcal{P}^r({}^m E; F)$. By characterizing regular m -homogeneous polynomials that are weakly sequentially continuous, we obtain necessary and sufficient conditions for which the monomial sequence forms an unconditional basis of $\mathcal{P}^r({}^m E; F)$. Moreover, if E and F are reflexive, then we show in Section 5.3 that $\mathcal{P}^r({}^m E; F)$ is reflexive if and only if the monomial sequence forms a basis of $\mathcal{P}^r({}^m E; F)$.

If E has an unconditional basis, then the monomial sequence forms an unconditional basis in both $\hat{\otimes}_{m,s,|\pi|} E$ and $\check{\otimes}_{m,s,|\epsilon|} E$. In Section 5.3, we obtain necessary and sufficient conditions for which the monomial sequence is a shrinking basis in both $\hat{\otimes}_{m,s,|\pi|} E$ and $\check{\otimes}_{m,s,|\epsilon|} E$.

2 MULTILINEAR OPERATORS ON BANACH SPACES AND BANACH LATTICES

2.1 Preliminaries

Definition 2.1.1. A real vector space E is said to be an ordered vector space if it is equipped with an order relation \geq , that is compatible with the algebraic structure of E in the sense that it satisfies the following two axioms:

- (i) If $x \geq y$, then $x + z \geq y + z$ holds for all $z \in E$.
- (ii) If $x \geq y$, then $\alpha x \geq \alpha y$ holds for all $\alpha \geq 0$.

Definition 2.1.2. An element x in an ordered vector space E is called positive if $x \geq 0$ holds. The set of all positive elements of E will be denoted by E^+ , i.e. $E^+ = \{x \in E : x \geq 0\}$. The set E^+ of positive vectors is called the positive cone of E .

Definition 2.1.3. A Riesz space (or a vector lattice) E is an ordered vector space with the additional property that for each pair of elements $x, y \in E$ the supremum and infimum of the set $\{x, y\}$ both exist in E . We shall write $x \vee y = \sup\{x, y\}$, and $x \wedge y = \inf\{x, y\}$.

Definition 2.1.4. A net $\{x_\alpha\}$ in a Riesz space is said to be decreasing (in symbols, $x_\alpha \downarrow$) if $\alpha > \beta$ implies $x_\alpha \leq x_\beta$. The notation $x_\alpha \downarrow x$ means that $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$ both hold.

For any vector x in a Riesz space we define

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = x \vee (-x)$$

The element x^+ is called the positive part, x^- the negative part and $|x|$ the absolute value of x .

Theorem 2.1.5. Let x be an element in a Riesz space. Then we have

(i) $x = x^+ - x^-$,

(ii) $|x| = x^+ + x^-$,

(iii) $x^+ \wedge x^- = 0$.

Moreover, the decomposition in (i) is unique in the sense that if $x = y - z$ holds with $y \geq 0, z \geq 0, y \wedge z = 0$, then $y = x^+$ and $z = x^-$.

Definition 2.1.6. In a Riesz space E the elements x and y are called disjoint (denoted by $x \perp y$) if $|x| \wedge |y| = 0$.

Definition 2.1.7. A subset in a Riesz space is called order bounded if it is bounded both from above and below.

Definition 2.1.8. A Riesz space is called Dedekind complete if every nonempty subset that is bounded above has a supremum.

Definition 2.1.9. A linear operator $T : E \rightarrow F$ between two ordered vector spaces is said to be positive (in symbols $T \geq 0$) if $T(x) \geq 0$ for all $x \geq 0$.

A linear operator $T : E \rightarrow F$ between two ordered vector spaces is positive if and only if $T(E^+) \subset F^+$ (and equivalently $T(x) \geq T(y)$ if $x \geq y$).

Definition 2.1.10. A linear operator $T : E \rightarrow F$ is said to be a regular operator if it can be written as the difference of two positive operators. The space of all regular operators from E to F is denoted by $\mathcal{L}^r(E, F)$.

If F is Dedekind complete, then the ordered vector space $\mathcal{L}^r(E, F)$ is a Dedekind complete Riesz space.

Definition 2.1.11. A subset D of a vector lattice is said to be upwards directed (in symbols $D \uparrow$) if for each pair $x, y \in D$ there exists some $z \in D$ with $x \leq z$ and $y \leq z$. The symbol $D \uparrow x$ means that D is upwards directed and $x = \sup D$ holds.

Definition 2.1.12. A Riesz subspace G of a vector lattice E is said to be order dense in E if for each $0 < x \in E$, there exists $y \in G$ such that $0 < y \leq x$.

Definition 2.1.13. Let E be a vector lattice. A norm on E is called a lattice norm if $|x| \leq |y|$ in E implies $\|x\| \leq \|y\|$. A Banach lattice is a vector lattice with a complete lattice norm.

Definition 2.1.14. A Banach lattice E is called order continuous if $\|x_\alpha\| \downarrow 0$ whenever $x_\alpha \downarrow 0$.

Definition 2.1.15. Let E be a vector lattice.

(i) A subspace U of E is called a sublattice of E if $x \vee y \in U$ and $x \wedge y \in U$ for all $x, y \in U$.

(ii) A subset A of E is called solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$.

(iii) A solid subspace I of E is called an ideal or order ideal in E .

(iv) An ideal B of E is called a band if $\sup(A) \in B$ for every subset $A \subset B$ which has a supremum in E .

(v) A band B of E is called a projection band if there is a linear projection $P : E \rightarrow B$ such that $0 \leq Px \leq x$ for all $x \in E^+$. Such a projection is called a band projection.

It is clear that every ideal in E is a sublattice of E . Moreover, the intersection of any two sublattices (ideals, or bands, respectively), is a sublattice (resp. ideal and band). The

sum of two ideals is also an ideal and the sum of two projection bands is a projection band.
However, the sum of two sublattices need not to be a sublattice.

2.2 Multilinear operators on Banach spaces

Definition 2.2.1. For vector spaces X_1, \dots, X_m and Y , an operator $T : X_1 \times \dots \times X_m \rightarrow Y$ is called an m -linear operator if it is linear in each variable. That is, for fixed k with $1 \leq k \leq m$, we have

$$\begin{aligned} T(x_1, \dots, x_{k-1}, \alpha x_k + \beta y_k, x_{k+1}, \dots, x_m) \\ = \alpha T(x_1, \dots, x_{k-1}, x_k, x_{k+1}, \dots, x_m) + \beta T(x_1, \dots, x_{k-1}, y_k, x_{k+1}, \dots, x_m), \end{aligned}$$

for $x_k, y_k \in X_k$, $k = 1, \dots, m$, $\alpha, \beta \in \mathbb{R}$.

For vector spaces X_1, \dots, X_m and Y , let $L(X_1, \dots, X_m; Y)$ denote the space of all m -linear operators from $X_1 \times \dots \times X_m$ to Y .

Definition 2.2.2. For each $x^1 \in X_1, \dots, x^m \in X_m$, define a linear functional $x^1 \otimes \dots \otimes x^m$ on $L(X_1, \dots, X_m; \mathbb{R})$ by

$$(x^1 \otimes \dots \otimes x^m)(\phi) = \phi(x^1, \dots, x^m),$$

for each $\phi \in L(X_1, \dots, X_m; \mathbb{R})$.

Let $X_1 \otimes \dots \otimes X_m$ denote the linear span of all $x^1 \otimes \dots \otimes x^m$ s as x^1, \dots, x^m range over X_1, \dots, X_m respectively, that is

$$X_1 \otimes \dots \otimes X_m = \left\{ \sum_{k=1}^n x_k^1 \otimes \dots \otimes x_k^m, x_k^1 \in X_1, \dots, x_k^m \in X_m, k = 1, \dots, n \right\}.$$

$X_1 \otimes \dots \otimes X_m$ is called the algebraic tensor product of X_1, \dots, X_m . Thus each element $u \in X_1 \otimes \dots \otimes X_m$ has a representation (not unique):

$$u = \sum_{k=1}^n x_k^1 \otimes \cdots \otimes x_k^m, \quad x_k^1 \in X_1, \dots, x_k^m \in X_m, \quad k = 1, \dots, n.$$

It is worthwhile to mention the Universal Property and Uniqueness Property of tensor products as follows.

We define an m -linear operator $\otimes : X_1 \times \cdots \times X_m \rightarrow X_1 \otimes \cdots \otimes X_m$, by

$$\otimes(x_1, \dots, x_m) = x_1 \otimes \cdots \otimes x_m, \quad \forall (x_1, \dots, x_m) \in X_1 \times \cdots \times X_m.$$

Theorem 2.2.3.(Universal Property). Let X_1, \dots, X_m and Y be vector spaces. For every m -linear operator $T : X_1 \times \cdots \times X_m \rightarrow Y$, there exists a unique linear operator $T^\otimes : X_1 \otimes \cdots \otimes X_m \rightarrow Y$, such that

$$T^\otimes(x_1 \otimes \cdots \otimes x_m) = T(x_1, \dots, x_m),$$

for all $x_1 \in X_1, \dots, x_m \in X_m$. The correspondence $T \leftrightarrow T^\otimes$ is an isomorphism between the vector spaces $L(X_1, \dots, X_m; Y)$ and $L(X_1 \otimes \cdots \otimes X_m; Y)$. That is, the following diagram commutes:

$$\begin{array}{ccc} X_1 \times \cdots \times X_m & \xrightarrow{T} & Y \\ \otimes \downarrow & \nearrow T^\otimes & \\ X_1 \otimes \cdots \otimes X_m & & \end{array}$$

Theorem 2.2.4.(Uniqueness of tensor products). Let X_1, \dots, X_m be vector spaces. Suppose there exists a vector space W and an m -linear mapping $A : X_1 \times \cdots \times X_m \rightarrow W$ with the property that, for every vector space Y and every m -linear mapping $T : X_1 \times \cdots \times X_m \rightarrow Y$, there is a unique linear mapping $B : W \rightarrow Y$ such that $T = B \circ A$. Then there is an isomorphism J from $X_1 \otimes \cdots \otimes X_m$ into W such that $J(x_1 \otimes \cdots \otimes x_m) = A(x_1, \dots, x_m)$, for every $x_1 \in X_1, \dots, x_m \in X_m$.

For a Banach space X over a real field \mathbb{R} , we denote by X^* the topological dual of X and by B_X the closed unit ball of X . Let X_1, \dots, X_m and Y be Banach spaces and $\mathcal{L}(X_1, \dots, X_m; Y)$ denote the space of all continuous m -linear operators from $X_1 \times \dots \times X_m$ to Y . For each $T \in \mathcal{L}(X_1, \dots, X_m; Y)$ the norm of a continuous m -linear operator is defined as follows:

$$\|T\| = \sup \left\{ \|T(x^1, \dots, x^m)\| : x^1 \in B_{X_1}, \dots, x^m \in B_{X_m} \right\}.$$

Then $\mathcal{L}(X_1, \dots, X_m; Y)$ with this norm is a Banach space.

Next we will introduce the m -fold projective tensor product of X_1, \dots, X_m .

Definition 2.2.5. The projective tensor norm on $X_1 \otimes \dots \otimes X_m$ is defined by

$$\|u\|_\pi = \inf \left\{ \sum_{k=1}^n \|x_{1,k}\| \dots \|x_{m,k}\| : u = \sum_{k=1}^n x_{1,k} \otimes \dots \otimes x_{m,k} \right\},$$

for every $u \in X_1 \otimes \dots \otimes X_m$. The completion of $X_1 \otimes \dots \otimes X_m$ with respect to this norm is denoted by $X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_m$ and called the m -fold projective tensor product of X_1, \dots, X_m .

Theorem 2.2.6. For every $T \in \mathcal{L}(X_1, \dots, X_m; Y)$, there exists a unique $T^\otimes \in \mathcal{L}(X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_m; Y)$ such that $T = T^\otimes \circ \otimes$ and $\|T\| = \|T^\otimes\|$. That is the following diagram commutes:

$$\begin{array}{ccc} X_1 \times \dots \times X_m & \xrightarrow{T} & Y \\ \otimes \downarrow & \nearrow T^\otimes & \\ X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_m & & \end{array}$$

Moreover, $\mathcal{L}(X_1, \dots, X_m; Y)$ is isometrically isomorphic to $\mathcal{L}(X_1 \hat{\otimes}_\pi \dots \hat{\otimes}_\pi X_m; Y)$ under the mapping $T \rightarrow T^\otimes$.

Next we will introduce the m -fold injective tensor product of X_1, \dots, X_m .

Definition 2.2.7. The injective tensor norm on $X_1 \otimes \cdots \otimes X_m$ is defined by

$$\|u\|_\epsilon = \sup \left\{ \left| \sum_{k=1}^n x_1^*(x_{1,k}) \cdots x_m^*(x_{m,k}) \right| : u = \sum_{k=1}^n x_{1,k} \otimes \cdots \otimes x_{m,k}, x_i^* \in B_{X_i^*}, i = 1, \dots, m \right\},$$

for every $u \in X_1 \otimes \cdots \otimes X_m$. The completion of $X_1 \otimes \cdots \otimes X_m$ with respect to this norm is denoted by $X_1 \check{\otimes}_\epsilon \cdots \check{\otimes}_\epsilon X_m$ and called the m -fold injective tensor product of X_1, \dots, X_m .

For $u = \sum_{k=1}^n x_{1,k} \otimes \cdots \otimes x_{m,k} \in X_1 \otimes \cdots \otimes X_m$, define $T_u : X_1^* \times \cdots \times X_m^* \rightarrow \mathbb{R}$ by

$$T_u(x_1^*, \dots, x_m^*) = \sum_{k=1}^n x_1^*(x_{1,k}) \cdots x_m^*(x_{m,k}), \quad \forall x_i^* \in X_i^*, i = 1, \dots, m.$$

Then T_u is a finite-rank operator which does not depend on the representations of u and hence, $T_u \in \mathcal{L}(X_1^*, \dots, X_m^*; \mathbb{R})$ with $\|T_u\| = \|u\|_\epsilon$. Therefore, $X_1 \check{\otimes}_\epsilon \cdots \check{\otimes}_\epsilon X_m$ is a closed subspace of $\mathcal{L}(X_1^*, \dots, X_m^*; \mathbb{R})$.

For the basic knowledge about m -linear operators, m -fold projective tensor products, m -fold injective tensor products, we refer to [16, 21, 22, 38, 42, 44].

2.3 Regular multilinear operators on Banach lattices

For a Banach lattice E over a real field \mathbb{R} , we denote by E^* the topological dual of E and by B_E the closed unit ball of E . The subset $E^+ = \{x \in E : x \geq 0\}$ is the positive cone of E . We denote by B_{E^+} the intersection of B_E and E^+ . Let E_1, \dots, E_m and F be vector lattices.

Definition 2.3.1. An m -linear operator $T : E_1 \times \dots \times E_m \rightarrow F$ is called positive if $T(x_1, \dots, x_m) \in F_+$ whenever $x_1 \in E_{1+}, \dots, x_m \in E_{m+}$. T is called regular if T is the difference of two positive m -linear operators.

Let $\mathcal{L}^r(E_1, \dots, E_m; F)$ denote the space of all regular m -linear operators from $E_1 \times \dots \times E_m$ to F . If F is Dedekind complete then $\mathcal{L}^r(E_1, \dots, E_m; F)$ is a Dedekind complete vector lattice, see [35, Lemma 2.12 and Proposition 2.14].

Definition 2.3.2. Let E_1, \dots, E_m, F be vector lattices. An m -linear operator $T : E_1 \times \dots \times E_m \rightarrow F$ is called a lattice m -morphism if $|T(x_1, \dots, x_m)| = T(|x_1|, \dots, |x_m|)$. In particular, lattice 1-morphism is called lattice homomorphism.

Let $(E_1 \bar{\otimes} \dots \bar{\otimes} E_m, \otimes)$ denote the m -fold vector lattice tensor product of E_1, \dots, E_m . The following facts are known:

(i) $E_1 \bar{\otimes} \dots \bar{\otimes} E_m$ is a vector lattice and \otimes is a lattice m -morphism from $E_1 \times \dots \times E_m$ to $E_1 \bar{\otimes} \dots \bar{\otimes} E_m$ defined by

$$\otimes(x_1, \dots, x_m) = x_1 \otimes \dots \otimes x_m,$$

for every $x_1 \in E_1, \dots, x_m \in E_m$.

(ii) For any vector lattice F , there is a one to one correspondence between lattice m -morphisms $T : E_1 \times \cdots \times E_m \rightarrow F$ and lattice homomorphisms $T^\otimes : E_1 \bar{\otimes} \cdots \bar{\otimes} E_m \rightarrow F$, given by $T = T^\otimes \circ \otimes$.

(iii) For uniformly complete vector lattice F , there is a one to one correspondence between positive m -linear operators $T : E_1 \times \cdots \times E_m \rightarrow F$ and increasing linear operators $T^\otimes : E_1 \bar{\otimes} \cdots \bar{\otimes} E_m \rightarrow F$, given by $T = T^\otimes \circ \otimes$.

(iv) $E_1 \otimes \cdots \otimes E_m$ is dense in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_m$, in the sense that for any $u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_m$ there exist $x_1 \in E_1^+, \dots, x_m \in E_m^+$ such that, for every $\delta > 0$, there is $v \in E_1 \otimes \cdots \otimes E_m$ with $|u - v| \leq \delta x_1 \otimes \cdots \otimes x_m$.

(v) If $u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_m$ then there exist $x_1 \in E_1^+, \dots, x_m \in E_m^+$ such that $|u| \leq x_1 \otimes \cdots \otimes x_m$.

(vi) $E_1 \otimes \cdots \otimes E_m$ is order dense in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_m$ in the sense that if $u > 0$ is in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_m$ then there exist $x_1 > 0$ in $E_1, \dots, x_m > 0$ in E_m such that $u \geq x_1 \otimes \cdots \otimes x_m > 0$.

Let E_1, \dots, E_m and F be Banach lattices. If F is Dedekind complete then

$$\mathcal{L}^r(E_1, \dots, E_m; F)$$

is a Banach lattice with the regular operator norm $\|T\|_r = \| |T| \|$ for every

$$T \in \mathcal{L}^r(E_1, \dots, E_m; F).$$

Lemma 2.3.3. Let E_1, \dots, E_m and F be Banach lattices with F Dedekind complete. Then for any $T \in \mathcal{L}^r(E_1, \dots, E_m; F)$,

$$\|T\|_r = \inf \left\{ \|S\| : S \in \mathcal{L}^r(E_1, \dots, E_m; F)^+ \right\},$$

$$|T(x_1, \dots, x_m)| \leq S(|x_1|, \dots, |x_m|), \quad \forall x_1 \in E_1, \dots, \forall x_m \in E_m \Big\}.$$

Moreover, $\|T\| \leq \|T\|_r$.

Definition 2.3.4. For Banach lattices E_1, \dots, E_m , the positive projective tensor norm $\|\cdot\|_{|\pi|}$ on $E_1 \bar{\otimes} \dots \bar{\otimes} E_m$ is defined by

$$\|u\|_{|\pi|} = \inf \left\{ \sum_{k=1}^n \|x_{1,k}\| \dots \|x_{m,k}\| : x_{i,k} \in E_i^+, |u| \leq \sum_{k=1}^n x_{1,k} \otimes \dots \otimes x_{m,k} \right\}$$

for every $u \in E_1 \bar{\otimes} \dots \bar{\otimes} E_m$. Then $\|\cdot\|_{|\pi|}$ is a lattice norm on $u \in E_1 \bar{\otimes} \dots \bar{\otimes} E_m$ and

(vii) $E_1 \otimes \dots \otimes E_m$ is norm dense in $E_1 \bar{\otimes} \dots \bar{\otimes} E_m$, and

(viii) the cone generated by $\{x_1 \otimes \dots \otimes x_m : x_k \in E_k^+, 1 \leq k \leq m\}$ is norm dense in $(E_1 \bar{\otimes} \dots \bar{\otimes} E_m)^+$.

Let $E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$ denote the completion of $E_1 \bar{\otimes} \dots \bar{\otimes} E_m$ under the lattice norm $\|\cdot\|_{|\pi|}$. Then $E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$ is a Banach lattice, called the m -fold Fremlin projective tensor product, or m -fold positive projective tensor product of E_1, \dots, E_m .

For each m -linear operator $T : E_1 \times \dots \times E_m \rightarrow F$, let $T^\otimes : E_1 \otimes \dots \otimes E_m \rightarrow F$ denote its linearization, that is,

$$T^\otimes(x^1 \otimes \dots \otimes x^m) = T(x^1, \dots, x^m), \quad \forall x^1 \in E_1, \dots, x^m \in E_m.$$

Then under the isometry: $T \rightarrow T^\otimes$,

$$\mathcal{L}^r(E_1, \dots, E_m; F) = \mathcal{L}^r(E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m; F).$$

That is the following diagram commutes:

$$\begin{array}{ccc}
E_1 \times \cdots \times E_m & \xrightarrow{T} & F \\
\downarrow \otimes & \nearrow T^\otimes & \\
E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m & &
\end{array}$$

Proposition 2.3.5. Let E_1, \dots, E_m, F be Banach lattices such that F is Dedekind complete. Then $\mathcal{L}^r(E_1, \dots, E_m; F)$ is lattice isometric to $\mathcal{L}^r(E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m; F)$ under the mapping $T \mapsto T^\otimes$.

Let E_1, \dots, E_m be Banach lattices. For $u = \sum_{k=1}^n x_{1,k} \otimes \cdots \otimes x_{m,k} \in E_1 \otimes \cdots \otimes E_m$, define $T_u : E_1^* \times \cdots \times E_m^* \rightarrow \mathbb{R}$ by

$$T_u(x_1^*, \dots, x_m^*) = \sum_{k=1}^n x_1^*(x_{1,k}) \cdots x_m^*(x_{m,k}), \quad \forall x_i^* \in E_i^*, i = 1, \dots, m.$$

Then T_u is a finite-rank m -linear operator which does not depend on the representations of u and hence, $T_u \in \mathcal{L}^r(E_1^*, \dots, E_m^*; \mathbb{R})$.

Definition 2.3.6. Let $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$ denote the closed sublattice generated by $E_1 \otimes \cdots \otimes E_m$ in $\mathcal{L}^r(E_1^*, \dots, E_m^*; \mathbb{R})$, called the m -fold positive injective tensor product of E_1, \dots, E_m . The norm on $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$ is denoted by $\|\cdot\|_{|\epsilon|}$, that is, for every $u \in E_1 \otimes \cdots \otimes E_m$, $\|u\|_{|\epsilon|} = \|T_u\|_r$.

By Lemma 2.3.3, $\|u\|_\epsilon \leq \|u\|_{|\epsilon|}$. In particular, for positive $u \in E_1 \otimes \cdots \otimes E_m$, we have

$$\|u\|_{|\epsilon|} = \sup \left\{ \left| \sum_{k=1}^n x_1^*(x_{1,k}) \cdots x_m^*(x_{m,k}) \right| : u = \sum_{k=1}^n x_{1,k} \otimes \cdots \otimes x_{m,k}, x_i^* \in B_{E_i^{*+}}, i = 1, \dots, m \right\}.$$

We denote by $\mathcal{K}^r(E_1, \dots, E_m; F)$ the sublattice of $\mathcal{L}^r(E_1, \dots, E_m; F)$ generated by all positive compact m -linear operators from $E_1 \times \cdots \times E_m$ to F , and by $\mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$

the sublattice of $\mathcal{L}^r(E_1, \dots, E_m; F)$ generated by all positive weakly sequentially continuous m -linear operators from $E_1 \times \dots \times E_m$ to F .

Lemma 2.3.7. If each of E_1, \dots, E_m contains no copy of ℓ_1 , then

$$\mathcal{L}_{wsc}^r(E_1, \dots, E_m; F) \subseteq \mathcal{K}^r(E_1, \dots, E_m; F).$$

Proof. Take $A \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$ and $x_n^i \in B_{E_i}$ for $i = 1, \dots, m$ and $n \in \mathbb{N}$. Since E_i contains no copy of ℓ_1 , there is a subsequence $x_{n_k}^i$ such that $x_{n_k}^i$ is a weakly Cauchy in E_i for $i = 1, \dots, m$. By [3, Corollary 2.5], $A(x_{n_k}^1, \dots, x_{n_k}^m)$ is norm Cauchy in F and hence, convergent in F . Thus $A \in \mathcal{K}^r(E_1, \dots, E_m; F)$. \square

Lemma 2.3.8. Let E be a Banach lattice, and let $x \in E^+$ be such that $x \neq 0$. Then there exists $f \in E^{*+}$ with $\|f\| = 1$, and $f(x) = \|x\|$.

Proof. It follows from [36, Corollary 1.9.7] that $\exists g \in E^*$ such that $\|g\| = 1$, and $g(x) = \|x\|$. If we denote $f = |g|$, then $|f| = |g|$, and hence $\|f\| = \|g\|$. It remains to show that $f(x) = \|x\|$. On one hand, $f(x) \leq \|f\| \cdot \|x\| = \|x\|$. On the other hand, since $f = |g| \geq g$, it follows that $f(x) \geq g(x) = \|x\|$. Thus we have that $f(x) = \|x\|$. \square

Lemma 2.3.9. Let $E_i, i = 1, \dots, m$, be Banach lattices. Then each $E_i, i = 1, \dots, m$, is lattice isometric to a complemented subspace of both $E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$ and $E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m$.

Proof. Without loss of generality we show that E_1 is a complemented subspace of

$$E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$$

and

$$E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m.$$

Take $x_2 \in E_2^+, \dots, x_m \in E_m^+$ such that $\|x_2\| = \dots = \|x_m\| = 1$. Then, by Lemma 2.3.8 there exist $f_2 \in E_2^{*+}, \dots, f_m \in E_m^{*+}$ such that $\|f_2\| = \dots = \|f_m\| = 1$, and $f_2(x_2) = \dots = f_m(x_m) = 1$.

Define $J : E_1 \rightarrow E_1 \otimes \dots \otimes E_m$, by

$$J(x) = x \otimes x_2 \otimes \dots \otimes x_m, \quad \forall x \in E_1.$$

Then

$$J(|x|) = |x| \otimes x_2 \otimes \dots \otimes x_m = |x \otimes x_2 \otimes \dots \otimes x_m| = |J(x)|,$$

Thus J is a lattice homomorphism. Moreover,

$$\|J(x)\|_{|\pi|} = \|x \otimes x_2 \otimes \dots \otimes x_m\|_{|\pi|} = \|x\| \cdot \|x_2\| \cdot \dots \cdot \|x_m\| = \|x\|,$$

and

$$\|J(x)\|_{|\epsilon|} = \|x \otimes x_2 \otimes \dots \otimes x_m\|_{|\epsilon|} = \|x\| \cdot \|x_2\| \cdot \dots \cdot \|x_m\| = \|x\|.$$

Thus E_1 is lattice isometric to a closed sublattice of both

$$E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$$

and

$$E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m,$$

respectively.

Next we show that $J[E_1]$ is complemented in both

$$E_1 \hat{\otimes}_{|\pi|} \dots \hat{\otimes}_{|\pi|} E_m$$

and

$$E_1 \overset{\check{\otimes}}{|\epsilon|} \cdots \overset{\check{\otimes}}{|\epsilon|} E_m.$$

Define $P : E_1 \otimes \cdots \otimes E_m \rightarrow J[E_1]$ by

$$P(u) = \sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot x_{1,k} \otimes x_2 \otimes \cdots \otimes x_m,$$

for $\forall u = \sum_{k=1}^n x_{1,k} \otimes \cdots \otimes x_{m,k} \in E_1 \otimes \cdots \otimes E_m$. Then P is positive and

$$\begin{aligned} P^2(u) &= P^2\left(\sum_{k=1}^n x_{1,k} \otimes \cdots \otimes x_{m,k}\right) \\ &= P\left(\sum_{k=1}^n P(x_{1,k} \otimes \cdots \otimes x_{m,k})\right) \\ &= P\left(\sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot x_{1,k} \otimes x_2 \otimes \cdots \otimes x_m\right) \\ &= \sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot P(x_{1,k} \otimes x_2 \otimes \cdots \otimes x_m) \\ &= \sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot f_2(x_2) \cdots f_m(x_m) \cdot x_{1,k} \otimes x_2 \otimes \cdots \otimes x_m \\ &= \sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot x_{1,k} \otimes x_2 \otimes \cdots \otimes x_m = P(u), \end{aligned}$$

and hence P is a projection.

Moreover, take any $u \in E_1 \otimes \cdots \otimes E_m$. For $\forall \varepsilon > 0$, there exist $x_{i,k} \in E_i^+$ such that

$$|u| \leq \sum_{k=1}^n x_{1,k} \otimes \cdots \otimes x_{m,k},$$

and

$$\sum_{k=1}^n \|x_{1,k}\| \cdots \|x_{m,k}\| \leq \|u\|_{|\pi|} + \varepsilon.$$

Thus

$$\begin{aligned}
P(|u|) &\leq \sum_{k=1}^n P(x_{1,k} \otimes \cdots \otimes x_{m,k}) \\
&= \sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot x_{1,k} \otimes x_2 \otimes \cdots \otimes x_m,
\end{aligned}$$

and

$$\begin{aligned}
\|P(u)\|_{|\pi|} &= \left\| |P(u)| \right\|_{|\pi|} \\
&\leq \left\| P(|u|) \right\|_{|\pi|} \\
&\leq \left\| \sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot x_{1,k} \otimes x_2 \otimes \cdots \otimes x_m \right\|_{|\pi|} \\
&\leq \sum_{k=1}^n |f_2(x_{2,k})| \cdots |f_m(x_{m,k})| \cdot \|x_{1,k}\| \cdot \|x_2\| \cdots \|x_m\| \\
&\leq \sum_{k=1}^n \|f_2\| \cdot \|x_{2,k}\| \cdots \|f_m\| \cdot \|x_{m,k}\| \cdot \|x_{1,k}\| \\
&= \sum_{k=1}^n \|x_{1,k}\| \cdots \|x_{m,k}\| \leq \|u\|_{|\pi|} + \varepsilon
\end{aligned}$$

Hence, $\|P(u)\|_{|\pi|} \leq \|u\|_{|\pi|}$, and P is continuous in positive projective norm.

Next we prove that P is continuous in positive injective norm. We take any $u \in E_1 \otimes \cdots \otimes E_m$ and discuss following two cases.

Case 1: $u \geq 0$.

Then $P(u) \geq 0$, and since $P(u) \in J[E_1]$, we have that for $\forall \varepsilon \geq 0$, there exist $x_i^* \in B_{E_i^*}$ and a representation of $P(u)$, say $P(u) = \sum_{k=1}^n z_{1,k} \otimes x_2 \otimes \cdots \otimes x_m$, such that

$$\|P(u)\|_{|\varepsilon|} \leq \left| \sum_{k=1}^n x_1^*(z_{1,k}) x_2^*(x_2) \cdots x_m^*(x_m) \right| + \varepsilon \quad (2.3.1)$$

Since $u \in E_1 \otimes \cdots \otimes E_m$, u has a representation: $u = \sum_{k=1}^n x_{1,k} \otimes \cdots \otimes x_{m,k}$, for some $x_{1,k} \in E_1^+ \dots, x_{m,k} \in E_m^+$. Hence

$$P(u) = \sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot x_{1,k} \otimes x_2 \otimes \cdots \otimes x_m.$$

Therefore, $z_{1,k} = f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot x_{1,k}$. By (2.3.1),

$$\begin{aligned} & \|P(u)\|_{|\epsilon|} \\ & \leq \left| \sum_{k=1}^n f_2(x_{2,k}) \cdots f_m(x_{m,k}) \cdot x_1^*(x_{1,k}) x_2^*(x_2) \cdots x_m^*(x_m) \right| + \epsilon \\ & = \left| \sum_{k=1}^n g(x_{1,k}) \cdot f_2(x_{2,k}) \cdots f_m(x_{m,k}) \right| + \epsilon \\ & \leq \|u\|_{|\epsilon|} + \epsilon, \end{aligned}$$

where $g = x_2^*(x_2) \cdots x_m^*(x_m) \cdot x_1^* \in B_{E_1^{*+}}$. It follows that $\|P(u)\|_{|\epsilon|} \leq \|u\|_{|\epsilon|}$, and hence P is continuous in positive injective norm for the case $u \geq 0$.

Case 2: u is an arbitrary element of $E_1 \otimes \cdots \otimes E_m$.

Then,

$$\|P(u)\|_{|\epsilon|} = \| |P(u)| \|_{|\epsilon|} \leq \|P(|u|)\|_{|\epsilon|} \leq \| |u| \|_{|\epsilon|} = \|u\|_{|\epsilon|}.$$

Hence P is continuous in positive injective norm also. □

Lemma 2.3.10. Let E_k and F_k be Banach lattices and $T_k : E_k \rightarrow F_k$ be a positive linear operator for each $k = 1, \dots, m$. Define $T_1 \otimes \cdots \otimes T_m : E_1 \otimes \cdots \otimes E_m \rightarrow F_1 \otimes \cdots \otimes F_m$ by

$$(T_1 \otimes \cdots \otimes T_m)(u) = \sum_{k=1}^n T_1(x_k^1) \otimes \cdots \otimes T_m(x_k^m)$$

for each $u = \sum_{k=1}^n x_k^1 \otimes \cdots \otimes x_k^m \in E_1 \otimes \cdots \otimes E_m$. Then $T_1 \otimes \cdots \otimes T_m$ is also a positive linear operator which can be extended to $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$ with $F_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} F_m$ as its range and

$$\|T_1 \otimes \cdots \otimes T_m\| \leq \|T_1\| \cdots \|T_m\|.$$

Proof. Take any positive $u \in E_1 \otimes \cdots \otimes E_m$, say $u = \sum_{k=1}^n x_k^1 \otimes \cdots \otimes x_k^m$.

Then,

$$\begin{aligned} \|(T_1 \otimes \cdots \otimes T_m)(u)\|_{|\epsilon|} &= \left\| \sum_{k=1}^n T_1(x_k^1) \otimes \cdots \otimes T_m(x_k^m) \right\|_{|\epsilon|} \\ &= \sup \left\{ \left| \sum_{k=1}^n \langle T_1(x_k^1), y_1^* \rangle \cdots \langle T_m(x_k^m), y_m^* \rangle \right| : y_i^* \in B_{F_i^{*+}}, i = 1, \dots, m \right\} \\ &= \sup \left\{ \left| \sum_{k=1}^n \langle x_k^1, T_1^*(y_1^*) \rangle \cdots \langle x_k^m, T_m^*(y_m^*) \rangle \right| : y_i^* \in B_{F_i^{*+}}, i = 1, \dots, m \right\} \\ &\leq \|T_1^*\| \cdots \|T_m^*\| \\ &\quad \cdot \sup \left\{ \left| \sum_{k=1}^n x_k^1(x_k^1) \cdots x_k^m(x_k^m) \right| : x_i^* \in B_{E_i^{*+}}, i = 1, \dots, m \right\} \\ &\leq \|T_1\| \cdots \|T_m\| \cdot \|u\|_{|\epsilon|} \end{aligned}$$

Thus $T_1 \otimes \cdots \otimes T_m$ can be continuously extended to $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$ with $F_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} F_m$ as its range and $\|T_1 \otimes \cdots \otimes T_m\| \leq \|T_1\| \cdots \|T_m\|$. \square

3 BASES IN THE SPACE OF REGULAR MULTILINEAR OPERATORS

3.1 Preliminaries

Definition 3.1.1. A sequence (x_n) in a Banach space X is said to be a Schauder basis (or simply a basis) if for each $x \in X$ there exists a unique sequence (α_n) of scalars satisfying $x = \sum_{n=1}^{\infty} \alpha_n x_n$, where the convergence of the series is assumed to be in the norm as usual.

Definition 3.1.2. A sequence of linear functionals (f_n) in X^* associated with a basis (x_n) in a Banach space X and defined by $f_n(x) = f_n(\sum_{i=1}^{\infty} \alpha_i x_i) = \alpha_n$ is called the associated sequence of coefficient functionals.

Definition 3.1.3. A sequence (x_n) in a Banach space X is a (Schauder) basic sequence if it is a (Schauder) basis for the closed subspace generated by $\{x_n : n \in \mathbb{N}\}$.

Definition 3.1.4. A basis (x_n) for a Banach space X is called boundedly complete, if whenever a sequence $(\alpha_n)_{n=1}^{\infty}$ of scalars is such that

$$\sup_m \left\| \sum_{n=1}^m \alpha_n x_n \right\| < \infty,$$

then the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges.

Definition 3.1.5. A basis (x_n) for a Banach space X is called shrinking if for any linear functional ϕ on X , we have that:

$$\sup \left\{ \left| \sum_{n=k}^{\infty} \phi(x_n) f_n(x) \right| : \|x\| \leq 1 \right\} \rightarrow 0,$$

as $k \rightarrow \infty$, where (f_n) is the sequence of coordinate functionals associated to the basis (x_n) .

Theorem 3.1.6. A sequence (x_n) is a shrinking basis in a Banach space X if and only if the associated sequence of coordinate functionals (f_n) is a boundedly complete basis in X^* .

Theorem 3.1.7. Let (x_n) be a basis of a Banach space X . Then (x_n) is both shrinking and boundedly complete if and only if X is reflexive.

Definition 3.1.8. A basis (x_n) for a Banach space X is unconditional if every convergent series of the form $\sum_{n=1}^{\infty} \alpha_n x_n$ is unconditionally convergent.

Theorem 3.1.9. For an unconditional basis $(x_n) \in X$, there exists a constant C such that for every sequence of scalars $(a_n)_{n=1}^{\infty}$ and $(\varepsilon_n)_{n=1}^{\infty}$ of modulus at most 1, we have:

$$\left\| \sum_{n=1}^m \varepsilon_n a_n x_n \right\| \leq C \left\| \sum_{n=1}^m a_n x_n \right\|, \quad \forall m \in \mathbb{N}. \quad (*)$$

$K = \inf\{C : C \text{ satisfies } (*)\}$ is called the unconditional basis constant of X , and (x_n) is called a K -unconditional basis of X .

Theorem 3.1.10. Let (x_n) be a 1-unconditional basis of a Banach space X . Then $0 < \alpha_n \leq \beta_n$ implies that $\left\| \sum_{n=1}^m \alpha_n x_n \right\| \leq \left\| \sum_{n=1}^m \beta_n x_n \right\|$ for $\forall m \in \mathbb{N}$. Thus, X has order defined coordinatewise, that is

$$x = \sum_{n=1}^{\infty} \alpha_n x_n \leq y = \sum_{n=1}^{\infty} \beta_n x_n \Leftrightarrow \alpha_n \leq \beta_n, \quad \forall n \in \mathbb{N}.$$

Therefore, X is a Banach lattice.

On the other hand, if a Banach lattice X has a basis, then this basis is 1-unconditional.

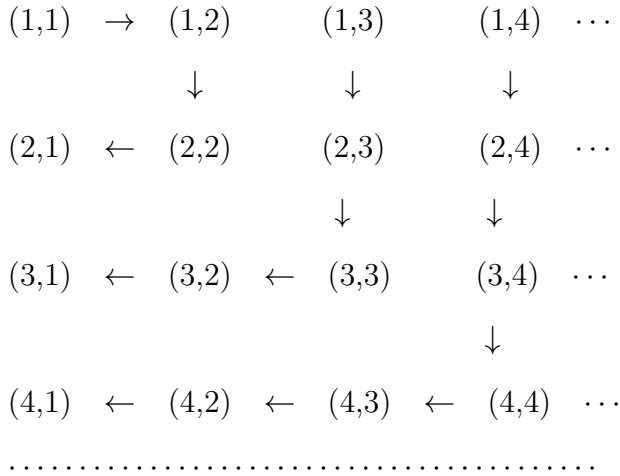
Lemma 3.1.11 Let $\{e_i : i \in \mathbb{N}\}$ be a basis of a Banach space X . A bounded subset B of X is relatively compact if and only if

$$\limsup_n \left\{ \left\| \sum_{i=n}^{\infty} e_i^*(x) e_i \right\| : x \in B \right\} = 0. \quad (3.1)$$

3.2 Bases in the space of vector valued regular multilinear operators

We consider the square order in \mathbb{N}^m . The square order in \mathbb{N}^m for $m = 2$ is (see, e.g., [26]): $(1, 1) \leq (1, 2) \leq (2, 2) \leq (2, 1) \leq (1, 3) \leq (2, 3) \leq (3, 3) \leq (3, 2) \leq (3, 1) \leq \dots$; the square order in \mathbb{N}^m for $m > 2$ was described inductively by Ryan in [42] as follows: given the order $s_1 \leq s_2 \leq s_3 \leq \dots$ of \mathbb{N}^{m-1} , the order in \mathbb{N}^m is: $(s_1, 1) \leq (s_1, 2) \leq (s_2, 2) \leq (s_2, 1) \leq (s_1, 3) \leq (s_2, 3) \leq (s_3, 3) \leq (s_3, 2) \leq (s_3, 1) \leq \dots$.

In a tabular form the square order for $m = 2$ on \mathbb{N}^2 may be presented in the following way.



Let E_1, \dots, E_m , and F be Banach lattices with 1-unconditional bases, and let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditional basis of E_k for $k = 1, \dots, m$ and $\{f_j : j \in \mathbb{N}\}$ be a 1-unconditional basis of F . With the square order in \mathbb{N}^m ,

$$\left\{ e_{i_1}^1 \otimes \dots \otimes e_{i_m}^m : (i_1, \dots, i_m) \in \mathbb{N}^m \right\}$$

forms a sequence in $E_1 \otimes \dots \otimes E_m$. Moreover, we have the following proposition (see, e.g., [12, 32]).

Proposition 3.2.1. Let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditional basis of E_k for $k = 1, \dots, m$. Then the sequence

$$\{e_{i_1}^1 \otimes \cdots \otimes e_{i_m}^m : (i_1, \dots, i_m) \in \mathbb{N}^m\} \quad (b1)$$

forms a 1-unconditional basis of both $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ and $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$.

We also consider the square order in \mathbb{N}^{m+1} . For each $(i_1, \dots, i_m, j) \in \mathbb{N}^{m+1}$, define a monomial

$$e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j : E_1 \times \cdots \times E_m \rightarrow F$$

by

$$\left(e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j \right) (x^1, \dots, x^m) = e_{i_1}^{1*}(x^1) \cdots e_{i_m}^{m*}(x^m) f_j$$

for each $x^1 \in E_1, \dots, x^m \in E_m$. Then $e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j$ is a positive m -linear operator.

Moreover, we have

Theorem 3.2.2. Let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditional basis of E_k for $k = 1, \dots, m$ and $\{f_j : j \in \mathbb{N}\}$ be a 1-unconditional basis of F . Then the monomial sequence

$$\left\{ e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j : (i_1, \dots, i_m, j) \in \mathbb{N}^{m+1} \right\} \quad (b2)$$

is a disjoint sequence in $\mathcal{L}^r(E_1, \dots, E_m; F)$ and hence, forms a 1-unconditional basic sequence in $\mathcal{L}^r(E_1, \dots, E_m; F)$.

Proof. To show that the monomial sequence (b2) is disjoint, we need to show that if

$$(i_1, \dots, i_m, j) \neq (k_1, \dots, k_m, k),$$

then

$$(e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j) \perp (e_{k_1}^{1*} \otimes \cdots \otimes e_{k_m}^{m*} \otimes f_k).$$

To do this, it suffices to show that for every $x^1 \in E_1^+, \dots, x^m \in E_m^+$,

$$\left((e_{i_1}^{1*} \otimes \dots \otimes e_{i_m}^{m*} \otimes f_j) \wedge (e_{k_1}^{1*} \otimes \dots \otimes e_{k_m}^{m*} \otimes f_k) \right) (x^1, \dots, x^m) = 0. \quad (3.2)$$

Let

$$\alpha_1 = e_{k_1}^{1*}(x^1), \alpha_2 = e_{k_2}^{2*}(x^2), \dots, \alpha_m = e_{k_m}^{m*}(x^m),$$

$$u_{1,1} = \alpha_1 e_{k_1}^1, u_{2,1} = \alpha_2 e_{k_2}^2, \dots, u_{m,1} = \alpha_m e_{k_m}^m,$$

and

$$u_{1,2} = x^1 - u_{1,1}, u_{2,2} = x^2 - u_{2,1}, \dots, u_{m,2} = x^m - u_{m,1}.$$

Then $(u_{1,1}, u_{1,2}), \dots, (u_{m,1}, u_{m,2})$ are partitions of x^1, \dots, x^m .

Recall that for a positive element x in a vector lattice, a partition of x is a finite sequence $(\omega_i)_{i=1}^p$ of positive elements such that $\omega_1 + \dots + \omega_p = x$. Let Πx denote the set of all partitions of x . Then Πx is a directed set. By [11, Proposition 2.1],

$$\begin{aligned} & \left((e_{i_1}^{1*} \otimes \dots \otimes e_{i_m}^{m*} \otimes f_j) \wedge (e_{k_1}^{1*} \otimes \dots \otimes e_{k_m}^{m*} \otimes f_k) \right) (x^1, \dots, x^m) \\ &= \inf \left\{ \sum_{j_1, \dots, j_m=1}^{p_1, \dots, p_m} \left(e_{i_1}^{1*} \otimes \dots \otimes e_{i_m}^{m*} \otimes f_j \right) (\omega_{1,j_1}, \dots, \omega_{m,j_m}) \right. \\ & \wedge \left(e_{k_1}^{1*} \otimes \dots \otimes e_{k_m}^{m*} \otimes f_k \right) (\omega_{1,j_1}, \dots, \omega_{m,j_m}) : (\omega_{1,j_1})_{j_1=1}^{p_1} \in \Pi x^1, \dots, (\omega_{m,j_m})_{j_m=1}^{p_m} \in \Pi x^m \\ & \leq \sum_{j_1, \dots, j_m=1}^2 \left(e_{i_1}^{1*} \otimes \dots \otimes e_{i_m}^{m*} \otimes f_j \right) (u_{1,j_1}, \dots, u_{m,j_m}) \wedge \left(e_{k_1}^{1*} \otimes \dots \otimes e_{k_m}^{m*} \otimes f_k \right) (u_{1,j_1}, \dots, u_{m,j_m}) \\ & = \sum_{j_1, \dots, j_m=1}^2 \left(e_{i_1}^{1*}(u_{1,j_1}) \dots e_{i_m}^{m*}(u_{m,j_m}) \cdot f_j \right) \wedge \left(e_{k_1}^{1*}(u_{1,j_1}) \dots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \right). \end{aligned} \quad (3.3)$$

Case 1: $(i_1, \dots, i_m) = (k_1, \dots, k_m)$ but $j \neq k$. In this case, the general term of (3.3)

is

$$\begin{aligned} & \left(e_{i_1}^{1*}(u_{1,j_1}) \cdots e_{i_m}^{m*}(u_{m,j_m}) \cdot f_j \right) \wedge \left(e_{k_1}^{1*}(u_{1,j_1}) \cdots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \right) \\ &= e_{i_1}^{1*}(u_{1,j_1}) \cdots e_{i_m}^{m*}(u_{m,j_m}) \cdot (f_j \wedge f_k) = 0 \end{aligned}$$

since $\{f_j : j \in \mathbb{N}\}$ is a basis of F . Thus (3.2) holds.

Case 2: $(i_1, \dots, i_m) \neq (k_1, \dots, k_m)$. In this case, if all j'_t s, $1 \leq t \leq m$, are 1, then the general term of (3.3) is

$$\begin{aligned} & \left(e_{i_1}^{1*}(u_{1,j_1}) \cdots e_{i_m}^{m*}(u_{m,j_m}) \cdot f_j \right) \wedge \left(e_{k_1}^{1*}(u_{1,j_1}) \cdots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \right) \\ &\leq e_{i_1}^{1*}(u_{1,j_1}) \cdots e_{i_m}^{m*}(u_{m,j_m}) \cdot f_j = e_{i_1}^{1*}(u_{1,1}) \cdots e_{i_m}^{m*}(u_{m,1}) \cdot f_j \\ &= e_{i_1}^{1*}(\alpha_1 e_{k_1}^1) \cdots e_{i_m}^{m*}(\alpha_m e_{k_m}^m) \cdot f_j = \alpha_1 \cdots \alpha_m \cdot e_{i_1}^{1*}(e_{k_1}^1) \cdots e_{i_m}^{m*}(e_{k_m}^m) \cdot f_j \\ &= 0 \end{aligned}$$

since $(i_1, \dots, i_m) \neq (k_1, \dots, k_m)$. Thus (3.2) holds.

If at least one of the j'_t s, $1 \leq t \leq m$, is 2, say $j_1 = 2$, then the general term of (3.3) is

$$\begin{aligned} & \left(e_{i_1}^{1*}(u_{1,j_1}) \cdots e_{i_m}^{m*}(u_{m,j_m}) \cdot f_j \right) \wedge \left(e_{k_1}^{1*}(u_{1,j_1}) \cdots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \right) \\ &\leq e_{k_1}^{1*}(u_{1,j_1}) \cdot e_{k_2}^{2*}(u_{2,j_2}) \cdots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \\ &= e_{k_1}^{1*}(u_{1,2}) \cdot e_{k_2}^{2*}(u_{2,j_2}) \cdots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \\ &= e_{k_1}^{1*}(x^1 - u_{1,1}) \cdot e_{k_2}^{2*}(u_{2,j_2}) \cdots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \\ &= (e_{k_1}^{1*}(x^1) - e_{k_1}^{1*}(u_{1,1})) \cdot e_{k_2}^{2*}(u_{2,j_2}) \cdots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \\ &= (\alpha_1 - \alpha_1) \cdot e_{k_2}^{2*}(u_{2,j_2}) \cdots e_{k_m}^{m*}(u_{m,j_m}) \cdot f_k \\ &= 0, \end{aligned}$$

Thus (3.2) holds. Therefore, the monomial sequence (b2) is disjoint and hence, forms a 1-unconditional basic sequence in $\mathcal{L}^r(E_1, \dots, E_m; F)$. \square

Recall that a subspace I of a vector lattice E is called an *order ideal* if $0 \leq x \leq y$ for some $y \in I$ implies that $x \in I$. In the following two theorems, first we will give characterizations of regular multilinear operators that are weakly sequentially continuous, and then show that $\mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$ is a closed order ideal of $\mathcal{L}^r(E_1, \dots, E_m; F)$.

Theorem 3.2.3. Let $\{e_i^k : i \in \mathbb{N}\}$ be 1-unconditionally shrinking basis of E_k for $k = 1, \dots, m$, and $\{f_j : j \in \mathbb{N}\}$ be a 1-unconditional basis of F . Then for any $T \in \mathcal{L}^r(E_1, \dots, E_m; F)$, the following statements are equivalent.

- (i) $|T|$ is weakly sequentially continuous.
- (ii) $|T|$ is compact and $\lim_{n \rightarrow \infty} \|T\|_{r,k,n} = 0$ for $k = 1, \dots, m$, where

$$\|T\|_{r,k,n} = \sup \left\{ \left\| |T| \left(x^1, \dots, x^{k-1}, \sum_{i=n}^{\infty} e_i^{k*}(x^k) e_i^k, x^{k+1}, \dots, x^m \right) \right\| : x^1 \in B_{E_1^+}, \dots, x^m \in B_{E_m^+} \right\}.$$

- (iii)

$$T = \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \sum_{j=1}^{\infty} \left\langle T(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \right\rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j$$

converges in $\mathcal{L}^r(E_1, \dots, E_m; F)$.

Proof. It needs to be mentioned that each E_i contains no copy of l_1 since each E_i has a shrinking basis for $i = 1, \dots, m$.

(i) \Rightarrow (ii): Take a sequence $(x_n^i)_{n=1}^{\infty}$ in B_{E_i} for $i = 1, \dots, m$. Since each E_i contains no copy of l_1 , there is a weakly Cauchy subsequence $(x_{n_k}^i)_{k=1}^{\infty}$ of $(x_n^i)_{n=1}^{\infty}$ for $i = 1, \dots, m$. By [3, Corollary 2.5], $(|T|(x_{n_k}^1, \dots, x_{n_k}^m))_{k=1}^{\infty}$ is a norm Cauchy sequence in F and hence, convergent in F . Thus $|T|$ is compact.

Now suppose that there exists k , without loss of generality, say $k = 1$, such that $\lim_{n \rightarrow \infty} \|T\|_{r,1,n} \neq 0$. Then there exist $\varepsilon > 0$, sequences $(x_j^1)_{j=1}^{\infty}$ in $B_{E_1^+}, \dots, (x_j^m)_{j=1}^{\infty}$ in $B_{E_m^+}$, and

$n_1 < n_2 < \dots$ such that

$$\left\| |T| \left(\sum_{i=n_j}^{\infty} e_i^{1*}(x_j^1) e_i^1, x_j^2, \dots, x_j^m \right) \right\| > \varepsilon, \quad j = 1, 2, \dots \quad (3.4)$$

Since each E_k contains no copy of ℓ_1 , each sequence $(x_j^k)_{j=1}^{\infty}$ has a weakly Cauchy subsequence. Without loss of generality, we may assume that each sequence $\{x_j^k\}_{j=1}^{\infty}$ is a weakly Cauchy sequence in E_k for $k = 2, \dots, m$. Since $\{e_i^1 : i \in \mathbb{N}\}$ is a shrinking basis of E_1 , we have that $\sum_{i=n_j}^{\infty} e_i^{1*}(x_j^1) e_i^1 \rightarrow 0$ weakly in E_1 . By [3, Lemma 2.4], (i) implies that

$$\lim_j |T| \left(\sum_{i=n_j}^{\infty} e_i^{1*}(x_j^1) e_i^1, x_j^2, \dots, x_j^m \right) = 0,$$

which contradicts (3.4). This contradiction shows that $\lim_{n \rightarrow \infty} \|T\|_{r,1,n} = 0$ and (ii) follows.

(ii) \Rightarrow (iii): Without loss of generality, for convenience, let $m = 2$. Since $|T|$ is compact, the set $\{|T|(x^1, x^2) : x^1 \in B_{E_1^+}, x^2 \in B_{E_2^+}\}$ is a relatively compact subset of F and hence,

$$A_n := \sup \left\{ \left\| \sum_{j=n}^{\infty} \langle |T|(x^1, x^2), f_j^* \rangle f_j \right\| : x^1 \in B_{E_1^+}, x^2 \in B_{E_2^+} \right\} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Now for each $n_1, n_2, n_3 \in \mathbb{N}$,

$$\begin{aligned} & \left\| T - \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=1}^{n_3} \langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \rangle e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes f_j \right\|_r \\ &= \left\| \left| T - \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=1}^{n_3} \langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \rangle e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes f_j \right| \right\| \\ &= \sup \left\{ \left\| \left| T - \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=1}^{n_3} \langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \rangle e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes f_j \right| (x^1, x^2) \right\| : \right. \\ & \qquad \qquad \qquad \left. x^1 \in B_{E_1^+}, x^2 \in B_{E_2^+} \right\}. \end{aligned} \quad (3.5)$$

For each fixed $x^1 \in B_{E_1^+}$ and $x^2 \in B_{E_2^+}$, by [9, p. 848, (2.10)],

$$\begin{aligned}
& \left| T - \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=1}^{n_3} \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes f_j \right| (x^1, x^2) \\
&= \sup \left\{ \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \left(T - \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=1}^{n_3} \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes f_j \right) (u_{j_1}^1, u_{j_2}^2) \right| : \right. \\
&\quad \left. u_{j_1}^1 \in E_1^+, \sum_{j_1=1}^{m_1} u_{j_1}^1 = x^1, \quad u_{j_2}^2 \in E_2^+, \sum_{j_2=1}^{m_2} u_{j_2}^2 = x^2 \right\}. \quad (3.6)
\end{aligned}$$

For each fixed $u_{j_1}^1 \in E_1^+$ with $\sum_{j_1=1}^{m_1} u_{j_1}^1 = x^1$ and each fixed $u_{j_2}^2 \in E_2^+$ with $\sum_{j_2=1}^{m_2} u_{j_2}^2 = x^2$,

$$\begin{aligned}
T(u_{j_1}^1, u_{j_2}^2) &= T\left(\sum_{i_1=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_1}^1, \sum_{i_2=1}^{\infty} e_{i_2}^{2*}(u_{j_2}^2) e_{i_2}^2\right) \\
&= \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) T(e_{i_1}^1, e_{i_2}^2) \\
&= \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \sum_{j=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle f_j.
\end{aligned}$$

Thus

$$\begin{aligned}
& \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \left(T - \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=1}^{n_3} \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes f_j \right) (u_{j_1}^1, u_{j_2}^2) \right| \\
&= \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \sum_{j=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle f_j \right. \\
&\quad \left. - \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=1}^{n_3} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle f_j \right| \\
&\leq \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=n_3+1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle f_j \right| \quad (:= I_1) \\
&+ \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \sum_{i_1=1}^{n_1} \sum_{i_2=n_2+1}^{\infty} \sum_{j=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle f_j \right| \quad (:= I_2) \\
&+ \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \sum_{i_1=n_1+1}^{\infty} \sum_{i_2=1}^{\infty} \sum_{j=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle f_j \right| \quad (:= I_3). \quad (3.7)
\end{aligned}$$

Here we have

$$\begin{aligned}
I_1 &= \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \sum_{j=n_3+1}^{\infty} \left\langle T \left(\sum_{i_1=1}^{n_1} e_{i_1}^{1*}(u_{j_1}^1) e_{i_1}^1, \sum_{i_2=1}^{n_2} e_{i_2}^{2*}(u_{j_2}^2) e_{i_2}^2 \right), f_j^* \right\rangle f_j \right| \\
&\leq \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \sum_{j=n_3+1}^{\infty} \left\langle |T| \left(\sum_{i_1=1}^{n_1} e_{i_1}^{1*}(u_{j_1}^1) e_{i_1}^1, \sum_{i_2=1}^{n_2} e_{i_2}^{2*}(u_{j_2}^2) e_{i_2}^2 \right), f_j^* \right\rangle f_j \\
&\leq \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \sum_{j=n_3+1}^{\infty} \left\langle |T| \left(\sum_{i_1=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_1}^1, \sum_{i_2=1}^{\infty} e_{i_2}^{2*}(u_{j_2}^2) e_{i_2}^2 \right), f_j^* \right\rangle f_j \\
&= \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \sum_{j=n_3+1}^{\infty} \left\langle |T| \left(u_{j_1}^1, u_{j_2}^2 \right), f_j^* \right\rangle f_j \\
&= \sum_{j=n_3+1}^{\infty} \left\langle |T| \left(\sum_{j_1=1}^{m_1} u_{j_1}^1, \sum_{j_2=1}^{m_2} u_{j_2}^2 \right), f_j^* \right\rangle f_j \\
&= \sum_{j=n_3+1}^{\infty} \left\langle |T| \left(x^1, x^2 \right), f_j^* \right\rangle f_j,
\end{aligned}$$

$$\begin{aligned}
I_2 &= \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \sum_{i_1=1}^{n_1} \sum_{i_2=n_2+1}^{\infty} \sum_{j=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle f_j \right| \\
&= \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| \sum_{i_1=1}^{n_1} \sum_{i_2=n_2+1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_2}^{2*}(u_{j_2}^2) T(e_{i_1}^1, e_{i_2}^2) \right| \\
&= \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} \left| T \left(\sum_{i_1=1}^{n_1} e_{i_1}^{1*}(u_{j_1}^1) e_{i_1}^1, \sum_{i_2=n_2+1}^{\infty} e_{i_2}^{2*}(u_{j_2}^2) e_{i_2}^2 \right) \right| \\
&\leq \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} |T| \left(\sum_{i_1=1}^{n_1} e_{i_1}^{1*}(u_{j_1}^1) e_{i_1}^1, \sum_{i_2=n_2+1}^{\infty} e_{i_2}^{2*}(u_{j_2}^2) e_{i_2}^2 \right) \\
&\leq \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} |T| \left(\sum_{i_1=1}^{\infty} e_{i_1}^{1*}(u_{j_1}^1) e_{i_1}^1, \sum_{i_2=n_2+1}^{\infty} e_{i_2}^{2*}(u_{j_2}^2) e_{i_2}^2 \right) \\
&= \sum_{j_1=1}^{m_1} \sum_{j_2=1}^{m_2} |T| \left(u_{j_1}^1, \sum_{i_2=n_2+1}^{\infty} e_{i_2}^{2*}(u_{j_2}^2) e_{i_2}^2 \right) \\
&= |T| \left(\sum_{j_1=1}^{m_1} u_{j_1}^1, \sum_{i_2=n_2+1}^{\infty} e_{i_2}^{2*} \left(\sum_{j_2=1}^{m_2} u_{j_2}^2 \right) e_{i_2}^2 \right) \\
&= |T| \left(x^1, \sum_{i_2=n_2+1}^{\infty} e_{i_2}^{2*}(x^2) e_{i_2}^2 \right),
\end{aligned}$$

and similarly,

$$I_3 \leq |T| \left(\sum_{i_1=n_1+1}^{\infty} e_{i_1}^{1*}(x^1)e_{i_1}^1, x^2 \right).$$

Combining (3.5), (3.6), and (3.7), we have

$$\begin{aligned} & \left\| T - \sum_{i_1=1}^{n_1} \sum_{i_2=1}^{n_2} \sum_{j=1}^{n_3} \left\langle T(e_{i_1}^1, e_{i_2}^2), f_j^* \right\rangle e_{i_1}^{1*} \otimes e_{i_2}^{2*} \otimes f_j \right\|_r \\ & \leq \sup \left\{ \left\| \sum_{j=n_3+1}^{\infty} \left\langle |T|(x^1, x^2), f_j^* \right\rangle f_j \right\| : x^1 \in B_{E_1^+}, x^2 \in B_{E_2^+} \right\} \\ & + \sup \left\{ \left\| |T| \left(x^1, \sum_{i_2=n_2+1}^{\infty} e_{i_2}^{2*}(x^2)e_{i_2}^2 \right) \right\| : x^1 \in B_{E_1^+}, x^2 \in B_{E_2^+} \right\} \\ & + \sup \left\{ \left\| |T| \left(\sum_{i_1=n_1+1}^{\infty} e_{i_1}^{1*}(x^1)e_{i_1}^1, x^2 \right) \right\| : x^1 \in B_{E_1^+}, x^2 \in B_{E_2^+} \right\} \\ & = A_{n_3+1} + \|T\|_{r,2,n_2+1} + \|T\|_{r,1,n_1+1} \\ & \rightarrow 0 \quad \text{as } n_1 \rightarrow \infty, n_2 \rightarrow \infty, n_3 \rightarrow \infty. \end{aligned}$$

Therefore, (iii) follows.

(iii) \Rightarrow (i): By Theorem 3.2.2, (iii) implies that for any $n_1, \dots, n_m, n \in \mathbb{N}$,

$$\begin{aligned} & \left\| |T| - \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} \sum_{j=1}^n \left\langle T(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \right\rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j \right\|_r \\ & \leq \left\| T - \sum_{i_1=1}^{n_1} \cdots \sum_{i_m=1}^{n_m} \sum_{j=1}^n \left\langle T(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \right\rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j \right\|_r \\ & \longrightarrow 0 \quad \text{as } n_1 \rightarrow \infty, \dots, n_m \rightarrow \infty, n \rightarrow \infty. \end{aligned}$$

Thus

$$|T| = \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \sum_{j=1}^{\infty} \left\langle T(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \right\rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j$$

converges in $\mathcal{L}^r(E_1, \dots, E_m; F)$, which implies that $|T|$ is approximable and hence, weakly sequentially continuous, and (i) follows. \square

Theorem 3.2.4. Suppose that each E_k has a 1-unconditionally shrinking basis for $k = 1, \dots, m$, and F has a 1-unconditional basis. Then $\mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$ is a closed order ideal of $\mathcal{L}^r(E_1, \dots, E_m; F)$.

Proof. Let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditionally shrinking basis of E_k for $k = 1, \dots, m$, and $\{f_j : j \in \mathbb{N}\}$ be a 1-unconditional basis of F . Take $S, T \in \mathcal{L}^r(E_1, \dots, E_m; F)$ such that $0 \leq S \leq T$ and $T \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$. Since T is weakly sequentially continuous, by Theorem 3.2.3,

$$T = \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \sum_{j=1}^{\infty} \langle T(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j$$

converges in $\mathcal{L}^r(E_1, \dots, E_m; F)$. For any $n_1, \dots, n_m, n \in \mathbb{N}$,

$$\begin{aligned} & \left\| \sum_{i_1=n_1}^{\infty} \cdots \sum_{i_m=n_m}^{\infty} \sum_{j=n}^{\infty} \langle S(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j \right\|_r \\ & \leq \left\| \sum_{i_1=n_1}^{\infty} \cdots \sum_{i_m=n_m}^{\infty} \sum_{j=n}^{\infty} \langle T(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j \right\|_r \\ & \longrightarrow 0 \quad \text{as } n_1 \rightarrow \infty, \dots, n_m \rightarrow \infty, n \rightarrow \infty. \end{aligned}$$

Thus the series

$$\sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \sum_{j=1}^{\infty} \langle S(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j$$

converges in $\mathcal{L}^r(E_1, \dots, E_m; F)$. Note that for any $x^1 \in E_1, \dots, x^m \in E_m$,

$$S(x^1, \dots, x^m) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \sum_{j=1}^{\infty} \left\langle S(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \right\rangle e_{i_1}^{1*}(x^1) \cdots e_{i_m}^{m*}(x^m) f_j.$$

Therefore,

$$S = \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \sum_{j=1}^{\infty} \left\langle S(e_{i_1}^1, \dots, e_{i_m}^m), f_j^* \right\rangle e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j,$$

which implies that $S \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$, and hence $\mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$ is an order ideal of $\mathcal{L}^r(E_1, \dots, E_m; F)$.

Now take $T_n \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$ and $T \in \mathcal{L}^r(E_1, \dots, E_m; F)$ such that $\|T_n - T\|_r \rightarrow 0$ as $n \rightarrow \infty$. For any $x^1, y^1 \in B_{E_1}, \dots, x^m, y^m \in B_{E_m}$,

$$\begin{aligned} & \left\| |T|(x^1, \dots, x^m) - |T|(y^1, \dots, y^m) \right\| \leq \left\| |T_n|(x^1, \dots, x^m) - |T_n|(y^1, \dots, y^m) \right\| \\ & + \left\| |T|(x^1, \dots, x^m) - |T_n|(x^1, \dots, x^m) \right\| + \left\| |T|(y^1, \dots, y^m) - |T_n|(y^1, \dots, y^m) \right\| \\ & \leq \left\| |T_n|(x^1, \dots, x^m) - |T_n|(y^1, \dots, y^m) \right\| + \left\| |T| - |T_n| \right\| (\|x^1\| \cdots \|x^m\| + \|y^1\| \cdots \|y^m\|) \\ & \leq \left\| |T_n|(x^1, \dots, x^m) - |T_n|(y^1, \dots, y^m) \right\| + 2\|T - T_n\|_r, \end{aligned}$$

which implies that $|T| \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$. Thus $T^+, T^- \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$ and hence, $T \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$, and $\mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$ is closed. \square

Note that each monomial $e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j$ belongs to $\mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$. As a consequence of Theorems 3.2.2, 3.2.3 and 3.2.4, we have the following.

Corollary 3.2.5. Let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditionally shrinking basis of E_k for $k = 1, \dots, m$, and $\{f_j : j \in \mathbb{N}\}$ be a 1-unconditional basis of F . Then the monomial sequence

$$\left\{ e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j : (i_1, \dots, i_m, j) \in \mathbb{N}^{m+1} \right\}$$

forms a 1-unconditional basis of $\mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$.

Corollary 3.2.6. Let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditional basis of E_k for $k = 1, \dots, m$, and $\{f_j : j \in \mathbb{N}\}$ be a 1-unconditional basis of F . Then the monomial sequence

$$\left\{ e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j : (i_1, \dots, i_m, j) \in \mathbb{N}^{m+1} \right\} \quad (b2)$$

forms a 1-unconditional basis of $\mathcal{L}^r(E_1, \dots, E_m; F)$ if and only if each $\{e_i^k : i \in \mathbb{N}\}$ is a shrinking basis of E_k for $k = 1, \dots, m$, and every positive m -linear operator from $E_1 \times \cdots \times E_m$ to F is weakly sequentially continuous.

Proof. By Corollary 3.2.5, we need only to prove the necessity. Suppose that the monomial sequence (b2) is a basis of $\mathcal{L}^r(E_1, \dots, E_m; F)$. Then

$$\left\{ e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} : (i_1, \dots, i_m) \in \mathbb{N}^m \right\}$$

is a basis of $\mathcal{L}^r(E_1, \dots, E_m; \mathbb{R}) = (E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m)^*$ and hence, the basis (b1) is a shrinking basis of $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$. Since each E_k is lattice isometric to a complemented sublattice of $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ by Lemma 2.3.9, it follows that each $\{e_i^k : i \in \mathbb{N}\}$ is a shrinking basis of E_k for $k = 1, \dots, m$.

Now take any positive $T \in \mathcal{L}^r(E_1, \dots, E_m; F)$. Then

$$T = \sum_{(i_1, \dots, i_m, j) \in \mathbb{N}^{m+1}} a_{i_1, \dots, i_m, j} e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \otimes f_j$$

converges in $\mathcal{L}^r(E_1, \dots, E_m; F)$. Thus T is approximable and hence, weakly sequentially continuous. \square

Remark 3.2.7. For each $i \in \mathbb{N}$, we use θ_i to denote the standard unit vectors in sequence spaces, that is,

$$\theta_i = (0, \dots, 0, \overset{i\text{-th}}{1}, 0, 0, \dots).$$

Then $\{\theta_i : i \in \mathbb{N}\}$ forms a standard basis for the Banach sequence lattices c_0 and ℓ_p for $1 \leq p < \infty$. For convenience, we denote c_0 by ℓ_p with $p = \infty$. Let $1 \leq p_1, \dots, p_m, q \leq \infty$ be such that $\frac{1}{p_1} + \dots + \frac{1}{p_m} < \frac{1}{q}$. Then every continuous m -linear operator from $\ell_{p_1} \times \dots \times \ell_{p_m}$ to ℓ_q is weakly sequentially continuous (see, e.g., [2, 19]). Thus with the square order in \mathbb{N}^{m+1} , the monomial sequence

$$\left\{ \theta_{i_1}^* \otimes \dots \otimes \theta_{i_m}^* \otimes \theta_j : (i_1, \dots, i_m, j) \in \mathbb{N}^{m+1} \right\}$$

forms a 1-unconditional basis of $\mathcal{L}^r(\ell_{p_1}, \dots, \ell_{p_m}; \ell_q)$. In particular, this monomial sequence forms a 1-unconditional basis of $\mathcal{L}^r(c_0, \dots, c_0; \ell_q)$ for any $1 \leq q < \infty$.

3.3 Bases in positive tensor products

With the square order in \mathbb{N}^m we have that the sequence

$$\left\{ e_{i_1}^1 \otimes \cdots \otimes e_{i_m}^m : (i_1, \dots, i_m) \in \mathbb{N}^m \right\} \quad (b1)$$

forms a 1-unconditional basis of both $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ and $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$ (see Proposition 3.2.1). In this section, we will give necessary and sufficient conditions for which the basis (b1) is shrinking or boundedly complete in both $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ and $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$. Here we need to mention that the sequence of coordinate functionals associated to (b1) is the following sequence

$$\left\{ e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} : (i_1, \dots, i_m) \in \mathbb{N}^m \right\}. \quad (b3)$$

Theorem 3.3.1. Let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditional basis of E_k for $k = 1, \dots, m$.

Then the following statements are equivalent.

- (i) The sequence (b1) is a shrinking basis of $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$.
- (ii) The sequence (b3) is a boundedly complete basis of $E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$.
- (iii) Each $\{e_i^{k*} : i \in \mathbb{N}\}$ is a basis of E_k^* for $k = 1, \dots, m$ and

$$(E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^* = E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*.$$

- (iv) Each $\{e_i^k : i \in \mathbb{N}\}$ is a shrinking basis of E_k for $k = 1, \dots, m$.

Proof. (iii) \Rightarrow (i): By Proposition 3.2.1, the sequence (b1) is a basis of $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$ and the sequence (b3) is a basis of $E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$. It follows from (iii) that the basis (b1) is shrinking and (i) follows.

(i) \Rightarrow (iv): Since each E_k is lattice isometric to a complemented sublattice of

$$E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$$

by Lemma 2.3.9, (iv) follows from (i).

(iv) \Rightarrow (ii): By (iv), each $\{e_i^{k*} : i \in \mathbb{N}\}$ is a boundedly complete basis of E_k^* for $k = 1, \dots, m$ and hence, each E_k^* is a KB-space. By [8, Theorem 7.5], $E_1^* \hat{\otimes}_{|\pi|} E_2^*$ is a KB-space and hence, $E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$ is a KB-space by mathematical induction. Moreover, Proposition 3.2.1 implies that the sequence (b3) is a basis of $E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$. Thus it is boundedly complete and (ii) follows.

(ii) \Rightarrow (iii): Since each E_k^* is lattice isometric to a complemented sublattice of $E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$ by Lemma 2.3.9, (ii) implies that each $\{e_i^{k*} : i \in \mathbb{N}\}$ is a boundedly complete basis of E_k^* for $k = 1, \dots, m$. Next we show that $(E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^* = E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$.

Note that $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$ is a sublattice of $\mathcal{L}^r(E_1^*, \dots, E_m^*; \mathbb{R}) = (E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*)^*$. Let $I_1 : E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m \mapsto (E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*)^*$ be the identity embedding, and let

$$I := I_1^* \Big|_{E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*} : E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^* \rightarrow (E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^*.$$

Then for each $u \in E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$,

$$\left\| I(u) \right\|_{(E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^*} \leq \|I\| \cdot \|u\|_{|\pi|} = \|u\|_{|\pi|}. \quad (3.9)$$

Take any $v \in (E_1^* \otimes \cdots \otimes E_m^*)^+$, say $v = \sum_{k=1}^n x_{1,k}^* \otimes \cdots \otimes x_{m,k}^*$. For any $\varepsilon > 0$, there exists $\phi \in \mathcal{L}^r(E_1^*, \dots, E_m^*; \mathbb{R})^+$ such that $\|\phi\| \leq 1$ and

$$\|v\|_{|\pi|} \leq \left| \sum_{k=1}^n \phi(x_{1,k}^*, \dots, x_{m,k}^*) \right| + \varepsilon. \quad (3.10)$$

For any $p \in \mathbb{N}$, let

$$w_p = \sum_{i_1, \dots, i_m=1}^p \phi(e_{i_1}^{1*}, \dots, e_{i_m}^{m*}) \cdot e_{i_1}^1 \otimes \dots \otimes e_{i_m}^m.$$

Then $w_p \geq 0$ and

$$\begin{aligned} \|w_p\|_{|\epsilon|} &= \sup \left\{ \left| \sum_{i_1, \dots, i_m=1}^p \phi(e_{i_1}^{1*}, \dots, e_{i_m}^{m*}) \cdot x_1^*(e_{i_1}^1) \dots x_m^*(e_{i_m}^m) \right| : x_1^* \in B_{E_1^{*+}}, \dots, x_m^* \in B_{E_m^{*+}} \right\} \\ &= \sup \left\{ \left| \phi \left(\sum_{i_1=1}^p x_1^*(e_{i_1}^1) e_{i_1}^{1*}, \dots, \sum_{i_m=1}^p x_m^*(e_{i_m}^m) e_{i_m}^{m*} \right) \right| : x_1^* \in B_{E_1^{*+}}, \dots, x_m^* \in B_{E_m^{*+}} \right\} \\ &\leq \sup \left\{ \|\phi\| \cdot \left\| \sum_{i_1=1}^p x_1^*(e_{i_1}^1) e_{i_1}^{1*} \right\| \dots \left\| \sum_{i_m=1}^p x_m^*(e_{i_m}^m) e_{i_m}^{m*} \right\| : x_1^* \in B_{E_1^{*+}}, \dots, x_m^* \in B_{E_m^{*+}} \right\} \\ &\leq \sup \left\{ \|\phi\| \cdot \left\| \sum_{i_1=1}^{\infty} x_1^*(e_{i_1}^1) e_{i_1}^{1*} \right\| \dots \left\| \sum_{i_m=1}^{\infty} x_m^*(e_{i_m}^m) e_{i_m}^{m*} \right\| : x_1^* \in B_{E_1^{*+}}, \dots, x_m^* \in B_{E_m^{*+}} \right\} \\ &= \sup \left\{ \|\phi\| \cdot \|x_1^*\| \dots \|x_m^*\| : x_1^* \in B_{E_1^{*+}}, \dots, x_m^* \in B_{E_m^{*+}} \right\} \\ &= \|\phi\| \leq 1. \end{aligned}$$

Thus

$$\begin{aligned} \|I(v)\|_{(E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m)^*} &\geq |\langle I(v), w_p \rangle| = |\langle v, w_p \rangle| \\ &= \left| \sum_{k=1}^n \langle x_{1,k}^* \otimes \dots \otimes x_{m,k}^*, w_p \rangle \right| \\ &= \left| \sum_{k=1}^n \sum_{i_1, \dots, i_m=1}^p \phi(e_{i_1}^{1*}, \dots, e_{i_m}^{m*}) \cdot x_{1,k}^*(e_{i_1}^1) \dots x_{m,k}^*(e_{i_m}^m) \right| \\ &= \left| \sum_{k=1}^n \phi \left(\sum_{i=1}^p x_{1,k}^*(e_i^1) e_i^{1*}, \dots, \sum_{i=1}^p x_{m,k}^*(e_i^m) e_i^{m*} \right) \right|. \end{aligned}$$

Letting $p \rightarrow \infty$,

$$\begin{aligned} \|I(v)\|_{(E_1 \check{\otimes}_{|\epsilon|} \dots \check{\otimes}_{|\epsilon|} E_m)^*} &\geq \left| \sum_{k=1}^n \phi \left(\sum_{i=1}^{\infty} x_{1,k}^*(e_i^1) e_i^{1*}, \dots, \sum_{i=1}^{\infty} x_{m,k}^*(e_i^m) e_i^{m*} \right) \right| \\ &= \left| \sum_{k=1}^n \phi(x_{1,k}^*, \dots, x_{m,k}^*) \right| \geq \|v\|_{|\pi|} - \epsilon. \end{aligned}$$

Therefore, $\left\| I(v) \right\|_{(E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^*} \geq \|v\|_{|\pi|}$. Since $E_1^* \otimes \cdots \otimes E_m^*$ is dense in $E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$, it follows that for each $u \in E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$,

$$\left\| I(u) \right\|_{(E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^*} \geq \|u\|_{|\pi|}. \quad (3.11)$$

Combining (3.9) and (3.11) we have shown that

$$I : E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^* \rightarrow (E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^*$$

is an isometry.

Next we show that I is onto. Take any $\xi \in (E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^{*+}$. Let

$$b_{i_1, \dots, i_m} = \langle \xi, e_{i_1}^1 \otimes \cdots \otimes e_{i_m}^m \rangle.$$

For any $n_1, \dots, n_m \in \mathbb{N}$ and any $w \in E_1 \otimes \cdots \otimes E_m$, say $w = \sum_{k=1}^n x_k^1 \otimes \cdots \otimes x_k^m$, we have

$$\begin{aligned} & \left| \left\langle I \left(\sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} b_{i_1, \dots, i_m} e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \right), w \right\rangle \right| \\ &= \left| \sum_{k=1}^n \sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} b_{i_1, \dots, i_m} e_{i_1}^{1*}(x_k^1) \cdots e_{i_m}^{m*}(x_k^m) \right| \\ &= \left| \sum_{k=1}^n \sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} \langle \xi, e_{i_1}^1 \otimes \cdots \otimes e_{i_m}^m \rangle e_{i_1}^{1*}(x_k^1) \cdots e_{i_m}^{m*}(x_k^m) \right| \\ &= \left| \sum_{k=1}^n \left\langle \xi, \left(\sum_{i_1=1}^{n_1} e_{i_1}^{1*}(x_k^1) e_{i_1}^1 \right) \otimes \cdots \otimes \left(\sum_{i_m=1}^{n_m} e_{i_m}^{m*}(x_k^m) e_{i_m}^m \right) \right\rangle \right| \\ &= \left| \left\langle \xi, \sum_{k=1}^n P_{n_1}^1(x_k^1) \otimes \cdots \otimes P_{n_m}^m(x_k^m) \right\rangle \right| \\ &= \left| \left\langle \xi, \sum_{k=1}^n (P_{n_1}^1 \otimes \cdots \otimes P_{n_m}^m)(x_k^1 \otimes \cdots \otimes x_k^m) \right\rangle \right| \\ &= \left| \left\langle \xi, (P_{n_1}^1 \otimes \cdots \otimes P_{n_m}^m) \left(\sum_{k=1}^n x_k^1 \otimes \cdots \otimes x_k^m \right) \right\rangle \right| \\ &= \left| \left\langle \xi, (P_{n_1}^1 \otimes \cdots \otimes P_{n_m}^m)(w) \right\rangle \right|, \end{aligned}$$

where each $P_{n_k}^k$ is a basis projection of E_k for $k = 1, \dots, m$. By Lemma 2.3.10,

$$\begin{aligned}
& \left\| \sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} b_{i_1, \dots, i_m} e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \right\|_{|\pi|} \\
&= \left\| I \left(\sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} b_{i_1, \dots, i_m} e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \right) \right\|_{(E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m)^*} \\
&= \sup \left\{ \left| \left\langle I \left(\sum_{i_1, \dots, i_m=1}^{n_1, \dots, n_m} b_{i_1, \dots, i_m} e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \right), w \right\rangle \right| : w \in E_1 \otimes \cdots \otimes E_m, \|w\|_{|\epsilon|} \leq 1 \right\} \\
&= \sup \left\{ \left| \left\langle \xi, (P_{n_1}^1 \otimes \cdots \otimes P_{n_m}^m)(w) \right\rangle \right| : w \in E_1 \otimes \cdots \otimes E_m, \|w\|_{|\epsilon|} \leq 1 \right\} \\
&\leq \sup \left\{ \|\xi\| \cdot \|P_{n_1}^1 \otimes \cdots \otimes P_{n_m}^m\| \cdot \|w\|_{|\epsilon|} : w \in E_1 \otimes \cdots \otimes E_m, \|w\|_{|\epsilon|} \leq 1 \right\} \\
&\leq \|\xi\| \cdot \|P_{n_1}^1\| \cdots \|P_{n_m}^m\| \leq \|\xi\| < \infty.
\end{aligned}$$

By (ii), the sequence (b3) is a boundedly complete basis of $E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$. Thus

$$u := \sum_{i_1, \dots, i_m=1}^{\infty} b_{i_1, \dots, i_m} e_{i_1}^{1*} \otimes \cdots \otimes e_{i_m}^{m*} \in E_1^* \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m^*$$

and $\langle I(u), w \rangle = \langle \xi, w \rangle$ for any $w \in E_1 \otimes \cdots \otimes E_m$. Therefore, $I(u) = \xi$ and I is onto. \square

Theorem 3.3.2. Let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditional basis of E_k for $k = 1, \dots, m$.

Then the following statements are equivalent.

- (i) The sequence (b1) is a shrinking basis of $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$.
- (ii) The sequence (b3) is a boundedly complete basis of $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$.
- (iii) Each $\{e_i^{k*} : i \in \mathbb{N}\}$ is a basis of E_k^* for $k = 1, \dots, m$ and

$$(E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m)^* = E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*.$$

(iv) Each $\{e_i^k : i \in \mathbb{N}\}$ is a shrinking basis of E_k for $k = 1, \dots, m$, and every positive m -linear operator from $E_1 \times \cdots \times E_m$ to \mathbb{R} is weakly sequentially continuous.

Proof. (iii) \Rightarrow (i): By Proposition 3.2.1, the sequence (b1) is a basis of $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ and the sequence (b3) is a basis of $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$. It follows from (iii) that the sequence (b1) is shrinking and (i) follows.

(i) \Rightarrow (iv): By (i), the sequence (b3) forms a basis of

$$(E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m)^* = \mathcal{L}^r(E_1, \dots, E_m; \mathbb{R}).$$

Thus (iv) follows from a special case of Corollary 3.2.6 with $F = \mathbb{R}$.

(iv) \Rightarrow (ii): By Corollary 3.2.6 with $F = \mathbb{R}$, the sequence (b3) is a basis of

$$\mathcal{L}^r(E_1, \dots, E_m; \mathbb{R}) = (E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m)^*$$

and hence, it is a boundedly complete basis of $\mathcal{L}^r(E_1, \dots, E_m; \mathbb{R})$. Since (iv) implies that each $\{e_i^{k*} : i \in \mathbb{N}\}$ is a basis of E_k^* for $k = 1, \dots, m$, the sequence (b3) is also a basis of $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$ by Proposition 3.2.1. Note that $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$ is a closed sublattice of $\mathcal{L}^r(E_1, \dots, E_m; \mathbb{R})$. Thus $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^* = \mathcal{L}^r(E_1, \dots, E_m; \mathbb{R})$ and hence, the sequence (b3) is a boundedly complete basis of $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$ and (ii) follows.

(ii) \Rightarrow (iii): Since each E_k^* is lattice isometric to a complemented sublattice of $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$ by Lemma 2.3.9, (ii) implies that each $\{e_i^{k*} : i \in \mathbb{N}\}$ is a boundedly complete basis of E_k^* for $k = 1, \dots, m$. Next we show that $(E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m)^* = E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$.

Note that $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$ is a closed sublattice of

$$\mathcal{L}^r(E_1, \dots, E_m; \mathbb{R}) = (E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m)$$

We only need to show that $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^* = \mathcal{L}^r(E_1, \dots, E_m; \mathbb{R})$. For convenience, without loss of generality, let $m = 2$. Take any $T \in \mathcal{L}^r(E_1, E_2; \mathbb{R})$. For each $x^1 \in E_1^+$, $x^2 \in E_2^+$, and each $n \in \mathbb{N}$,

$$\begin{aligned}
& \left| \sum_{(i,j) \in \mathbb{N}^2, (i,j) < (n,n)} T(e_i^1, e_j^2) e_i^{1*} \otimes e_j^{2*} \right| (x^1, x^2) \\
&= \left| \sum_{i=1}^{n-1} \sum_{j=1}^n T(e_i^1, e_j^2) e_i^{1*} \otimes e_j^{2*} \right| (x^1, x^2) \quad (\text{square order in } \mathbb{N}^2) \\
&\leq \sum_{i=1}^{n-1} \sum_{j=1}^n |T|(e_i^1, e_j^2) e_i^{1*}(x^1) e_j^{2*}(x^2) \leq \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |T|(e_i^1, e_j^2) e_i^{1*}(x^1) e_j^{2*}(x^2) \\
&= |T| \left(\sum_{i=1}^{\infty} e_i^{1*}(x^1) e_i^1, \sum_{j=1}^{\infty} e_j^{2*}(x^2) e_j^2 \right) = |T|(x^1, x^2) \leq \|x^1\| \|x^2\| \|T\|_r.
\end{aligned}$$

Thus

$$\left\| \sum_{(i,j) \in \mathbb{N}^2, (i,j) < (n,n)} T(e_i^1, e_j^2) e_i^{1*} \otimes e_j^{2*} \right\|_r \leq \|T\|_r.$$

By (ii), the series $\sum_{(i,j) \in \mathbb{N}^2} T(e_i^1, e_j^2) e_i^{1*} \otimes e_j^{2*}$ converges in $E_1^* \check{\otimes}_{|\varepsilon|} E_2^*$. Note that for each $x^1 \in E_1$ and each $x^2 \in E_2$,

$$\begin{aligned}
T(x^1, x^2) &= T \left(\sum_{i=1}^{\infty} e_i^{1*}(x^1) e_i^1, \sum_{j=1}^{\infty} e_j^{2*}(x^2) e_j^2 \right) = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} T(e_i^1, e_j^2) e_i^{1*}(x^1) e_j^{2*}(x^2) \\
&= \lim_n \sum_{i=1}^{n-1} \sum_{j=1}^n T(e_i^1, e_j^2) e_i^{1*}(x^1) e_j^{2*}(x^2) \\
&= \lim_n \sum_{(i,j) \in \mathbb{N}^2, (i,j) < (n,n)} T(e_i^1, e_j^2) e_i^{1*}(x^1) e_j^{2*}(x^2) \\
&= \left(\sum_{(i,j) \in \mathbb{N}^2} T(e_i^1, e_j^2) e_i^{1*} \otimes e_j^{2*} \right) (x^1, x^2).
\end{aligned}$$

Hence, $T = \sum_{(i,j) \in \mathbb{N}^2} T(e_i^1, e_j^2) e_i^{1*} \otimes e_j^{2*} \in E_1^* \check{\otimes}_{|\varepsilon|} E_2^*$ and (iii) follows. \square

For each $i \in \mathbb{N}$, let θ_i denote the standard unit vectors in the sequence spaces c_0 and ℓ_p for $1 \leq p < \infty$. For convenience, we denote c_0 by ℓ_p with $p = \infty$. With the square order

in \mathbb{N}^m , Proposition 3.2.1 implies that the sequence

$$\left\{ \theta_{i_1} \otimes \cdots \otimes \theta_{i_m} : (i_1, \dots, i_m) \in \mathbb{N}^m \right\} \quad (b4)$$

forms a 1-unconditional basis of both $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_m}$ and $\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_m}$ for any $1 \leq p_1, \dots, p_m \leq \infty$. If, moreover, $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$, then every continuous m -linear operator from $\ell_{p_1} \times \cdots \times \ell_{p_m}$ to \mathbb{R} is weakly sequentially continuous (see, e.g., [2, 19]). By Theorems 3.3.1 and 3.3.2, we have the following examples.

Example 3.3.3. Let $1 \leq p_1, \dots, p_m \leq \infty$. Then

- (i) The basis (b4) is boundedly complete in $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_m}$ if $1 \leq p_1, \dots, p_m < \infty$.
- (ii) The basis (b4) is shrinking in $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_m}$ if $\frac{1}{p_1} + \cdots + \frac{1}{p_m} < 1$.
- (iii) The basis (b4) is shrinking in $\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_m}$ if $1 < p_1, \dots, p_m \leq \infty$.
- (iv) The basis (b4) is boundedly complete in $\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_m}$ if $\frac{1}{p_1} + \cdots + \frac{1}{p_m} > m - 1$.

3.4 Reflexivity of regular multilinear operators

Let E_1, \dots, E_m and F be any Banach lattices (with or without bases). For each $(m + 1)$ -linear operator $T : E_1 \times \dots \times E_m \times F \rightarrow \mathbb{R}$, define an m -linear operator $A : E_1 \times \dots \times E_m \rightarrow F^*$ by

$$\langle A(x^1, \dots, x^m), y \rangle = T(x^1, \dots, x^m, y), \quad x^1 \in E_1, \dots, x^m \in E_m, y \in F \quad (3.12)$$

Then we have the following.

Lemma 3.4.1. Let E_1, \dots, E_m and F be Banach lattices such that F^* is order continuous. Then $\mathcal{L}^r(E_1, \dots, E_m, F; \mathbb{R})$ is lattice isometric to $\mathcal{L}^r(E_1, \dots, E_m; F^*)$ under the mapping $T \mapsto A$ defined in (3.12). Moreover, if F is reflexive then $\mathcal{L}_{wsc}^r(E_1, \dots, E_m, F; \mathbb{R}) = \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F^*)$.

Proof. It is easy to see that the mapping $T \mapsto A$ defined in (3.12) is linear, one to one, and onto. Next we show that it is a lattice homomorphism.

Take any $x^1 \in E_1^+, \dots, x^m \in E_m^+$, and $y \in F^+$. Let $\Pi x^1, \dots, \Pi x^m$, and Πy denote the sets of all partitions of x_1, \dots, x_m , and y . By [9, p.848, (2.10)],

$$\begin{aligned} & |T|(x^1, \dots, x^m, y) \\ = & \lim \left\{ \sum_{i_1, \dots, i_m=1}^{p_1, \dots, p_m} \sum_{j=1}^p |T(u_{i_1}^1, \dots, u_{i_m}^m, y_j)| : (u_{i_k}^k)_{i_k=1}^{p_k} \in \Pi x^k, 1 \leq k \leq m, (y_j)_{j=1}^p \in \Pi y \right\} \\ = & \lim \left\{ \sum_{i_1, \dots, i_m=1}^{p_1, \dots, p_m} \lim \left\{ \sum_{j=1}^p \left| \langle A(u_{i_1}^1, \dots, u_{i_m}^m), y_j \rangle \right| : (y_j)_{j=1}^p \in \Pi y \right\} : (u_{i_k}^k)_{i_k=1}^{p_k} \in \Pi x^k, \right. \\ & \left. 1 \leq k \leq m \right\} \\ = & \lim \left\{ \sum_{i_1, \dots, i_m=1}^{p_1, \dots, p_m} \langle |A(u_{i_1}^1, \dots, u_{i_m}^m)|, y \rangle : (u_{i_k}^k)_{i_k=1}^{p_k} \in \Pi x^k, 1 \leq k \leq m \right\} \end{aligned}$$

$$\begin{aligned}
&= \left\langle \lim \left\{ \sum_{i_1, \dots, i_m=1}^{p_1, \dots, p_m} |A(u_{i_1}^1, \dots, u_{i_m}^m)| : (u_{i_k}^k)_{i_k=1}^{p_k} \in \Pi x^k, 1 \leq k \leq m \right\}, y \right\rangle \\
&= \left\langle |A|(x^1, \dots, x^m), y \right\rangle.
\end{aligned}$$

Thus the mapping $T \mapsto A$ is a lattice homomorphism. Moreover,

$$\begin{aligned}
\|T\|_r &= \sup \left\{ |T|(x^1, \dots, x^m, y) : x^1 \in B_{E_1^+}, \dots, x^m \in B_{E_m^+}, y \in B_{F^+} \right\} \\
&= \sup \left\{ \left\langle |A|(x^1, \dots, x^m), y \right\rangle : x^1 \in B_{E_1^+}, \dots, x^m \in B_{E_m^+}, y \in B_{F^+} \right\} \\
&= \|A\|_r.
\end{aligned}$$

Now take any $A \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F^*)$, and take sequences $(x_n^i)_{n=1}^\infty$ in E_i and $(y_n)_{n=1}^\infty$ in F such that $x_n^i \rightarrow x^i$ weakly in E_i , $i = 1, \dots, m$, and $y_n \rightarrow y$ weakly in F . Let $M = \sup_{n \geq 1} \|y_n\|$. Then

$$\begin{aligned}
&\left| T(x_n^1, \dots, x_n^m, y_n) - T(x^1, \dots, x^m, y) \right| \\
&= \left| \left\langle A(x_n^1, \dots, x_n^m), y_n \right\rangle - \left\langle A(x^1, \dots, x^m), y \right\rangle \right| \\
&\leq \left| \left\langle A(x_n^1, \dots, x_n^m) - A(x^1, \dots, x^m), y_n \right\rangle \right| + \left| \left\langle A(x^1, \dots, x^m), y_n - y \right\rangle \right| \\
&\leq M \cdot \left\| A(x_n^1, \dots, x_n^m) - A(x^1, \dots, x^m) \right\| + \left| \left\langle A(x^1, \dots, x^m), y_n - y \right\rangle \right| \\
&\rightarrow 0 \text{ as } n \rightarrow \infty
\end{aligned}$$

and hence, $T \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m, F; \mathbb{R})$.

Next we assume that F is reflexive. Take any $T \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m, F; \mathbb{R})$. To show that $A \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F^*)$, it suffices to show that for any sequences $(x_n^i)_{n=1}^\infty$ in E_i such that $x_n^i \rightarrow x^i$ weakly in E_i as $n \rightarrow \infty$ for $i = 1, \dots, m$, we have

$$\begin{aligned} & \left\| A(x_n^1, \dots, x_n^m) - A(x^1, \dots, x^m) \right\| \\ &= \sup \left\{ \left| \left\langle A(x_n^1, \dots, x_n^m) - A(x^1, \dots, x^m), z \right\rangle \right| : z \in B_F \right\} \\ &\rightarrow 0 \quad \text{as } n \rightarrow \infty \end{aligned} \tag{3.13}$$

Suppose (3.13) does not hold. Then there exist $\varepsilon_0 > 0$, subsequences $(x_{n_k}^i)_{k=1}^\infty$ of $(x_n^i)_{n=1}^\infty$, $i = 1, \dots, m$, and a sequence $(z_k)_{k=1}^\infty$ in B_F such that

$$\left| \left\langle A(x_{n_k}^1, \dots, x_{n_k}^m) - A(x^1, \dots, x^m), z_k \right\rangle \right| > \varepsilon_0, \quad k = 1, 2, \dots \tag{3.14}$$

Since F is reflexive, $(z_k)_{k=1}^\infty$ has a weakly convergent subsequence, without loss of generality, we may assume that $(z_k)_{k=1}^\infty$ is a weakly convergent sequence, say $z_k \rightarrow z$ weakly in F . Thus

$$\begin{aligned} & \left| \left\langle A(x_{n_k}^1, \dots, x_{n_k}^m) - A(x^1, \dots, x^m), z_k \right\rangle \right| = \left| T(x_{n_k}^1, \dots, x_{n_k}^m, z_k) - T(x^1, \dots, x^m, z_k) \right| \\ &\leq \left| T(x_{n_k}^1, \dots, x_{n_k}^m, z_k) - T(x^1, \dots, x^m, z_k) \right| + \left| T(x^1, \dots, x^m, z_k) - T(x^1, \dots, x^m, z) \right| \\ &= \left| T(x_{n_k}^1, \dots, x_{n_k}^m, z_k) - T(x^1, \dots, x^m, z) \right| + \left| \left\langle A(x^1, \dots, x^m), z_k - z \right\rangle \right| \\ &\rightarrow 0 \quad \text{as } k \rightarrow \infty, \end{aligned}$$

which contradicts (3.14). This contradiction shows that $A \in \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F^*)$. □

In the following theorem, we characterize reflexivity of $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ and

$E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m$ under the condition for which every E_1, \dots, E_m has a basis.

Theorem 3.4.2. Let E_1, \dots, E_m be a reflexive Banach lattices with 1-unconditional bases. Then the following statements are equivalent.

- (i) $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ is reflexive.
- (ii) $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$ is reflexive.
- (iii) $\mathcal{L}^r(E_1, \dots, E_m; \mathbb{R})$ is reflexive.
- (iv) $\mathcal{L}^r(E_1, \dots, E_m; \mathbb{R})$ has a monomial basis.
- (v) $\mathcal{L}^r(E_1, \dots, E_m; \mathbb{R}) = \mathcal{L}_{wsc}^r(E_1, \dots, E_m; \mathbb{R})$.

Proof. (iv) \Leftrightarrow (v) follows from Corollary 3.2.6, and (i) \Leftrightarrow (iii) follows from the fact that $(E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m)^* = \mathcal{L}^r(E_1, \dots, E_m; \mathbb{R})$. Let $\{e_i^k : i \in \mathbb{N}\}$ be a 1-unconditional basis of E_k for $k = 1, \dots, m$. Suppose (i) holds. Then the sequence (b1) is a shrinking basis of $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$. Thus (ii) and (v) follows from Theorem 3.3.2. Suppose (ii) holds. Then the sequence (b3) is a boundedly complete basis of $E_1^* \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_m^*$ and hence, (i) follows from Theorem 3.3.2. Suppose (v) holds. Then the sequence (b1) is a shrinking basis of $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ by Theorem 3.3.2. Since each $\{e_i^k : i \in \mathbb{N}\}$ is a boundedly complete basis of E_k , it follows from Theorem 3.3.1 that the sequence (b1) is also a boundedly complete basis of $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$. Thus $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_m$ is reflexive and (i) follows. \square

As a consequence of Lemma 3.4.1 and Theorem 3.4.2, we have the following.

Theorem 3.4.3. Let E_1, \dots, E_m and F be reflexive Banach lattices with 1-unconditional bases. Then the following statements are equivalent.

- (i) $\mathcal{L}^r(E_1, \dots, E_m; F)$ is reflexive.
- (ii) $\mathcal{L}^r(E_1, \dots, E_m; F)$ has a monomial basis.
- (iii) $\mathcal{L}^r(E_1, \dots, E_m; F) = \mathcal{L}_{wsc}^r(E_1, \dots, E_m; F)$.

4 COMPLEMENTATION IN VECTOR LATTICE SYMMETRIC TENSOR PRODUCTS

4.1 Preliminaries

For a vector space X and $n \in \mathbb{N}$, let $\otimes_n X$ denote the n -fold algebraic tensor product of X . For $x_1 \otimes \cdots \otimes x_n \in \otimes_n X$, let $x_1 \otimes_s \cdots \otimes_s x_n$ denote its symmetrization, that is,

$$x_1 \otimes_s \cdots \otimes_s x_n = \frac{1}{n!} \sum_{\sigma \in \pi(n)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}, \quad (4.1)$$

where $\pi(n)$ is the group of permutations of $\{1, \dots, n\}$. Let $\otimes_{n,s} X$ denote the n -fold algebraic symmetric tensor product of X , that is, the linear span of $\{x_1 \otimes_s \cdots \otimes_s x_n : x_1, \dots, x_n \in X\}$ in $\otimes_n X$. Define $\theta_n : X \rightarrow \otimes_{n,s} X$ by

$$\theta_n(x) = x \otimes \overbrace{\cdots}^n \otimes x = x \otimes_s \overbrace{\cdots}^n \otimes_s x, \quad \forall x \in X. \quad (4.2)$$

Then θ_n is an n -homogeneous polynomial. It is known that each $u \in \otimes_{n,s} X$ admits a representation $u = \sum_{i=1}^m \lambda_i \theta_n(x_i)$, where $\lambda_1, \dots, \lambda_m$ are scalars and x_1, \dots, x_m are vectors in X .

Let X and Y be vector spaces. For an n -homogeneous polynomial $P : X \rightarrow Y$, let $T_P : X^n \rightarrow Y$ denote its associated symmetric n -linear operator, which is related to P by

the following *Polarization Formula*

$$T_P(x_1, \dots, x_n) = \frac{1}{2^n n!} \sum_{\delta_i = \pm 1} \delta_1 \cdots \delta_n P\left(\sum_{i=1}^n \delta_i x_i\right), \quad x_1, \dots, x_n \in X. \quad (4.3)$$

Each n -homogeneous polynomial $P : X \rightarrow Y$ induces a unique linear operator $\tilde{P} : \otimes_{n,s} X \rightarrow Y$, called the *linearization* of P , such that $P = \tilde{P} \circ \theta_n$. Moreover,

$$\tilde{P}(u) = \sum_{i=1}^m \lambda_i P(x_i), \quad \forall u = \sum_{i=1}^m \lambda_i \theta_n(x_i) \in \otimes_{n,s} X. \quad (4.4)$$

For the basic knowledge about homogeneous polynomials and symmetric tensor products, we refer to [21, 22, 38, 42].

For a vector lattice E , let E_+ denote its positive cone and E^\sim denote its order dual. A sequence (x_n) in E is called *uniformly convergent* to $x \in E$ (or *uniformly Cauchy*) if there exist $u \in E_+$ and a scalar null sequence $(\alpha_n) > 0$ such that

$$|x_n - x| \leq \alpha_n u \quad (\text{or } |x_m - x_n| \leq |\alpha_m - \alpha_n| u), \quad m, n = 1, 2, \dots$$

A vector lattice E is called *uniformly complete* if every uniformly Cauchy sequence in E is uniformly convergent.

Let E and F be vector lattices. An n -linear operator $T : E^n \rightarrow F$ is called (i) *positive* if $T(x_1, \dots, x_n) \in F_+$ whenever $x_1, \dots, x_n \in E_+$; (ii) *regular* if it is a difference of two positive n -linear operators; and (iii) *lattice n -morphism* if $|T(x_1, \dots, x_n)| = T(|x_1|, \dots, |x_n|)$ for every $x_1, \dots, x_n \in E$. An n -homogeneous polynomial $P : E \rightarrow F$ is called (i) *positive* if its associated symmetric n -linear operator T_P is positive; (ii) *regular* if it is a difference of two positive n -homogeneous polynomials; and (iii) *lattice homomorphism* if its associated symmetric n -linear operator T_P is a lattice n -morphism.

Next we introduce the Fremlin vector lattice symmetric tensor product. Let E be a vector lattice. Then $\otimes_{n,s}E$ with the positive cone generated by $\{\theta_n(x) : x \in E_+\}$ is an ordered vector space. The n -fold Fremlin vector lattice symmetric tensor product of E is a pair $(\bar{\otimes}_{n,s}E, \theta_n)$ such that

(a) $\bar{\otimes}_{n,s}E$ is a vector lattice in which $\otimes_{n,s}E$ is embedded as a linear subspace, and the n -homogeneous polynomial $\theta_n : E \rightarrow \bar{\otimes}_{n,s}E$ is a lattice homomorphism,

(b) for any vector lattice F and any n -homogeneous polynomial $P : E \rightarrow F$ that is a lattice homomorphism, there exists a unique linear operator $\tilde{P} : \bar{\otimes}_{n,s}E \rightarrow F$, called the *linearization* of P , such that $P = \tilde{P} \circ \theta_n$ and \tilde{P} is also a lattice homomorphism,

(c) for any uniformly complete vector lattice F and any positive n -homogeneous polynomial $P : E \rightarrow F$, there exists a unique positive linear operator $\tilde{P} : \bar{\otimes}_{n,s}E \rightarrow F$, called the *linearization* of P , such that $P = \tilde{P} \circ \theta_n$,

(d) $\otimes_{n,s}E$ is dense in $\bar{\otimes}_{n,s}E$ in the sense that for any $u \in \bar{\otimes}_{n,s}E$ there exists $x \in E_+$ for which for every $\delta > 0$, there exists $v \in \otimes_{n,s}E$ such that $|u - v| \leq \delta \theta_n(x)$,

(e) $\otimes_{n,s}E$ is order dense in $\bar{\otimes}_{n,s}E$ in the sense that for any $u \in \bar{\otimes}_{n,s}E$ with $u > 0$, there exists $v \in \otimes_{n,s}E$ such that $0 < v < u$, and

(f) for any $u \in \bar{\otimes}_{n,s}E$ there exists $x \in E_+$ such that $|u| \leq \theta_n(x)$.

Note that the symmetric n -linear operator $T_{\theta_n} : E^n \rightarrow \bar{\otimes}_{n,s}E$ associated to $\theta_n : E \rightarrow \bar{\otimes}_{n,s}E$ is as follows.

$$T_{\theta_n}(x_1, \dots, x_n) = x_1 \otimes_s \cdots \otimes_s x_n, \quad \forall x_1, \dots, x_n \in E.$$

Since the n -homogeneous polynomial θ_n is a lattice homomorphism, it follows that T_{θ_n} is a lattice n -morphism and hence,

$$|x_1 \otimes_s \cdots \otimes_s x_n| = |x_1| \otimes_s \cdots \otimes_s |x_n|, \quad \forall x_1, \dots, x_n \in E. \quad (4.5)$$

For the basic knowledge about Fremlin vector lattice symmetric tensor products, we refer to [23, 24, 45] (also see [32]).

Recall that two elements $x, y \in E$ are called disjoint, denoted by $x \perp y$, if $|x| \wedge |y| = 0$. It is known that $x \perp y$ if and only if $|x + y| = |x - y|$. The following lemma is needed in section 4.2.

Lemma 4.1.1. Let $x, y \in E$ be such that $x \perp y$, and let $u, v \in \otimes_{n-1, s} E$. Then $(x \otimes_s u) \perp (y \otimes_s v)$ in $\bar{\otimes}_{n, s} E$.

Proof. Note that

$$|x \otimes_s u| = |x| \otimes_s |u| \leq |x| \otimes_s w, \quad |y \otimes_s v| = |y| \otimes_s |v| \leq |y| \otimes_s w,$$

where $w = |u| + |v|$. Since $x \perp y$, it follows that

$$|x \otimes_s w + y \otimes_s w| = |x + y| \otimes_s w = |x - y| \otimes_s w = |x \otimes_s w - y \otimes_s w|,$$

which implies that $(x \otimes_s w) \perp (y \otimes_s w)$ and hence, $(x \otimes_s u) \perp (y \otimes_s v)$. □

4.2 Vector lattice tensor product case

In this section, E is a vector lattice.

Theorem 4.2.1. Let E be a vector lattice and $n \in \mathbb{N}$. Then $\bar{\otimes}_{n,s}E$ is lattice isomorphic to a sublattice of $\bar{\otimes}_{n+1,s}E$.

Proof. Take $e \in E_+$ and define an n -homogeneous polynomial $P : E \rightarrow \bar{\otimes}_{n+1,s}E$ by $P(x) = e \otimes_s \theta_n(x)$ for every $x \in E$. Then its associated symmetric n -linear operator T_P is as follows.

$$T_P(x_1, \dots, x_n) = e \otimes_s x_1 \otimes_s \cdots \otimes_s x_n$$

for every $x_1, \dots, x_n \in E$. It follows from (4.5) that T_P is a lattice n -morphism and hence, P is a lattice homomorphism. Thus its linearization $\tilde{P} : \bar{\otimes}_{n,s}E \rightarrow \bar{\otimes}_{n+1,s}E$ is also a lattice homomorphism. Next we show that \tilde{P} is injective.

First suppose that $\tilde{P}(v) = 0$ for some $v \in \bar{\otimes}_{n,s}E$, say $v = \sum_{i=1}^m \lambda_i \theta_n(x_i)$. By (4.4),

$$0 = \tilde{P}(v) = \sum_{i=1}^m \lambda_i P(x_i) = \sum_{i=1}^m \lambda_i e \otimes_s \theta_n(x_i) = e \otimes_s v,$$

which implies that $v = 0$. Now suppose that $\tilde{P}(u) = 0$ for some $u \in \bar{\otimes}_{n,s}E$. If $u \neq 0$, then $|u| > 0$. Since $\bar{\otimes}_{n,s}E$ is order dense in $\bar{\otimes}_{n+1,s}E$ in the sense (e) in section 4.1, there exists $v \in \bar{\otimes}_{n,s}E$ such that $0 < v < |u|$. Thus

$$0 = \tilde{P}(0) \leq \tilde{P}(v) \leq \tilde{P}(|u|) = |\tilde{P}(u)| = 0,$$

which implies that $\tilde{P}(v) = 0$ and hence, $v = 0$. This contradiction shows that $u = 0$. Therefore, \tilde{P} is injective and the proof is complete. \square

It is of interest to ask under what conditions the image of $\bar{\otimes}_{n,s}E$ is complemented in $\bar{\otimes}_{n+1,s}E$. From now on, we assume that E is a vector lattice with $E^\sim \neq \{0\}$. Take $e \in E_+$

and $\phi \in (E^*)_+$ such that $\phi(e) = 1$. Recall that for $x, y \in X$ and $m, n \in \mathbb{N}$,

$$\theta_n(x) = x \otimes \overbrace{\cdots}^n \otimes x = x \otimes_s \overbrace{\cdots}^n \otimes_s x \quad (4.6)$$

and

$$\theta_n(x) \otimes_s \theta_m(y) = x \otimes_s \overbrace{\cdots}^n \otimes_s x \otimes_s y \otimes_s \overbrace{\cdots}^m \otimes_s y. \quad (4.7)$$

The mappings $J_{n \rightarrow n+1}$ and $K_{n+1 \rightarrow n}$ are introduced by Blasco in [6] as follows. Define

$J_{n \rightarrow n+1} : \otimes_{n,s} E \rightarrow \otimes_{n+1,s} E$ by

$$J_{n \rightarrow n+1}(\theta_n(x)) = \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \phi(x)^{k-1} \theta_k(e) \otimes_s \theta_{n-k+1}(x) \quad (4.8)$$

for every $x \in E$ and by

$$J_{n \rightarrow n+1}(u) = \sum_{i=1}^m \lambda_i J_{n \rightarrow n+1}(\theta_n(x_i)) \quad (4.9)$$

for every $u = \sum_{i=1}^m \lambda_i \theta_n(x_i) \in \otimes_{n,s} E$. Then $J_{n \rightarrow n+1}$ is linear and injective. Define $K_{n+1 \rightarrow n} : \otimes_{n+1,s} E \rightarrow \otimes_{n,s} E$ by

$$K_{n+1 \rightarrow n} \left(\sum_{i=1}^m \lambda_i \theta_{n+1}(x_i) \right) = \sum_{i=1}^m \lambda_i \phi(x_i) \theta_n(x_i) \quad (4.10)$$

for every $v = \sum_{i=1}^m \lambda_i \theta_{n+1}(x_i) \in \otimes_{n+1,s} E$. Then $K_{n+1 \rightarrow n}$ is linear and surjective such that

$$K_{n+1 \rightarrow n} \circ J_{n \rightarrow n+1} = id_{\otimes_{n,s} E}. \quad (4.11)$$

Moreover, for every $x \in E$,

$$J_{n \rightarrow n+1}(\theta_n(x)) \phi(x) = \theta_{n+1}(x) - \theta_{n+1}(x - \phi(x)e). \quad (4.12)$$

If, in addition, $x \in \ker \phi$, the kernel of ϕ , then

$$J_{n \rightarrow n+1}(\theta_n(x)) = (n+1)e \otimes_s \theta_n(x). \quad (4.13)$$

All these properties of $J_{n \rightarrow n+1}$ and $K_{n+1 \rightarrow n}$ come from [6].

From positivity perspective, $\otimes_{n,s}E$ is an ordered subspace of the vector lattice $\bar{\otimes}_{n,s}E$. Note that each positive $u \in \otimes_{n,s}E$ has a representation $u = \sum_{i=1}^m \lambda_i \theta_n(x_i)$, where $\lambda_i > 0$ and $x_i > 0$ for $i = 1, \dots, m$. It follows from (4.10), (4.12), and (4.13) that J and K are positive.

Theorem 4.2.2. Let E be a vector lattice such that $E^\sim \neq \{0\}$ and $\bar{\otimes}_{n+1,s}E$ is uniformly complete. Then $\bar{\otimes}_{n,s}E$ is positively isomorphic to a complemented subspace of $\bar{\otimes}_{n+1,s}E$.

Proof. First we show that $J_{n \rightarrow n+1} : \otimes_{n,s}E \rightarrow \otimes_{n+1,s}E$ defined in (4.8) and (4.9) can be positively extended to $\tilde{J}_{n \rightarrow n+1} : \bar{\otimes}_{n,s}E \rightarrow \bar{\otimes}_{n+1,s}E$. Take any $u \in \bar{\otimes}_{n,s}E$. Since $\otimes_{n,s}E$ is dense in $\bar{\otimes}_{n,s}E$ in the sense of (d) in section 4.1, there exists $x \in E_+$ for which for any $k \in \mathbb{N}$ there exists $v_k \in \otimes_{n,s}E$ such that $|u - v_k| \leq \frac{1}{k} \theta_n(x)$. It is easy to see that $J_{n \rightarrow n+1}(v_k)$ is a uniformly Cauchy sequence in $\bar{\otimes}_{n+1,s}E$ and hence, its limit exists, which is defined to be $\tilde{J}_{n \rightarrow n+1}(u)$. Thus $J_{n \rightarrow n+1}$ is extended to $\tilde{J}_{n \rightarrow n+1}$. Since $J_{n \rightarrow n+1} : \otimes_{n,s}E \rightarrow \otimes_{n+1,s}E$ is injective and $\otimes_{n,s}E$ is order dense in $\bar{\otimes}_{n,s}E$ in the sense of (e) in section 4.1, it follows that $\tilde{J}_{n \rightarrow n+1} : \bar{\otimes}_{n,s}E \rightarrow \bar{\otimes}_{n+1,s}E$ is also injective (please refer to the proof of Theorem 4.2.1). Therefore, $\bar{\otimes}_{n,s}E$ is linearly isomorphic to $\tilde{J}_{n \rightarrow n+1}[\bar{\otimes}_{n,s}E]$, the image of $\bar{\otimes}_{n,s}E$ under the mapping $\tilde{J}_{n \rightarrow n+1}$. Next we show that $\tilde{J}_{n \rightarrow n+1}[\bar{\otimes}_{n,s}E]$ is a complemented subspace of $\bar{\otimes}_{n+1,s}E$.

Note that the uniform completeness of $\bar{\otimes}_{n+1,s}E$ implies the uniform completeness of $\bar{\otimes}_{n,s}E$. Similarly $K_{n+1 \rightarrow n} : \otimes_{n+1,s}E \rightarrow \otimes_{n,s}E$ defined in (4.10) can be positively extended to $\tilde{K}_{n+1 \rightarrow n} : \bar{\otimes}_{n+1,s}E \rightarrow \bar{\otimes}_{n,s}E$. Take any $u \in \bar{\otimes}_{n,s}E$. Since $\otimes_{n,s}E$ is dense in $\bar{\otimes}_{n,s}E$ in the sense of (d) in section 4.1, there exists $x \in E_+$ for which for any $\delta > 0$ there exists $v \in \otimes_{n,s}E$

such that $|u - v| \leq \delta\theta_n(x)$. It follows from (4.11) that

$$\begin{aligned}
|(\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1})(u) - u| &= |(\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1})(u) - v + v - u| \\
&= |(\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1})(u) - (\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1})(v) - (u - v)| \\
&\leq |(\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1})(u - v)| + |u - v| \\
&\leq (\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1})\left(|(u - v)|\right) + |u - v| \\
&\leq (\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1})(\delta\theta_n(x)) + \delta\theta_n(x) = 2\delta\theta_n(x),
\end{aligned}$$

which implies that $(\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1})(u) = u$ and hence, $\tilde{K}_{n+1 \rightarrow n} \circ \tilde{J}_{n \rightarrow n+1} = id_{\bar{\otimes}_{n,s}E}$. Therefore, $\tilde{J}_{n \rightarrow n+1} \circ \tilde{K}_{n+1 \rightarrow n}$ is a projection on $\bar{\otimes}_{n+1,s}E$ and $\tilde{J}_{n \rightarrow n+1}[\bar{\otimes}_{n,s}E]$ is a complemented subspace of $\bar{\otimes}_{n+1,s}E$. \square

It also is of interest to ask under what conditions the image of $\bar{\otimes}_{n,s}E$ is a band in $\bar{\otimes}_{n+1,s}E$. In this case, we need that $\ker(\phi)$ is a band in E . By [37, Proposition 1.4.8], ϕ is a lattice homomorphism if and only if $\ker(\phi)$ is an ideal in E . Moreover, if $\ker(\phi)$ is a projection band in E , then $E = \ker(\phi) \oplus \ker(\phi)^\perp$. Thus for any $x \in E$, $(x - \phi(x)e) \perp \phi(x)e$, which implies that $|x| = |x - \phi(x)e| + |\phi(x)e|$. If, in addition, $x \geq 0$, then $x - \phi(x)e = |x| - |\phi(x)e| = |x - \phi(x)e| \geq 0$. In summary, if $\ker(\phi)$ is a projection band in E , then

$$|x - \phi(x)e| = |x| - |\phi(x)e|, \quad \forall x \in E \quad (4.14)$$

and

$$x - \phi(x)e \geq 0, \quad \forall x \in E_+. \quad (4.15)$$

We need to consider two cases of even n and odd n separately. We will show that $J_{2n \rightarrow 2n+1} : \otimes_{2n,s}E \rightarrow \otimes_{2n+1,s}E$ and $J_{2n-1 \rightarrow 2n} : \otimes_{2n-1,s}E \rightarrow \otimes_{2n,s}E$ are lattice homomorphisms. To do so, we first reformulate $J_{2n \rightarrow 2n+1}$ and $J_{2n-1 \rightarrow 2n}$ in terms of $x - \phi(x)e$.

In order to construct the formula for $J_{n \rightarrow n+1}$ we introduce the coefficients of the corresponding binoms of each term of $J_{n \rightarrow n+1}$ in a form of a double sequence $A_i^{(k)}$. (k) denotes the number of step and i denotes the number of the corresponding binom.

$$A_1^{(1)} = A_2^{(1)} = \dots = 1$$

$$A_1^{(2)} = A_1^{(1)} = 1,$$

$$A_2^{(2)} = A_1^{(1)} + A_2^{(1)} = 1 + 1 = 2,$$

...

$$A_i^{(2)} = \sum_{j=1}^i A_j^{(1)} = 1 + \dots + 1 = i$$

$$A_1^{(3)} = A_1^{(2)} = 1,$$

$$A_2^{(3)} = A_1^{(2)} + A_2^{(2)} = 1 + 2 = 3,$$

$$A_3^{(3)} = A_1^{(2)} + A_2^{(2)} + A_3^{(2)} = 1 + 2 + 3 = 6,$$

...

$$A_i^{(3)} = \sum_{j=1}^i A_j^{(2)} = 1 + 2 + \cdots + i = \frac{i(i+1)}{2}$$

$$A_1^{(4)} = A_1^{(3)} = 1,$$

$$A_2^{(4)} = A_1^{(3)} + A_2^{(3)} = 1 + 3 = 4,$$

$$A_3^{(4)} = A_1^{(3)} + A_2^{(3)} + A_3^{(3)} = 1 + 3 + 6 = 10,$$

...

$$A_i^{(4)} = \sum_{j=1}^i A_j^{(3)} = 1 + 3 + 6 + 10 + 15 + \cdots + \frac{i(i+1)}{2}$$

The general formula for the i -th coefficient of the k -th step is

$$A_i^{(k)} = \sum_{j=1}^i A_j^{(k-1)}, \quad k = 2, 3, \dots \quad \text{and} \quad i = 1, 2, \dots$$

For convenience, let $y = \phi(x)e$. Before passing to the general formula for the even case we give the particular case for $J_{8 \rightarrow 9}(\theta_8(x))$.

$$\begin{aligned} J_{8 \rightarrow 9}(\theta_8(x)) &= \sum_{k=1}^9 \binom{9}{k} (-1)^{k+1} \phi(x)^{k-1} \cdot \theta_k(e) \otimes_s \theta_{9-k}(x) \\ &= \binom{9}{1} e \otimes_s \theta_8(x) - \binom{9}{2} \phi(x) \theta_2(e) \otimes_s \theta_7(x) \\ &+ \binom{9}{3} \phi(x)^2 \theta_3(e) \otimes_s \theta_6(x) - \binom{9}{4} \phi(x)^3 \theta_4(e) \otimes_s \theta_5(x) \\ &+ \binom{9}{5} \phi(x)^4 \theta_5(e) \otimes_s \theta_4(x) - \binom{9}{6} \phi(x)^5 \theta_6(e) \otimes_s \theta_3(x) \end{aligned}$$

$$\begin{aligned}
& + \binom{9}{7} \phi(x)^6 \theta_7(e) \otimes_s \theta_2(x) - \binom{9}{8} \phi(x)^7 \theta_8(e) \otimes_s x \\
& + \phi(x)^8 \theta_9(e) =
\end{aligned}$$

$$\begin{aligned}
& = 9e \otimes_s \theta_8(x) - 36\phi(x)\theta_2(e) \otimes_s \theta_7(x) \\
& + 84\phi(x)^2\theta_3(e) \otimes_s \theta_6(x) - 126\phi(x)^3\theta_4(e) \otimes_s \theta_5(x) \\
& + 126\phi(x)^4\theta_5(e) \otimes_s \theta_4(x) - 84\phi(x)^5\theta_6(e) \otimes_s \theta_3(x) \\
& + 36\phi(x)^6\theta_7(e) \otimes_s \theta_2(x) - 9\phi(x)^7\theta_8(e) \otimes_s x \\
& + \phi(x)^8\theta_9(e) =
\end{aligned}$$

$$\begin{aligned}
& = e \otimes_s \left[9\theta_8(x) - 36y \otimes_s \theta_7(x) + 84\theta_2(y) \otimes_s \theta_6(x) \right. \\
& - 126\theta_3(y) \otimes_s \theta_5(x) + 126\theta_4(y) \otimes_s \theta_4(x) \\
& - 84\theta_5(y) \otimes_s \theta_3(x) + 36\theta_6(y) \otimes_s \theta_2(x) \\
& \left. - 9\theta_7(y) \otimes_s x + \theta_8(y) \right] =
\end{aligned}$$

$$\begin{aligned}
& = e \otimes_s \left[9\theta_7(x) \otimes_s (x - y) \right. \\
& - 27y \otimes_s \theta_6(x) \otimes_s (x - y) \\
& \left. + 57\theta_2(y) \otimes_s \theta_5(x) \otimes_s (x - y) \right]
\end{aligned}$$

$$\begin{aligned}
& - 69\theta_3(y) \otimes_s \theta_4(x) \otimes_s (x - y) \\
& + 57\theta_4(y) \otimes_s \theta_3(x) \otimes_s (x - y) \\
& - 27\theta_5(y) \otimes_s \theta_2(x) \otimes_s (x - y) \\
& + 9\theta_6(y) \otimes_s x \otimes_s (x - y) \\
& + \theta_7(y) \Big] =
\end{aligned}$$

$$\begin{aligned}
& = e \otimes_s \Big[9\theta_6(x) \otimes_s \theta_2(x - y) \\
& - 18y \otimes_s \theta_5(x) \otimes_s \theta_2(x - y) \\
& + 39\theta_2(y) \otimes_s \theta_4(x) \otimes_s \theta_2(x - y) \\
& - 30\theta_3(y) \otimes_s \theta_3(x) \otimes_s \theta_2(x - y) \\
& + 27\theta_4(y) \otimes_s \theta_2(x) \otimes_s \theta_2(x - y) \\
& + 9\theta_6(y) \otimes_s x \otimes_s (x - y) \\
& + \theta_7(y) \Big] =
\end{aligned}$$

$$\begin{aligned}
& = e \otimes_s \Big[9\theta_5(x) \otimes_s \theta_3(x - y) \\
& - 9y \otimes_s \theta_4(x) \otimes_s \theta_3(x - y) \\
& + 30\theta_2(y) \otimes_s \theta_3(x) \otimes_s \theta_3(x - y) \\
& + 27\theta_4(y) \otimes_s \theta_2(x) \otimes_s \theta_2(x - y) \\
& + 9\theta_6(y) \otimes_s x \otimes_s (x - y) \\
& + \theta_7(y) \Big] =
\end{aligned}$$

$$\begin{aligned}
&= e \otimes_s \left[9\theta_4(x) \otimes_s \theta_4(x-y) \right. \\
&\quad + 30\theta_2(y) \otimes_s \theta_3(x) \otimes_s \theta_3(x-y) \\
&\quad + 27\theta_4(y) \otimes_s \theta_2(x) \otimes_s \theta_2(x-y) \\
&\quad + 9\theta_6(y) \otimes_s x \otimes_s (x-y) \\
&\quad \left. + \theta_7(y) \right] =
\end{aligned}$$

$$\begin{aligned}
&= e \otimes_s \left[\binom{9}{1} \theta_4(x) \otimes_s \theta_4(x-y) \right. \\
&\quad + \left(\binom{9}{3} - 3\binom{9}{2} + 6\binom{9}{1} \right) \theta_2(y) \otimes_s \theta_3(x) \otimes_s \theta_3(x-y) \\
&\quad + \left(\binom{9}{5} - 2\binom{9}{4} + 3\binom{9}{3} - 4\binom{9}{2} + 5\binom{9}{1} \right) \theta_4(y) \otimes_s \theta_2(x) \otimes_s \theta_2(x-y) \\
&\quad + \left(\binom{9}{7} - \binom{9}{6} + \binom{9}{5} - \binom{9}{4} + \binom{9}{3} - \binom{9}{2} + \binom{9}{1} \right) \theta_6(y) \otimes_s x \otimes_s (x-y) \\
&\quad \left. + \theta_7(y) \right]
\end{aligned}$$

$$\begin{aligned}
J_{8 \rightarrow 9}(\theta_8(x)) &= \left\{ \left[\sum_{j=1}^7 (-1)^{j+1} A_j^{(1)} \binom{9}{9-(j+1)} \right] \otimes_{1,s} x \otimes_{1,s} (x-y) \otimes_{6,s} y \right. \\
&\quad \left. + \left[\sum_{j=1}^5 (-1)^{j+3} A_j^{(2)} \binom{9}{9-(j+3)} \right] \otimes_{2,s} x \otimes_{2,s} (x-y) \otimes_{4,s} y \right\}
\end{aligned}$$

$$\begin{aligned}
& + \left[\sum_{j=1}^3 (-1)^{j+5} A_j^{(3)} \binom{9}{9-(j+5)} \right] \otimes_{3,s} x \otimes_{3,s} (x-y) \otimes_{2,s} y \\
& + \left[\sum_{j=1}^1 (-1)^{j+7} A_j^{(4)} \binom{9}{9-(j+7)} \right] \otimes_{4,s} x \otimes_{4,s} (x-y) + \otimes_{8,s} y \} \otimes_s e.
\end{aligned}$$

After substituting back $y = \phi(x)e$ into the formula for $J_{8 \rightarrow 9}(\theta_8(x))$, we get

$$\begin{aligned}
J_{8 \rightarrow 9}(\theta_8(x)) & = \theta_8(\phi(x)e) \otimes_s e \\
& + 9x \otimes_s (x - \phi(x)e) \otimes_s \theta_6(\phi(x)e) \otimes_s e \\
& + 27\theta_2(x) \otimes_s \theta_2(x - \phi(x)e) \otimes_s \theta_4(\phi(x)e) \otimes_s e \\
& + 30\theta_3(x) \otimes_s \theta_3(x - \phi(x)e) \otimes_s \theta_2(\phi(x)e) \otimes_s e \\
& + 9\theta_4(x) \otimes_s \theta_4(x - \phi(x)e) \otimes_s e.
\end{aligned}$$

The general formula for $J_{2n \rightarrow 2n+1}(\theta_{2n}(x))$ (even case) will have the following form.

$$\begin{aligned}
J_{2n \rightarrow 2n+1}(\theta_{2n}(x)) & = \left\{ \left[\sum_{j=1}^{2n-1} (-1)^{j+1} A_j^{(1)} \binom{2n+1}{2n+1-(j+1)} \right] \otimes_{1,s} x \otimes_{1,s} (x-y) \otimes_{2n-2,s} y \right. \\
& + \left[\sum_{j=1}^{2n-3} (-1)^{j+3} A_j^{(2)} \binom{2n+1}{2n+1-(j+3)} \right] \otimes_{2,s} x \otimes_{2,s} (x-y) \otimes_{2n-4,s} y \\
& + \dots + \\
& + \left[\sum_{j=1}^{2n-(2k-1)} (-1)^{j+(2k-1)} A_j^{(k)} \binom{2n+1}{2n+1-(j+2k-1)} \right] \\
& \quad \otimes_{k,s} x \otimes_{k,s} (x-y) \otimes_{2n-2k,s} y \\
& + \dots + \\
& + \left. \left[\sum_{j=1}^1 (-1)^{j+(2n-1)} A_j^{(n)} \binom{2n+1}{2-j} \right] \otimes_{n,s} x \otimes_{n,s} (x-y) + \otimes_{2n,s} y \right\} \otimes_s e.
\end{aligned}$$

$$J_{2n \rightarrow 2n+1}(\theta_{2n}(x)) = \left\{ \sum_{k=1}^n \left[\sum_{j=1}^{2n-2k+1} (-1)^{j+2k-1} A_j^{(k)} \binom{2n+1}{2n-2k-j+2} \right] \right. \\ \left. \otimes_{k,s} x \otimes_{k,s} (x-y) \otimes_{2n-2k,s} y + \otimes_{2n,s} y \right\} \otimes_s e.$$

After making the notation

$$i = j + 2k - 1 \quad (j = i - 2k + 1),$$

we get

$$J_{2n \rightarrow 2n+1}(\theta_{2n}(x)) = \left\{ \sum_{k=1}^n \left[\sum_{i=2k}^{2n} (-1)^i A_{i-2k+1}^{(k)} \binom{2n+1}{2n+1-i} \right] \right. \\ \left. \otimes_{k,s} x \otimes_{k,s} (x-y) \otimes_{2n-2k,s} y + \otimes_{2n,s} y \right\} \otimes_s e.$$

Finally, by (4.6) and (4.7), $J_{2n \rightarrow 2n+1} : \otimes_{2n,s} E \rightarrow \otimes_{2n+1,s} E$ can be reformulated as follows.

$$J_{2n \rightarrow 2n+1}(\theta_{2n}(x)) = \theta_{2n}(\phi(x)e) \otimes_s e \\ + \sum_{k=1}^n b_k \theta_k(x) \otimes_s \theta_k(x - \phi(x)e) \otimes_s \theta_{2n-2k}(\phi(x)e) \otimes_s e, \quad (4.16)$$

where

$$b_k = \sum_{i=2k}^{2n} (-1)^i A_{i-2k+1}^{(k)} \binom{2n+1}{2n+1-i} > 0, \quad k = 1, 2, \dots,$$

Now we need the general formula for the odd case also and before passing to it we give the particular case for $J_{9 \rightarrow 10}(\theta_9(x))$.

$$J_{9 \rightarrow 10}(\theta_9(x)) = \sum_{k=1}^{10} \binom{10}{k} (-1)^{k+1} \phi(x)^{k-1} \cdot \theta_k(e) \otimes_s \theta_{10-k}(x) \\ = \binom{10}{1} e \otimes_s \theta_9(x) - \binom{10}{2} \phi(x) \theta_2(e) \otimes_s \theta_8(x)$$

$$\begin{aligned}
& + \binom{10}{3} \phi(x)^2 \theta_3(e) \otimes_s \theta_7(x) - \binom{10}{4} \phi(x)^3 \theta_4(e) \otimes_s \theta_6(x) \\
& + \binom{10}{5} \phi(x)^4 \theta_5(e) \otimes_s \theta_5(x) - \binom{10}{6} \phi(x)^5 \theta_6(e) \otimes_s \theta_4(x) \\
& + \binom{10}{7} \phi(x)^6 \theta_7(e) \otimes_s \theta_3(x) - \binom{10}{8} \phi(x)^7 \theta_8(e) \otimes_s \theta_2(x) \\
& + \binom{10}{9} \phi(x)^8 \theta_9(e) \otimes_s x - \phi(x)^9 \theta_{10}(e) = \\
\\
& = 10e \otimes_s \theta_9(x) - 45\phi(x)\theta_2(e) \otimes_s \theta_8(x) \\
& + 120\phi(x)^2\theta_3(e) \otimes_s \theta_7(x) - 210\phi(x)^3\theta_4(e) \otimes_s \theta_6(x) \\
& + 252\phi(x)^4\theta_5(e) \otimes_s \theta_5(x) - 210\phi(x)^5\theta_6(e) \otimes_s \theta_4(x) \\
& + 120\phi(x)^6\theta_7(e) \otimes_s \theta_3(x) - 45\phi(x)^7\theta_8(e) \otimes_s \theta_2(x) \\
& + 10\phi(x)^8\theta_9(e) \otimes_s x - \phi(x)^9\theta_{10}(e) = \\
\\
& = e \otimes_s \left[\theta_9(x) + 9\theta_9(x) - 45y \otimes_s \theta_8(x) + 120\theta_2(y) \otimes_s \theta_7(x) \right. \\
& - 210\theta_3(y) \otimes_s \theta_6(x) + 252\theta_4(y) \otimes_s \theta_5(x) \\
& - 210\theta_5(y) \otimes_s \theta_4(x) + 120\theta_6(y) \otimes_s \theta_3(x) \\
& \left. - 45\theta_7(y) \otimes_s \theta_2(x) + 10\theta_8(y) \otimes_s x - \theta_9(y) \right] =
\end{aligned}$$

$$\begin{aligned}
&= e \otimes_s \left[\theta_9(x) + 9\theta_8(x) \otimes_s (x - y) \right. \\
&- 36y \otimes_s \theta_7(x) \otimes_s (x - y) \\
&+ 84\theta_2(y) \otimes_s \theta_6(x) \otimes_s (x - y) \\
&- 126\theta_3(y) \otimes_s \theta_5(x) \otimes_s (x - y) \\
&+ 126\theta_4(y) \otimes_s \theta_4(x) \otimes_s (x - y) \\
&- 84\theta_5(y) \otimes_s \theta_3(x) \otimes_s (x - y) \\
&+ 36\theta_6(y) \otimes_s \theta_2(x) \otimes_s (x - y) \\
&- 9\theta_7(y) \otimes_s x \otimes_s (x - y) \\
&\left. + \theta_8(y) \otimes_s (x - y) \right] =
\end{aligned}$$

$$\begin{aligned}
&= e \otimes_s \left[\theta_9(x) + 9\theta_7(x) \otimes_s \theta_2(x - y) \right. \\
&- 27y \otimes_s \theta_6(x) \otimes_s \theta_2(x - y) \\
&+ 57\theta_2(y) \otimes_s \theta_5(x) \otimes_s \theta_2(x - y) \\
&- 69\theta_3(y) \otimes_s \theta_4(x) \otimes_s \theta_2(x - y) \\
&+ 57\theta_4(y) \otimes_s \theta_3(x) \otimes_s \theta_2(x - y) \\
&- 27\theta_5(y) \otimes_s \theta_2(x) \otimes_s \theta_2(x - y) \\
&+ 9\theta_6(y) \otimes_s x \otimes_s \theta_2(x - y) \\
&\left. + \theta_8(y) \otimes_s (x - y) \right] =
\end{aligned}$$

$$\begin{aligned}
&= e \otimes_s \left[\theta_9(x) + 9\theta_6(x) \otimes_s \theta_3(x - y) \right. \\
&- 18y \otimes_s \theta_5(x) \otimes_s \theta_3(x - y) \\
&+ 39\theta_2(y) \otimes_s \theta_4(x) \otimes_s \theta_3(x - y) \\
&- 30\theta_3(y) \otimes_s \theta_3(x) \otimes_s \theta_3(x - y) \\
&+ 27\theta_4(y) \otimes_s \theta_2(x) \otimes_s \theta_3(x - y) \\
&+ 9\theta_6(y) \otimes_s x \otimes_s \theta_2(x - y) \\
&\left. + \theta_8(y) \otimes_s (x - y) \right] =
\end{aligned}$$

$$\begin{aligned}
&= e \otimes_s \left[\theta_9(x) + 9\theta_5(x) \otimes_s \theta_4(x - y) \right. \\
&- 9y \otimes_s \theta_4(x) \otimes_s \theta_4(x - y) \\
&+ 30\theta_2(y) \otimes_s \theta_3(x) \otimes_s \theta_4(x - y) \\
&+ 27\theta_4(y) \otimes_s \theta_2(x) \otimes_s \theta_3(x - y) \\
&+ 9\theta_6(y) \otimes_s x \otimes_s \theta_2(x - y) \\
&\left. + \theta_8(y) \otimes_s (x - y) \right] =
\end{aligned}$$

$$\begin{aligned}
&= e \otimes_s \left[\theta_9(x) + 9\theta_4(x) \otimes_s \theta_5(x - y) \right. \\
&+ 30\theta_2(y) \otimes_s \theta_3(x) \otimes_s \theta_4(x - y) \\
&\left. + 27\theta_4(y) \otimes_s \theta_2(x) \otimes_s \theta_3(x - y) \right]
\end{aligned}$$

$$\begin{aligned}
& + 9\theta_6(y) \otimes_s x \otimes_s \theta_2(x - y) \\
& + \theta_8(y) \otimes_s (x - y) \Big] =
\end{aligned}$$

$$\begin{aligned}
= & e \otimes_s \left[\theta_9(x) + \left(\binom{10}{1} - \binom{10}{0} \right) \theta_4(x) \otimes_s \theta_5(x - y) \right. \\
& + \left(\binom{10}{3} - 4\binom{10}{2} + 10\binom{10}{1} - 10\binom{10}{0} \right) \theta_2(y) \otimes_s \theta_3(x) \otimes_s \theta_4(x - y) \\
& + \left(\binom{10}{5} - 3\binom{10}{4} + 6\binom{10}{3} - 10\binom{10}{2} + 15\binom{10}{1} - 15\binom{10}{0} \right) \theta_4(y) \otimes_s \theta_2(x) \otimes_s \theta_3(x - y) \\
& + \left(\binom{10}{7} - 2\binom{10}{6} + 3\binom{10}{5} - 4\binom{10}{4} + 5\binom{10}{3} - 6\binom{10}{2} + 7\binom{10}{1} \right. \\
& \left. - 7\binom{10}{0} \right) \theta_6(y) \otimes_s x \otimes_s \theta_2(x - y) \\
& \left. + \theta_8(y) \otimes_s (x - y) \right]
\end{aligned}$$

$$\begin{aligned}
J_{9 \rightarrow 10}(\theta_9(x)) & = \left\{ \left[\sum_{j=1}^9 (-1)^{j+1} A_j^{(1)} \binom{10}{10-j} - A_9^{(1)} \right] \otimes_{1,s} (x - y) \otimes_{8,s} y \right. \\
& + \left[\sum_{j=1}^7 (-1)^{j+3} A_j^{(2)} \binom{10}{10-(j+2)} - A_7^{(2)} \right] \otimes_{1,s} x \otimes_{2,s} (x - y) \otimes_{6,s} y \\
& + \left[\sum_{j=1}^5 (-1)^{j+5} A_j^{(3)} \binom{10}{10-(j+4)} - A_5^{(3)} \right] \otimes_{2,s} x \otimes_{3,s} (x - y) \otimes_{4,s} y \\
& + \left[\sum_{j=1}^3 (-1)^{j+7} A_j^{(4)} \binom{10}{10-(j+6)} - A_3^{(4)} \right] \otimes_{3,s} x \otimes_{4,s} (x - y) \otimes_{2,s} y \\
& \left. + \left[\sum_{j=1}^1 (-1)^{j+9} A_j^{(5)} \binom{10}{10-(j+8)} - A_1^{(5)} \right] \otimes_{4,s} x \otimes_{5,s} (x - y) + \otimes_{9,s} x \right\} \otimes_s e.
\end{aligned}$$

By (4.6) and (4.7) the formula for $J_{9 \rightarrow 10} : \otimes_{9,s} E \rightarrow \otimes_{10,s} E$ is reformulated as follows after substituting back $y = \phi(x)e$.

$$\begin{aligned}
J_{9 \rightarrow 10}(\theta_9(x)) &= \theta_9(x) \otimes_s e + (x - \phi(x)e) \otimes_s \theta_8(\phi(x)e) \otimes_s e \\
&+ 9x \otimes_s \theta_2(x - \phi(x)e) \otimes_s \theta_6(\phi(x)e) \otimes_s e \\
&+ 27\theta_2(x) \otimes_s \theta_3(x - \phi(x)e) \otimes_s \theta_4(\phi(x)e) \otimes_s e \\
&+ 30\theta_3(x) \otimes_s \theta_4(x - \phi(x)e) \otimes_s \theta_2(\phi(x)e) \otimes_s e \\
&+ 9\theta_4(x) \otimes_s \theta_5(x - \phi(x)e) \otimes_s e.
\end{aligned}$$

The general formula for $J_{2n-1 \rightarrow 2n}(\theta_{2n-1}(x))$ (odd case) has the following form.

$$\begin{aligned}
J_{2n-1 \rightarrow 2n}(\theta_{2n-1}(x)) &= \left\{ \left[\sum_{j=1}^{2n-1} (-1)^{j+1} A_j^{(1)} \binom{2n}{2n-j} - A_{2n-1}^{(1)} \right] \otimes_{1,s} (x-y) \otimes_{2n-2,s} y \right. \\
&+ \left[\sum_{j=1}^{2n-3} (-1)^{j+3} A_j^{(2)} \binom{2n}{2n-(j+2)} - A_{2n-3}^{(2)} \right] \otimes_{1,s} x \otimes_{2,s} (x-y) \\
&\quad \otimes_{2n-4,s} y \\
&+ \dots + \\
&+ \left[\sum_{j=1}^{2n-(2k-1)} (-1)^{j+(2k-1)} A_j^{(k)} \binom{2n}{2n-(j+2k-2)} - A_{2n-(2k-1)}^{(k)} \right] \\
&\quad \otimes_{k-1,s} x \otimes_{k,s} (x-y) \otimes_{2n-2k,s} y \\
&+ \dots + \\
&+ \left[\sum_{j=1}^1 (-1)^{j+(2n-1)} A_j^{(n)} \binom{2n}{2-j} - A_1^{(n)} \right] \otimes_{n-1,s} x \otimes_{n,s} (x-y) \\
&+ \left. \otimes_{2n-1,s} x \right\} \otimes_s e.
\end{aligned}$$

$$J_{2n-1 \rightarrow 2n}(\theta_{2n-1}(x)) = \left\{ \sum_{k=1}^n \left[\sum_{j=1}^{2n-2k+1} (-1)^{j+2k-1} A_j^{(k)} \binom{2n}{2n-2k-j+2} - A_{2n-2k+1}^{(k)} \right] \otimes_{k-1,s} x \otimes_{k,s} (x-y) \otimes_{2n-2k,s} y + \otimes_{2n-1,s} x \right\} \otimes_s e.$$

After making the notation

$$i = j + 2k - 1 \quad (j = i - 2k + 1),$$

we finally get

$$J_{2n-1 \rightarrow 2n}(\theta_{2n-1}(x)) = \left\{ \sum_{k=1}^n \left[\sum_{i=2k}^{2n} (-1)^i A_{i-2k+1}^{(k)} \binom{2n}{2n-i+1} - A_{2n-(2k-1)}^{(k)} \right] \otimes_{k-1,s} x \otimes_{k,s} (x-y) \otimes_{2n-2k,s} y + \otimes_{2n-1,s} x \right\} \otimes_s e.$$

By (4.6) and (4.7) the formula for $J_{2n-1 \rightarrow 2n} : \otimes_{2n-1,s} E \rightarrow \otimes_{2n,s} E$ can be reformulated as follows.

$$\begin{aligned} J_{2n-1 \rightarrow 2n}(\theta_{2n-1}(x)) &= \theta_{2n-1}(x) \otimes_s e \\ &+ \sum_{k=1}^n c_k \theta_{k-1}(x) \otimes_s \theta_k(x - \phi(x)e) \otimes_s \theta_{2n-2k}(\phi(x)e) \otimes_s e, \end{aligned} \quad (4.17)$$

where

$$c_k = \sum_{i=2k}^{2n} (-1)^i A_{i-2k+1}^{(k)} \binom{2n}{2n-i+1} - A_{2n-(2k-1)}^{(k)} > 0, \quad k = 1, 2, \dots$$

Theorem 4.2.3. Let E be a vector lattice with $E^\sim \neq \{0\}$. If there exists $\phi \in (E^\sim)_+$ such that $\ker(\phi)$ is a projection band in E , then $\bar{\otimes}_{n,s} E$ is lattice isomorphic to a projection band of $\bar{\otimes}_{n+1,s} E$.

Proof. Let $P_n : E \rightarrow \bar{\otimes}_{n+1,s} E$ denote the n -homogeneous polynomial induced by $J_{n \rightarrow n+1} : \otimes_{n,s} E \rightarrow \otimes_{n+1,s} E$ introduced in (4.8) and (4.9), that is, $P_n(x) = J_{n \rightarrow n+1}(\theta_n(x))$ for each $x \in E$. To show that P_n is a lattice homomorphism for any $n \in \mathbb{N}$, we consider two cases of even n and odd n separately and show that $P_{2n} : E \rightarrow \bar{\otimes}_{2n+1,s} E$ and $P_{2n-1} : E \rightarrow \bar{\otimes}_{2n,s} E$ are lattice homomorphisms for any $n \in \mathbb{N}$.

According to the reformulated forms of $J_{2n \rightarrow 2n+1}$ in (4.16) and $J_{2n-1 \rightarrow 2n}$ in (4.17), the symmetric $(2n)$ -linear operator $T_{2n} : E^{2n} \rightarrow \bar{\otimes}_{2n+1,s} E$ associated to P_{2n} and the symmetric $(2n-1)$ -linear operator $T_{2n-1} : E^{2n-1} \rightarrow \bar{\otimes}_{2n,s} E$ associated to P_{2n-1} are, respectively, as follows

$$\begin{aligned}
T_{2n}(x_1, \dots, x_{2n}) &= \phi(x_1)e \otimes_s \cdots \otimes_s \phi(x_{2n})e \otimes_s e \\
&+ \frac{1}{(2n)!} \sum_{\sigma \in \pi(2n)} \left[\sum_{k=1}^n b_k x_{\sigma(1)} \otimes_s \cdots \otimes_s x_{\sigma(k)} \otimes_s \right. \\
&\quad \left(x_{\sigma(k+1)} - \phi(x_{\sigma(k+1)})e \right) \otimes_s \cdots \otimes_s \left(x_{\sigma(2k)} - \phi(x_{\sigma(2k)})e \right) \otimes_s \\
&\quad \left. \phi(x_{\sigma(2k+1)})e \otimes_s \cdots \otimes_s \phi(x_{\sigma(2n)})e \otimes_s e \right], \tag{4.18}
\end{aligned}$$

and

$$\begin{aligned}
T_{2n-1}(x_1, \dots, x_{2n-1}) &= x_1 \otimes_s \cdots \otimes_s x_{2n-1} \otimes_s e \\
&+ \frac{1}{(2n-1)!} \sum_{\sigma \in \pi(2n-1)} \left[\sum_{k=1}^n c_k x_{\sigma(1)} \otimes_s \cdots \otimes_s x_{\sigma(k-1)} \otimes_s \right. \\
&\quad \left(x_{\sigma(k)} - \phi(x_{\sigma(k)})e \right) \otimes_s \cdots \otimes_s \left(x_{\sigma(2k-1)} - \phi(x_{\sigma(2k-1)})e \right) \otimes_s \\
&\quad \left. \phi(x_{\sigma(2k)})e \otimes_s \cdots \otimes_s \phi(x_{\sigma(2n-1)})e \otimes_s e \right]. \tag{4.19}
\end{aligned}$$

Note that all terms but the first term in (4.18) and (4.19) contain an element $x_k - \phi(x_k)e$, which is in $\ker(\phi)$, and contain an element e , which is in $\text{span}\{e\}$. By Lemma 4.1.1, they are disjoint pairwise. While the first term in both (4.18) and (4.19) contains an element e . By Lemma 4.1.1 again, it is disjoint to all other terms. Therefore, all terms in both (4.18)

and (4.19) are disjoint pairwise. Thus

$$\begin{aligned}
\left| T_{2n}(x_1, \dots, x_{2n}) \right| &= \left| \phi(x_1)e \otimes_s \cdots \otimes_s \phi(x_{2n})e \otimes_s e \right| \\
&+ \frac{1}{(2n)!} \sum_{\sigma \in \pi(2n)} \sum_{k=1}^n \left| b_k x_{\sigma(1)} \otimes_s \cdots \otimes_s x_{\sigma(k)} \otimes_s \right. \\
&\quad \left(x_{\sigma(k+1)} - \phi(x_{\sigma(k+1)})e \right) \otimes_s \cdots \otimes_s \left(x_{\sigma(2k)} - \phi(x_{\sigma(2k)})e \right) \otimes_s \\
&\quad \left. \phi(x_{\sigma(2k+1)})e \otimes_s \cdots \otimes_s \phi(x_{\sigma(2n)})e \otimes_s e \right|.
\end{aligned}$$

It follows from (4.5) and (4.14) that

$$\begin{aligned}
\left| T_{2n}(x_1, \dots, x_{2n}) \right| &= \phi(|x_1|)e \otimes_s \cdots \otimes_s \phi(|x_{2n}|)e \otimes_s e \\
&+ \frac{1}{(2n)!} \sum_{\sigma \in \pi(2n)} \left[\sum_{k=1}^n b_k |x_{\sigma(1)}| \otimes_s \cdots \otimes_s |x_{\sigma(k)}| \otimes_s \right. \\
&\quad \left(|x_{\sigma(k+1)}| - \phi(|x_{\sigma(k+1)}|)e \right) \otimes_s \cdots \otimes_s \left(|x_{\sigma(2k)}| - \phi(|x_{\sigma(2k)}|)e \right) \otimes_s \\
&\quad \left. \phi(|x_{\sigma(2k+1)}|)e \otimes_s \cdots \otimes_s \phi(|x_{\sigma(2n)}|)e \otimes_s e \right] \\
&= T_{2n}(|x_1|, \dots, |x_{2n}|).
\end{aligned}$$

Consequently, T_{2n} is a lattice $(2n)$ -morphism. Similarly we can prove that T_{2n-1} is a lattice $(2n-1)$ -morphism. Therefore, T_n is a lattice n -morphism and hence, $P_n : E \rightarrow \bar{\otimes}_{n+1,s} E$ is a lattice homomorphism. Consequently, its linearization $\tilde{P}_n : \bar{\otimes}_{n,s} E \rightarrow \bar{\otimes}_{n+1,s} E$ is a lattice homomorphism.

Note that for each $x \in E$, $P_n(x) = J_{n \rightarrow n+1}(\theta_n(x))$, which, by (4.4), implies that $\tilde{P}_n(u) = J_{n \rightarrow n+1}(u)$ for each $u \in \bar{\otimes}_{n,s} E$. Since $J_{n \rightarrow n+1} : \bar{\otimes}_{n,s} E \rightarrow \bar{\otimes}_{n+1,s} E$ is injective and $\bar{\otimes}_{n,s} E$ is order dense in $\bar{\otimes}_{n,s} E$ in the sense of (e) in section 4.1, it follows that $\tilde{P}_n : \bar{\otimes}_{n,s} E \rightarrow \bar{\otimes}_{n+1,s} E$ is also injective (please refer to the proof of Theorem 4.2.1). Therefore, \tilde{P}_n is a lattice isomorphism.

Now let $R_{n+1} : E \rightarrow \bar{\otimes}_{n,s} E$ denote the $(n+1)$ -homogeneous polynomial induced by $K_{n+1 \rightarrow n} : \bar{\otimes}_{n+1,s} E \rightarrow \bar{\otimes}_{n,s} E$ introduced in (4.10), that is, $R_{n+1}(x) = K_{n+1 \rightarrow n}(\theta_{n+1}(x))$

for each $x \in E$. To show that R_{n+1} is a lattice homomorphism, we need to show that its symmetric $(n+1)$ -linear operator $S_{n+1} : E^{n+1} \rightarrow \bar{\otimes}_{n,s} E$ is a lattice $(n+1)$ -morphism. It follows from the definition of $K_{n+1 \rightarrow n}$ in (4.10) that

$$R_{n+1}(x) = \phi(x)\theta_n(x) = \phi(x)x \otimes_s \overbrace{\cdots}^n \otimes_s x, \quad \forall x \in E.$$

Thus its symmetric $(n+1)$ -linear operator $S_{n+1} : E^{n+1} \rightarrow \bar{\otimes}_{n,s} E$ is as follows

$$\begin{aligned} & S_{n+1}(x_1, \dots, x_n, x_{n+1}) \\ &= \frac{1}{n+1} \sum_{i=1}^{n+1} \phi(x_i)x_1 \otimes_s \cdots \otimes_s x_{i-1} \otimes_s x_{i+1} \otimes_s \cdots \otimes_s x_{n+1}. \end{aligned}$$

It is obvious that S_{n+1} is positive. Now

$$\begin{aligned} & (n+1) \left| S_{n+1}(x_1, \dots, x_n, x_{n+1}) \right| \otimes_s e \\ &= \left| \sum_{i=1}^{n+1} \phi(x_i)x_1 \otimes_s \cdots \otimes_s x_{i-1} \otimes_s x_{i+1} \otimes_s \cdots \otimes_s x_n \otimes_s x_{n+1} \otimes_s e \right| \\ &= \left| \sum_{i=1}^n \left(\phi(x_i)x_{n+1} - \phi(x_{n+1})x_i \right) \otimes_s x_1 \otimes_s \cdots \otimes_s x_{i-1} \otimes_s x_{i+1} \otimes_s \cdots \otimes_s x_n \otimes_s e \right. \\ &+ \left. (n+1)\phi(x_{n+1})x_1 \otimes_s \cdots \otimes_s x_n \otimes_s e \right|. \end{aligned}$$

Note that all but the last terms contain an element $\phi(x_i)x_{n+1} - \phi(x_{n+1})x_i$, which is in $\ker(\phi)$, and contain an element e , which is in $\text{span}\{e\}$; and the last term contains an element e . By Lemma 4.1.1, they all are disjoint pairwise. Thus

$$\begin{aligned} & (n+1) \left| S_{n+1}(x_1, \dots, x_n, x_{n+1}) \right| \otimes_s e \\ &= \sum_{i=1}^n \left| \left(\phi(x_i)x_{n+1} - \phi(x_{n+1})x_i \right) \otimes_s x_1 \otimes_s \cdots \otimes_s x_{i-1} \otimes_s x_{i+1} \otimes_s \cdots \otimes_s x_n \otimes_s e \right| \\ &+ (n+1) \left| \phi(x_{n+1})x_1 \otimes_s \cdots \otimes_s x_n \otimes_s e \right| \end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^n \left| \phi(x_i)x_{n+1} - \phi(x_{n+1})x_i \right| \otimes_s |x_1| \otimes_s \cdots \otimes_s |x_{i-1}| \otimes_s |x_{i+1}| \otimes_s \cdots \otimes_s |x_n| \otimes_s e \\
&+ (n+1)\phi(|x_{n+1}|)|x_1| \otimes_s \cdots \otimes_s |x_n| \otimes_s e \\
&\geq \sum_{i=1}^n \left(\phi(|x_i|)|x_{n+1}| - \phi(|x_{n+1}|)|x_i| \right) \otimes_s |x_1| \otimes_s \cdots \otimes_s |x_{i-1}| \otimes_s |x_{i+1}| \otimes_s \cdots \otimes_s |x_n| \otimes_s e \\
&+ (n+1)\phi(|x_{n+1}|)|x_1| \otimes_s \cdots \otimes_s |x_n| \otimes_s e \\
&= \sum_{i=1}^{n+1} \phi(|x_i|) \cdot |x_1| \otimes_s \cdots \otimes_s |x_{i-1}| \otimes_s |x_{i+1}| \otimes_s \cdots \otimes_s |x_n| \otimes_s |x_{n+1}| \otimes_s e \\
&= (n+1)S_{n+1}(|x_1|, \dots, |x_n|, |x_{n+1}|) \otimes_s e,
\end{aligned}$$

which implies that

$$\left| S_{n+1}(x_1, \dots, x_n, x_{n+1}) \right| = S_{n+1}(|x_1|, \dots, |x_n|, |x_{n+1}|).$$

Therefore, S_{n+1} is a lattice $(n+1)$ -morphism and hence, $R_{n+1} : E \rightarrow \bar{\otimes}_{n,s}E$ is a lattice homomorphism. Consequently, its linearization $\tilde{R}_{n+1} : \bar{\otimes}_{n+1,s}E \rightarrow \bar{\otimes}_{n,s}E$ is a lattice homomorphism.

Note that for each $x \in E$, $R_{n+1}(x) = K_{n+1 \rightarrow n}(\theta_{n+1}(x))$, which, by (4.4), implies that $\tilde{R}_{n+1}(u) = K_{n+1 \rightarrow n}(u)$ for each $u \in \bar{\otimes}_{n+1,s}E$. Also note that $\tilde{P}_n(u) = J_{n \rightarrow n+1}(u)$ for each $u \in \bar{\otimes}_{n,s}E$. Thus by (4.11), $(\tilde{R}_{n+1} \circ \tilde{P}_n)(u) = u$ for each $u \in \bar{\otimes}_{n,s}E$. Since $\bar{\otimes}_{n,s}E$ is dense in $\bar{\otimes}_{n,s}E$ in the sense of (d) in section 4.1, it follows that $\tilde{R}_{n+1} \circ \tilde{P}_n = id_{\bar{\otimes}_{n,s}E}$ (please refer to the proof of Theorem 4.2.2). Therefore, $\tilde{P}_n \circ \tilde{R}_{n+1}$ is a projection on $\bar{\otimes}_{n+1,s}E$.

Next we show that the image of $\bar{\otimes}_{n,s}E$ under the mapping \tilde{P}_n is a projection band in $\bar{\otimes}_{n+1,s}E$. Take any positive $v \in \bar{\otimes}_{n+1,s}E$. Then v admits a representation $v = \sum_{i=1}^m \lambda_i \theta_{n+1}(x_i)$, where $\lambda_i > 0$ and $x_i > 0$ for $i = 1, \dots, m$. Since $\ker(\phi)$ is a projection band in E , $x_i - \phi(x_i)e \geq 0$. Combining (4.10), (4.11), (4.12) and (4.15) yields that

$$\begin{aligned}
(\tilde{P}_n \circ \tilde{R}_{n+1})(v) &= (J_{n \rightarrow n+1} \circ K_{n+1 \rightarrow n})(v) = \sum_{i=1}^m \lambda_i (J_{n \rightarrow n+1} \circ K_{n+1 \rightarrow n})(\theta_{n+1}(x_i)) \\
&= \sum_{i=1}^m \lambda_i J_{n \rightarrow n+1}(\phi(x_i)\theta_n(x_i))
\end{aligned}$$

$$\begin{aligned}
&= \sum_{i=1}^m \lambda_i(\theta_{n+1}(x_i) - \theta_{n+1}(x_i - \phi(x_i)e)) \\
&\leq \sum_{i=1}^m \lambda_i \theta_{n+1}(x_i) = v.
\end{aligned}$$

Now take any $u \in \bar{\otimes}_{n+1,s}E$. Since $\otimes_{n+1,s}E$ is dense in $\bar{\otimes}_{n+1,s}E$ in the sense of (d) in section 4.1, there is $x \in E$ for which for any $\delta > 0$ there exists $v \in \otimes_{n+1,s}E$ such that $|u - v| \leq \delta \theta_{n+1}(x)$. Since $|v| - |u| \leq |u - v| \leq \delta \theta_{n+1}(x)$, it follows that $|v| \leq |u| + \delta \theta_{n+1}(x)$. On the other hand, since $|u| - |v| \leq |u - v| \leq \delta \theta_{n+1}(x)$, it follows that

$$\begin{aligned}
&(\tilde{P}_n \circ \tilde{R}_{n+1})(|u|) - (\tilde{P}_n \circ \tilde{R}_{n+1})(|v|) \\
&\leq (\tilde{P}_n \circ \tilde{R}_{n+1})(|u - v|) \\
&\leq \delta(\tilde{P}_n \circ \tilde{R}_{n+1})(\theta_{n+1}(x))
\end{aligned}$$

and hence,

$$\begin{aligned}
(\tilde{P}_n \circ \tilde{R}_{n+1})(|u|) &\leq (\tilde{P}_n \circ \tilde{R}_{n+1})(|v|) + \delta(\tilde{P}_n \circ \tilde{R}_{n+1})(\theta_{n+1}(x)) \\
&\leq |v| + \delta(\tilde{P}_n \circ \tilde{R}_{n+1})(\theta_{n+1}(x)) \\
&\leq |u| + \delta \theta_{n+1}(x) + \delta(\tilde{P}_n \circ \tilde{R}_{n+1})(\theta_{n+1}(x)),
\end{aligned}$$

which implies that $(\tilde{P}_n \circ \tilde{R}_{n+1})(|u|) \leq |u|$. Therefore, $\tilde{P}_n \circ \tilde{R}_{n+1}$ is a band projection and by [37, Lemma 1.2.8], $\tilde{P}_n[\bar{\otimes}_{n,s}E]$ is a projection band in $\bar{\otimes}_{n+1,s}E$. \square

For $n \in \mathbb{N}$, let us consider $J_{n \rightarrow n+1} : \otimes_{n,s}E \rightarrow \otimes_{n+1,s}E$ and $K_{n+1 \rightarrow n} : \otimes_{n+1,s}E \rightarrow \otimes_{n,s}E$ introduced in (4.8)-(4.10). Then for $m, n \in \mathbb{N}$ with $m > n$,

$$J_{n \rightarrow m} := J_{m-1 \rightarrow m} \circ J_{m-2 \rightarrow m-1} \circ \cdots \circ J_{n \rightarrow n+1} : \otimes_{n,s}E \rightarrow \otimes_{m,s}E$$

is a positive, linear, injective map, and

$$K_{m \rightarrow n} := J_{n+1 \rightarrow n} \circ J_{n+2 \rightarrow n+1} \circ \cdots \circ K_{m \rightarrow m-1} : \otimes_{m,s} E \rightarrow \otimes_{n,s} E$$

is a positive, linear, surjective map, and $K_{m \rightarrow n} \circ J_{n \rightarrow m} = id_{\otimes_{n,s} E}$. Consequently, we have

Corollary 4.2.4. Let $m, n \in \mathbb{N}$ with $m > n$ and let E be a vector lattice with $E^\sim \neq \{0\}$.

(i) If $\bar{\otimes}_{m,s} E$ is uniformly complete, then $\bar{\otimes}_{n,s} E$ is positively isomorphic to a complemented subspace of $\bar{\otimes}_{m,s} E$.

(ii) If there exists $\phi \in (E^*)_+$ such that $ker(\phi)$ is a projection band in E , then $\bar{\otimes}_{n,s} E$ is lattice isomorphic to a projection band of $\bar{\otimes}_{m,s} E$.

For vector lattices E and F with F Dedekind complete, let $\mathcal{P}^r({}^n E; F)$ denote the space of all regular n -homogeneous polynomials from E to F . Then $\mathcal{P}^r({}^n E; F)$ is lattice isomorphic to $\mathcal{L}^r(\bar{\otimes}_{n,s} E; F)$, the space of all regular linear operators from $\bar{\otimes}_{n,s} E$ to F (see, e.g., [9]). Corollary 4.5 yields the following.

Corollary 4.2.5. Let $m, n \in \mathbb{N}$ with $m > n$ and let F be a Dedekind complete vector lattice and E be a vector lattice with $E^\sim \neq \{0\}$.

(i) If $\bar{\otimes}_{m,s} E$ is uniformly complete, then $\mathcal{P}^r({}^n E; F)$ is positively isomorphic to a complemented subspace of $\mathcal{P}^r({}^m E; F)$.

(ii) If there exists $\phi \in (E^*)_+$ such that $ker(\phi)$ is a projection band in E , then $\mathcal{P}^r({}^n E; F)$ is lattice isomorphic to a projection band of $\mathcal{P}^r({}^m E; F)$.

Remark 4.2.6. (i) Let E be a Banach sequence lattice with an order defined coordinatewise.

Define $\phi : E \rightarrow \mathbb{R}$ by

$$\phi(x) = (x_1, 0, 0, \dots), \quad \forall x = (x_1, x_2, \dots) \in E.$$

Then ϕ is a lattice homomorphism in E^* such that $ker(\phi)$ is a projection band in E .

(ii) For any $t \in [0, 1]$, define $\delta_t : C[0, 1] \rightarrow \mathbb{R}$ by $\delta_t(f) = f(t)$ for any $f \in C[0, 1]$.

Then δ_t is a lattice homomorphism in $C[0, 1]^*$.

(iii) There is no lattice homomorphism in $L^p[0, 1]^*$ for $1 \leq p < \infty$.

4.3 Banach lattice tensor product case

In this section, E is a Banach lattice. The *positive projective symmetric tensor norm* on $\bar{\otimes}_{n,s}E$ is defined by

$$\|u\|_{s,|\pi|} = \inf \left\{ \sum_{i=1}^m \lambda_i \|x_i\|^n : \lambda_i > 0, x_i \in E_+, |u| \leq \sum_{i=1}^m \lambda_i \theta_n(x_i) \right\}$$

for each $u \in \bar{\otimes}_{n,s}E$. Then $\|\cdot\|_{s,|\pi|}$ is a lattice norm on $\bar{\otimes}_{n,s}E$ and

(g) $\otimes_{n,s}E$ is norm dense in $\bar{\otimes}_{n,s}E$, and

(h) the cone generated by $\{\theta_n(x) : x \in E_+\}$ is norm dense in $(\bar{\otimes}_{n,s}E)_+$.

Let $\hat{\otimes}_{n,s,|\pi|}E$ denote the completion of $\bar{\otimes}_{n,s}E$ under the lattice norm $\|\cdot\|_{s,|\pi|}$. Then $\hat{\otimes}_{n,s,|\pi|}E$ is a Banach lattice, called the *n-fold Fremlin projective symmetric tensor product*, or the *n-fold positive projective symmetric tensor product* of E . Moreover,

(i) for any Banach lattice F and any regular n -homogeneous polynomial $P : E \rightarrow F$, there exists a unique regular linear operator $\tilde{P} : \hat{\otimes}_{n,s,|\pi|}E \rightarrow F$, called the *linearization* of P , such that $P = \tilde{P} \circ \theta_n$ and $\|P\|_r = \|\tilde{P}\|_r$ (see, e.g., [9, 23, 24, 45]).

Theorem 4.3.1. Let E be a Banach lattice and let $m, n \in \mathbb{N}$ with $m > n$. Then $\hat{\otimes}_{n,s,|\pi|}E$ is positively isomorphic to a complemented subspace of $\hat{\otimes}_{m,s,|\pi|}E$. Moreover, if there exists $\phi \in (E^*)_+$ such that $\ker(\phi)$ is a projection band in E , then $\hat{\otimes}_{n,s,|\pi|}E$ is lattice isomorphic to a projection band of $\hat{\otimes}_{m,s,|\pi|}E$.

Proof. Without loss of generality, we only need to consider $m = n + 1$ and we also need to recall $J_{n \rightarrow n+1} : \otimes_{n,s}E \rightarrow \otimes_{n+1,s}E$ and $K_{n+1 \rightarrow n} : \otimes_{n+1,s}E \rightarrow \otimes_{n,s}E$ introduced in (4.8)-(4.10). Take any $u \in \otimes_{n,s}E$ and any $\varepsilon > 0$. Then there exist $\lambda_i > 0$ and $x_i > 0$ for $i = 1, \dots, m$ such that $|u| \leq \sum_{i=1}^m \lambda_i \theta_n(x_i)$ and $\sum_{i=1}^m \lambda_i \|x_i\|^n \leq \|u\|_{s,|\pi|} + \varepsilon$. Since $J_{n \rightarrow n+1}$ is positive, it

follows that

$$\begin{aligned}
\|J_{n \rightarrow n+1}(u)\|_{s,|\pi|} &= \| |J_{n \rightarrow n+1}(u)| \|_{s,|\pi|} \leq \| J_{n \rightarrow n+1}(|u|) \|_{s,|\pi|} \leq \sum_{i=1}^m \lambda_i \|J_{n \rightarrow n+1}(\theta_n(x_i))\|_{s,|\pi|} \\
&= \sum_{i=1}^m \lambda_i \left\| \sum_{k=1}^{n+1} (-1)^{k+1} \binom{n+1}{k} \phi(x_i)^{k-1} \theta_k(e) \otimes_s \theta_{n+1-k}(x_i) \right\|_{s,|\pi|} \\
&\leq \sum_{i=1}^m \lambda_i \sum_{k=1}^{n+1} \binom{n+1}{k} \|\phi\|^{k-1} \|x_i\|^{k-1} \|e\|^k \|x_i\|^{n+1-k} \\
&= C \sum_{i=1}^m \lambda_i \|x_i\|^n \leq C(\|u\|_{s,|\pi|} + \varepsilon),
\end{aligned}$$

where $C = \sum_{k=1}^{n+1} \binom{n+1}{k} \|\phi\|^{k-1} \|e\|^k$. Consequently, $\|J_{n \rightarrow n+1}(u)\|_{s,|\pi|} \leq C\|u\|_{s,|\pi|}$ and hence, $J_{n \rightarrow n+1} : (\otimes_{n,s} E, \|\cdot\|_{s,|\pi|}) \rightarrow (\otimes_{n+1,s} E, \|\cdot\|_{s,|\pi|})$ is continuous.

Now take any $v \in \otimes_{n+1,s} E$ and any $\varepsilon > 0$. Then there exist $\lambda_i > 0$ and $x_i > 0$ for $i = 1, \dots, m$ such that $|v| \leq \sum_{i=1}^m \lambda_i \theta_{n+1}(x_i)$ and $\sum_{i=1}^m \lambda_i \|x_i\|^{n+1} \leq \|v\|_{s,|\pi|} + \varepsilon$. Since $K_{n+1 \rightarrow n}$ is positive, it follows that

$$\begin{aligned}
\|K_{n+1 \rightarrow n}(v)\|_{s,|\pi|} &= \| |K_{n+1 \rightarrow n}(v)| \|_{s,|\pi|} \leq \| K_{n+1 \rightarrow n}(|v|) \|_{s,|\pi|} \\
&\leq \left\| \sum_{i=1}^m K_{n+1 \rightarrow n}(\lambda_i \theta_{n+1}(x_i)) \right\|_{s,|\pi|} = \left\| \sum_{i=1}^m \lambda_i \phi(x_i) \theta_n(x_i) \right\|_{s,|\pi|} \\
&\leq \sum_{i=1}^m \lambda_i \|\phi\| \|x_i\|^{n+1} \leq \|\phi\| (\|v\|_{s,|\pi|} + \varepsilon).
\end{aligned}$$

Consequently, $\|K_{n+1 \rightarrow n}(v)\|_{s,|\pi|} \leq \|\phi\| \|v\|_{s,|\pi|}$ and hence, $K_{n+1 \rightarrow n} : (\otimes_{n+1,s} E, \|\cdot\|_{s,|\pi|}) \rightarrow (\otimes_{n,s} E, \|\cdot\|_{s,|\pi|})$ is continuous.

Note that $\otimes_{n,s} E$ and $\otimes_{n+1,s} E$ are dense in $\hat{\otimes}_{n,s,|\pi|} E$ and $\hat{\otimes}_{n+1,s,|\pi|} E$, respectively. Thus $J_{n \rightarrow n+1} : \otimes_{n,s} E \rightarrow \otimes_{n+1,s} E$ and $K_{n+1 \rightarrow n} : \otimes_{n+1,s} E \rightarrow \otimes_{n,s} E$ can be boundedly, positively, and linearly extended to $\hat{J}_{n \rightarrow n+1} : \hat{\otimes}_{n,s,|\pi|} E \rightarrow \hat{\otimes}_{n+1,s,|\pi|} E$ and $\hat{K}_{n+1 \rightarrow n} : \hat{\otimes}_{n+1,s,|\pi|} E \rightarrow \hat{\otimes}_{n,s,|\pi|} E$, respectively. Moreover, (4.11) implies that $\hat{K}_{n+1 \rightarrow n} \circ \hat{J}_{n \rightarrow n+1} = id_{\hat{\otimes}_{n,s,|\pi|} E}$ and hence, $\hat{J}_{n \rightarrow n+1} \circ \hat{K}_{n+1 \rightarrow n}$ is a projection on $\hat{\otimes}_{n+1,s,|\pi|} E$. The first part of the theorem is proved.

Now assume that $\ker(\phi)$ is a projection band in E . In the proof of Theorem 4.2.3, we have an n -homogeneous polynomial $P_n : E \rightarrow \otimes_{n+1,s} E \subseteq \hat{\otimes}_{n+1,s,|\pi|} E$ induced by $J_{n \rightarrow n+1}$, and an $(n+1)$ -homogeneous polynomial $R_{n+1} : E \rightarrow \otimes_{n,s} E \subseteq \hat{\otimes}_{n,s,|\pi|} E$ induced by $K_{n+1 \rightarrow n}$ such that P_n and R_{n+1} are lattice homomorphisms. Thus their linearizations $\tilde{P}_n : \hat{\otimes}_{n,s,|\pi|} E \rightarrow \hat{\otimes}_{n+1,s,|\pi|} E$ and $\tilde{R}_{n+1} : \hat{\otimes}_{n+1,s,|\pi|} E \rightarrow \hat{\otimes}_{n,s,|\pi|} E$ are lattice homomorphisms. Moreover, in the proof of Theorem 4.2.3, \tilde{P}_n is injective in $\bar{\otimes}_{n,s} E$, $\tilde{R}_{n+1} \circ \tilde{P}_n = id_{\bar{\otimes}_{n,s} E}$, and $(\tilde{P}_n \circ \tilde{R}_{n+1})(u) \leq u$ for each $u \in (\bar{\otimes}_{n+1,s} E)_+$. Since $\bar{\otimes}_{n,s} E$ is dense in $\hat{\otimes}_{n,s,|\pi|} E$, it follows that \tilde{P}_n is injective in $\hat{\otimes}_{n,s,|\pi|} E$, $\tilde{R}_{n+1} \circ \tilde{P}_n = id_{\hat{\otimes}_{n,s,|\pi|} E}$, and $(\tilde{P}_n \circ \tilde{R}_{n+1})(u) \leq u$ for each $u \in (\hat{\otimes}_{n+1,s,|\pi|} E)_+$. Therefore, $\tilde{P}_n : \hat{\otimes}_{n,s,|\pi|} E \rightarrow \hat{\otimes}_{n+1,s,|\pi|} E$ is a lattice isomorphism and $\tilde{P}_n \circ \tilde{R}_{n+1} : \hat{\otimes}_{n+1,s,|\pi|} E \rightarrow \hat{\otimes}_{n+1,s,|\pi|} E$ is a band projection. By [37, Lemma 1.2.8], $\tilde{P}_n[\hat{\otimes}_{n,s,|\pi|} E]$ is a projection band in $\hat{\otimes}_{n+1,s,|\pi|} E$. \square

For Banach lattices E and F with F Dedekind complete, $\mathcal{P}^r({}^n E; F)$ is a Banach lattice, which is lattice isometric to $\mathcal{L}^r(\hat{\otimes}_{n,s,|\pi|} E; F)$, the space of all regular linear operators from $\hat{\otimes}_{n,s,|\pi|} E$ to F (see, e.g., [9]).

Corollary 4.3.2. Let E and F be Banach lattices with F Dedekind complete, and let $m, n \in \mathbb{N}$ with $m > n$. Then $\mathcal{P}^r({}^n E; F)$ is positively isomorphic to a complemented subspace of $\mathcal{P}^r({}^m E; F)$. Moreover, if there exists $\phi \in (E^*)_+$ such that $\ker(\phi)$ is a projection band in E , then $\mathcal{P}^r({}^n E; F)$ is lattice isomorphic to a projection band of $\mathcal{P}^r({}^m E; F)$.

5 BASES IN THE SPACES OF REGULAR HOMOGENEOUS POLYNOMIALS ON BANACH LATTICES

5.1 Preliminaries

For Banach lattices E and F over the real field \mathbb{R} or the complex field \mathbb{C} , and for each positive integer $m \geq 2$, we denote by $\mathcal{P}(^m E; F)$ the space of all continuous m -homogeneous polynomials from E into F , by $\mathcal{P}_K(^m E; F)$ the space of all compact m -homogeneous polynomials from E into F , and by $\mathcal{P}_w(^m E; F)$ the subspace of all P in $\mathcal{P}(^m E; F)$ that are weakly continuous on bounded sets. In particular, if $F = \mathbb{R}$ or \mathbb{C} , then $\mathcal{P}(^m E; F)$ and $\mathcal{P}_w(^m E; F)$ are simply denoted by $\mathcal{P}(^m E)$ and $\mathcal{P}_w(^m E)$ respectively.

Let $\otimes_m E$ denote the m -fold algebraic tensor product of E . It is known that each $u \in \otimes_{m,s} E$ has a representation $u = \sum_{k=1}^n \lambda_k x_k \otimes \cdots \otimes x_k$ where λ_k , $k = 1, \dots, n$, are scalars and x_k , $k = 1, \dots, n$, are vectors in E . The symmetric injective tensor norm on $\otimes_{m,s} E$ is defined by

$$\|u\|_{s,\epsilon} = \sup \left\{ \left| \sum_{k=1}^n \lambda_k (x^*(x_k))^m \right| : u = \sum_{k=1}^n \lambda_k \cdot x_k \otimes \cdots \otimes x_k, x^* \in \mathcal{B}_{E^*} \right\}$$

for every $u \in \otimes_{m,s} E$. The completion of $\otimes_{m,s} E$ with respect to this norm is denoted by $\check{\otimes}_{m,s,\epsilon} E$ and called the m -fold symmetric injective tensor product of E . For every $u \in \check{\otimes}_{m,s,\epsilon} E$ [22], we have

$$\|s(u)\|_{s,\epsilon} \leq \|s(u)\|_\epsilon \leq \frac{m^m}{m!} \|s(u)\|_{s,\epsilon}.$$

For each $u \in \otimes_{m,s} E$, $u = \sum_{k=1}^n \lambda_k x_k \otimes \cdots \otimes x_k$, define $P_u : E^* \mapsto \mathbb{R}$ by

$$P_u(x^*) = \sum_{k=1}^n \lambda_k (x^*(x_k))^m, \quad \forall x^* \in E^*. \quad (*)$$

Then P_u is an m -homogeneous polynomial which does not depend on the representations of u and $P_u \in \mathcal{P}(^m E^*; \mathbb{R})$ with $\|P_u\| = \|u\|_{s,\epsilon}$.

For the basic knowledge about the symmetric injective tensor products $\check{\otimes}_{m,\epsilon} E$, $\check{\otimes}_{m,s,\epsilon} E$, we refer to [16, 21, 22, 39, 44].

For each m -homogeneous polynomial $P : E \rightarrow F$, let $A_P : \otimes_{m,s} E \rightarrow F$ denote its linearization, that is,

$$A_P(x \otimes \cdots \otimes x) = P(x), \quad \forall x \in E.$$

Then under the isometry: $P \rightarrow A_P$, $\mathcal{P}(^m E; F) = \mathcal{L}(\hat{\otimes}_{m,s,\pi} E; F)$ and $\mathcal{P}_K(^m E; F) = \mathcal{K}(\hat{\otimes}_{m,s,\pi} E; F)$ (see [39, 42]).

For the basic knowledge about homogeneous polynomials, we refer to [21, 22, 39].

Let $\mathcal{P}^r(^m E; F)$ denote the space of all regular m -homogeneous polynomials from E to F . If F is Dedekind complete then $\mathcal{P}^r(^m E; F)$ is a Banach lattice with the regular polynomial norm $\|P\|_r = \| |P| \|$ for every $P \in \mathcal{P}^r(^m E; F)$.

If E is a Banach lattice, then for any $u \in \otimes_{m,s} E$, the polynomial $P_u : E^* \mapsto \mathbb{R}$ defined in (*) is a regular m -homogeneous polynomial and hence, $P_u \in \mathcal{P}^r(^m E^*; \mathbb{R})$. Let $\check{\otimes}_{m,s,|\epsilon|} E$ denote the closed sublattice generated by $\otimes_{m,s} E$ in $\mathcal{P}^r(^m E^*; \mathbb{R})$, called the m -fold positive symmetric injective tensor product of E . The norm on $\check{\otimes}_{m,s,|\epsilon|} E$ is denoted by $\| \cdot \|_{s,|\epsilon|}$, that is, for every $u \in \otimes_{m,s} E$, $\|u\|_{s,|\epsilon|} = \|P_u\|_r$ and $\|u\|_{s,\epsilon} \leq \|u\|_{s,|\epsilon|}$. In particular, if u is a positive element in $\otimes_{m,s} E$, then

$$\|u\|_{s,|\epsilon|} = \sup \left\{ \left| \sum_{k=1}^n \lambda_k (x^*(x_k))^m \right| : u = \sum_{k=1}^n \lambda_k x_k \otimes \cdots \otimes x_k, \quad x^* \in \mathcal{B}_{E^{**}} \right\}.$$

The symmetric projective tensor norm on $\otimes_{m,s}E$ is defined by

$$\|u\|_{s,\pi} = \inf \left\{ \sum_{k=1}^n |\lambda_k| \cdot \|x_k\|^m : u = \sum_{k=1}^n \lambda_k x_k \otimes_s \cdots \otimes_s x_k \in \otimes_{m,s}E \right\}, \quad u \in \otimes_{m,s}E.$$

Let $\hat{\otimes}_{m,s,\pi}E$ denote the completion of $(\otimes_{m,s}E, \|\cdot\|_{s,\pi})$, called the m -fold symmetric projective tensor product of E . For a Banach lattice E , let $\bar{\otimes}_{m,s}E$ denote the m -fold vector lattice tensor product of E . The positive symmetric tensor norm on $\bar{\otimes}_{m,s}E$ is defined by

$$\|u\|_{s,|\pi|} = \inf \left\{ \sum_{k=1}^n \|x_k\|^m : x_k \in E^+, |u| \leq \sum_{k=1}^n x_k \otimes \cdots \otimes x_k \right\}$$

for $\forall u \in \bar{\otimes}_{m,s}E$. Then $\|\cdot\|_{s,|\pi|}$ is a lattice norm on $\bar{\otimes}_{m,s}E$. Let $\hat{\otimes}_{m,s,|\pi|}E$ denote the completion of $\bar{\otimes}_{m,s}E$ under the lattice norm $\|\cdot\|_{s,|\pi|}$. Then $\hat{\otimes}_{m,s,|\pi|}E$ is a Banach lattice called the m -fold Fremlin symmetric tensor product or the m -fold positive symmetric projective tensor product of E . For the Banach lattices E and F with F Dedekind complete, $\mathcal{P}^r({}^mE; F)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|}E; F)$. In particular, $(\hat{\otimes}_{m,s,|\pi|}E)^* = \mathcal{P}^r({}^mE; \mathbb{R})$.

5.2 Bases in the space of regular homogeneous polynomials

In this section we generalize the results of section 3.1 for regular homogeneous polynomials.

Theorem 5.2.1 Let $\{e_i : i \in \mathbb{N}\}$ and $\{f_j : j \in \mathbb{N}\}$ be 1-unconditional bases of E and F , respectively. Then the sequence

$$\{e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* \otimes f_j : (i_1, \dots, i_m, j) \in \mathbb{N}^{m+1}\}$$

of monomials is a disjoint sequence in $\mathcal{P}^r({}^m E; F)$ and hence, forms a 1-unconditional basic sequence in $\mathcal{P}^r({}^m E; F)$.

Proof. The proof follows from Theorem 3.2.2. □

Next we get a result for regular homogeneous polynomials, analogical to Theorem 3.2.3.

Theorem 5.2.2. Let $\{e_i : i \in \mathbb{N}\}$ be a 1-unconditionally shrinking basis of E , and $\{f_j : j \in \mathbb{N}\}$ be a (1-unconditional) basis of F . Then for any $P \in \mathcal{P}^r({}^m E; F)$, the following statements are equivalent:

- (i) $|P|$ is weakly sequentially continuous.
- (ii) $|P|$ is compact and $\lim_n \|P\|_{r,n} = 0$, where

$$\|P\|_{r,n} = \sup \left\{ \left\| |T_P| \left(\sum_{i=n}^{\infty} e_i^*(x) e_i, x, \dots, x \right) \right\| : x \in \mathcal{B}_{E^+} \right\}$$

and T_P is the associated multilinear symmetric operator.

- (iii)

$$P(x) = \sum_{i_1=1}^{\infty} \cdots \sum_{i_m=1}^{\infty} \sum_{j=1}^{\infty} e_{i_1}^*(x) \cdots e_{i_m}^*(x) \langle T(e_{i_1}, \dots, e_{i_m}), f_j^* \rangle \cdot f_j$$

converges in F uniformly for $\forall x \in \mathcal{B}_{E^+}$.

Theorem 5.2.3. Suppose that E has a 1-unconditionally shrinking basis, and F has a 1-unconditional basis. Then $\mathcal{P}_{wsc}^r({}^m E; F)$ is a closed order ideal of $\mathcal{P}^r({}^m E; F)$.

Proof. See Theorem 3.2.4. □

Theorems 5.2.2 and 5.2.3 follow from the facts that $|P|$ is weakly sequentially continuous if and only if $|T_P|$ is weakly sequentially continuous, and that $|P|$ is compact if and only if $|T_P|$ is compact.

As consequences of Theorems 5.2.1, 5.2.2, 5.2.3, we have the following corollaries.

Corollary 5.2.4. Let $\{e_i : i \in \mathbb{N}\}$ be 1-unconditionally shrinking basis of E , and $\{f_j : j \in \mathbb{N}\}$ be a 1-unconditional basis of F . Then the monomial sequence

$$\{e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* \otimes f_j : (i_1, \dots, i_m, j) \in \mathbb{N}^{m+1}\}$$

forms a 1-unconditional basis of $\mathcal{P}_{wsc}^r({}^m E; F)$.

Corollary 5.2.5. Let $\{e_i : i \in \mathbb{N}\}$ and $\{f_j : j \in \mathbb{N}\}$ be 1-unconditional bases of E and F , respectively. Then the monomial sequence

$$\{e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* \otimes f_j : (i_1, \dots, i_m, j) \in \mathbb{N}^{m+1}\}$$

forms a 1-unconditional basis of $\mathcal{P}^r({}^m E; F)$ if and only if each $\{e_i : i \in \mathbb{N}\}$ is a shrinking basis of E , and every positive m -homogeneous polynomial from E to F is weakly sequentially continuous.

Proof. By Corollary 5.2.4, we need only to prove the necessity. Suppose that the monomial sequence $\{e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* \otimes f_j : (i_1, \dots, i_m, j) \in \mathbb{N}^{m+1}\}$ is a basis of $\mathcal{P}^r({}^m E; F)$. Then $\{e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* : (i_1, \dots, i_m) \in \mathbb{N}^m\}$ is a basis of $\mathcal{P}^r({}^m E; \mathbb{R}) = (\hat{\otimes}_{m,s,|\pi|} E)^*$, and hence the

basis $\{e_{i_1} \otimes \cdots \otimes e_{i_m} : (i_1, \dots, i_m) \in \mathbb{N}^m\}$ is a shrinking basis of $\hat{\otimes}_{m,s,|\pi|} E$. Since E is lattice isometric to a complemented sublattice of $\hat{\otimes}_{m,s,|\pi|} E$ by chapter 4, it follows that $\{e_i : i \in \mathbb{N}\}$ is a shrinking basis of E .

Now take any positive $P \in \mathcal{P}^r({}^m E; F)$. Then $T_P \in \mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; F)$, and

$$T_P = \sum_{(i_1, \dots, i_m, j) \in \mathbb{N}^{m+1}} a_{i_1, \dots, i_m, j} e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* \otimes f_j$$

converges in $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; F)$. Thus T_P is approximable and hence, weakly sequentially continuous. It follows that P is weakly sequentially continuous. \square

5.3 Bases in positive symmetric tensor products and reflexivity

Similar to section 3.3, the following results hold for symmetric positive tensor products.

Theorem 5.3.1. Let $\{e_i : i \in \mathbb{N}\}$ be a 1-unconditional basis of E . Then the following statements are equivalent.

- (i) $\{e_{i_1} \otimes \cdots \otimes e_{i_m} : (i_1, \dots, i_m) \in \mathbb{N}^m\}$ is a shrinking basis of $\check{\otimes}_{m,s,|\epsilon|} E$.
- (ii) $\{e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* : (i_1, \dots, i_m) \in \mathbb{N}^m\}$ is a boundedly complete basis of $\hat{\otimes}_{m,s,|\pi|} E^*$.
- (iii) Each $\{e_i^* : i \in \mathbb{N}\}$ is a basis of E^* and

$$(\check{\otimes}_{m,s,|\epsilon|} E)^* = \hat{\otimes}_{m,s,|\pi|} E^*.$$

- (iv) Each $\{e_i : i \in \mathbb{N}\}$ is a shrinking basis of E .

Theorem 5.3.2. Let $\{e_i : i \in \mathbb{N}\}$ be a 1-unconditional basis of E . Then the following statements are equivalent.

- (i) $\{e_{i_1} \otimes \cdots \otimes e_{i_m} : (i_1, \dots, i_m) \in \mathbb{N}^m\}$ is a shrinking basis of $\hat{\otimes}_{m,s,|\pi|} E$.
- (ii) $\{e_{i_1}^* \otimes \cdots \otimes e_{i_m}^* : (i_1, \dots, i_m) \in \mathbb{N}^m\}$ is a boundedly complete basis of $\check{\otimes}_{m,s,|\epsilon|} E^*$.
- (iii) Each $\{e_i^* : i \in \mathbb{N}\}$ is a basis of E^* and

$$(\hat{\otimes}_{m,s,|\pi|} E)^* = \check{\otimes}_{m,s,|\epsilon|} E^*.$$

(iv) Each $\{e_i : i \in \mathbb{N}\}$ is a shrinking basis of E , and every positive symmetric m -linear operator from $E \times \cdots \times E$ to \mathbb{R} is weakly sequentially continuous.

Analogous results on reflexivity hold also for symmetric regular operator spaces.

Theorem 5.3.3. Let E be a reflexive Banach lattice with a 1-unconditional basis. Then the following statements are equivalent:

- (i) $\hat{\otimes}_{m,s,|\pi|} E$ is reflexive.
- (ii) $\check{\otimes}_{m,s,|\epsilon|} E^*$ is reflexive.
- (iii) $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; \mathbb{R})$ is reflexive.
- (iv) $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; \mathbb{R})$ has a monomial basis.
- (v) $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; \mathbb{R}) = \mathcal{L}_{wsc}^r(\hat{\otimes}_{m,s,|\pi|} E; \mathbb{R})$.

Lemma 5.3.4. Let E and F be Banach lattices such that F^* is order continuous. Then $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E, F; \mathbb{R})$ is lattice isometric to $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; F^*)$ under the mapping $T \rightarrow A$ defined in (3.12) for $x_1 = \cdots = x_m$. Moreover, if F is reflexive then $\mathcal{L}_{wsc}^r(\hat{\otimes}_{m,s,|\pi|} E; F^*) = \mathcal{L}_{wsc}^r(\hat{\otimes}_{m,s,|\pi|} E, F; \mathbb{R})$.

Theorem 5.3.5. Let E , and F be reflexive Banach lattices with 1-unconditional bases respectively. Then the following statements are equivalent.

- (i) $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; F)$ is reflexive.
- (ii) $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; F)$ has a monomial basis.
- (iii) $\mathcal{L}^r(\hat{\otimes}_{m,s,|\pi|} E; F) = \mathcal{L}_{wsc}^r(\hat{\otimes}_{m,s,|\pi|} E; F)$.

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