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Complex Vector Lattices: Tensor Products And Multilinear Maps

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COMPLEX VECTOR LATTICES: TENSOR PRODUCTS AND MULTILINEAR MAPS

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ABSTRACT

In this thesis, we study completions of Archimedean real vector lattices relative to any nonempty set of continuous positively homogeneous functions defined on $\mathbb{R}^n$. Examples of such completions include square mean closed vector lattices and geometric mean closed vector lattices. These functional completions lead to a vector lattice complexification of any Archimedean real vector lattice. Unlike the vector space complexification of an Archimedean real vector lattice, the vector lattice complexification always results in an Archimedean complex vector lattice. For example, we prove that the vector space complexification of the Fremlin tensor product $C(X)\bar{\otimes}C(Y)$ is not a complex vector lattice when X and Y are uncountable metrizable compact spaces. The vector lattice complexification is employed to construct an Archimedean complex vector lattice tensor product, powers of Archimedean complex vector lattices, and the symmetric (antisymmetric) Archimedean complex vector lattice tensor product. We use tensor products and powers to develop a theory for various multilinear maps between Archimedean complex vector lattices. Finally, we prove the Cauchy-Schwarz Inequality for sesquilinear maps from a complex vector space to various types of Archimedean complex vector lattices.

DEDICATION

To my loving parents, Mike and Caroline Schwanke

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1 INTRODUCTION

The study of polynomials on infinite-dimensional vector lattices implicitly began in 2001 with a paper by Buskes and van Rooij in which the notion of orthosymmetric maps was introduced ([14], Definition 1). Indeed, Loane proved in Proposition 4.38 of [25] that a homogeneous polynomial between Archimedean vector lattices is orthogonally additive if and only if its associated symmetric multilinear map is orthosymmetric. In his 2007 thesis ([25]), Loane explicitly studied polynomials between Archimedean vector lattices, and a theory of polynomials between Archimedean vector lattices has undergone a rapid development since. As Loane states in his abstract of [25], the motivation for studying polynomials on vector lattices includes a desire to gain an insight into complex holomorphic functions. However, the study of polynomials on vector lattices has so far been limited to real vector lattices. A reason is the fact that the theory of complex vector lattices is less developed than the theory of real vector lattices. This thesis is devoted to developing a theory of complex vector lattices and establishing a foundation for the study of homogeneous polynomials on complex vector lattices.

The idea of a complex modulus in the vector space complexification $E + iE$ of a Banach lattice E, which led to the notion of a complex vector lattice, dates back to a 1963 paper by Rieffel (see [34], page 812 and [35], page 35) on complex AL-spaces. Lotz ([26]) defined in 1968 for Banach lattices E the modulus $|f + ig|$ of an element $f + ig \in E + iE$ by

$$|f + ig| = \sup\{f \cos\theta + g \sin\theta : 0 \leq \theta \leq 2\pi\}. \ (*)$$

Luxemburg and Zaanen ([27], Section 3) in 1971 extend formula $(*)$ above to the vector space complexification $E+iE$ of a uniformly complete vector lattice E while studying order bounded maps and integral operators. In 1973, de Schipper defines a *complex vector lattice* to be a complex vector space of the form $E+iE$, where E is a real vector lattice which is closed under the supremum in $(*)$ above ([19], page 356).

In spite of the fact that uniform completeness is not mentioned in de Schipper's definition, the assumption of uniform completeness has proliferated in studies on complex vector lattices. In particular, complex vector lattices have almost invariably been identified with complexifications $E+iE$ of uniformly complete vector lattices E. For instance, Schaefer defines in [37] complex vector lattices axiomatically and derives formula $(*)$, but includes uniform completeness in the axioms. The choice of definition for complex vector lattices in [37] as well as the standard assumption of uniform completeness in results for complex vector lattices in Sections 91 and 92 of [46] appear to have codified that practice. In fact, a development of the theory of complex vector lattices has suffered from this almost universal blanket assumption of uniform completeness in order to have a modulus available.

One year after de Schipper defined complex vector lattices, Mittelmeyer and Wolff define, for $\mathbb{K}=\mathbb{R}$ and $\mathbb{K}=\mathbb{C}$, what we call Archimedean vector lattices over $\mathbb{K}$ by axiomatizing a modulus on a vector space over $\mathbb{K}$ ([29], Definition 1.1). They prove that a real vector space E is a real vector lattice if and only if E is equipped with a modulus ([29], Proposition 1.3). They also prove that a complex vector space E is an Archimedean complex vector lattice if and only if E is equipped with an Archimedean modulus ([29], Proposition 1.5, Theorem 2.2). (Their use of representation theory in the proof of Theorem 2.2 confines them to Archimedean complex vector lattices.) An Archimedean vector lattice over $\mathbb{K}$ is a vector space over $\mathbb{K}$ that is equipped with an Archimedean modulus.

Despite having the ready-made utility of Mittelmeyer and Wolff available, rewriting all of the theory for results in Archimedean real vector lattices and Archimedean complex vector lattices alike, seems a rather Herculean, and at times, uninteresting task. We hasten to add that fundamental results for real vector lattices exist that are not valid for complex ones. An example is the Riesz decomposition property (see [45], Remark 2). In the opposite direction, Kalton recently (see [23], e.g. Theorem 4.2) proved results for complex Banach lattices that fail for real Banach lattices. Nonetheless, there exists a large body of results that both theories have in common. But even with complex vector lattices satisfactorily defined, these results that are in common lack a proper transfer mechanism, a more or less mechanical procedure that transfers real results into their complex analogues.

We present exactly such a mechanism in this thesis.

This transfer mechanism transforms any Archimedean real vector lattice E into the smallest Archimedean complex vector lattice that contains E as a real vector sublattice. We call this transformation the *vector lattice complexification* of E. Given an Archimedean vector lattice E, its vector lattice complexification coincides with $E + iE$ if and only if E is closed under the supremum in $(*)$ above. As Azouzi states on page 3 of his 2008 thesis ([2]), a theory of complex vector lattices can be built on real vector lattices that are closed under the supremum in $(*)$. He calls such vector lattices *square mean closed*. Various constructs such as the Fremlin tensor product are important to the theory of vector lattices however, and these constructs often do not preserve the property of closure under the supremum in $(*)$. For example, if X and Y are uncountable compact metrizable spaces then $C(X)$ and $C(Y)$ are uniformly complete vector lattices and are therefore square mean closed. The Fremlin tensor product $C(X)\bar{\otimes}C(Y)$ is not a square mean closed vector lattice however (Theorem 4.14). Therefore, the vector lattice complexification of $C(X)\bar{\otimes}C(Y)$ is a complex vector lattice and $(C(X)\bar{\otimes}C(Y)) + i(C(X)\bar{\otimes}C(Y))$ is not a complex vector lattice.

Given an Archimedean real vector lattice E, Azouzi calls the smallest square mean closed vector lattice inside its Dedekind completion that contains E the *square mean closure* of E. He proves that there exists a unique square mean closure for every Archimedean real vector lattice ([2], Remark 4). We prove in Chapter 3 that vector lattice homomorphisms defined on an Archimedean real vector lattice can often be uniquely extended to its square mean closure (Theorem 3.19(2)). Therefore, a universal property for the square mean closure is obtained. We define the *square mean completion* of an Archimedean real vector lattice via this universal property. We choose *completion* over *closure* because the universal definition of the square mean completion makes no reference to a specific surrounding vector lattice. As we studied the square mean completion, we found a host of similar completions, which we call functional completions, that are equally useful in applications to existing literature in vector lattice theory. Indeed, for any continuous, positively homogeneous, real-valued function h on $\mathbb{R}^n$ one can define a functional calculus with respect to h on any uniformly complete vector lattice. The smallest vector lattice in the uniform completion of an Archimedean vector lattice E on which such a calculus can be defined will be called the *h-completion* of E. The reason for the choice of the word *completion* is due to the existence of a universal property for h-completions that generalizes the universal property of the square mean completion (Theorem 3.19(2)). The square mean completion is associated with

$$h(x,y) = \frac{1}{\sqrt{2}}\sqrt{x^2+y^2} \quad (x, y \in \mathbb{R})$$

and proves to be useful in our theory of tensor products and multilinear maps on complex vector lattices, developed in Chapter 4.

The theory of functional completions clarifies and extends previous results in the literature for very specific positively homogeneous functions like the square mean and the geometric mean (see, e.g., [2], Proposition 2.20, [3], Corollary 4.7). Indeed, the modulus formula ($*$) above and formulas for the square mean and geometric mean found in [2] and [3] are all connected to functional calculus. We illustrate this fact by connecting the use of differential calculus as first seen in Theorem 4.2 of [4] by Beukers, Huijsmans, and de Pagter to h-completions for convex or concave h (Theorem 3.8). Our results yield concrete formulas for operations on Archimedean vector lattices that are abstractly defined via functional calculus. These results sharpen a special case of Kusraev's Theorem 5.5 in [24] (while keeping the structure of its proof largely intact) in three ways. We weaken the assumption of uniform completeness, verify that the proof of Theorem 5.5 in [24] in our special case does not (contrary to Kusraev's proof) require more than the Countable Axiom of Choice, and provide more concrete formulas that directly link to Lemmas 4.2 and 4.3 in [3].

We use the square mean completion in Chapter 4 to construct the vector lattice complexification alluded to above for any Archimedean real vector lattice (Theorem 4.2). Along with the vector lattice complexification, corresponding complexifications for various types of linear and multilinear maps between Archimedean real vector lattices are introduced. The vector lattice complexification is employed to obtain an Archimedean complex vector lattice tensor product (Theorem 4.10(1)), the symmetric (antisymmetric) Archimedean complex vector lattice tensor product (Theorems 4.20 and 4.22), and powers of Archimedean complex vector lattices (Theorem 4.25). Using the Archimedean complex vector lattice tensor product, we obtain results for complex maps of order bounded variation (Theorem 4.30). The complex Banach lattice tensor product is introduced (Theorem 4.32), although the vector space complexification suffices for its construction.

The final chapter is devoted to the Cauchy-Schwarz Inequality for sesquilinear maps from a complex vector space to various types of Archimedean complex vector lattices. Much of Chapter 3 is the content of [11] and most of Sections 4.1, 4.2, 4.4, and 4.5 can also be found in [10]. We next turn to Chapter 2 for a collection of preliminary definitions and results in the theory of vector lattices that will be needed throughout the thesis.

2 PRELIMINARIES

We refer the reader to the following standard texts [1], [28], and [46] for a more detailed account of real and complex vector lattices. Throughout, $\mathbb{R}$ is used for the standard ordered field of real numbers, $\mathbb{C}$ denotes the standard field of complex numbers, and $\mathbb{K}$ stands for either $\mathbb{R}$ or $\mathbb{C}$. The symbol for the set of (nonzero) positive integers will be $\mathbb{N}$. For $n \in \mathbb{N}$ and sets $A_1, ..., A_n$, we write $\times_{k=1}^{n} A_k$ for the Cartesian product $A_1 \times \cdots \times A_n$, abbreviated by $\times_n A$ if $A_k = A$ for every $k \in \{1, ..., n\}$. Real vector lattices are the subject of Section 2.1, while Section 2.2 contains preliminary information regarding complex vector lattices.

2.1 Real Vector Lattices

Here we present standard facts about real vector lattices. Specific examples will illustrate the definitions.

We call a vector space V over $\mathbb{R}$ a *real ordered vector space* if it is equipped with a partial ordering $\leq$ with the following properties.

(1) If $u \leq v$ then $u + w \leq v + w \quad (u, v, w \in V)$.

(2) If $0 \leq u$ then $0 \leq \lambda u$ $(u \in V)$ for every $\lambda \in [0, \infty)$.

For a real ordered vector space V and $u, v \in V$ we write $u < v$ when $u \leq v$ and $u \neq v$.

Let A be a partially ordered set and suppose that S is a subset of A. We say that $a \in A$ is an *upper bound (lower bound)* of S if $s \leq a$ $(a \leq s)$ for every $s \in S$. The set S is called *bounded above (bounded below)* if there exists $a \in A$ such that a is an upper bound (lower bound) of S. We call S *order bounded* if S is bounded below and bounded above. If there exists a least upper bound of S in A, we denote it by $\sup S$. Analogously, we denote the greatest lower bound of S by $\inf S$, if it exists. For $s, t \in S$ such that $\sup\{s,t\}$ and $\inf\{s,t\}$ exist in A, we write $s \vee t := \sup\{s,t\}$ and $s \wedge t := \inf\{s,t\}$. Similarly, we set $\bigvee_{k=1}^{n} s_k := \sup\{s_1, ..., s_n\}$ (respectively $\bigwedge_{k=1}^{n} s_k := \inf\{s_1, ..., s_n\}$) when there exists a least upper bound (respectively greatest lower bound) of $\{s_1, ..., s_n\}$ in A.

A real ordered vector space E is called a *real vector lattice* if for every $f, g \in E$ we have that $f \vee g$ and $f \wedge g$ both exist in E.

A real vector lattice E is called *Archimedean* if

$$\inf\{\frac{1}{n} f : n \in \mathbb{N}\} = 0 \quad (0 \leq f \in E).$$

Example 2.1. *(1) $\mathbb{R}$ is an Archimedean real vector lattice.*

(2) The set of all real-valued functions on a nonempty set A is denoted by $\mathbb{R}^A$. For $f, g \in \mathbb{R}^A$, we define the pointwise addition $f + g \in \mathbb{R}^A$ by $(f+g)(a) := f(a) + g(a)$ for every $a \in A$. Similarly, for $\lambda \in \mathbb{R}$, we define the pointwise scalar multiplication $\lambda f \in \mathbb{R}^A$ by $(\lambda f)(a) := \lambda f(a)$ for each $a \in A$. In addition, the pointwise ordering is given by

$$f \leq g \text{ if } f(a) \leq g(a) \text{ for every } a \in A \quad (f, g \in \mathbb{R}^A).$$

Then $\mathbb{R}^A$ is an Archimedean real vector lattice with respect to pointwise addition, pointwise scalar multiplication, and pointwise ordering.

For the special case $\mathbb{R}^n$ $(n \in \mathbb{N})$, the pointwise addition, scalar multiplication, and ordering are often referred to as the coordinatewise addition, scalar multiplication, and ordering. For $x = (x_1, ..., x_n), y = (y_1, ..., y_n) \in \mathbb{R}^n$, and for $\lambda \in \mathbb{R}$, the coordinatewise *addition and coordinatewise scalar multiplication on* $\mathbb{R}^n$ *are*

$$x + y = (x_1 + y_1, ..., x_n + y_n), \text{ and}$$
$$\lambda x = (\lambda x_1, ..., \lambda x_n),$$

respectively. The coordinatewise ordering on $\mathbb{R}^n$ *is given by* $x \leq y$ *when* $x_k \leq y_k$ *for every* $k \in \{1, .., n\}$.

(3) *The lexicographical ordering is defined on* $\mathbb{R}^2$ *by writing* $(x_1, x_2) \leq (y_1, y_2)$ *when* $x_1 < y_1$ *or when* $x_1 = y_1$ *and* $x_2 \leq y_2$. *With respect to coordinatewise addition, coordinatewise scalar multiplication, and the lexicographical ordering,* $\mathbb{R}^2$ *is a non-Archimedean real vector lattice. Indeed, we have* $(0, 0) < (0, 1) \leq \frac{1}{n}(1, 1)$ *for every* $n \in \mathbb{N}$.

Let E be a real vector lattice. For $f \in E$ we use the shorthand

$$f^+ := f \vee 0 \text{ and } f^- := (-f) \vee 0.$$

Theorem 2.2. (see [28], Theorem 11.7) *If* E *is a real vector lattice and* $f \in E$ *then*

(1) $0 \leq f^+$ *and* $0 \leq f^-$,

(2) $-f^- \leq f \leq f^+$, *and*

(3) $f = f^+ - f^-$.

Let E be a real vector lattice and let $f \in E$. The absolute value $|f|$ of f is defined by

$$|f| := f \vee (-f).$$

We record some facts involving the absolute value that will be used throughout the thesis.

Theorem 2.3. (see [28], Theorems 11.7 and 12.1) *Let E be a real vector lattice. If $f, g \in E$ and $\lambda \in \mathbb{R}$ then*

(1) $0 \leq |f|$,

(2) $|f| = f^+ + f^-$,

(3) $f^+ \leq |f|$ *and* $f^- \leq |f|$,

(4) $|\,|f|\,| = |f|$,

(5) $|f| = 0$ *if and only if* $f = 0$,

(6) $|\lambda f| = |\lambda||f|$, *and*

(7) $||f| - |g|| \leq |f + g| \leq |f| + |g|$.

We define the *positive cone* of a real vector lattice E by

$$E^+ := \{f \in E : f \geq 0\}.$$

Then $E = \{f - g : f, g \in E^+\}$ by Theorem 2.2.

We call a vector subspace L of a real vector lattice E a *vector sublattice* of E if $|f| \in L$ for every $f \in L$. A subset $A \subseteq E$ is said to be *order dense* in E if for every $0 < g \in E^+$ there exists $f \in A \cap E^+$ such that $0 < f \leq g$. We say that $A \subseteq E$ is *majorizing* in E if for every $f \in E$ there exists $a \in A$ such that $f \leq a$. A vector subspace L of E is called an *ideal* of E if $g \in L$ whenever $f \in L, g \in E$, and $|g| \leq |f|$.

Example 2.4. *(1) For a topological space X we define*

$$C(X) := \{f : X \to \mathbb{R} : f \textit{ is continuous}\}.$$

Then $C(X)$ *is a vector sublattice of* $\mathbb{R}^X$*, but not necessarily an ideal of* $\mathbb{R}^X$*. Indeed,* $C[0,1]$ *is not an ideal of* $\mathbb{R}^{[0,1]}$*. (We write* $C[0,1]$ *instead of* $C([0,1])$ *throughout this thesis.)*

(2) *The vector sublattice* $\{f \in C[0,1] : f(0) = 0\}$ *of* $C[0,1]$ *is an ideal of* $C[0,1]$ *that is order dense in* $C[0,1]$ *and not majorizing in* $C[0,1]$*.*

(3) *The real vector lattice of all real-valued constant functions on* $[0,1]$ *is a majorizing vector sublattice of* $C[0,1]$ *that is not order dense in* $C[0,1]$*.*

(4) *Define* $c := \{f : \mathbb{N} \to \mathbb{R} : f \text{ is convergent}\}$ *and* $c_0 := \{f : \mathbb{N} \to \mathbb{R} : f \text{ converges to } 0\}$*. With respect to its natural vector space structure and pointwise ordering,* l^∞ *is an Archimedean real vector lattice. Moreover,* c_0 *is an ideal of* c*, and* c *is a vector sublattice of* l^∞ *that is not an ideal of* l^∞*.*

(5) *Every ideal of a real vector lattice* E *is a vector sublattice of* E*.*

The rest of this section is mostly devoted to maps between real vector lattices.

Let E and F be real vector lattices. We call a map $T : E \to F$ a *linear map* if $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g)$ for every $f, g \in E$ and for every $\alpha, \beta \in \mathbb{R}$. A linear map $T : E \to F$ is called a *vector lattice homomorphism* if

$$T(|f|) = |T(f)| \quad (f \in E).$$

A bijective vector lattice homomorphism is called a *vector lattice isomorphism.* If there exists a vector lattice isomorphism between E and F we say that E and F are *isomorphic as real vector lattices.*

Let $E_1, ..., E_n, F$ be real vector lattices. A map $T : \times_{k=1}^n E_k \to F$ is called *positive* if $T(f_1, ..., f_n) \in F^+$ whenever $f_k \in E_k^+$ for all $k \in \{1, ..., n\}$. An *n-linear map* $T : \times_{k=1}^n E_k \to F$ is a map which is linear in each variable separately. An n-linear map

$T : \times_{k=1}^{n} E_k \to F$ is called a *vector lattice n-morphism* if for each $k \in \{1, ..., n\}$ the mapping $f_k \mapsto T(f_1, ..., f_k, ..., f_n)$ $(f_k \in E_k)$ is a vector lattice homomorphism for fixed $f_j \in E_j^+$ $(j \neq k)$. The notions of vector lattice n-morphisms and positive n-linear maps (for $n = 2$) date back at least to a 1972 paper by Fremlin ([20], Definitions 3.1). Every vector lattice n-morphism is a positive n-linear map.

Example 2.5. *(1) Let $X_1, ..., X_n$ be topological spaces, and let $x_k \in X_k$ $(k \in \{1, ..., n\})$. The map $T : \times_{k=1}^{n} C(X_k) \to \mathbb{R}$ defined by*

$$T(f_1, ..., f_n) = f_1(x_1) \cdots f_n(x_n) \quad (f_k \in C(X_k),\ k \in \{1, ..., n\})$$

is a vector lattice n-morphism.

(2) The map $T : L^1[0,1] \to \mathbb{R}$ defined by $T(f) = \int_0^1 f(x)dx \quad (f \in L^1[0,1])$ is a positive linear map that is not a vector lattice homomorphism.

The following definition can be found in Definition 32.1 of [28].

Let E be a real vector lattice. If $\sup A$ exists in E for every nonempty order bounded subset A of E, we say that E is *Dedekind complete*. We call a real vector lattice E^δ the *Dedekind completion* of E if the following hold.

(1) E^δ is Dedekind complete.

(2) There exists a vector sublattice L of E^δ and a vector lattice isomorphism $\phi : E \to L$.

(3) Every $g \in E^\delta$ satisfies

$$g = \sup\{\phi(f) : f \in E, \phi(f) \leq g\} = \inf\{\phi(f) : f \in E, g \leq \phi(f)\}.$$

Every Archimedean real vector lattice has a Dedekind completion ([28], Theorem 32.5).

2.2 Complex Vector Lattices

We collect a variety of facts involving complex vector lattices in this section that will be used throughout Chapters 4 and 5.

Let V be a vector space over $\mathbb{R}$. We call the vector space

$$V_{\mathbb{C}} := V + iV = \{u + iv : u, v \in V\}$$

over $\mathbb{C}$ the *vector space complexification* of V. Note that $\{u + i0 : u \in V\} \subset V_{\mathbb{C}}$ and that $\{u + i0 : u \in V\}$ is a vector space over $\mathbb{R}$ with respect to the operations

$$(u + i0) + (v + i0) = (u + v) + i0 \quad (u, v \in V), \text{ and}$$
$$\lambda(u + i0) = \lambda u + i0 \quad (u \in V, \lambda \in \mathbb{R}).$$

Moreover, the map $\phi : V \to \{u + i0 : u \in V\}$ defined by $\phi(u) = u + i0$ is an isomorphism of real vector spaces. Therefore, we consider V to be a subset of $V_{\mathbb{C}}$.

Of course, $\mathbb{C} = \mathbb{R}_{\mathbb{C}}$ is an example. Indeed, the addition and scalar multiplication on $\mathbb{R}$ extend to corresponding vector space operations on $\mathbb{C}$. Most importantly for us, the absolute value on $\mathbb{R}$ extends to a complex modulus on $\mathbb{C}$ as well. From ordinary calculus, one obtains

$$\begin{aligned}|x + iy| &= \sup\{x\cos\theta + y\sin\theta : \theta \in [0, 2\pi]\}\\ &= \sup\{\mathrm{Re}(e^{-i\theta}(x + iy)) : \theta \in [0, 2\pi]\} \quad (x + iy \in \mathbb{C}).\end{aligned}$$

In an analogous manner, the absolute value on a real vector lattice E can be extended to $E_{\mathbb{C}}$ as follows

$$\begin{aligned} |f+ig| &:= \sup\{f\cos\theta + g\sin\theta : \theta \in [0,2\pi]\} \qquad (*) \\ &= \sup\{\mathrm{Re}(e^{-i\theta}(f+ig)) : \theta \in [0,2\pi]\} \quad (f+ig \in E_{\mathbb{C}}) \end{aligned}$$

when the supremum above exists. Indeed, suppose that E is a real vector lattice that is closed under the supremum in $(*)$. We have for $f \in E$ that

$$\begin{aligned} |f+i0| &= \sup\{f\cos\theta + 0\sin\theta : \theta \in [0,2\pi]\} \\ &= \sup\{f\cos\theta : \theta \in [0,2\pi]\} \\ &= f \vee (-f) = |f|. \end{aligned}$$

The content above leads to the notion of a complex vector lattice.

A vector space E over $\mathbb{C}$ is called a *complex vector lattice* if

(1) $E = E_\rho + iE_\rho$ for some real vector lattice E_ρ, and

(2) $\sup\{f\cos\theta + g\sin\theta : \theta \in [0,2\pi]\} \in E_\rho$ for every $f, g \in E_\rho$.

As mentioned in the introduction, the definition of a complex vector lattice dates back to a paper by de Schipper ([19], page 356). We next state some standard properties of complex vector lattices.

Proposition 2.6. (see [46], Theorem 91.2) *Let E be a complex vector lattice. Suppose that $f+ig, h+il \in E$ and that $\lambda \in \mathbb{C}$. Define $|a+ib| := \sup\{a\cos\theta + b\sin\theta : \theta \in [0,2\pi]\}$ for every $a+ib \in E$. The following hold.*

(1) $|f| \leq |f+ig|$ and $|g| \leq |f+ig|$.

(2) $|f + ig| = 0$ *if and only if* $f + ig = 0$.

(3) $|\lambda(f + ig)| = |\lambda||f + ig|$.

(4) $||f + ig| - |h + il|| \leq |f + ig + h + il| \leq |f + ig| + |h + il|$.

We say that a complex vector lattice E is *Archimedean* if

$$\inf\{\frac{1}{n}|h| : n \in \mathbb{N}\} = 0 \quad (h \in E).$$

A complex vector lattice $E = E_\rho + iE_\rho$ is Archimedean if and only E_ρ is Archimedean.

Example 2.7. *(1)* $\mathbb{C}$ *is an Archimedean complex vector lattice.*

(2) If X *is a topological space then* $C(X)_\mathbb{C}$ *is an Archimedean complex vector lattice. Given* $f+ig \in C(X)$, *the element* $|f+ig|$ *is given by* $|f+ig|(x) = \sqrt{[f(x)]^2 + [g(x)]^2} \quad (x \in X)$ ([46], page 188).

We next record some basic definitions involving complex vector lattices that are analogous to corresponding definitions for real vector lattices.

Let E be a complex vector lattice. We define $E^+ := \{f \in E : |f| = f\}$ and call E^+ the *positive cone* of E. A subset A of E is said to be *order bounded* if there exists $f \in E^+$ such that $|a| \leq f$ for every $a \in A$. We call E *Dedekind complete* if $\sup A$ exists in E^+ for every nonempty order bounded subset A of E^+.

Given complex vector lattices E and F, we call a map $T : E \to F$ a $\mathbb{C}$-*linear* map if $T(\alpha f + \beta g) = \alpha T(f) + \beta T(g) \quad (f, g \in E, \alpha, \beta \in \mathbb{C})$. An $\mathbb{R}$-*linear* map is defined analogously. A $\mathbb{C}$-linear map $T : E \to F$ is called a *vector lattice homomorphism* if

$$T(|f|) = |T(f)| \quad (f \in E).$$

A bijective vector lattice homomorphism is called a *vector lattice isomorphism*. If there exists a vector lattice isomorphism between E and F we say that E and F are *isomorphic as complex vector lattices*.

Let E be an Archimedean complex vector lattice. We call a vector subspace L of E a *vector sublattice* of E if $|f| \in L$ for every $f \in L$. A subset $A \subseteq E$ is said to be *order dense* in E if for every $0 < g \in E^+$ there exists $f \in A \cap E^+$ such that $0 < f \leq g$. We say that $A \subseteq E$ is *majorizing* in E if for every $f \in E$ there exists $a \in A$ such that $|f| \leq |a|$. If a vector subspace L of E has the property that $g \in L$ whenever $f \in L, g \in E$, and $|g| \leq |f|$ we call L an *ideal* of E. Every ideal of E is a vector sublattice of E.

We next discuss a few classes of maps between complex vector lattices.

Let V be a vector space over $\mathbb{R}$. We call V the *real part* of its vector space complexification $V + iV$ and write $V = (V + iV)_\rho$. Given vector spaces $V_1, ..., V_n, W$ over $\mathbb{R}$ and a map $T : \times_{k=1}^n (V_k)_\mathbb{C} \to W_\mathbb{C}$, we say that T *is real* if we have that $T(f_1, ..., f_n) \in W$ for every $f_k \in V_k$ $(k \in \{1, ..., n\})$. A map $T : \times_{k=1}^n V_k \to W$ is called an $n_\mathbb{R}$*-linear* map if T is an $\mathbb{R}$-linear map in each variable separately. An $n_\mathbb{C}$*-linear* map is defined similarly. The following formula ([6], Theorem 3) uniquely extends an $n_\mathbb{R}$-linear map $T : \times_{k=1}^n V_k \to W$ to an $n_\mathbb{C}$-linear map $T_\mathbb{C} : \times_{k=1}^n (V_k)_\mathbb{C} \to W_\mathbb{C}$:

$$T_\mathbb{C}(f_0^1 + if_1^1, ..., f_0^n + if_1^n) = \sum_{\epsilon_k \in \{0,1\}} T(f_{\epsilon_1}^1, ..., f_{\epsilon_n}^n) i^{\sum_{k=1}^n \epsilon_k}$$

where $(f_0^1 + if_1^1, ..., f_0^n + if_1^n) \in \times_{k=1}^n (V_k)_\mathbb{C}$. We will say that $T_\mathbb{C}$ is the *complexification of* T. Conversely, when $T : \times_{k=1}^n V_{k\mathbb{C}} \to W_\mathbb{C}$ is a real map we write T_ρ for the restriction of T to $\times_{k=1}^n V_k$. It follows that $(T_\rho)_\mathbb{C} = T$ whenever T is a real $n_\mathbb{C}$-linear map. We point out (see [36], page 364) that every vector space over $\mathbb{C}$ can be written as $V + iV$ for some vector space V over $\mathbb{R}$. A complex vector lattice, however, contains a canonical real part that is determined by its positive cone. This fact has a variety of consequences because much of the basic theory of real vector lattices is encoded via vector lattice homomorphisms and positive linear maps and runs parallel with the theory of complex vector lattices. We collect some examples of this phenomenon that will be used repeatedly.

Let $E_1, ..., E_n, F$ be complex vector lattices. A map $T : \times_{k=1}^n E_k \to F$ is called *positive* if $T(f_1, ..., f_n) \in F^+$ whenever $f_k \in E_k^+$ for all $k \in \{1, ..., n\}$. We declare an $n_{\mathbb{C}}$-linear map $T : \times_{k=1}^n E_k \to F$ to be a *vector lattice n-morphism* if the map

$$f_k \mapsto T(f_1, ..., f_k, ..., f_n) \quad (f_k \in E_k)$$

is a vector lattice homomorphism for each $k \in \{1, ..., n\}$ and for fixed $f_j \in E_j^+$ $(j \neq k)$. Every vector lattice n-morphism is positive, and every positive $n_{\mathbb{C}}$-linear map is real. For emphasis, we will at times refer to a vector lattice n-morphism between Archimedean complex vector lattices as a vector lattice $n_{\mathbb{C}}$*-morphism* or a vector lattice $\mathbb{C}$*-homomorphism* when $n = 1$. Analogously, we will at times refer to a vector lattice n-morphism between real vector lattices as a *vector lattice $n_{\mathbb{R}}$-morphism* or a vector lattice $\mathbb{R}$*-homomorphism* when $n = 1$.

The following definitions and theorem of Mittelmeyer and Wolff unify Archimedean real vector lattices with Archimedean complex vector lattices.

A *modulus* on a vector space E over $\mathbb{K}$ is an idempotent mapping m on E that satisfies

(1) $m(\alpha f) = |\alpha| m(f)$ for every $\alpha \in \mathbb{K}$ and for every $f \in E$,

(2) $m\Big(m\big(m(f) + m(g)\big) - m(f+g)\Big) = m(f) + m(g) - m(f+g)$ for every $f, g \in E$, and

(3) E is in the $\mathbb{K}$-linear hull of $m(E)$.

A modulus m is said to be *Archimedean* if for $f, g \in E$ it follows from

$$m\big(m(g) - nm(f)\big) = m(g) - nm(f)$$

for every $n \in \mathbb{N}$ that $f = 0$ ([29], Definition 1.1).

We summarize some facts obtained by Mittelmeyer and Wolff in the following theorem.

Theorem 2.8. ([29], Lemma 1.2, Corollary 1.4, Proposition 1.5, Theorem 2.2)

(1) If m is a modulus on a vector space E over $\mathbb{R}$ then $m(E)$ is a cone in E. Moreover, E is a vector lattice (as defined in Section 2.1) under the partial ordering induced by $m(E)$. Furthermore, we have that $m(f) = f \vee (-f)$ for every $f \in E$.

(2) If m is an Archimedean modulus on a vector space E over $\mathbb{C}$ then E is of the form $E_\rho + iE_\rho$, where

$$E_\rho := m(E) - m(E) = \{f - g : f, g \in m(E)\}$$

and E_ρ is an Archimedean vector lattice under the partial ordering induced by $m(E)$. Moreover, $\sup\{f \cos\theta + g \sin\theta : \theta \in [0, 2\pi]\}$ exists in E_ρ for every $f, g \in E_\rho$. Also, $m(f + ig) = \sup\{f \cos\theta + g \sin\theta : \theta \in [0, 2\pi]\}$ for every $f + ig \in E$.

In light of Mittelmeyer and Wolff's unifying theorem above, one can define real and complex Archimedean vector lattices simultaneously. We will use the following definition periodically throughout this thesis.

A vector space E over $\mathbb{K}$ is called an *Archimedean vector lattice over* $\mathbb{K}$ if E when equipped with an Archimedean modulus.

3 FUNCTIONAL COMPLETIONS

This chapter is devoted to a theory of functional completions for Archimedean real vector lattices with respect to positively homogeneous functions on $\mathbb{R}^n$. We use transfinite induction in Section 3.1 to concretely construct the uniform completion of an Archimedean real vector lattice that will be needed in the development of functional completions (Propositions 3.1 and 3.2). In Section 3.2, we sharpen a special case of Kusraev's Theorem 5.5 in [24], as mentioned in the introduction (Theorem 3.8). Finally, we introduce functional completions in Section 3.3 and prove their existence and uniqueness (Theorem 3.19 and Corollary 3.20).

3.1 Uniformly Complete Vector Lattices and the Uniform Completion

We remind the reader of the definition of relatively uniform convergence of sequences in Archimedean real vector lattices and the definition of a uniformly complete Archimedean real vector lattice.

Given an Archimedean real vector lattice E, a sequence (f_n) in E is said to converge *relatively uniformly* to f in E if there exists $0 < p \in E$ such that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ for which $|f_n - f| < \epsilon p$ for every $n \geq N$. In this case, we write $f_n \overset{ru}{\to} f$. We call a sequence (f_n) in E a *relatively uniformly Cauchy sequence* if there exists $0 < p \in E$ such that for every $\epsilon > 0$ there exists $N \in \mathbb{N}$ for which $|f_m - f_n| < \epsilon p$ for every $m, n \geq N$. We say that E is *uniformly complete* if every relatively uniformly Cauchy sequence in E converges relatively uniformly to an element of E.

Given an Archimedean real vector lattice E and a relatively uniformly Cauchy sequence (f_n) in E, there exists at most one f such that $f_n \overset{ru}{\to} f$ ([28], Theorem 16.2(i)). There exist various ways of introducing uniform completions of Archimedean real vector lattices in the literature, (see [42], Definition 8.6, [33], Definition 2.12, [44], page 894). For our purposes, we choose the definition by van Haandel in Definition 8.6 of [42].

Given an Archimedean real vector lattice E, we call a pair (E^u, ϕ) a *uniform completion* of E if the following hold.

(1) E^u is a uniformly complete Archimedean real vector lattice.

(2) $\phi : E \to E^u$ is an injective vector lattice homomorphism.

(3) For every uniformly complete Archimedean real vector lattice F and for every vector lattice homomorphism $T : E \to F$ there exists a unique vector lattice homomorphism $T^u : E^u \to F$ such that $T^u \circ \phi = T$.

We will also use the following definition, which was introduced (with slightly different notation) on page 85 of [28]. For an Archimedean real vector lattice E and for $A \subseteq E$, we define

$$\bar{A} := \{f \in E : \text{ there exists a sequence } (f_n) \text{ in } A \text{ such that } f_n \overset{ru}{\to} f\}$$

and call $\bar{A}$ the *pseudo uniform closure* of A. We declare A to be *relatively uniformly closed* if $\bar{A} = A$. The relatively uniformly closed sets are the closed sets in the *relatively uniform topology*, defined on page 84 of [28].

Finally, we iterate the pseudo uniform closure of a subset A of E as follows via transfinite induction.

$$A_1 := A,$$
$$A_\alpha := \overline{A_{\alpha-1}} \text{ when } \alpha > 1 \text{ is not a limit ordinal, and}$$
$$A_\alpha := \bigcup_{\beta<\alpha} A_\beta \text{ when } \alpha \text{ is a limit ordinal.}$$

Since [42] is somewhat inaccessible and the proof of the existence of the uniform completion in [42] skips the use of the iterated pseudo-closures, we provide a different proof. We start by extending positive linear maps on vector sublattices of an Archimedean real vector lattice to their pseudo-closures as follows.

Proposition 3.1. *Let L be a vector sublattice of an Archimedean real vector lattice E. The following hold.*

(1) L_α is a vector sublattice of E for every ordinal α.

(2) L_{ω_1} is relatively uniformly closed in E.

(3) If E is relatively uniformly complete then so is L_{ω_1}.

(4) L is dense in L_{ω_1} in the relatively uniform topology.

(5) For every uniformly complete Archimedean real vector lattice F, for every ordinal $1 \leq \alpha \leq \omega_1$, and for every positive linear map $T : L \to F$ there exists a unique positive linear map $T_\alpha : L_\alpha \to F$ such that $T_\alpha|_L = T$. Moreover, if T is a vector lattice homomorphism then so is T_α.

Proof. Statement (1) follows from transfinite induction and uses elementary calculus of relatively uniformly convergent sequences, (see [28], Theorem 16.2). Part (2) is an immediate consequence of the fact that every sequence in L_{ω_1} resides in an L_α for some $\alpha < \omega_1$. Statement (3) follows directly from (2), whereas (4) follows directly from the definition of the relatively uniform topology. To prove (5), let $T : L \to F$ be a positive linear map and define $T_1 := T$. Let $1 < \alpha \leq \omega_1$ be an ordinal and assume that T can be uniquely extended to a positive linear map $T_\beta : L_\beta \to F$ for every ordinal $1 \leq \beta < \alpha$. Let $f \in L_\alpha$. Suppose that α is not a limit ordinal. There exists a relatively uniformly Cauchy sequence (f_n) in $L_{\alpha-1}$ such that $f_n \overset{ru}{\to} f$. Since $T_{\alpha-1}$ is positive, we have from Proposition 1.3 in [7] that

$$|T_{\alpha-1}(g)| \leq T_{\alpha-1}(|g|) \text{ for all } g \in L_{\alpha-1}. \tag{$*$}$$

Therefore, $(T_{\alpha-1}(f_n))$ is a relatively uniformly Cauchy sequence in the uniformly complete vector lattice F. Hence there exists (a unique) $h \in F$ such that $T_{\alpha-1}(f_n) \overset{ru}{\to} h$. Define $T_\alpha : L_\alpha \to F$ by $T_\alpha(f) = h$. It follows from $(*)$ that T_α is well-defined. If α is a limit ordinal then define $T_\alpha(f) = T_\beta(f)$ ($f \in L_\beta$ and $\beta < \alpha$). By the induction hypothesis, T_α is well-defined.

It is readily checked by using elementary calculus of relatively uniformly convergent sequences that T_α is a positive linear map for every ordinal $1 \leq \alpha \leq \omega_1$, and that T_α is a vector lattice homomorphism when T is a vector lattice homomorphism. That T_α is indeed the unique positive linear extension of T to L_α follows from uniform density and transfinite induction. □

It is evident that if a uniform completion of an Archimedean real vector lattice exists then it is unique. We use the previous proposition to prove that every Archimedean real vector lattice has a uniform completion. The reader should compare Proposition 3.2 with Theorem 3.3 of [41], where Triki deals with Quinn's definition of uniform completion (see [33], Definition 2.12). A small adaptation of Theorem 3.3 of [41] to vector lattice homomorphisms rather than positive linear maps shows, in effect, that Quinn's definition of uniform completion is equivalent to van Haandel's definition above. In addition, we generalize Theorem 3.3 of [41] to multilinear maps.

Proposition 3.2. (1) *If E is an Archimedean real vector lattice then there exists a uniform completion (E^u, ϕ) of E.*

(2) Let $E_1, ..., E_n, F$ be Archimedean real vector lattices, and suppose that F is uniformly complete. If $T : \times_{k=1}^{n} E_k \to F$ is a positive n-linear map then there exist injective vector lattice homomorphisms $\phi_k : E_k \to E_k^u$ $(k \in \{1, ..., n\})$ and a unique positive n-linear map $T^u : \times_{k=1}^{n} E_k^u \to F$ such that $T^u(\phi_1(f_1), ..., \phi_n(f_n)) = T(f_1, ..., f_n)$ for every $(f_1, ..., f_n) \in \times_{k=1}^{n} E_k$. If T is a vector lattice n-morphism then so is T^u.

Proof. (1) Assume that E and F are Archimedean real vector lattices. The natural embedding ϕ of E into E^δ yields an injective vector lattice homomorphism. Define

$$E^u := \phi(E)_{\omega_1}.$$

Since we have $\phi(E) \subseteq E^u$, we may consider ϕ as a map from E to E^u. Let $T : E \to F$ be a positive linear map. Then the map $\tilde{T} : \phi(E) \to F$ defined by $\tilde{T}(\phi(f)) = T(f)$ is also a positive linear map. If T is a vector lattice homomorphism then so $\tilde{T}$. By Proposition 3.1(5), there exists a unique positive linear extension $\tilde{T}_{\omega_1} : E^u \to F$ of $\tilde{T}$, and if $\tilde{T}$ is a vector lattice homomorphism then so is $\tilde{T}_{\omega_1}$. Moreover, we have $\tilde{T}_{\omega_1} \circ \phi = T$.

(2) Let $E_1, ..., E_n, F$ be Archimedean real vector lattices. For each $k \in \{1, ..., n\}$, let ϕ_k be the natural embedding of E_k into E_k^δ, considered as a map from E_k to $\phi_k(E_k)$. Define

$$E_k^u := \phi_k(E_k)_{\omega_1}$$

for each $k \in \{1, ..., n\}$. Suppose $T : \times_{k=1}^{n} E_k \to F$ be a positive n-linear map and consider T as a map from $\times_{k=1}^{n} \phi_k(E_k)$ to F by identifying $\phi_k(f_k)$ with f_k for every $f_k \in E_k$ and for all $k \in \{1, ..., n\}$. For each $g_k \in E_k^+$ $(k \in \{2, ..., n\})$ we define

$$T_{g_2,...,g_n}(x) := T(x, g_2, ..., g_n) \ (x \in E_1).$$

By Proposition 3.1(5), there exists a unique positive linear map $T^u_{g_2,...,g_n} : E_1^u \to F$ such that $T^u_{g_2,...,g_n}(x) = T_{g_2,...,g_n}(x)$ $(x \in E_1)$. Moreover, if $T_{g_2,...,g_n}$ is a vector lattice homomorphism then so is $T^u_{g_2,...,g_n}$. Next we define

$$T_+(g_1, ..., g_n) = T^u_{g_2,...,g_n}(g_1) \ (g_1 \in E_1^u \text{ and } g_k \in E_k^+ \ (k \in \{2, ..., n\})).$$

Let $j \in \{2, ..., n\}$, and let $g_j, g_j' \in E_j^+$. Note that

$$T^u_{g_2,\dots g_j+g_j',\dots,g_n} \text{ and } T^u_{g_2,\dots g_j,\dots,g_n} + T^u_{g_2,\dots g_j',\dots,g_n}$$

are both positive linear extensions of $T_{g_2,...,g_j+g_j',...,g_n}$ from E_1^u to F. It follows from the uniqueness of such extensions that $T^u_{g_2,\dots g_j+g_j',\dots,g_n} = T^u_{g_2,\dots g_j,\dots,g_n} + T^u_{g_2,\dots g_j',\dots,g_n}$. Therefore, T_+ is additive in each variable separately. By routine reasoning, T_+ extends to a positive n-linear map from $E_1^u \times E_2 \times \cdots \times E_n$ to F which is a vector lattice n-morphism in the case that T is a vector lattice n-morphism. By repeating this argument for the remaining $n-1$ variables, we obtain the desired result. □

3.2 Functional Calculus: Basic Facts, Examples, and Convexity

We first review the functional calculus for Archimedean real vector lattices introduced by Buskes, de Pagter, and van Rooij in Section 3 of [9]. To do so, we write $\mathcal{H}(\mathbb{R}^m)$ for the space of all continuous, real-valued functions h on $\mathbb{R}^m$ that are *positively homogeneous*, i.e. $h(\lambda x) = \lambda h(x)$ for every $\lambda \in \mathbb{R}^+$ and all $x \in \mathbb{R}^m$. The space of all nonzero real-valued vector lattice homomorphisms on an Archimedean real vector lattice E is denoted by $H(E)$. For a nonempty subset A of an Archimedean real vector lattice E, we denote by $\langle A \rangle$ the vector sublattice generated by A in E. Given $a_1, ..., a_m, b \in E$ and $h \in \mathcal{H}(\mathbb{R}^m)$, we write $h(a_1, ..., a_m) = b$ when $h(\omega(a_1), ..., \omega(a_m)) = \omega(b)$ for every $\omega \in H(\langle a_1, ..., a_m, b \rangle)$.

Let $E^\#$ denote the set of all linear functionals on an Archimedean real vector lattice E. We say that $G \subseteq E^\#$ *separates the points* of E if for every $f \in E \setminus \{0\}$ there exists $T \in G$ such that $T(f) \neq 0$.

Example 3.3 will prove useful in the proof of Theorem 3.8(1).

Example 3.3. *Let A be a nonempty set, and for every $a \in A$ define $\hat{a} \in (\mathbb{R}^A)^\#$ by $\hat{a}(f) = f(a)$ $(f \in \mathbb{R}^A)$. Then $\{\hat{a} : a \in A\}$ separates the points of $\mathbb{R}^A$.*

To see that $\{\hat{a} : a \in A\}$ separates the points of $\mathbb{R}^A$, let $f \in \mathbb{R}^A \setminus \{0\}$. Since $f \neq 0$ there exists $a \in A$ such that $f(a) \neq 0$, and therefore $\hat{a}(f) \neq 0$.

The Stolarsky and Gini means (see [40], respectively [30]) are examples of elements of $\mathcal{H}(\mathbb{R}^2)$. Though they are often defined on $(0, \infty)^2$, they can be extended continuously to all of $\mathbb{R}^2$ as follows.

Example 3.4. *For real numbers $r \neq s$ and $s \neq 0$, define*

$$\mu_{r,s}(x,y) = \begin{cases} \left(\frac{r(|x|^s - |y|^s)}{s(|x|^r - |y|^r)}\right)^{\frac{1}{s-r}} & if\ x \neq y \\ |x| & if\ x = y \end{cases}$$

for $x, y \in \mathbb{R}$. We call $\mu_{r,s}$ the (r,s)-Stolarsky mean. Particularly, $\mu_{2,4}(x,y) = \sqrt{\frac{|x|^2 + |y|^2}{2}}$ for $x, y \in \mathbb{R}$ and $\mu_{1,-1}(x,y) = \sqrt{|xy|}$ for $x, y \in \mathbb{R}$. We call $\mu_{2,4}$ the square mean and $\mu_{1,-1}$ the geometric mean.

For short, we denote the square mean by μ and the geometric mean by γ throughout this thesis. The reader will periodically be reminded of this shorthand and does not need to memorize the meaning of μ or γ.

Example 3.5. *For real numbers $r \neq s$, define*

$$\nu_{r,s}(x,y) = \begin{cases} \left(\frac{|x|^s+|y|^s}{|x|^r+|y|^r}\right)^{\frac{1}{s-r}} & if\ (x,y) \neq (0,0) \\ 0 & if\ (x,y) = (0,0) \end{cases}$$

for $x, y \in \mathbb{R}$. We call $\nu_{r,s}$ the (r,s)-Gini mean.

In Section 4.1 we will use functional calculus with respect to the square mean μ in Example 3.4 to construct the vector lattice complexification of an Archimedean real vector lattice. We will first need to identify elements of an Archimedean real vector lattice E of the form $h(a_1, ..., a_m)$ $(a_1, ..., a_m \in E^+)$ for convex or concave $h \in \mathcal{H}(\mathbb{R}^m)$ with elements of E that are defined via differential calculus. To this end, we follow the idea to use tangents by Beukers, Huijsmans, and de Pagter in Theorem 4.2 of [4]. Some notations are needed.

Notations 3.6. Let E be an Archimedean real vector lattice. The Euclidean norm on $\mathbb{R}^m$ is denoted by $||\ ||$. For $h \in \mathcal{H}(\mathbb{R}^m)$ we set

$$\Delta_h = \{\boldsymbol{c} \in (\mathbb{R}^+)^m : h \text{ is differentiable at } \boldsymbol{c} \text{ and } ||\boldsymbol{c}|| = 1\}.$$

For $h \in \mathcal{H}(\mathbb{R}^m), \boldsymbol{c} \in \Delta_h$, and $\boldsymbol{a} := (a_1, ..., a_m) \in E^m$ we define $\nabla h(\boldsymbol{c}) \cdot \boldsymbol{a} := \sum_{k=1}^{m} \frac{\partial h(\boldsymbol{c})}{\partial x_k} a_k$. Given $a_1, ..., a_m \in E^+$ $(m \geq 2)$ and $\theta = (\theta_1, ..., \theta_{m-1}) \in [0, \pi]^{m-1}$ we define

$$s_\theta(a_1, ..., a_m) :=$$

$$\cos\theta_1 a_1 + \sum_{k=2}^{m-2}\Big(\prod_{j=1}^{k-1} \sin\theta_j\Big)\cos\theta_k a_k + \Big(\prod_{j=1}^{m-2} \sin\theta_j\Big)\cos\theta_{m-1} a_{m-1} + \Big(\prod_{j=1}^{m-1} \sin\theta_j\Big)a_m,$$

where $\sum_{k=2}^{m-2}\Big(\prod_{j=1}^{k-1} \sin\theta_j\Big)\cos\theta_k$ is taken to equal zero for $m \in \{2,3\}$, and $\Big(\prod_{j=1}^{m-2} \sin\theta_j\Big)\cos\theta_{m-1}$ is taken to equal zero for $m = 2$. For short, we denote $s_\theta(e_1, ..., e_m)$ by n_θ, where e_k is the kth element of the standard orthonormal basis of $\mathbb{R}^m$. We note that s_θ appears in the derivation of hyperspherical coordinates on pages 64–65 in [5]. Finally, for $n \in \mathbb{N}$ we set

$$P_n = \left\{ \left(\frac{l_1 \pi}{2^{n+1}}, ..., \frac{l_{m-1}\pi}{2^{n+1}} \right) : l_1, ..., l_{m-1} \in \{1, ..., 2^n\} \right\}.$$

As announced in the introduction, we sharpen a special case of Kusraev's Theorem 5.5 in [24] while closely following Kusraev's proof (Theorem 3.8(1)). We need the following lemma. Parts (1) and (2) are known, but we were unable to find a reference for (3).

Lemma 3.7. *Let $h \in \mathcal{H}(\mathbb{R}^m)$ be convex (concave) on $(\mathbb{R}^+)^m$. The following hold.*

(1) h is differentiable almost everywhere with respect to Lebesgue measure on $(0, \infty)^m$.

(2) If h is differentiable at $\boldsymbol{x} \in \mathbb{R}^m$ then $\nabla h(\lambda \boldsymbol{x}) = \nabla h(\boldsymbol{x})$ for every $0 < \lambda \in \mathbb{R}$.

(3) $h(\boldsymbol{x}) = \sup\{\nabla h(\boldsymbol{c}) \cdot \boldsymbol{x} : \boldsymbol{c} \in \Delta_h\}$ for every $\boldsymbol{x} \in (\mathbb{R}^+)^m$ ($h(\boldsymbol{x}) = \inf\{\nabla h(\boldsymbol{c}) \cdot \boldsymbol{x} : \boldsymbol{c} \in \Delta_h\}$ for every $\boldsymbol{x} \in (\mathbb{R}^+)^m$).

Proof. (1) By Exercise 1.17 in [32], which follows from Radamacher's Theorem (also see Exercise 1.18 in [32]), h is differentiable on $(0, \infty)^m$ outside a set of Lebesgue measure zero.

(2) Suppose that h is differentiable at $\boldsymbol{x} \in \mathbb{R}^m$ and let $0 < \lambda \in \mathbb{R}$. It suffices to show that

$$\frac{\partial h(\lambda \boldsymbol{x})}{\partial x_k} = \frac{\partial h(\boldsymbol{x})}{\partial x_k}$$

for every $k \in \{1, ..., m\}$. To this end, let $k \in \{1, ..., m\}$. Setting $t' := \frac{t}{\lambda}$ for $t \in \mathbb{R}$, we obtain

$$\begin{aligned} \frac{\partial h(\lambda \boldsymbol{x})}{\partial x_k} &= \lim_{t \to 0} \frac{h(\lambda x_1, ..., \lambda x_k + t, ..., \lambda x_m)}{t} = \lim_{\lambda t' \to 0} \frac{h(\lambda x_1, ..., \lambda x_k + \lambda t', ..., \lambda x_m)}{\lambda t'} \\ &= \lim_{t' \to 0} \frac{h(x_1, ..., x_k + t', ..., x_m)}{t'} = \frac{\partial h(\boldsymbol{x})}{\partial x_k}. \end{aligned}$$

(3) Suppose that h is convex on $(\mathbb{R}^+)^m$. It follows from Euler's Homogeneous Function Theorem (for instance, Exercise 2-34 in [39]) that $\nabla h(\boldsymbol{c}) \cdot \boldsymbol{c} = h(\boldsymbol{c})$ whenever $\boldsymbol{c} \in \mathbb{R}^m$ and h

is differentiable at $\boldsymbol{c}$. From this observation as well as the convexity of h, we obtain

$$h(\boldsymbol{x}) = \sup\{\nabla h(\boldsymbol{c}) \cdot \boldsymbol{x} : \boldsymbol{c} \in \Delta_h\}$$

for every $\boldsymbol{x} \in (\mathbb{R}^+)^m$ where h is differentiable. That $\nabla h(\boldsymbol{c}) \cdot \boldsymbol{x} \leq h(\boldsymbol{x})$ for every $\boldsymbol{c} \in \Delta_h$ and for every $\boldsymbol{x} \in (\mathbb{R}^+)^m$ is obtained as well. Suppose that h is not differentiable at $\boldsymbol{b} \in (\mathbb{R}^+)^m$ and let $\epsilon > 0$. We just need to show that there exists $\boldsymbol{c} \in \Delta_h$ such that $h(\boldsymbol{b}) - \nabla h(\boldsymbol{c}) \cdot \boldsymbol{b} < \epsilon$. To this end, note that there exists $\delta_1 > 0$ such that $||h(\boldsymbol{c}) - h(\boldsymbol{b})|| < \epsilon/2$ whenever $||\boldsymbol{c} - \boldsymbol{b}|| < \delta_1$ since h is continuous. Since h is convex and continuous, it is locally Lipschitz (see [32], Proposition 1.6). Let

$$B_{\delta_2}(\boldsymbol{b}) = \{\boldsymbol{x} \in (\mathbb{R}^+)^m : ||\boldsymbol{x} - \boldsymbol{b}|| < \delta_2\}$$

be a neighborhood of $\boldsymbol{b}$ where h is Lipschitz, say with Lipschitz constant M. Then each partial derivative satisfies $|\frac{\partial h(\boldsymbol{c})}{\partial x_k}| \leq M$ for every $\boldsymbol{c} \in B_{\delta_2}(\boldsymbol{b})$ where h is differentiable. Furthermore, there exists $\delta_3 > 0$ such that $\sum_{k=1}^m |c_k - b_k| < \frac{\epsilon}{2mM}$ whenever $||\boldsymbol{c} - \boldsymbol{b}|| < \delta_3$. In this case we get

$$||\nabla h(\boldsymbol{c}) \cdot \boldsymbol{c} - \nabla h(\boldsymbol{c}) \cdot \boldsymbol{b}|| = ||\nabla h(\boldsymbol{c}) \cdot (\boldsymbol{c} - \boldsymbol{b})|| \leq mM \sum_{k=1}^m |c_k - b_k| < \epsilon/2.$$

Since h is differentiable almost everywhere on $(0, \infty)^m$, we may choose $\boldsymbol{c} \in \Delta_h$ and $0 < \lambda \in \mathbb{R}$ such that $||\lambda \boldsymbol{c} - \boldsymbol{b}|| < \delta_1 \wedge \delta_2 \wedge \delta_3$. From part (2) and the identity $\nabla h(\boldsymbol{c}) \cdot \boldsymbol{c} = h(\boldsymbol{c})$ for $\boldsymbol{c} \in \mathbb{R}^m$ where h is differentiable, we obtain

$$\begin{aligned} ||h(\boldsymbol{b}) - \nabla h(\boldsymbol{c}) \cdot \boldsymbol{b}|| &= ||h(\boldsymbol{b}) - h(\lambda \boldsymbol{c}) + \nabla h(\boldsymbol{c}) \cdot \lambda \boldsymbol{c} - \nabla h(\boldsymbol{c}) \cdot \boldsymbol{b}|| \\ &\leq ||h(\boldsymbol{b}) - h(\lambda \boldsymbol{c})|| + ||\nabla h(\boldsymbol{c}) \cdot \lambda \boldsymbol{c} - \nabla h(\boldsymbol{c}) \cdot \boldsymbol{b}|| \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

The case where h is concave on $(\mathbb{R}^+)^m$ is handled in a similar manner. $\square$

We now turn to the result promised before Lemma 3.7. To avoid possible confusion when working with vector sublattices, we use the following notation. Let E be an Archimedean real vector lattice, and let L be a vector sublattice of E. For a nonempty subset A of L, we write L-sup A for the least upper bound of A in L and E-sup A for the least upper bound of A in E.

Theorem 3.8. *Let E be an Archimedean real vector lattice, let $h \in \mathcal{H}(\mathbb{R}^m)$, and let $a_1, ..., a_m \in E^+$.*

(1) If h is convex (concave) on $(\mathbb{R}^+)^m$ then $h(a_1, ..., a_m) = b$ for some $b \in E$ if and only if

$$b = E\text{-}\sup\{\nabla h(\boldsymbol{c}) \cdot \boldsymbol{a} : \boldsymbol{c} \in \Delta_h\} \ (b = E\text{-}\inf\{\nabla h(\boldsymbol{c}) \cdot \boldsymbol{a} : \boldsymbol{c} \in \Delta_h\}).$$

(2) If h is convex (concave) on $(\mathbb{R}^+)^m$, if $m \geq 2$, and if all the partial derivatives of h are uniformly continuous on $\{s_\theta : \theta \in \bigcup_{n\in\mathbb{N}} P_n\}$ then the sequence

$$\big(\sup\{\nabla h(s_\theta) \cdot \boldsymbol{a} : \theta \in P_n\}\big) \text{ (the sequence } \big(\inf\{\nabla h(s_\theta) \cdot \boldsymbol{a} : \theta \in P_n\}\big))$$

converges relatively uniformly to $h(a_1, ..., a_m)$.

Proof. (1) Let $\boldsymbol{a} = (a_1, ..., a_m)$ and set $A := \{\nabla h(\boldsymbol{c}) \cdot \boldsymbol{a} : \boldsymbol{c} \in \Delta_h\}$. Moreover, define $L := \langle a_1, ..., a_m, E\text{-}\sup A\rangle$. Suppose that E-sup A exists in E. To prove that

$$E - \sup A = h(a_1, ..., a_m),$$

it suffices to verify that

$$h(\omega(a_1), ..., \omega(a_m)) = \omega(E\text{-}\sup A)$$

for every ω in a point separating subset of $H(L)$ ([9], Lemma 3.3). To this end, note that there exists a metrizable space Y, an order-dense vector sublattice F of $C(Y)$, and a vector lattice isomorphism $\phi : L \to F$ (see [12], (ii) on page 526 and Theorem 2.4(i)). Consequently, we can take the point separating set mentioned above to be $\{\hat{y} \circ \phi : y \in Y\}$ (recall Example 3.3). Moreover, we have $F\text{-sup}\,\phi(A) = C(Y)\text{-sup}\,\phi(A)$ (see [17], Lemma 13.21(i)). Since ϕ is an isomorphism and since $E\text{-sup}\,A \in L$ we get

$$\phi(E\text{-}\sup A) = \phi(L\text{-}\sup A) = F\text{-}\sup \phi(A) = C(Y)\text{-}\sup \phi(A).$$

Define

$$b(y) = h(\phi(a_1)(y), ..., \phi(a_m)(y)) \ (y \in Y),$$

and note that $b \in C(Y)$. Since $a_1, ..., a_m \in E^+$ we have $\phi(a_k) \in C(Y)^+$ $(k \in \{1, ..., m\})$. From Lemma 3.7(3) above we obtain

$$b(y) = \sup\Big\{\sum_{k=1}^{m} \frac{\partial h(\boldsymbol{c})}{\partial x_k}\phi(a_k)(y) : \boldsymbol{c} \in \Delta_h\Big\} \ (y \in Y).$$

Therefore, $b = C(Y)\text{-}\sup \phi(A)$ and thus $b = \phi(E\text{-}\sup A)$. Moreover, we have

$$\hat{y}(\phi(E\text{-}\sup A)) = b(y) = h(\phi(a_1)(y), ..., \phi(a_m)(y)) = h(\hat{y}(\phi(a_1)), ..., \hat{y}(\phi(a_m))) \ (y \in Y).$$

We conclude that $h(a_1, ..., a_m) = E\text{-}\sup A$.

Conversely, suppose that $h(a_1, ..., a_m) = b$ for some $b \in E$ and let $\boldsymbol{c} \in \Delta_h$. Lemma 3.7(3) implies that $\frac{\partial h(\boldsymbol{c})}{\partial x_k} \leq h(e_k)$ for every $k \in \{1, ..., m\}$, and thus we have

$$\sum_{k=1}^{m} \frac{\partial h(\boldsymbol{c})}{\partial x_k} a_k \leq \sum_{k=1}^{m} h(e_k) a_k.$$

Since A is bounded above, E^δ-$\sup A$ exists in the Dedekind completion E^δ of E. Thus $h(a_1,..,a_m) = E^\delta$-$\sup A$. Moreover, we have E^δ-$\sup A = E$-$\sup A$ since $h(a_1,...,a_m) \in E$.

(2) Assume that $m \geq 2$ and that all the partial derivatives of h exist and are uniformly continuous on $\{s_\theta : \theta \in \bigcup_{n\in\mathbb{N}} P_n\}$. It follows from the derivation of hyperspherical coordinates on pages 64–65 in [5] that $\boldsymbol{d} = s_\theta$ for some

$$\theta \in [0,\pi]^{m-2} \times [0, 2\pi] \ (\theta \in [0, 2\pi], \text{ for } m = 2)$$

whenever $\boldsymbol{d} \in \mathbb{R}^m$ and $||\boldsymbol{d}|| = 1$. In particular, if $\boldsymbol{d} \in (\mathbb{R}^+)^m$ and $||\boldsymbol{d}|| = 1$ then $\boldsymbol{d} = s_\theta$ for some $\theta \in [0, \frac{\pi}{2}]^{m-1}$. A standard induction argument verifies that $||s_\theta|| = 1$ for every $\theta \in [0, \frac{\pi}{2}]^{m-1}$. Evidently, the sequence (σ_n) defined by $\sigma_n = \sup\{\nabla h(s_\theta) \cdot \boldsymbol{a} : \theta \in P_n\}$ for $n \in \mathbb{N}$ is increasing and $\sup\{\sigma_n : n \in \mathbb{N}\} = h(a_1,...,a_m)$. Let $r, n \in \mathbb{N}$ and assume $r < n$. Next we set

$$I_j = \Big[\frac{(l_j - 1)\pi}{2^{r+1}}, \frac{l_j\pi}{2^{r+1}}\Big] \ (j \in \{1,...,m-1\}).$$

By Exercise 91.10 in [46] we get

$$\begin{aligned} &|\sup\{\nabla h(s_\theta)\cdot \boldsymbol{a} : \theta \in P_n\} - \sup\{\nabla h(s_\phi)\cdot \boldsymbol{a} : \phi \in P_r\}| \\ &\qquad \leq \sup\{|\nabla h(s_\theta - s_\phi)\cdot \boldsymbol{a}| : \phi \in P_r, \theta \in \times_{j=1}^{m-1} I_j\} \\ &\qquad \leq \sup\{|\nabla h(s_\theta - s_\phi)|\cdot |\boldsymbol{a}| : \phi \in P_r, \theta \in \times_{j=1}^{m-1} I_j\}. \end{aligned}$$

Note that $||\phi - \theta|| \leq \sqrt{m-1}\frac{\pi}{2^{r+1}}$ for every $\phi \in P_r$ and every $\theta \in \times_{j=1}^{m-1} I_j$. Thus given $\epsilon > 0$ we have for sufficiently large r that

$$\sup\{|\nabla h(s_\theta - s_\phi)|\cdot|\boldsymbol{a}| : \ \phi \in P_r, \theta \in \times_{j=1}^{m-1} I_j\} \leq \epsilon \sum_{k=1}^{m} |a_k|.$$

It follows that $\sigma_n \overset{ru}{\to} h(a_1,...,a_m)$. The proof of the concave case is similar. □

In particular, Theorem 3.8(2) holds for all the *pth power means*. The pth power mean is the Stolarsky mean $\mu_{p,2p}$ ($p \in \mathbb{N} \setminus \{1\}$). Indeed, all pth power means are continuously differentiable on the compact set $\{s_\theta : \theta \in [0, \frac{\pi}{2}]^{m-1}\}$.

Two special cases of Theorem 3.8(1) follow as corollaries.

Corollary 3.9. *For $m \in \mathbb{N} \setminus \{1\}$, define $h(x_1, ..., x_m) = \Big(\sum_{k=1}^{m} x_k^2\Big)^{\frac{1}{2}}$ ($x_1, ..., x_m \in \mathbb{R}$). Let E be an Archimedean real vector lattice and suppose that $a_1, ..., a_m \in E^+$. We have that $h(a_1, ..., a_m) \in E$ if and only if*

$$h(a_1, ..., a_m) = \sup\{s_\theta(a_1, ..., a_m) : \theta \in [0, \frac{\pi}{2}]^{m-1}\}.$$

Proof. Evidently h is a member of $\mathcal{H}(\mathbb{R}^m)$ and h is convex on $(\mathbb{R}^+)^m$. It is readily checked that $\Delta_h = \{\boldsymbol{c} \in (\mathbb{R}^+)^m : ||\boldsymbol{c}|| = 1\}$. For $a_1, ..., a_m \in E^+$ we have from Theorem 3.8(1) that

$$\begin{aligned} h(a_1, ..., a_m) &= \sup\Big\{\sum_{k=1}^{m} \frac{c_k}{\sqrt{c_1^2 + \cdots + c_m^2}} a_k : c \in (\mathbb{R}^+)^m, ||\boldsymbol{c}|| = 1\Big\} \\ &= \sup\Big\{\sum_{k=1}^{m} d_k a_k : \boldsymbol{d} \in (\mathbb{R}^+)^m, ||\boldsymbol{d}|| = 1\Big\} \\ &= \sup\{s_\theta(a_1, ..., a_m) : \theta \in [0, \frac{\pi}{2}]^{m-1}\}. \end{aligned}$$

□

Corollary 3.10. *For $m \in \mathbb{N} \setminus \{1\}$, define $h(x_1, ..., x_m) = m\Big(\prod_{k=1}^{m} |x_k|\Big)^{\frac{1}{m}}$ ($x_1, ..., x_m \in \mathbb{R}$). Suppose that E is an Archimedean real vector lattice and let $a_1, ..., a_m \in E^+$. We have that $h(a_1, ..., a_m) \in E$ if and only if*

$$h(a_1, ..., a_m) = \inf\{\theta_1 a_1 + \cdots + \theta_m a_m : \theta_1, ..., \theta_m \in (0, \infty), \theta_1 \cdots \theta_m = 1\}.$$

Proof. Note that h is a member of $\mathcal{H}(\mathbb{R}^m)$ that is concave on $(\mathbb{R}^+)^m$. From ordinary calculus we have $\Delta_h = \{\boldsymbol{c} \in (0,\infty)^m : ||\boldsymbol{c}|| = 1\}$. It follows from Theorem 3.8(1) that

$$h(a_1, ..., a_m) = \inf\Big\{\sum_{k=1}^{m} \frac{c_1 \cdots \hat{c}_k \cdots c_m}{(c_1 \cdots c_m)^{\frac{m-1}{m}}} a_k : \boldsymbol{c} \in (0,\infty)^m, ||\boldsymbol{c}|| = 1\Big\}$$
$$= \inf\{\theta_1 a_1 + \cdots + \theta_m a_m : \theta_1, ..., \theta_m \in (0,\infty), \theta_1 \cdots \theta_m = 1\}.$$

□

3.3 Functional Completions

In Remark 4 of [2], Azouzi constructs what he calls the *square mean closure* of a given Archimedean real vector lattice inside its Dedekind completion. Although Azouzi does not mention functional calculus, it turns out that his square mean closure is with respect to the square mean (via functional calculus)in Example 3.4. Indeed, Azouzi calls an Archimedean real vector lattice E *square mean closed* if

$$f \boxplus g := \sup\{f \cos\theta + g \sin\theta : \theta \in [0, 2\pi]\}$$

exists in E for every $f, g \in E^+$. For a square mean closed Archimedean real vector lattice E and the square mean $\mu \in \mathcal{H}(\mathbb{R}^m)$ we thus have

$$\mu(f,g) = \frac{1}{\sqrt{2}}(f \boxplus g) \ (f, g \in E^+)$$

by Corollary 3.9. Therefore, in a way, Theorem 3.8(2) generalizes the Beukers-Huijsmans-de Pagter circle approximation theorem (see section 2 of [4]) for the existence of a modulus in the vector space complexification of a uniformly complete Archimedean real vector lattice. Indeed, the aforementioned circle approximation theorem was later reformulated in Lemma 2.8 of [2] for square mean closed Archimedean real vector lattices. The circle approximation for the square mean is generalized to all pth-power means in Theorem 3.8(2). The circle approximation theorem of Beukers, Huijsmans, and de Pagter is particularly noteworthy because its proof did not depend on the Axiom of Choice.

The authors of [3] call an Archimedean real vector lattice E *geometric mean closed* if

$$f \boxtimes g := \frac{1}{2}\inf\{\theta f + \theta^{-1} g : \theta \in (0,\infty)\}$$

exists in E for every $f, g \in E^+$. For the geometric mean $\gamma \in \mathcal{H}(\mathbb{R}^2)$ and a geometric mean closed Archimedean real vector lattice E we have $\gamma(f,g) = f \boxtimes g$ for every $f, g \in E^+$ (Corollary 3.10).

Hence square mean closed Archimedean real vector lattices and geometric mean closed Archimedean real vector lattices are examples of Archimedean real vector lattices in which one can freely use functional calculus with respect to specific continuous, real-valued, positively homogeneous functions on $\mathbb{R}^m$.

For an Archimedean real vector lattice E and for $h \in \mathcal{H}(\mathbb{R}^m)$, we say that E is *h-complete* if for every $a_1, ..., a_m \in E$ there exists $b \in E$ such that $h(a_1, ..., a_m) = b$. For a subset $\mathcal{D}$ of $\bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$, we say that E is *$\mathcal{D}$-complete* if E is h-complete for every $h \in \mathcal{D}$.

Example 3.11. *The Archimedean real vector lattice $S[0,1]$ of all step functions on $[0,1]$ is h-complete for every $h \in \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$.*

Indeed, let $h \in \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$ and let $s, t \in S[0,1]$. There exists a partition $(A_k)_{k=1}^n$ of $[0,1]$, where A_k is an interval for every $k \in \{1, ..., n\}$, as well as $\alpha_k, \beta_k \in \mathbb{R}$ $(k \in \{1, ..., n\})$ such that

$$s(x) = \sum_{k=1}^{n} \alpha_k \chi_{A_k}(x)\ (x \in [0,1]) \text{ and } t(x) = \sum_{k=1}^{n} \beta_k \chi_{A_k}(x)\ (x \in [0,1]).$$

In the line above, χ_{A_k} denotes the characteristic function of A_k defined for $x \in [0,1]$ by

$$\chi_{A_k}(x) = \begin{cases} 1 & : x \in A_k \\ 0 & : x \notin A_k. \end{cases}$$

We will verify that

$$h(s,t) = \sum_{k=1}^{n} h(\alpha_k, \beta_k)\chi_{A_k}.$$

To this end, set $G := \{\hat{a} : a \in [0,1]\}$. We have $G \subseteq H(\langle s, t, \sum_{k=1}^{m} h(\alpha_k, \beta_k)\chi_{A_k}\rangle)$. Moreover, G separates the points of $S[0,1]$ (Example 3.3). Let $\hat{a} \in G$ and note that $a \in A_j$ for exactly one $j \in \{1, ..., n\}$. It follows that

$$\begin{aligned} h(\hat{a}(s), \hat{a}(t)) &= h(\sum_{k=1}^{n} \alpha_k \chi_{A_k}(a), \sum_{k=1}^{n} \beta_k \chi_{A_k}(a)) = h(\alpha_j, \beta_j) \\ &= \sum_{k=1}^{n} h(\alpha_k, \beta_k)\chi_{A_k}(a) = \hat{a}(\sum_{k=1}^{n} h(\alpha_k, \beta_k)\chi_{A_k}). \end{aligned}$$

Then $h(s,t) = \sum_{k=1}^{n} h(\alpha_k, \beta_k)\chi_{A_k}$ by Lemma 3.3 of [9].

Let A be a nonempty subset of a vector space V over $\mathbb{R}$. We denote by $[A]$ the vector subspace of V generated by A.

Example 3.12. *The vector sublattice $\langle \sin, \cos \rangle$ of $C(\mathbb{R})$ is not square mean complete.*

To verify the claim in Example 3.12, suppose that $\langle \sin, \cos \rangle$ is square mean complete. Then by Corollary 3.9 we have for $x \in \mathbb{R}$ that

$$\begin{aligned} \mu(\sin x, \cos x) &= \frac{1}{\sqrt{2}} \sup\{\sin x \cos\theta + \cos x \sin\theta : \theta \in [0, 2\pi]\} \\ &= \frac{1}{\sqrt{2}} \sup\{\sin(x+\theta) : \theta \in [0, 2\pi]\} = \frac{1}{\sqrt{2}}\mathbf{1}, \end{aligned}$$

where $\mathbf{1}$ denotes the constant function on $\mathbb{R}$ with value 1. It follows that $\mu(\sin, \cos) = \frac{1}{\sqrt{2}}\mathbf{1}$. Therefore, we have $\mathbf{1} \in \langle \sin, \cos \rangle$. Note that $\langle \sin, \cos \rangle = \langle [\sin, \cos] \rangle$. Therefore, every $f \in \langle \sin, \cos \rangle$ is of the form

$$f = \bigwedge_{i=1}^{n} \bigvee_{j=1}^{m} t_{j,k}$$

where $t_{j,k} \in [\sin, \cos]$ for each j and each k ([1], Exercise 4.8). Let $t_1, t_2 \in [\sin, \cos]$. By the continuity of t_1 and t_2 there exists an open interval (a, b) of $\mathbb{R}$ such that $t_1 \wedge t_2 = t_i$ on (a, b) for some $i \in \{1, 2\}$. Using induction we find an open interval $(c, d) \subseteq (a, b)$ such that $\bigwedge_{j=1}^{n} (\bigvee_{k=1}^{m} t_{j,k}) = \bigvee_{k=1}^{m} t_{j_0,k}$ on (c, d) for some $j_0 \in \{1, ..., n\}$. Repeating this argument yields a nonempty open interval $(r, s) \subseteq (c, d)$ such that $\bigvee_{k=1}^{m} t_{j_0,k} = t_{j_0,k_0}$ on (r, s) for some $k_0 \in \{1, ..., m\}$. In particular, there exists an open interval (r, s) in $\mathbb{R}$ and $t \in [\sin, \cos]$ such that $t|_{(r,s)} = \mathbf{1}|_{(r,s)}$. Write $t(x) = \sum_{j=1}^{n} \alpha_j \sin x + \sum_{k=1}^{m} \beta_k \cos x$ $(x \in \mathbb{R})$. It follows that

$$\Big(\sum_{j=1}^{n} \alpha_j \sin x + \sum_{k=1}^{m} \beta_k \cos x\Big)|_{(r,s)} = \mathbf{1}|_{(r,s)}.$$

But differentiating both sides of the above equation twice yields

$$-\Big(\sum_{j=1}^{n} \alpha_j \sin x + \sum_{k=1}^{m} \beta_k \cos x\Big)|_{(r,s)} = \mathbf{0}|_{(r,s)},$$

where $\mathbf{0}$ denotes the constant function with value 0. We have arrived at a contradiction.

Next we expand on Azouzi's idea of a square mean closure by completing Archimedean vector lattices with respect to any nonempty subset of $\bigcup_{m \in \mathbb{N}} \mathcal{H}(\mathbb{R}^m)$.

For $\mathcal{D} \subseteq \bigcup_{m \in \mathbb{N}} \mathcal{H}(\mathbb{R}^m)$ $(\mathcal{D} \neq \varnothing)$ and an Archimedean real vector lattice E, we call a pair $(E^{\mathcal{D}}, \phi)$ a *$\mathcal{D}$-completion* of E if the following hold.

(1) $E^{\mathcal{D}}$ is a $\mathcal{D}$-complete Archimedean real vector lattice.

(2) $\phi : E \to E^{\mathcal{D}}$ is an injective vector lattice homomorphism.

(3) For every $\mathcal{D}$-complete Archimedean real vector lattice F and for every vector lattice homomorphism $T : E \to F$ there exists a unique vector lattice homomorphism $T^{\mathcal{D}} : E^{\mathcal{D}} \to F$ such that $T^{\mathcal{D}} \circ \phi = T$.

Given $h \in \mathcal{H}(\mathbb{R}^m)$, we denote a pair that satisfies (1)-(3) above for $\mathcal{D} = \{h\}$ by (E^h, ϕ) and call (E^h, ϕ) an *h-completion* of E. We refer to $\mathcal{D}$-completions as *functional completions* when the specificity of the set $\mathcal{D}$ is not present.

We will prove the existence and the uniqueness of the functional completion of an Archimedean real vector lattice for which we need several prerequisite results. The first of these, in a way, captures the idea of functional calculus (see [9], Section 3) via a property of vector lattice homomorphisms. We note that a proof of the first part of the theorem can be found in Proposition 3.6 of [24] for uniformly complete vector lattices and for $\mathcal{D} = \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$. For convenience, we write $\delta(h)$ instead of m when $h \in \mathcal{H}(\mathbb{R}^m)$.

Theorem 3.13. *Let $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$ be nonempty, and let E and F be $\mathcal{D}$-complete Archimedean real vector lattices. If $T : E \to F$ is a vector lattice homomorphism then*

$$T(h(a_1, ..., a_{\delta(h)})) = h(T(a_1), ..., T(a_{\delta(h)}))$$

for every $h \in \mathcal{D}$ and for every $a_1, ..., a_{\delta(h)} \in E$. Moreover, suppose there exists $h \in \mathcal{D}$ such that $h(\epsilon_1 x, ..., \epsilon_{\delta(h)} x) = \lambda|x|$ $(x \in \mathbb{R})$ for some $\epsilon_1, ..., \epsilon_{\delta(h)} \in \mathbb{R}$ and some $\lambda \in \mathbb{R} \setminus \{0\}$. Then every linear map $S : E \to F$ such that $S(h(a_1, ..., a_{\delta(h)})) = h(S(a_1), ..., S(a_{\delta(h)}))$ for every $a_1, ..., a_{\delta(h)} \in E$ is a vector lattice homomorphism.

Proof. Let $T : E \to F$ be a vector lattice homomorphism. Since the theorem is trivial when T is the zero map, we assume that $T \neq \mathbf{0}$. Let $h \in \mathcal{D}$ and suppose that $a_1, ..., a_{\delta(h)} \in E$. Define $G_1 := \langle a_1, ..., a_{\delta(h)}, h(a_1, ..., a_{\delta(h)}) \rangle$ and $G_2 := \langle T(a_1), ..., T(a_{\delta(h)}), T(h(a_1, ..., a_{\delta(h)})) \rangle$.

Let $\omega \in H(G_2)$. Then $\omega \circ T|_{G_1} \in H(G_1)$ since $T \neq \mathbf{0}$. Since E and F are h-complete we have

$$h(\omega(T(a_1)), ..., \omega(T(a_{\delta(h)}))) = \omega \circ T(h(a_1, ..., a_{\delta(h)})).$$

Thus we have

$$h(T(a_1), ..., T(a_{\delta(h)})) = T(h(a_1, ..., a_{\delta(h)})).$$

Conversely, suppose that there exist $\epsilon_1, ..., \epsilon_{\delta(h)} \in \mathbb{R}$ as well as $\lambda \in \mathbb{R} \setminus \{0\}$ such that $h(\epsilon_1 x, ..., \epsilon_{\delta(h)} x) = \lambda |x|$ for every $x \in \mathbb{R}$. Let $S : E \to F$ be a linear map such that $S(h(a_1, ..., a_{\delta(h)})) = h(S(a_1), ..., S(a_{\delta(h)}))$ for every $a_1, ..., a_{\delta(h)} \in E$. Suppose that $a \in E$ and that $\omega \in H(\langle a \rangle)$. Then we have

$$h(\omega(\epsilon_1 a), ..., \omega(\epsilon_m a)) = h(\epsilon_1 \omega(a), ..., \epsilon_m \omega(a)) = \lambda |\omega(a)| = \omega(\lambda |a|).$$

Thus we have $h(\epsilon_1 a, ..., \epsilon_{\delta(h)} a) = \lambda |a|$. Similarly, we get $h(S(\epsilon_1 a), ..., S(\epsilon_{\delta(h)} a)) = \lambda |S(a)|$. Hence we obtain

$$S(\lambda |a|) = S(h(\epsilon_1 a, ..., \epsilon_{\delta(h)} a)) = h(S(\epsilon_1 a), ..., S(\epsilon_{\delta(h)} a)) = \lambda |S(a)|.$$

We conclude that $S(|a|) = |S(a)|$ since $\lambda \neq 0$ and S is linear. $\square$

As a particular case of theorem above, suppose that E and F are both h-complete Archimedean real vector lattices for some Stolarsky mean or Gini mean h. A linear map $T : E \to F$ is a vector lattice homomorphism if and only if $T(h(f, g)) = h(T(f), T(g))$ for every $f, g \in E$. Thus Theorem 3.13 generalizes Corollary 4.7 by Azouzi, Boulabiar, and Buskes in [3] as well as Proposition 2.20 of Azouzi in [2]. We point out that Corollary 3.14 below is a generalization of Lemma 4.3 in [3] and corrects a mistake (first noted in [15]) in its proof. For the proof of Corollary 3.14, respectively Corollary 3.15, apply Corollary 3.9, respectively Corollary 3.10, and Theorem 3.13.

Corollary 3.14. ([3], Corollary 4.7) *For square mean complete Archimedean vector lattices E and F and a linear map $T : E \to F$, the following are equivalent.*

(1) T is a vector lattice homomorphism.

(2) $T(f \boxplus g) = T(f) \boxplus T(g)$ for every $f, g \in E$.

Corollary 3.15. ([2], Proposition 2.20) *For geometric mean complete Archimedean real vector lattices E and F and a linear map $T : E \to F$, the following are equivalent.*

(1) T is a vector lattice homomorphism.

(2) $T(f \boxtimes g) = T(f) \boxtimes T(g)$ for every $f, g \in E^+$.

The following theorem is needed for our construction of functional completions.

Theorem 3.16. ([9], Theorem 3.7) *Every uniformly complete Archimedean real vector lattice is $\bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$-complete.*

We remind the reader that $h \in \mathcal{H}(\mathbb{R}^m)$ is *positive* if $h(x_1, ..., x_m) \in \mathbb{R}^+$ for every $x_1, ..., x_m \in \mathbb{R}^+$. If $h(x_1, ..., x_m) = h(|x_1|, ..., |x_m|)$ for every $x_1, ..., x_m \in \mathbb{R}$, we call h *absolutely invariant*. We denote the set of all $h \in \mathcal{H}(\mathbb{R}^m)$ that are positive and absolutely invariant by $\mathcal{H}^+_{|\,|}(\mathbb{R}^m)$. Examples of members of $\mathcal{H}^+_{|\,|}(\mathbb{R}^2)$ include the Stolarsky and Gini means from Examples 3.4 and 3.5. We first manufacture a $\mathcal{D}$-completion $E^{\mathcal{D}}$ of E for any Archimedean real vector lattice E. That this is indeed the $\mathcal{D}$-completion will subsequently be proved. Let E be an Archimedean real vector lattice and assume that $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$ is nonempty. Let (E^u, ϕ) be the uniform completion of E. Following the lead of Azouzi in Remark 4 of [2], define

$$E_1 := \phi(E), \text{ and for every } r \in \mathbb{N},$$
$$E_{r+1} := \left\langle E_r \cup \{h(a_1, ..., a_{\delta(h)}) : h \in \mathcal{D}, a_1, ..., a_{\delta(h)} \in E_r\} \right\rangle,$$

where the latter is the vector lattice generated in E^u. We define

$$E^{\mathcal{D}} := \bigcup_{r\in\mathbb{N}} E_r.$$

Clearly $E^{\mathcal{D}}$ is a vector sublattice of E^u. Let $h \in \mathcal{D}$ and let $a_1, ..., a_{\delta(h)} \in E^{\mathcal{D}}$. There exists $r \in \mathbb{N}$ such that $a_1, ..., a_{\delta(h)} \in E_r$, and thus $h(a_1, ..., a_{\delta(h)}) \in E_{r+1}$. We have just proved the following.

Proposition 3.17. *$E^{\mathcal{D}}$ is a $\mathcal{D}$-complete Archimedean vector sublattice of E^u.*

By using Proposition 3.2(2) one can, alternatively to the definition of a $\mathcal{D}$-completion, replace the homomorphisms in that definition by positive maps if the range space is required to be uniformly complete. This is the content of the next proposition.

Proposition 3.18. *Let $E_1, ..., E_n, F$ be Archimedean real vector lattices, and suppose that F is uniformly complete. Let $\mathcal{D}$ be a nonempty subset of $\bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$. If $T : \times_{k=1}^n E_k \to F$ is a positive n-linear map then there exists a unique positive n-linear map $T^{\mathcal{D}} : \times_{k=1}^n E_k^{\mathcal{D}} \to F$ such that $T^{\mathcal{D}} \circ \phi = T$.*

We prove that $E^{\mathcal{D}}$ is the $\mathcal{D}$-completion of E by proving a more general theorem that involves multilinear maps. Given a nonempty subset A of an Archimedean real vector lattice E and $h \in \mathcal{H}(\mathbb{R}^m)$, we define $h(A) := \{f \in E : f = h(a_1, ..., a_m) \text{ for some } a_1, ..., a_m \in A\}$.

Theorem 3.19. *Let $E_1, ..., E_n, F$ be Archimedean real vector lattices, and assume that F is $\mathcal{D}$-complete ($\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$, $\mathcal{D} \neq \varnothing$). Also let $T : \times_{k=1}^n E_k \to F$ be a vector lattice n-morphism. Denoting for every $k \in \{1, ..., n\}$ the natural embedding of E_k into E_k^u by ϕ_k, the following hold.*

(1) If $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}^+_{|\,|}(\mathbb{R}^m)$ and then there exists a uniquely determined vector lattice n-morphism $T^{\mathcal{D}} : \times_{k=1}^n E_k^{\mathcal{D}} \to F$ such that

$$T^{\mathcal{D}}(\phi_1(f_1), ..., \phi_n(f_n)) = T(f_1, ..., f_n) \ (f_k \in E_k \ (k \in \{1, ..., n\})).$$

(2) If $n = 1$ then statement (1) *holds for all nonempty* $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$*.*

Proof. We will prove statements (1) and (2) simultaneously, since the positivity and the absolute invariance of elements of $\mathcal{D}$ is not used in our proof of (1) in case $n = 1$. Suppose that $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}^+_{|\ |}(\mathbb{R}^m)$ is nonempty. Let $T : \times_{k=1}^n E_k \to F$ be a vector lattice n-morphism and let $j \in \{1, ..., n\}$. Then $E_j^{\mathcal{D}}$, as defined preceding Proposition 3.17, is a $\mathcal{D}$-complete Archimedean real vector lattice. Denote the natural embedding into the uniform completion of E_j by ϕ_j. Clearly we have $\phi_j(E_j) \subseteq E_j^{\mathcal{D}}$ and ϕ_j is an injective vector lattice homomorphism from E_j into $E_j^{\mathcal{D}}$. We prove that there exists a uniquely determined vector lattice n-morphism $T^{\mathcal{D}} : \times_{k=1}^n E_k^{\mathcal{D}} \to F$ such that

$$T^{\mathcal{D}}(\phi_1(f_1), ..., \phi_n(f_n)) = T(f_1, ..., f_n) \ (f_k \in E_k \ , k \in \{1, ..., n\}).$$

To this end, we consider T as a vector lattice n-morphism from $\times_{k=1}^n E_k$ to F^u. By Proposition 3.2(2), there exists a unique vector lattice n-morphism $T^u : \times_{k=1}^n E_k^u \to F^u$ such that

$$T^u(\phi_1(f_1), ..., \phi_n(f_n)) = T(f_1, ..., f_n) \ (f_k \in E_k, k \in \{1, ..., n\}).$$

Define $T^{\mathcal{D}} := T^u|_{\times_{k=1}^n E_k^{\mathcal{D}}}$. To prove that $T^{\mathcal{D}}(\times_{k=1}^n E_k^{\mathcal{D}}) \subseteq F$, we write $E_{k,r}$ for $(E_k)_r$, where $(E_k)_r$ is defined as preceding Proposition 3.17, and we use induction with respect to r. We again write $\delta(h)$ instead of m when $h \in \mathcal{H}(\mathbb{R}^m)$.

Obviously $T^{\mathcal{D}}(\times_{k=1}^n E_{k,1}) \subseteq F$. Let $r \in \mathbb{N}$ and suppose that $T^{\mathcal{D}}(\times_{k=1}^n E_{k,r}) \subseteq F$. Assume that $h_1, ..., h_n \in \mathcal{D}$ and that $a_1^k, ..., a_{\delta(h_k)}^k \in E_{k,r}$ $(k \in \{1, ..., n\})$. Write

$$x = T^{\mathcal{D}}\big(h_1(a_1^1, ..., a_{\delta(h_1)}^1), ..., h_n(a_1^n, ..., a_{\delta(h_n)}^n)\big).$$

Since each h_k is absolutely invariant, we may assume that $a_1^k, ..., a_{\delta(h_k)}^k \in E_{k,r}^+$ for every $k \in \{1, ..., n\})$. It follows that $x \in T^{\mathcal{D}}\big(h_1(E_{1,r}^+) \times \cdots \times h_n(E_{n,r}^+)\big)$. Since $h_1, ..., h_n$ are all positive, we can repeatedly employ Theorem 3.13 to obtain

$$\begin{aligned}
x &\in h_1\Big(\big(T^{\mathcal{D}}(E_{1,r}^{+} \times h_2(E_{2,r}^{+}) \times \cdots \times h_n(E_{n,r}^{+}))\big)^{\delta(h_1)}\Big) \\
&= h_1\Bigg(\bigg(h_2\Big(\big(T^{\mathcal{D}}(E_{1,r}^{+} \times E_{2,r}^{+} \times h_3(E_{3,r}^{+}) \times \cdots \times h_n(E_{n,r}^{+}))\big)^{\delta(h_2)}\Big)\bigg)^{\delta(h_1)}\Bigg) \\
&= h_1\Bigg(\bigg(\ldots h_{n-1}\bigg(\Big(h_n\Big(\big(T^{\mathcal{D}}(E_{1,r}^{+} \times \cdots \times E_{n,r}^{+})\big)^{\delta(h_n)}\Big)\Big)^{\delta(h_{n-1})}\bigg)\ldots\bigg)^{\delta(h_1)}\Bigg) \\
&\subseteq F,
\end{aligned}$$

where the last inclusion follows from the induction hypothesis and the assumption that F is $\mathcal{D}$-complete. Moreover, from the n-linearity of $T^{\mathcal{D}}$ we get

$$T^{\mathcal{D}}(\times_{k=1}^{n}[E_{k,r} \bigcup \{h(a_1^k, ..., a_{\delta(h)}^k) : h \in \mathcal{D}, a_1^k, ..., a_{\delta(h)}^k \in E_{k,r}\}]) \subseteq F.$$

By Exercise 4.1.8 in [1], every element of $E_{k,r+1}^{+}$ can be expressed as $\bigwedge_{j=1}^{p_k} \bigvee_{l=1}^{q_k} u_{k,j,l}$ for some $u_{k,j,l} \in [E_{k,r} \bigcup \{h(a_1^k, ..., a_{\delta(h)}^k) : h \in \mathcal{D}, a_1^k, ..., a_{\delta(h)}^k \in E_{k,r}\}]$. We may assume that each $u_{k,j,l}$ is positive. Since $T^{\mathcal{D}}$ is a vector lattice n-morphism, we have

$$T^{\mathcal{D}}(\bigwedge_{j=1}^{p_1} \bigvee_{l=1}^{q_1} u_{1,j,l}, ..., \bigwedge_{j=1}^{p_n} \bigvee_{l=1}^{q_n} u_{n,j,l}) = \bigwedge_{j=1}^{p_1} \bigvee_{l=1}^{q_1} \cdots \bigwedge_{j=1}^{p_n} \bigvee_{l=1}^{q_n} T^{\mathcal{D}}(u_{1,j,l}, ..., u_{n,j,l})$$

for every $u_{k,j,l} \in E_{k,r+1}^{+}$. Since $T^{\mathcal{D}}(u_{1,j,l}, ..., u_{n,j,l}) \in F$ for each j and each l, it follows that

$$\bigwedge_{j=1}^{p_1} \bigvee_{l=1}^{q_1} \cdots \bigwedge_{j=1}^{p_n} \bigvee_{l=1}^{q_n} T^{\mathcal{D}}(u_{1,j,l}, ..., u_{n,j,l}) \in F.$$

Hence we get $T^{\mathcal{D}}(\times_{k=1}^{n} E_{k,r+1}^{+}) \subseteq F$. We have $T^{\mathcal{D}}(\times_{k=1}^{n} E_{k,r+1}) \subseteq F$ because $T^{\mathcal{D}}$ is n-linear. This completes the proof. □

Corollary 3.20. *If E is an Archimedean real vector lattice and $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$ is nonempty then $E^{\mathcal{D}}$ with the natural embedding from E into $E^{\mathcal{D}}$ is the unique $\mathcal{D}$-completion of E.*

Proof. Let E be an Archimedean vector lattice and suppose that $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}(\mathbb{R}^m)$ is nonempty. We proved in the previous theorem that $E^{\mathcal{D}}$ with the natural embedding from E into $E^{\mathcal{D}}$ is a $\mathcal{D}$-completion of E. Next, we prove the uniqueness of $E^{\mathcal{D}}$. Suppose $(E_1^{\mathcal{D}}, \phi_1)$ and $(E_2^{\mathcal{D}}, \phi_2)$ are $\mathcal{D}$-completions of E. Since $\phi_1 : E \to E_1^{\mathcal{D}}$ is a vector lattice homomorphism, there exists a unique vector lattice homomorphism $\phi_1^{\mathcal{D}} : E_2^{\mathcal{D}} \to E_1^{\mathcal{D}}$ such that $\phi_1^{\mathcal{D}} \circ \phi_2 = \phi_1$. Likewise, there exists a unique vector lattice homomorphism $\phi_2^{\mathcal{D}} : E_1^{\mathcal{D}} \to E_2^{\mathcal{D}}$ such that $\phi_2^{\mathcal{D}} \circ \phi_1 = \phi_2$. Then we have $\phi_2^{\mathcal{D}} \circ \phi_1^{\mathcal{D}} \circ \phi_2 = \phi_2^{\mathcal{D}} \circ \phi_1 = \phi_2$. Thus we have $\phi_2^{\mathcal{D}} \circ \phi_1^{\mathcal{D}} = I$. Similarly, we get $\phi_1^{\mathcal{D}} \circ \phi_2^{\mathcal{D}} = I$. $\square$

4 COMPLEXIFICATIONS AND COMPLEX TENSOR PRODUCTS

We use the theory of functional completions developed in Chapter 3 to complexify any Archimedean real vector lattice into an Archimedean complex vector lattice (Theorem 4.2). This chapter also contains corresponding complexifications for various types of linear and multilinear maps between Archimedean real vector lattices. Using the complexifications alluded to above, we prove the existence of an Archimedean complex vector lattice tensor product (Theorem 4.10(1)), symmetric (antisymmetric) Archimedean complex vector lattice tensor product (Theorems 4.20 and 4.22), and powers of Archimedean complex vector lattices (Theorem 4.25). A one-to-one correspondence between order bounded maps on the Archimedean complex vector lattice tensor product and complex maps of order bounded variation is given (Theorem 4.30). We also prove the existence of a complex Banach lattice tensor product (Theorem 4.32).

4.1 The Vector Lattice Complexification

We discuss a specific case of Proposition 3.18 and Theorem 3.19 in Section 3.3 that we will use to complexify Archimedean real vector lattices. As in Chapter 3, let $\mu \in \mathcal{H}(\mathbb{R}^2)$ be the square mean. If E is a square mean closed Archimedean real vector lattice then, as noted in Section 3.3, we have

$$\mu(f, g) = \frac{1}{\sqrt{2}} \sup\{f \cos\theta + g \sin\theta : \theta \in [0, 2\pi]\} \ (f, g \in E).$$

Therefore, a vector space $E+iE$ over $\mathbb{C}$ is an Archimedean complex vector lattice if and only if E is a square mean complete Archimedean real vector lattice. Noting that the square mean is positive and absolutely invariant (defined following Theorem 3.16), we mention a special corollary of Proposition 3.18 and Theorem 3.19.

Corollary 4.1. *If E is an Archimedean real vector lattice then there exists a unique square mean completion (E^{μ},ϕ) of E. Let $E_1,...,E_n,F$ be Archimedean real vector lattices with square mean completions (E_k^{μ},ϕ_k) $(k\in\{1,...,n\})$, and suppose that F is square mean complete. For every vector lattice n-morphism $T:\times_{k=1}^{n}E_k\to F$ there exists a unique vector lattice n-morphism $T^{\mu}:\times_{k=1}^{n}E_k^{\mu}\to F$ such that*

$$T^{\mu}(\phi_1(f_1),...,\phi_n(f_n))=T(f_1,...,f_n)\ (f_{k\in E_k},k\in\{1,...,n\}).$$

If F is uniformly complete and $T:\times_{k=1}^{n}E_k\to F$ is a positive n-linear map then there exists a unique positive n-linear map $T^{\mu}:\times_{k=1}^{n}E_k^{\mu}\to F$ such that

$$T^{\mu}(\phi_1(f_1),...,\phi_n(f_n)=T(f_1,...,f_n)\ (f_k\in E_k,k\in\{1,...,n\}).$$

Here ϕ_k is the natural embedding of E_k into E_k^{u}.

We next turn to complexifications of Archimedean real vector lattices.

For an Archimedean real vector lattice E we define a pair $(E_{|\mathbb{C}|},\phi)$ to be a *vector lattice complexification* of E if the following hold.

(1) $E_{|\mathbb{C}|}$ is an Archimedean complex vector lattice.

(2) $\phi:E\to(E_{|\mathbb{C}|})_{\rho}$ is an injective vector lattice $\mathbb{R}$-homomorphism.

(3) For every Archimedean complex vector lattice F and vector lattice $\mathbb{R}$-homomorphism $T:E\to F_{\rho}$ there exists a unique vector lattice $\mathbb{C}$-homomorphism $T_{|\mathbb{C}|}:E_{|\mathbb{C}|}\to F$ such that $T_{|\mathbb{C}|}\circ\phi=T$.

We next prove the existence and uniqueness of the vector lattice complexification.

Theorem 4.2. *If E is an Archimedean real vector lattice then there exists a vector lattice complexification of E, unique up to vector lattice isomorphism.*

Proof. Let E be an Archimedean real vector lattice. By Corollary 4.1 there exists a unique square mean completion (E^μ, ϕ) of E. Define $E_{|\mathbb{C}|} := (E^\mu)_\mathbb{C}$. Observe that $E_{|\mathbb{C}|}$ is an Archimedean complex vector lattice and that $(E_{|\mathbb{C}|})_\rho = E^\mu$. Next let F be an Archimedean complex vector lattice, and let $T : E \to F_\rho$ be a vector lattice $\mathbb{R}$-homomorphism. Since F_ρ is square mean complete, there exists a unique vector lattice $\mathbb{R}$-homomorphism $T^\mu : E^\mu \to F_\rho$ such that $T^\mu \circ \phi = T$. Define $T_{|\mathbb{C}|} : E_{|\mathbb{C}|} \to F$ by

$$T_{|\mathbb{C}|}(f + ig) = T^\mu(f) + iT^\mu(g) \ (f + ig \in E_{|\mathbb{C}|}).$$

Then $T_{|\mathbb{C}|} \circ \phi = T$. We have from Corollary 3.14 (see also Proposition 3.4 of [2]) that

$$T_{|\mathbb{C}|}(|f + ig|) = T^\mu(f \boxplus g) = T^\mu(f) \boxplus T^\mu(g) = |T_{|\mathbb{C}|}(f + ig)| \ (f + ig \in E_{|\mathbb{C}|}).$$

Thus $T_{|\mathbb{C}|}$ is a vector lattice $\mathbb{C}$-homomorphism. Therefore, $(E_{|\mathbb{C}|}, \phi)$ is a vector lattice complexification of E. Next, we prove the uniqueness. To this end, suppose $(E_{1|\mathbb{C}|}, \phi_1)$ and $(E_{2|\mathbb{C}|}, \phi_2)$ are vector lattice complexifications of E. Then $((E_{1|\mathbb{C}|})_\rho, \phi_1)$ and $((E_{2|\mathbb{C}|})_\rho, \phi_2)$ are both square mean completions of E. Hence there exists a vector lattice isomorphism $\psi : (E_{1|\mathbb{C}|})_\rho \to (E_{2|\mathbb{C}|})_\rho$. Similar to $T_{|\mathbb{C}|}$ above, the map $\psi_\mathbb{C} : E_{1|\mathbb{C}|} \to E_{2|\mathbb{C}|}$ defined by $\psi_\mathbb{C}(f + ig) = \psi(f) + i\psi(g)$ is a vector lattice $\mathbb{C}$-homomorphism. The bijectivity of $\psi_\mathbb{C}$ is evident. □

For the square mean completion (E^μ, ϕ) of E, we will from now on identify E with $\phi(E)$. Using this identification, we complexify positive $n_\mathbb{R}$-linear maps (respectively, vector lattice $n_\mathbb{R}$-morphisms) to positive $n_\mathbb{C}$-linear maps (respectively, vector lattice $n_\mathbb{C}$-morphisms) as follows. Let $E_1, ..., E_n, F$ be Archimedean real vector lattices such that F is square mean complete. Suppose that $T : \times_{k=1}^n E_k \to F$ is a vector lattice $n_{\mathbb{R}}$-morphism. We define the map $T_{|\mathbb{C}|} : \times_{k=1}^n E_{k|\mathbb{C}|} \to F_{\mathbb{C}}$ by

$$T_{|\mathbb{C}|}(f_0^1 + if_1^1, ..., f_0^n + if_1^n) := \sum_{\epsilon_k \in \{0,1\}} T^\mu(f_{\epsilon_1}^1, ..., f_{\epsilon_n}^n) i^{\sum_{k=1}^n \epsilon_k}.$$

for every $(f_0^1 + if_1^1, ..., f_0^n + if_1^n) \in \times_{k=1}^n E_{k|\mathbb{C}|}$. If F is uniformly complete and T above is any positive $n_{\mathbb{R}}$-linear map, we define $T_{|\mathbb{C}|}$ in a similar manner. We collect a few facts regarding this complexification in the following proposition. Statement (3) and the statement that $T_{|\mathbb{C}|} = (T^\mu)_{\mathbb{C}}$ in (1) and (2) are evident. The proof of (2) follows from Corollary 4.1, and the proof of (1) is contained in the proof of Theorem 4.2 above for $n = 1$. The proof for general $n \in \mathbb{N}$ is similar.

Proposition 4.3. *Let $E_1, ..., E_n, F$ be Archimedean real vector lattices such that F is square mean complete.*

(1) If a map $T : \times_{k=1}^n E_k \to F$ is a vector lattice $n_{\mathbb{R}}$-morphism then $T_{|\mathbb{C}|}$ is a vector lattice $n_{\mathbb{C}}$-morphism and $T_{|\mathbb{C}|} = (T^\mu)_{\mathbb{C}}$.

(2) If F is uniformly complete and $T : \times_{k=1}^n E_k \to F$ is a positive $n_{\mathbb{R}}$-linear map then $T_{|\mathbb{C}|}$ is a positive $n_{\mathbb{C}}$-linear map and $T_{|\mathbb{C}|} = (T^\mu)_{\mathbb{C}}$.

(3) If in (1) or (2) all $E_1, ..., E_n$ are square mean complete then $T_{|\mathbb{C}|} = T_{\mathbb{C}}$.

4.2 The Complex Archimedean Vector Lattice Tensor Product

We define the Archimedean complex vector lattice tensor product in this section and prove its existence by applying the vector lattice complexification from Section 4.1 to the Archimedean real vector lattice tensor product (Theorem 4.10(1)). We also prove that the vector space complexification of the Archimedean real vector lattice tensor product is not necessarily a complex vector lattice (Theorem 4.14).

The definitions of relatively uniformly convergent sequences and relatively uniformly Cauchy sequences in an Archimedean complex vector lattice are identical to the corresponding definitions for Archimedean real vector lattices given in Section 3.1, modulo replacing $\mathbb{R}$ with $\mathbb{C}$. The definitions of a uniformly complete Archimedean complex vector lattice and a uniform completion of an Archimedean complex vector lattice are also analogous to the corresponding definitions for Archimedean real vector lattices found in Section 3.1. Propositions 3.1 and 3.2 in Section 3.1 also hold for Archimedean complex vector lattices. In particular, there exists a uniform completion of every Archimedean complex vector lattice, unique up to vector lattice isomorphism.

Example 4.4. *If E is a uniformly complete Archimedean real vector lattice then $E_{\mathbb{C}}$ is a uniformly complete Archimedean complex vector lattice. If E is a uniformly complete Archimedean complex vector lattice then E_ρ is a uniformly complete Archimedean real vector lattice. Dedekind complete real vector lattices and Dedekind complete complex vector lattices are uniformly complete.*

Let E be an Archimedean complex vector lattice and let A be a subset of E. Like in Section 3.1 for real vector lattices, we define

$$\bar{A} := \{f \in E : \text{ there exists a sequence } (f_n) \text{ in } A \text{ such that } f_n \overset{ru}{\to} f\}$$

and call $\bar{A}$ the *pseudo uniform closure* of A. We call A *relatively uniformly closed* if $\bar{A} = A$, and we say that A is *uniformly dense* in E if $\bar{A} = E$. The *relatively uniform topology* on an Archimedean complex vector lattice E is the collection of subsets of E that are complements of relatively uniformly closed subsets of E.

Like in Section 3.1, we use transfinite induction to iterate the pseudo uniform closure of a subset A of an Archimedean complex vector lattice E. For an Archimedean complex vector lattice E and a subset A of E, we define

$$A_1 := A,$$
$$A_\alpha := \overline{A_{\alpha-1}} \text{ when } \alpha > 1 \text{ is not a limit ordinal, and}$$
$$A_\alpha := \bigcup_{\beta<\alpha} A_\beta \text{ when } \alpha \text{ is a limit ordinal.}$$

Given a subset A of an Archimedean complex vector lattice E, we know from the complex analogue of Proposition 3.1 in Section 3.1 that A_{ω_1} is uniformly closed in E. It is possible however that there exists an ordinal $\alpha < \omega_1$ such that A_α is relatively uniformly dense in E. Motivated by this observation, we define the *density number* $\tau(A,E)$ *of* A *in* E by $\tau(A,E) := \min\{\alpha : \bar{A}_\alpha = E\}$.

Next we define the Archimedean complex vector lattice tensor product. For $n = 2$ the definition is analogous to Fremlin's Archimedean real vector lattice tensor product (see [20], Theorem 4.2).

Given Archimedean complex vector lattices $E_1, ..., E_n$, we define a pair $(\bar{\otimes}_{k=1}^n E_k, \bar{\otimes})$ to be an *Archimedean complex vector lattice tensor product* of $E_1, ..., E_n$ if the following hold.

(1) $\bar{\otimes}_{k=1}^n E_k$ is an Archimedean complex vector lattice.

(2) $\bar{\otimes}$ is a vector lattice n-morphism.

(3) For every Archimedean complex vector lattice F and every vector lattice n-morphism $T : \times_{k=1}^n E_k \to F$, there exists a uniquely determined vector lattice homomorphism $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^n E_k \to F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.

As noted by Schep in Section 2 of [38], one can extend Fremlin's Theorem 4.2 in [20] to obtain an Archimedean real vector lattice tensor product of any number of factors by following Fremlin's original proof. Below and throughout the rest of this section, $(\otimes_{k=1}^n V_k, \otimes)$ denotes the algebraic tensor product of vector spaces $V_1, ..., V_n$ over $\mathbb{K}$.

Lemma 4.5. *Let $E_1, ..., E_n$ be Archimedean real vector lattices.*

(1) There exists an essentially unique Archimedean real vector lattice $\bar{\otimes}_{k=1}^n E_k$ and a vector lattice n-morphism $\bar{\otimes} : \times_{k=1}^n E_k \to \bar{\otimes}_{k=1}^n E_k$ such that for every Archimedean vector lattice F over $\mathbb{R}$ and every vector lattice n-morphism $T : \times_{k=1}^n E_k \to F$, there exists a unique vector lattice homomorphism $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^n E_k \to F$ for which $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.

(2) There exists an injective linear map $S : \otimes_{k=1}^n E_k \to \bar{\otimes}_{k=1}^n E_k$ such that $S \circ \otimes = \bar{\otimes}$.

(3) For every $w \in \bar{\otimes}_{k=1}^n E_k$ there exist $x_k \in E_k^+$ $(k \in \{1, ..., n\})$ such that for every $\epsilon > 0$, there exists $v \in \otimes_{k=1}^n E_k$ for which $|w - v| \leq \epsilon(x_1 \otimes \cdots \otimes x_n)$. That is, $\otimes_{k=1}^n E_k$ is relatively uniformly dense in $\bar{\otimes}_{k=1}^n E_k$.

(4) For every $0 < w \in \bar{\otimes}_{k=1}^n E_k$ there exist $x_k \in E_k^+$ $(k \in \{1, ..., n\})$ such that

$$0 < (x_1 \otimes \cdots \otimes x_n) \leq w.$$

That is, $\otimes_{k=1}^n E_k$ is order dense in $\bar{\otimes}_{k=1}^n E_k$.

For the main result of this section (Theorem 4.10) we need several prerequisite results. The next lemma surely is known but we could only find an explicit reference in the literature for a special case in the thesis [43].

Lemma 4.6. *If $V_1, ..., V_n$ are vector spaces over $\mathbb{R}$ then $\otimes_{k=1}^n (V_{k\mathbb{C}})$ and $(\otimes_{k=1}^n V_k)_{\mathbb{C}}$ are isomorphic as vector spaces over $\mathbb{C}$.*

Proof. Since the algebraic tensor product is associative, we only need to prove the result for $n = 2$ and use induction. The case $n = 2$ is the content of Theorem 2.1.2 in [43], but we provide a sketch of van Zyl's proof to correct some potential confusion caused by an accumulation of minor misprints. First let U and V be vector spaces over $\mathbb{R}$, and let $(U \otimes V, \otimes)$ and $(U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}}, \otimes_1)$ be the algebraic tensor products of U and V, respectively

$U_{\mathbb{C}}$ and $V_{\mathbb{C}}$. Since $\otimes_{\mathbb{C}} : U_{\mathbb{C}} \times V_{\mathbb{C}} \to (U \otimes V)_{\mathbb{C}}$ is a bilinear map over $\mathbb{C}$, it induces a unique $\mathbb{C}$-linear map $T : U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}} \to (U \otimes V)_{\mathbb{C}}$. It is easy to see that T is surjective. To show that T is injective, let $w = \sum_{k=1}^{n} (u_k + iu'_k) \otimes_1 (v_k + iv'_k) \in U_{\mathbb{C}} \otimes_1 V_{\mathbb{C}}$ and suppose that $T(w) = 0$. Note that

$$T(w) = \sum_{k=1}^{n} (u_k \otimes v_k - u'_k \otimes v'_k + iu'_k \otimes v_k + iu_k \otimes v'_k).$$

Thus for any $\mathbb{R}$-linear functionals ϕ on U and ψ on V we have

$$\sum_{k=1}^{n} \big(\phi(u_k)\psi(v_k) - \phi(u'_k)\psi(v'_k)\big) = 0 \text{ and } \sum_{k=1}^{n} \big(\phi(u'_k)\psi(v_k) + \phi(u_k)\psi(v'_k)\big) = 0 \,. \tag{$*$}$$

Let $\xi = \xi_r + i\xi_c$ be a $\mathbb{C}$-linear functional on $U_{\mathbb{C}}$ and let $\eta = \eta_r + i\eta_c$ be a $\mathbb{C}$-linear functional on $V_{\mathbb{C}}$, both written in their natural decompositions. Then ξ_r, ξ_c are $\mathbb{R}$-linear functionals on U and η_r, η_c are $\mathbb{R}$-linear functionals on V. It follows that

$$\begin{aligned}
&\sum_{k=1}^{n} \xi(u_k + iu'_k)\eta(v_k + iv'_k) \\
&= \sum_{k=1}^{n} \big(\xi_r(u_k)\eta_r(v_k) - \xi_r(u'_k)\eta_r(v'_k)\big) - \sum_{k=1}^{n} \big(\xi_r(u'_k)\eta_c(v_k) + \xi_r(u_k)\eta_c(v'_k)\big) \\
&+ i \sum_{k=1}^{n} \big(\xi_r(u'_k)\eta_r(v_k) + \xi_r(u_k)\eta_r(v'_k)\big) + i \sum_{k=1}^{n} \big(\xi_r(u_k)\eta_c(v_k) - \xi_r(u'_k)\eta_c(v'_k)\big) \\
&- \sum_{k=1}^{n} \big(\xi_c(u'_k)\eta_r(v_k) + \xi_c(u_k)\eta_r(v'_k)\big) - \sum_{k=1}^{n} \big(\xi_c(u_k)\eta_c(v_k) - \xi_c(u'_k)\eta_c(v'_k)\big) \\
&+ i \sum_{k=1}^{n} \big(\xi_c(u_k)\eta_r(v_k) - \xi_c(u'_k)\eta_r(v'_k)\big) - i \sum_{k=1}^{n} \big(\xi_c(u'_k)\eta_c(v_k) + \xi_c(u_k)\eta_c(v'_k)\big).
\end{aligned}$$

Applying $(*)$ again to each of these eight summands, we get

$$\sum_{k=1}^{n} \xi(u_k + iu'_k)\eta(v_k + iv'_k) = 0.$$

Thus $w = 0$ and T is injective. Therefore, T is a vector space isomorphism. $\square$

In light of the previous lemma, we will from now on identify $(\otimes_{k=1}^{n} V_{k\mathbb{C}})_{\rho}$ with $\otimes_{k=1}^{n} V_k$ for vector spaces $V_1, ..., V_n$ over $\mathbb{R}$.

We next note that there exists a simpler construction of the square mean completion than the construction for general functional completions preceding Proposition 3.17. In Remark 4 of [2], Azouzi constructs the square mean closure of an Archimedean real vector lattice E essentially as follows. Let $E_1 := E$, and for every $r \in \mathbb{N}$ define

$$E_{r+1} := E_r \cup [\{\mu(f, g) : f, g \in E_r\}],$$

where $[\{\mu(f, g) : f, g \in E_r\}]$ is the vector subspace of E^{δ} generated by $\{\mu(f, g) : f, g \in E_r\}$. Define $E^{\boxplus} := \bigcup_{r\in\mathbb{N}} E_r$. To verify that $E^{\boxplus}$ is a vector sublattice of E^{δ}, note that for every $f \in E^{\boxplus}$ there exists $r \in \mathbb{N}$ such that $f \in E_r$. It follows that

$$|f| = f \vee (-f) = \sup\{f \cos\theta : \theta \in [0, 2\pi]\} = \sqrt{2}\mu(f, 0) \in E_{r+1}.$$

Let F be a square mean complete Archimedean real vector lattice. It follows from Corollary 3.14 and Theorem 3.19 in Section 3.3 that every vector lattice homomorphism $T : E \to F$ extends uniquely to a vector lattice homomorphism on $E^{\boxplus}$. By the uniqueness of the square mean completion, we have that $E^{\boxplus}$ and E^{μ} are isomorphic as vector lattices. From the identity $\lambda\mu(f, g) = \mu(\lambda f, \lambda g)$ for every $\lambda \in \mathbb{R}^{+}$ and every $f, g \in E$, we have $E_{r+1}^{+} = \{\sum_{k=1}^{m} \mu(f_k, g_k) : f_k, g_k \in E_r\}$ for every $r \in \mathbb{N}$. We use this fact in the first of the two following lemmas that are needed for Proposition 4.9.

Lemma 4.7. *For an Archimedean real vector lattice E and for every $f \in (E^{\mu})^{+}$ there exists $u_1, ..., u_n \in E^{+}$ and $t_{k,1}, ..., t_{k,p_k} \in \{\cos, \sin\}$ $(k \in \{1, ..., n\})$ such that*

$$f = \sup_{\theta_k \in [0, \frac{\pi}{2}]} \Big\{ \sum_{k=1}^{n} \prod_{j=1}^{p_k} t_{k,j}(\theta_k) u_k \Big\}.$$

Proof. Our proof runs via mathematical induction. Let $f \in E_2^{+}$ and first suppose that $f = \mu(u, v)$ for some $u, v \in E^{+}$. Then we have $f = \sup\{u \cos\theta + v \sin\theta : \theta \in [0, \frac{\pi}{2}]\}$. Next suppose that $f = \sum\limits_{k=1}^{n} \mu(u_k, v_k)$. We get that

$$f = \sum_{k=1}^{n} \sup_{\theta_k \in [0, \frac{\pi}{2}]} \{u_k \cos\theta_k + v_k \sin\theta_k\} = \sup_{\theta_k \in [0, \frac{\pi}{2}]} \Big\{ \sum_{k=1}^{n} (u_k \cos\theta_k + v_k \sin\theta_k) \Big\}.$$

This completes the base step of the induction argument. For the inductive step, suppose that for every $f \in E_r^{+}$ there exists $u_1, ..., u_n \in E^{+}$ and $t_1, ..., t_{p_k} \in \{\cos, \sin\}$ $(k \in \{1, ..., n\})$ such that

$$f = \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \Big\{ \sum_{k=1}^{n} \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \Big\}.$$

Let $f \in E_{r+1}^{+}$. From the argument in the base step above, we may assume that $f = \mu(u, v)$ for some $u, v \in E_r^{+}$. By the induction hypothesis there exists $u_1, ..., u_n, v_1, ..., v_n \in E_r^{+}$ and $t_{k,1}, ..., t_{k,p_k}, s_{k,1}, ..., s_{k,q_k} \in \{\cos, \sin\}$ $(k \in \{1, ..., n\})$ such that

$$u = \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \Big\{ \sum_{k=1}^{n} \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \Big\} \text{ and } v = \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \Big\{ \sum_{k=1}^{m} \prod_{j=1}^{q_k} s_{k,j}(\theta_{k,j}) v_k \Big\}.$$

Then we have

$$
\begin{aligned}
f &= \mu\Big(\sup_{\theta_{k,j}\in[0,\frac{\pi}{2}]}\Big\{\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k\Big\}, \sup_{\theta_{k,j}\in[0,\frac{\pi}{2}]}\Big\{\sum_{k=1}^{m}\prod_{j=1}^{q_k} s_{k,j}(\theta_{k,j})v_k\Big\}\Big) \\
&= \sup_{\phi\in[0,\frac{\pi}{2}]}\Bigg\{\sup_{\theta_{k,j}\in[0,\frac{\pi}{2}]}\Big\{\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k\Big\}\cos\phi + \sup_{\theta_{k,j}\in[0,\frac{\pi}{2}]}\Big\{\sum_{k=1}^{m}\prod_{j=1}^{r_k} s_{k,j}(\theta_{k,j})v_k\Big\}\sin\phi\Bigg\} \\
&= \sup_{\phi,\theta_{j,k}\in[0,\frac{\pi}{2}]}\Big\{\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})\cos\phi u_k + \sum_{k=1}^{m}\prod_{j=1}^{q_k} s_{k,j}(\theta_{k,j})\sin\phi v_k\Big\}.
\end{aligned}
$$

□

Lemma 4.8. *Let $t_1, ..., t_n$ be Lipschitz functions on $\mathbb{R}$ with Lipschitz constant 1. Assume that $|t_k(x)| \leq 1$ for every $k \in \{1, ..., n\}$ and every $x \in \mathbb{R}$. Then we have*

$$|\prod_{k=1}^{n} t_k(x_k) - \prod_{k=1}^{n} t_k(y_k)| \leq \sum_{k=1}^{n} |x_k - y_k|$$

for every $x_k, y_k \in \mathbb{R}$ $(k \in \{1, ..., n\})$.

Proof. We prove this lemma via mathematical induction. First, if t is a Lipschitz function on $\mathbb{R}$ with Lipschitz constant 1 then $|t(x) - t(y)| \leq |x - y|$. Next suppose that

$$|\prod_{k=1}^{n} t_k(x_k) - \prod_{k=1}^{n} t_k(y_k)| \leq \sum_{k=1}^{n} |x_k - y_k|$$

whenever $t_1, ..., t_n$ are Lipschitz functions with Lipschitz constant 1 with the property that $|t_k(x)| \leq 1$ $(k \in \{1, ..., n\}, x \in \mathbb{R})$. Let $t_1, ..., t_n, t_{n+1}$ be Lipschitz functions with Lipschitz constant 1, and suppose that $|t_k(x)| \leq 1$ $(k \in \{1, ..., n+1\}, x \in \mathbb{R})$. Using the inductive hypothesis, we have that

$$
\begin{aligned}
&\Big|\prod_{k=1}^{n+1} t_k(x_k) - \prod_{k=1}^{n+1} t_k(y_k)\Big| \\
&\quad \leq \Big|\prod_{k=1}^{n+1} t_k(x_k) - t_{n+1}(y_{n+1})\prod_{k=1}^{n} t_k(x_k)\Big| + \Big|t_{n+1}(y_{n+1})\prod_{k=1}^{n} t_k(x_k) - \prod_{k=1}^{n+1} t_k(y_k)\Big| \\
&\quad = \big|t_{n+1}(x_{n+1}) - t_{n+1}(y_{n+1})\big|\Big|\prod_{k=1}^{n} t_k(x_k)\Big| + \big|t_{n+1}(y_{n+1})\big|\Big|\prod_{k=1}^{n} t_k(x_k) - \prod_{k=1}^{n} t_k(y_k)\Big| \\
&\quad \leq \big|t_{n+1}(x_{n+1}) - t_{n+1}(y_{n+1})\big| + \Big|\prod_{k=1}^{n} t_k(x_k) - \prod_{k=1}^{n} t_k(y_k)\Big| \\
&\quad \leq \sum_{k=1}^{n+1} |x_k - y_k|.
\end{aligned}
$$

□

The idea of the proof for the following proposition derives from Lemma 2.8 in [2] by Azouzi, which is a reformulation of the Beukers-Huijsmans-de Pagter circle approximation theorem (see section 2 of [4]) mentioned in the beginning of Section 3.3.

Proposition 4.9. *If E is an Archimedean real vector lattice and (E^μ, ϕ) is the square mean completion of E then E is relatively uniformly dense in E^μ.*

Proof. Let E be an Archimedean real vector lattice and first suppose that $f \in (E^\mu)^+$. From Lemma 4.7 we know that

$$
f = \sup_{\theta_{k,j} \in [0, \frac{\pi}{2}]} \Big\{ \sum_{k=1}^{n} \prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) u_k \Big\}
$$

for some $u_1, ..., u_n \in E^+$ and some $t_{k,1}, ..., t_{k,p_k} \in \{\cos, \sin\}$ ($k \in \{1, ..., n\}$). For every $\theta_{k,j} \in [0, \frac{\pi}{2}]$ and every $m \in \mathbb{N}$ there exist $l_{k,j} \in \mathbb{N}$ such that $|\frac{l_{k,j}\pi}{2^m} - \theta_{k,j}| \leq \frac{\pi}{2^m}$. Noting that sine and cosine are both Lipschitz functions with Lipschitz constant 1, it follows from

Lemma 4.8 that

$$\begin{aligned}\Big|\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k - \sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)u_k\Big| &\leq \sum_{k=1}^{n}\Big|\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j}) - \prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)\Big||u_k| \\ &\leq \sum_{k=1}^{n}\sum_{j=1}^{p_k}|\theta_{k,j} - \frac{l_{k,j}\pi}{2^m}||u_k| \\ &\leq \frac{\pi}{2^m}\sum_{k=1}^{n} p_k|u_k|.\end{aligned}$$

Thus we have

$$\begin{aligned}\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}(\theta_{k,j})u_k &\leq \sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)u_k + \frac{\pi}{2^m}\sum_{k=1}^{n} p_k|u_k| \\ &\leq \bigvee_{l_{k,j}=1}^{2^m}\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_{k,j}\left(\frac{l_{k,j}\pi}{2^m}\right)u_k + \frac{\pi}{2^m}\sum_{k=1}^{n} p_k|u_k|.\end{aligned}$$

Since the inequality above holds for all $\theta_{k,j} \in [0, \frac{\pi}{2}]$ ($k \in \{1, ..., n\}, j \in \{1, ..., p_k\}$), we get

$$0 \leq f - \bigvee_{l_{k,j}=1}^{2^m}\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_k\left(\frac{l_{k,j}\pi}{2^m}\right) \leq \frac{\pi}{2^m}\sum_{k=1}^{n} p_k|u_k|.$$

It follows that $\sigma_m := \bigvee_{l_{k,j}=1}^{2^m}\sum_{k=1}^{n}\prod_{j=1}^{p_k} t_k\left(\frac{l_{k,j}\pi}{2^m}\right)$ converges relatively uniformly to f. Finally, for $f \in E$ there exist sequences $(a_n), (b_n)$ in E such that $a_n \overset{ru}{\to} f^+$ and $b_n \overset{ru}{\to} f^-$. Then we have that $a_n - b_n \overset{ru}{\to} f$. □

We are ready to construct the Archimedean complex vector lattice tensor product. Parts (1), (2), and (4) of the following theorem are analogous to the corresponding parts of Theorem 4.2 in [20] and Lemma 4.5. Parts (2) and (4) correspond to results by Schep for Archimedean real vector lattices in Section 2 of [38]. Part (3) is slightly weaker than the real analogues found in [20] and [38], but it is sufficient for obtaining our results for maps of order bounded variation in Section 4.5. We do not yet know if it is possible to strengthen (3).

Theorem 4.10. *Let $E_1, ..., E_n$ be Archimedean complex vector lattices.*

(1) There exists an essentially unique Archimedean complex vector lattice $\bar{\otimes}_{k=1}^n E_k$ and a vector lattice n-morphism $\bar{\otimes} : \times_{k=1}^n E_k \to \bar{\otimes}_{k=1}^n E_k$ such that for every Archimedean complex vector lattice F and every vector lattice n-morphism $T : \times_{k=1}^n E_k \to F$, there exists a unique vector lattice homomorphism $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^n E_k \to F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.

(2) There exists an injective $\mathbb{C}$-linear map $S : \otimes_{k=1}^n E_k \to \bar{\otimes}_{k=1}^n E_k$ such that $S \circ \otimes = \bar{\otimes}$.

(3) $\tau(\otimes_{k=1}^n E_k, \bar{\otimes}_{k=1}^n E_k) \leq 2$. Thus, $\otimes_{k=1}^n E_k$ is dense in $\bar{\otimes}_{k=1}^n E_k$ in the relatively uniform topology.

(4) For every $w \in (\bar{\otimes}_{k=1}^n E_k) \setminus \{0\}$ there exists $x_k \in E_k^+$ $(k \in \{1, ..., n\})$ such that

$$0 < (x_1 \otimes \cdots \otimes x_n) \leq |w|.$$

That is, $\otimes_{k=1}^n E_k$ is order dense in $\bar{\otimes}_{k=1}^n E_k$.

Proof. (1) Let $E_1, ..., E_n, F$ be Archimedean complex vector lattices. We will denote by $(\bar{\otimes}_{k=1}^n E_{k\rho}, \bar{\otimes})$ the Archimedean real vector lattice tensor product of $E_{1\rho}, ..., E_{n\rho}$ (see Lemma 4.5). We prove that the pair $((\bar{\otimes}_{k=1}^n E_{k\rho})_{|\mathbb{C}|}, \bar{\otimes}_{|\mathbb{C}|})$ is the unique Archimedean complex vector lattice tensor product of $E_1, ..., E_n$. To this end, let $T : \times_{k=1}^n E_k \to F$ be a vector lattice n-morphism. From Lemma 4.5(1) the map $\bar{\otimes}$ induces a unique vector lattice homomorphism $T_\rho^{\bar{\otimes}}$ on $\bar{\otimes}_{k=1}^n E_{k\rho}$ such that $T_\rho^{\bar{\otimes}} \circ \bar{\otimes} = T_\rho$. The map $T_\rho^{\bar{\otimes}}$ extends uniquely to a vector lattice homomorphism $(T_\rho^{\bar{\otimes}})^\mu$ on $(\bar{\otimes}_{k=1}^n E_{k\rho})^\mu$ (Corollary 4.1). By Proposition 4.3(1) the map $\bar{\otimes}_{|\mathbb{C}|}$ is a vector lattice n-morphism and $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}$ is a vector lattice homomorphism. We will prove that the map $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}$ is the unique vector lattice homomorphism such that $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|} \circ \bar{\otimes}_{|\mathbb{C}|} = T$.

Indeed, for every $(f_0^1+if_1^1,...,f_0^n+if_1^n)\in\times_{k=1}^n E_k$ we have

$$\begin{aligned}(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}\circ\bar{\otimes}_{|\mathbb{C}|}(f_0^1+if_1^1,...,f_0^n+if_1^n)&=(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}(\sum_{\epsilon_k\in\{0,1\}}\bar{\otimes}(f_{\epsilon_1}^1,...,f_{\epsilon_n}^n)i^{\sum_{k=1}^n\epsilon_k})\\&=\sum_{\epsilon_k\in\{0,1\}}T_\rho^{\bar{\otimes}}\circ\bar{\otimes}(f_{\epsilon_1}^1,...,f_{\epsilon_n}^n)i^{\sum_{k=1}^n\epsilon_k}\\&=\sum_{\epsilon_k\in\{0,1\}}T_\rho(f_{\epsilon_1}^1,...,f_{\epsilon_n}^n)i^{\sum_{k=1}^n\epsilon_k}\\&=T(f_0^1+if_1^1,...,f_0^n+if_1^n).\end{aligned}$$

Since every vector lattice $\mathbb{C}$-homomorphism is real, the uniqueness of $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}$ follows from the uniqueness of $T_\rho^{\bar{\otimes}}$.

The proof of uniqueness of the Archimedean complex vector lattice tensor product is not different from the real case.

(2) Consider the newly minted tensor product $(\bar{\otimes}_{k=1}^n E_k,\bar{\otimes})$, constructed in (1). By Lemma 4.5(1) there exists an Archimedean real vector lattice G and a vector lattice n-morphism $T:\times_{k=1}^n E_{k\rho}\to G$ such that the induced linear map $T^{\otimes}:\otimes_{k=1}^n E_{k\rho}\to G$ is injective. By taking the square mean completion of G if necessary, we will assume that G is square mean complete. Taking vector space complexifications, we find an injective vector lattice homomorphism $(T^{\otimes})_{\mathbb{C}}:(\otimes_{k=1}^n E_{k\rho})_{\mathbb{C}}\to G_{\mathbb{C}}$, or equivalently, $(T^{\otimes})_{\mathbb{C}}:\otimes_{k=1}^n E_k\to G_{\mathbb{C}}$ (see Lemma 4.6). If $(T_{\mathbb{C}})^{\otimes}:\otimes_{k=1}^n E_k\to G$ is the unique linear map induced by $T_{\mathbb{C}}$ then for every $f_0^k+if_1^k\in E_k$ $(k\in\{1,...,n\})$ we have

$$\begin{aligned}(T_{\mathbb{C}})^{\otimes}((f_0^1+if_1^1)\otimes\cdots\otimes(f_0^n+if_1^n))&=T_{\mathbb{C}}(f_0^1+if_1^1,...,f_0^n+if_1^n)\\&=\sum_{\epsilon_k\in\{0,1\}}T(f_{\epsilon_1}^1,...,f_{\epsilon_n}^n)i^{\sum_{k=1}^n\epsilon_k}\\&=\sum_{\epsilon_k\in\{0,1\}}T^{\otimes}(f_{\epsilon_1}^1\otimes\cdots\otimes f_{\epsilon_n}^n)i^{\sum_{k=1}^n\epsilon_k}.\end{aligned}$$

Then $(T_{\mathbb{C}})^{\otimes}$ is a real map and $((T_{\mathbb{C}})^{\otimes})_{\rho} = T^{\otimes}$. Therefore, we have $(T_{\mathbb{C}})^{\otimes} = (T^{\otimes})_{\mathbb{C}}$. From part (1) of this theorem there exists a uniquely determined vector lattice $\mathbb{C}$-homomorphism $(T_{\mathbb{C}})^{\bar{\otimes}} : \bar{\otimes}_{k=1}^{n} E_k \to G_{\mathbb{C}}$ such that $(T_{\mathbb{C}})^{\bar{\otimes}} \circ \bar{\otimes} = T_{\mathbb{C}}$. Moreover, there exists a unique $\mathbb{C}$-linear map $S : \otimes_{k=1}^{n} E_k \to \bar{\otimes}_{k=1}^{n} E_k$ such that $S \circ \otimes = \bar{\otimes}$. Then $(T_{\mathbb{C}})^{\bar{\otimes}} \circ S \circ \otimes = T_{\mathbb{C}}$, and hence $(T_{\mathbb{C}})^{\bar{\otimes}} \circ S = (T_{\mathbb{C}})^{\otimes} = (T^{\otimes})_{\mathbb{C}}$. It follows from the injectivity of $(T^{\otimes})_{\mathbb{C}}$ that S is injective.

(3) By Lemma 4.5(3) we know that $\otimes_{k=1}^{n} E_{k\rho}$ is relatively uniformly dense in $\bar{\otimes}_{k=1}^{n} E_{k\rho}$. We also know from Proposition 4.9 that $\bar{\otimes}_{k=1}^{n} E_{k\rho}$ is relatively uniformly dense in $(\bar{\otimes}_{k=1}^{n} E_{k\rho})^{\mu}$. It follows that $\tau(\otimes_{k=1}^{n} E_k, \bar{\otimes}_{k=1}^{n} E_k) \leq 2$.

(4) Suppose $w \in (\bar{\otimes}_{k=1}^{n} E_k) \setminus \{0\}$. Then $0 < |w| \in (\bar{\otimes}_{k=1}^{n} E_{k\rho})^{\mu}$. It follows immediately from the definition of the Dedekind completion that $\bar{\otimes}_{k=1}^{n} E_{k\rho}$ is order dense in $(\bar{\otimes}_{k=1}^{n} E_{k\rho})^{\delta}$ (see Section 2.1). Thus $\bar{\otimes}_{k=1}^{n} E_{k\rho}$ is order dense in $(\bar{\otimes}_{k=1}^{n} E_{k\rho})^{\mu}$. Therefore, there exists $w_0 \in \bar{\otimes}_{k=1}^{n} E_{k\rho}$ such that $0 < w_0 \leq |w|$. From Lemma 4.5(4) there exists $x_1 \otimes \cdots \otimes x_n \in \otimes_{k=1}^{n} E_{k\rho}$ with $x_k \in E_k^{+}$ $(k \in \{1, ..., n\})$ such that $0 < (x_1 \otimes \cdots \otimes x_n) \leq w_0$. $\square$

In the proof of Theorem 4.10(1) it is necessary to take the vector lattice complexification of $\bar{\otimes}_{k=1}^{n} E_{k\rho}$ to ensure that $(\bar{\otimes}_{k=1}^{n} E_{k\rho})_{|\mathbb{C}|}$ is an Archimedean complex vector lattice. For instance, Theorems 4.13 and 4.14 furnish examples where the vector space complexification $(\bar{\otimes}_{k=1}^{n} E_{k\rho})_{\mathbb{C}}$ does not suffice. We need two lemmas first.

Lemma 4.11. *Let X and Y be nonempty subsets of $\mathbb{R}$ without isolated points. The function $S : (x, y) \mapsto \sqrt{x^2 + y^2}$ $((x, y) \in X \times Y)$ is in the square mean completion of $C(X)\bar{\otimes}C(Y)$ but for all nonempty open subsets U of X and W of Y we have $S|_{U \times W} \notin C(U) \otimes C(W)$.*

Proof. For $f \in C(X)$ and $g \in C(Y)$ we identify $f \otimes g$ with the function $(x, y) \mapsto f(x)g(y)$ $((x, y) \in X \times Y)$. Consider the element S of the square mean completion of $C(X)\bar{\otimes}C(Y)$ defined by

$$(x, y) \mapsto \sqrt{x^2 + y^2} \ ((x, y) \in X \times Y).$$

Let U and W be open nonempty subsets of X and Y, respectively. We will show that the vector subspace of $C(U)$ generated by $\{S(\cdot, y) : y \in W\}$, whose elements are considered as functions on U, is not finite-dimensional. It follows that $S|_{U\times W} \notin C(U) \otimes C(W)$ ([22], Proposition 1). Since W is open and nonempty and Y has no isolated points, we can choose $\alpha_k \in W$ (for all $k \in \mathbb{N}$) for which $\alpha_i^2 \neq \alpha_j^2$ when $i \neq j$. Let $n \in \mathbb{N}$ and let $\lambda_1, ..., \lambda_n \in \mathbb{R}$ for which

$$\lambda_k \sqrt{x^2 + \alpha_k^2} = \lambda_k S(x, y_k) = 0 \ (x \in U).$$

Since the function $x \mapsto \lambda_k \sqrt{x^2 + \alpha_k^2}$ $(x \in \mathbb{R})$ is n times differentiable at every $x \in X \setminus \{0\}$, a routine calculation shows that the $n \times n$ matrix $A(x)$ defined by

$$A(x)_{ij} = \frac{1}{(x^2 + \alpha_j^2)^{\frac{2i-1}{2}}}$$

when evaluated at the vector $(\lambda_1, ..., \lambda_n)$ yields the vector $(0, ..., 0)$ for every nonzero $x \in U$. However, we have that

$$\prod_{k=1}^{n} \sqrt{x^2 + \alpha_k^2} \det(A(x)) = \det(B(x)),$$

where the $n \times n$ matrix $B(x)$ is defined by

$$B(x)_{ij} = \frac{1}{(x^2 + \alpha_j^2)^{i-1}}.$$

Note that $B(x)$ has (Vandermonde) determinant

$$\prod_{1 \leq j < k \leq n} \Big(\frac{1}{x^2 + \alpha_j^2} - \frac{1}{x^2 + \alpha_k^2} \Big) = \prod_{1 \leq j < k \leq n} \frac{\alpha_j^2 - \alpha_k^2}{(x^2 + \alpha_j^2)(x^2 + \alpha_k^2)} \neq 0.$$

Thus $\det(A(x)) \neq 0$ for every non-zero $x \in U$, and the vector subspace of $C(U)$ generated by $\{S(\cdot, y) : y \in Y\}$ (as functions on U) is infinite dimensional. Therefore, we have that $S|_{U\times W} \notin C(U) \otimes C(W)$. □

Lemma 4.12. *Let X and Y be nonempty subsets of $\mathbb{R}$ without isolated points, and let $f \in C(X)\bar{\otimes}C(Y)$. There exists a nonempty open subset V of $X \times Y$ and $g \in C(X) \otimes C(Y)$ such that $f|_V = g|_V$.*

Proof. Note that $C(X)\bar{\otimes}C(Y)$ is the vector lattice generated by $C(X) \otimes C(Y)$ in $C(X \times Y)$ ([20], Section 4). Therefore, every $f \in C(X)\bar{\otimes}C(Y)$ is of the form $f = \bigwedge_{j=1}^{n} \bigvee_{k=1}^{m} f_{j,k}$, where $f_{j,k} \in C(X) \otimes C(Y)$ for each j and each k ([1], Exercise 4.8). By following the proof of the statement given in Example 3.12 of Section 3.3, one finds a nonempty open subset V of $X \times Y$ such that $\bigwedge_{j=1}^{n} \bigvee_{k=1}^{m} f_{j,k} = f_{j_0,k_0}$ on V for some $j_0 \in \{1, ..., n\}$ and some $k_0 \in \{1, ..., m\}$. □

Theorem 4.13. *If X and Y are nonempty subsets of $\mathbb{R}$ without isolated points then $C(X)\bar{\otimes}C(Y)$ is not square mean complete. Therefore, the vector space $(C(X)\bar{\otimes}C(Y))_{\mathbb{C}}$ over $\mathbb{C}$ is not an Archimedean complex vector lattice.*

Proof. Assume that the element S of Lemma 4.11 is in $C(X)\bar{\otimes}C(Y)$. By Lemma 4.12 there exists a nonempty open set V in $X \times Y$ and an element $g \in C(X) \otimes C(Y)$ such that $g|_V = S|_V$. However, the open set V contains a nonempty open subset of the form $U \times W$ with $0 \notin U$. This contradicts Lemma 4.11. □

We use Theorem 4.13 to prove the following.

Theorem 4.14. *If X and Y are uncountable compact metrizable spaces then $C(X)\bar{\otimes}C(Y)$ is not square mean complete. Therefore, $(C(X)\bar{\otimes}C(Y))_{\mathbb{C}}$ is not an Archimedean complex vector lattice.*

Proof. By Theorem 1 in [31], we know that both X and Y contain a closed subset homeomorphic with the Cantor set $\mathbb{D}$. Then $\mathbb{D} \times \mathbb{D}$ can be viewed as a closed subset of $X \times Y$, and the function $F_0 : (x, y) \longmapsto \sqrt{x^2 + y^2}$ $((x, y) \in \mathbb{D} \times \mathbb{D})$ is continuous. By Tietze's Extension Theorem, the function $x \longmapsto x$ $(x \in \mathbb{D})$ can be extended to continuous functions f and g on X and Y, respectively. Then the function $F : (x, y) \longmapsto \sqrt{f(x)^2 + g(y)^2}$ $((x, y) \in X \times Y)$ is a continuous function in the square mean completion of $C(X)\bar{\otimes}C(Y)$ that extends F_0. If F were in $C(X)\bar{\otimes}C(Y)$ then its restriction to $\mathbb{D} \times \mathbb{D}$ would be in $C(\mathbb{D})\bar{\otimes}C(\mathbb{D})$, which by Lemma 4.11 is impossible. This proves the theorem. □

It is certainly tempting to conjecture the following.

Conjecture 4.15. *If X and Y are infinite compact metrizable spaces then $C(X)\bar{\otimes}C(Y)$ is not square mean complete.*

The above two theorems show that the old way of complexifying Archimedean real vector lattices via vector space complexifications is inadequate for pursuing complex analysis on Archimedean complex vector lattices.

We remark that the Archimedean complex vector lattice tensor product, like its real counterpart ([20], Theorem 5.3, [38], Section 2), possesses a universal property with respect to positive multilinear maps with uniformly complete Archimedean complex vector lattices as range.

Theorem 4.16. *Let $E_1, ..., E_n, F$ be Archimedean complex vector lattices such that F is uniformly complete. If $T : \times_{k=1}^n E_k \to F$ is a positive $n_{\mathbb{C}}$-linear map then there exists a unique positive $\mathbb{C}$-linear map $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^n E_k \to F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.*

Proof. As in the proof of Theorem 4.10(1), denote by $(\bar{\otimes}_{k=1}^n E_{k\rho}, \bar{\otimes})$ the Archimedean real vector lattice tensor product of $E_{1\rho}, ..., E_{n\rho}$. Let $T : \times_{k=1}^n E_k \to F$ be a positive $n_{\mathbb{C}}$-linear map. From Section 2 of [38] (also see Proposition 5.1 in [20]), there exists a unique positive $\mathbb{R}$-linear map $T_\rho^{\bar{\otimes}} : \bar{\otimes}_{k=1}^n E_{k\rho} \to F$ such that $T_\rho^{\bar{\otimes}} \circ \bar{\otimes} = T_\rho$. The map $T_\rho^{\bar{\otimes}}$ extends uniquely to a positive $\mathbb{R}$-linear map on $(T_\rho^{\bar{\otimes}})^\mu$ on $(\bar{\otimes}_{k=1}^n E_{k\rho})^\mu$ (Corollary 4.1). Moreover, $\bar{\otimes}_{|\mathbb{C}|}$ is a vector lattice $n_{\mathbb{C}}$-morphism and $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}$ is a positive $\mathbb{C}$-linear map (Proposition 4.3). That $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|} \circ \bar{\otimes}_{|\mathbb{C}|} = T$ follows from the fact that $(T_\rho^{\bar{\otimes}}) \circ \bar{\otimes} = T_\rho$ (see [38], Section 2). Since every positive $\mathbb{C}$-linear map is real, the uniqueness of $(T_\rho^{\bar{\otimes}})_{|\mathbb{C}|}$ follows from the uniqueness of $T_\rho^{\bar{\otimes}}$. □

A reformulation of Theorem 4.10(1) in terms of Archimedean real vector lattices and vector lattice complexifications is the following.

Theorem 4.17. *Let $E_1, ...E_n, F$ be Archimedean real vector lattices, and suppose that a map $T : \times_{k=1}^n E_k \to F$ is a vector lattice $n_{\mathbb{R}}$-morphism. There exists a unique vector lattice $n_{\mathbb{C}}$-morphism $(T_{|\mathbb{C}|})^{\bar{\otimes}} : \bar{\otimes}_{k=1}^n E_{k|\mathbb{C}|} \to F_{|\mathbb{C}|}$ such that $(T_{|\mathbb{C}|})^{\bar{\otimes}} \circ \bar{\otimes}|_{\times_{k=1}^n E_k} = T$.*

Proof. Consider T to be a vector lattice $n_{\mathbb{R}}$-morphism from $\times_{k=1}^n E_k$ to F^μ. By Proposition 4.3(1), there exists a unique vector lattice $n_{\mathbb{C}}$-morphism $T_{|\mathbb{C}|} : \times_{k=1}^n E_{k|\mathbb{C}|} \to F_{|\mathbb{C}|}$ such that $T_{|\mathbb{C}|}|_{\times_{k=1}^n E_k} = T$. If $(T_{|\mathbb{C}|})^{\bar{\otimes}}$ is the unique vector lattice $\mathbb{C}$-homomorphism induced by $T_{\mathbb{C}}$ then $(T_{|\mathbb{C}|})^{\bar{\otimes}} \circ \bar{\otimes} = T_{|\mathbb{C}|}$. In particular, we have $(T_{|\mathbb{C}|})^{\bar{\otimes}} \circ \bar{\otimes}|_{\times_{k=1}^n E_k} = T$. □

4.3 The Symmetric Archimedean Complex Vector Lattice Tensor Product and the Antisymmetric Archimedean Complex Vector Lattice Tensor Product

The symmetric (antisymmetric) Archimedean real vector lattice tensor product was introduced by Loane in Section 4.4 (Section 4.9) of [25]. Both of these tensor products are reviewed in this section. We prove a universal property for each of these tensor products that facilitates their complexification into a corresponding symmetric (antisymmetric) Archimedean complex vector lattice tensor product.

For an Archimedean real vector lattice E, we call $\otimes_n E := E \otimes \cdots \otimes E$ (n-times) the *n-fold algebraic tensor product* of E, and we call $\bar{\otimes}_n E := E \bar{\otimes} \ldots \bar{\otimes} E$ (n-times) the *n-fold Archimedean vector lattice tensor product* of E.

Let E and F be Archimedean vector lattices over $\mathbb{K}$, and denote the set of all permutations on $\{1, ..., n\}$ by S_n. We call an n-linear map $T : \times_n E \to F$ *symmetric* if

$$T(f_1, ..., f_n) = T(f_{\sigma(1)}, ..., f_{\sigma(n)})$$

for every $\sigma \in S_n$ and every $f_1, ..., f_n \in E$. For an n-linear map $T : \times_n E \to F$ and $\sigma \in S_n$ we define

$$T_\sigma(f_1, ..., f_n) = T(f_{\sigma(1)}, ..., f_{\sigma(n)}) \ (f_1, ..., f_n \in E),$$

and we set

$$T_s := \frac{1}{n!} \sum_{\sigma \in S_n} T_\sigma.$$

The map T_s is called the *symmetrization* of T. We call a map $R : \times_n E \to F$ a *symmetrized vector lattice n-morphism* if there exists a vector lattice n-morphism $T : \times_n E \to F$ such that $R = T_s$. Every symmetrized vector lattice n-morphism is a symmetric, positive, n-linear map. The following example shows that a symmetrized vector lattice n-morphism is not necessarily a vector lattice n-morphism. We refer to a vector lattice 2-morphism as a *vector lattice bimorphism*.

Example 4.18. *The map $T : C[0,1] \times C[0,1] \to \mathbb{R}$ defined by $T(f, g) = f(0)g(1)$ is a T is a vector lattice bimorphism, but T_s is not a vector lattice bimorphism.*

We clearly have that T is a vector lattice bimorphism and that

$$T_s(f, g) = \frac{f(0)g(1) + g(0)f(1)}{2} \ (f, g \in C[0,1]).$$

Define $a(x) = 1$ ($x \in [0,1]$) and $b(x) = 2x - 1$ ($x \in [0,1]$). We have that $T_s(|a|, |b|) = 1$ and $|T_s(a,b)| = 0$. But T_s is a vector lattice bimorphism if and only if $T_s(|f|, |g|) = |T_s(f,g)|$ for every $f, g \in C[0,1]$ ([7], Proposition 1.3(ii)). Thus T_s is not a vector lattice bimorphism.

Loane introduced the n-fold symmetric Archimedean real vector lattice tensor product (for $n = 2$) in Section 4.4 of [25]. We give a summary some facts regarding the n-fold symmetric Archimedean real vector lattice tensor product that appear (with some difference in notation) in [25].

Let E be an Archimedean real vector lattice. Define $S : \otimes_n E \to \otimes_n E$ by

$$S(\sum_{k=1}^{m} f_{1_k} \otimes \cdots \otimes f_{n_k}) = \sum_{k=1}^{m} \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)_k} \otimes \cdots \otimes f_{\sigma(n)_k}.$$

Note that $\bar{\otimes}_\sigma : \times_n E \to \bar{\otimes}_n E$ is a vector lattice n-morphism for every $\sigma \in S_n$. Thus there exists a vector lattice homomorphism $\psi_\sigma : \bar{\otimes}_n E \to \bar{\otimes}_n E$ such that $\psi_\sigma \circ \bar{\otimes} = \bar{\otimes}_\sigma$ for every $\sigma \in S_n$ (Lemma 4.5(1)). Define $\bar{S} := \frac{1}{n!} \sum_{\sigma \in S_n} \psi_\sigma$ and denote the vector sublattice $\langle S(\otimes_n E) \rangle$ of $\bar{\otimes}_n E$ by $\bar{\otimes}_{n,s} E$. Loane proves (for $n = 2$) in Section 4.4 of [25] that $\bar{S}(\bar{\otimes}_n E) = \bar{\otimes}_{n,s} E$. Set $\bar{\otimes}_s := \bar{S} \circ \bar{\otimes}$. We call $(\bar{\otimes}_{n,s} E, \bar{\otimes}_s)$ the *n-fold symmetric Archimedean vector lattice tensor product* of E.

We next prove that Loane's symmetric Archimedean real vector lattice tensor product is the universal space that factors symmetrized vector lattice n-morphisms.

Theorem 4.19. *Let E be an Archimedean real vector lattice and let $n \in \mathbb{N} \setminus \{1\}$. The following hold.*

(1) $\bar{\otimes}_{n,s} E$ is an Archimedean real vector lattice.

(2) $\bar{\otimes}_s$ is a symmetrized vector lattice n-morphism.

(3) For every Archimedean real vector lattice F and for every symmetrized vector lattice n-morphism $T_s : \times_n E \to F$ there exists a uniquely determined vector lattice homomorphism $T_s^{\bar{\otimes}_s} : \bar{\otimes}_{n,s} E \to F$ such that $T_s^{\bar{\otimes}_s} \circ \bar{\otimes}_s = T_s$.

Furthermore, suppose that G is an Archimedean real vector lattice and that $\odot$ is a symmetrized vector lattice n-morphism. If $(G, \odot)$ satisfies (1)–(3) above then there exists a vector lattice isomorphism $\phi : \bar{\otimes}_{n,s} E \to G$ such that $\odot = \phi \circ \bar{\otimes}_s$.

Proof. By definition $\bar{\otimes}_{n,s} E$ is an Archimedean real vector lattice. Using the notation preceding the statement of this theorem, we have

$$\bar{\otimes}_s = \bar{S} \circ \bar{\otimes} = \frac{1}{n!} \sum_{\sigma \in S_n} \psi_\sigma \circ \bar{\otimes} = \frac{1}{n!} \sum_{\sigma \in S_n} \bar{\otimes}_\sigma.$$

Thus $(\bar{\otimes}_{n,s} E, \bar{\otimes}_s)$ satisfies (1) and (2) above. To prove that $(\bar{\otimes}_{n,s} E, \bar{\otimes}_s)$ satisfies (3), let F be an Archimedean real vector lattice, and let $T_s : \times_n E \to F$ be a symmetrized vector lattice n-morphism. There exists a vector lattice n-morphism $T : \times_n E \to F$ such that

$$T_s := \frac{1}{n!} \sum_{\sigma \in S_n} T_\sigma.$$

Define $T_s^{\bar{\otimes}_s} := T^{\bar{\otimes}}|_{\bar{\otimes}_{n,s} E}$, where $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^n E_k \to F$ is the unique vector lattice homomorphism induced by T (Lemma 4.5(1)). Then $T_s^{\bar{\otimes}_s}$ is a vector lattice homomorphism. For $f_1, ..., f_n \in E$ we have

$$\begin{aligned} T_s^{\bar{\otimes}_s} \circ \bar{\otimes}_s(f_1, ..., f_n) &= T_s^{\bar{\otimes}_s}\Big(\frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}\Big) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} T^{\bar{\otimes}}(f_{\sigma(1)} \otimes \cdots \otimes f_{\sigma(n)}) \\ &= \frac{1}{n!} \sum_{\sigma \in S_n} T(f_{\sigma(1)}, ..., f_{\sigma(n)}) \\ &= T_s(f_1, ..., f_n). \end{aligned}$$

To see that $T_s^{\bar{\otimes}_s}$ is unique, note that $\langle S(\otimes_n E)\rangle = \langle [S(\otimes_n E)]\rangle$. Define $\otimes_s := S \circ \otimes$. Every element of $\bar{\otimes}_{n,s}E$ is of the form

$$\bigwedge_{i=1}^{m}\bigvee_{j=1}^{p}\sum_{k=1}^{q}\lambda_k f_{1_{i,j,k}} \otimes_s \cdots \otimes_s f_{n_{i,j,k}}$$

for some $f_{1_{i,j,k}}, ..., f_{n_{i,j,k}} \in E$ ([1], Exercise 4.8). Suppose that $R : \bar{\otimes}_{n,s}E \to F$ is a vector lattice homomorphism such that $R(f_1 \otimes_s \cdots \otimes_s f_n) = T_s(f_1, ..., f_n)$ for every $f_1, ..., f_n \in E$. Then we have for every $\bigwedge_{i=1}^{m}\bigvee_{j=1}^{p}\sum_{k=1}^{q}\lambda_k f_{1_{i,j,k}} \otimes_s \cdots \otimes_s f_{n_{i,j,k}} \in \bar{\otimes}_{n,s}E$ that

$$\begin{aligned}
R(\bigwedge_{i=1}^{m}\bigvee_{j=1}^{p}\sum_{k=1}^{q}\lambda_k f_{1_{i,j,k}} \otimes_s \cdots \otimes_s f_{n_{i,j,k}}) &= \bigwedge_{i=1}^{m}\bigvee_{j=1}^{p}\sum_{k=1}^{q}\lambda_k R(f_{1_{i,j,k}} \otimes_s \cdots \otimes_s f_{n_{i,j,k}})\\
&= \bigwedge_{i=1}^{m}\bigvee_{j=1}^{p}\sum_{k=1}^{q}\lambda_k T_s(f_{1_{i,j,k}}, ..., f_{n_{i,j,k}})\\
&= \bigwedge_{i=1}^{m}\bigvee_{j=1}^{p}\sum_{k=1}^{q}\lambda_k T_s^{\bar{\otimes}_s}(f_{1_{i,j,k}} \otimes_s \cdots \otimes_s f_{n_{i,j,k}})\\
&= T_s^{\bar{\otimes}_s}(\bigwedge_{i=1}^{m}\bigvee_{j=1}^{p}\sum_{k=1}^{q}\lambda_k f_{1_{i,j,k}} \otimes_s \cdots \otimes_s f_{n_{i,j,k}}).
\end{aligned}$$

Thus $T_s^{\bar{\otimes}_s} : \bar{\otimes}_{n,s}E \to F$ is the unique vector lattice homomorphism such that $T_s^{\bar{\otimes}_s} \circ \bar{\otimes}_s = T_s$. The proof of the uniqueness of $(\bar{\otimes}_{n,s}E, \bar{\otimes}_s)$ is routine. □

With a universal property for Loane's n-fold symmetric Archimedean real vector lattice tensor product established, we define the n-fold symmetric Archimedean complex vector lattice tensor product via an analogous universal property.

For an Archimedean complex vector lattice E, we call a pair $(\bar{\otimes}_{n,s}E, \bar{\otimes}_s)$ an *n-fold symmetric Archimedean complex vector lattice tensor product* of E if the following hold.

(1) $\bar{\otimes}_{n,s}E$ is an Archimedean complex vector lattice.

(2) $\bar{\otimes}_s$ is a symmetrized vector lattice $n_{\mathbb{C}}$-morphism.

(3) For every Archimedean complex vector lattice F and for every symmetrized vector lattice $n_{\mathbb{C}}$-morphism $T_s : \times_n E \to F$ there exists a unique vector lattice $\mathbb{C}$-homomorphism $T_s^{\bar{\otimes}_s} : \bar{\otimes}_{n,s} E \to F$ such that $T_s^{\bar{\otimes}_s} \circ \bar{\otimes}_s = T_s$.

It is evident that if an n-fold symmetric Archimedean complex vector lattice tensor product exists then it is unique. The existence of the n-fold symmetric Archimedean complex vector lattice tensor product will be proved (Theorem 4.20). We first need to complexify symmetrized vector lattice $n_{\mathbb{R}}$-morphisms. Let E and F be Archimedean real vector lattices such that F is square mean complete. Given the symmetrization T_s of a vector lattice $n_{\mathbb{R}}$-morphism $T : \times_n E \to F$, we define

$$(T_s)_{|\mathbb{C}|} := \frac{1}{n!} \sum_{\sigma \in S_n} (T_\sigma)_{|\mathbb{C}|}.$$

We remind the reader that the complexification $R_{|\mathbb{C}|}$ of a vector lattice $n_{\mathbb{R}}$-morphism R is defined in Section 4.1. Thus $(T_s)_{|\mathbb{C}|}$ is the unique extension of T_s to a symmetrized vector lattice $n_{\mathbb{C}}$-morphism. We next prove the existence of the n-fold symmetric Archimedean complex vector lattice tensor product.

Theorem 4.20. *If E is an Archimedean complex vector lattice and $n \in \mathbb{N} \setminus \{1\}$ then there exists an n-fold symmetric Archimedean complex vector lattice tensor product of E, unique up to vector lattice isomorphism.*

Proof. Let E be an Archimedean complex vector lattice. Suppose that $n \in \mathbb{N} \setminus \{1\}$ and that T_s is the symmetrization of a vector lattice n-morphism T. Then T_s is positive and n-linear. In particular, T_s is real and $(T_s)_\rho = \frac{1}{n!} \sum_{\sigma \in S_n} (T_\sigma)_\rho$. Let $(\bar{\otimes}_{n,s} E_\rho, \bar{\otimes}_s)$ denote the n-fold symmetric Archimedean vector lattice tensor product of E_ρ. From Theorem 4.19 the map $\bar{\otimes}_s$ induces a uniquely determined vector lattice $\mathbb{R}$-homomorphism $((T_s)_\rho)^{\bar{\otimes}_s} : \bar{\otimes}_{n,s} E_\rho \to F_\rho$ such that $((T_s)_\rho)^{\bar{\otimes}_s} \circ \bar{\otimes}_s = (T_s)_\rho$. The map $(\bar{\otimes}_s)_{|\mathbb{C}|} := \frac{1}{n!} \sum_{\sigma \in S_n} (\bar{\otimes}_\sigma)_{|\mathbb{C}|}$ is a symmetrized vector lattice $n_{\mathbb{C}}$-morphism, and $(((T_s)_\rho)^{\bar{\otimes}_s})_{|\mathbb{C}|}$ is a vector lattice $\mathbb{C}$-homomorphism (Proposition 4.3(1)).

That $(((T_s)_\rho)^{\bar{\otimes}_s})_{|\mathbb{C}|} \circ (\bar{\otimes}_s)_{|\mathbb{C}|} = T_s$ follows from the fact that $(((T_s)_\rho)^{\bar{\otimes}_s}) \circ (\bar{\otimes}_s) = (T_s)_\rho$ (see Theorem 4.19). The uniqueness of $(((T_s)_\rho)^{\bar{\otimes}_s})_{|\mathbb{C}|}$ follows from the uniqueness of $((T_s)_\rho)^{\bar{\otimes}_s}$. □

In the rest of this section, we discuss the antisymmetrization of n-linear maps as well as the n-fold antisymmetric Archimedean vector lattice tensor product for real and complex Archimedean vector lattices.

Let E and F be Archimedean vector lattices over $\mathbb{K}$. An n-linear map $T : \times_n E \to F$ is called *antisymmetric* if

$$T(f_1, ..., f_n) = -T(f_{\sigma(1)}, ..., f_{\sigma(n)})$$

for every transposition $\sigma \in S_n$. For a permutation $\sigma \in S_n$ we define

$$p(\sigma) = \begin{cases} 1 & : \sigma \text{ is odd} \\ 0 & : \sigma \text{ is even.} \end{cases}$$

Given an n-linear map $T : \times_n E \to F$ and $\sigma \in S_n$, we set

$$T_a := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{p(\sigma)} T_\sigma.$$

The map T_a is called the *antisymmetrization* of T, and a map $R : \times_n E \to F$ is said to be an *antisymmetrized vector lattice n-morphism* if there exists a vector lattice n-morphism $T : \times_n E \to F$ such that $R = T_a$. Every antisymmetrized vector lattice n-morphism is antisymmetric and n-linear. Every antisymmetrized vector lattice n-morphism is a *regular n-linear map*, that is, the difference of two positive n-linear maps. Thus every antisymmetrized vector lattice $n_{\mathbb{C}}$-morphism is real.

Loane introduced in Section 4.9 of [25] the n-fold antisymmetric Archimedean real vector lattice tensor product (for $n = 2$). He did not give a formal definition, but he stated that the antisymmetric Archimedean real vector lattice tensor product is constructed in a way that is analogous to the symmetric Archimedean real vector lattice tensor product. We outline some of the details.

Let E be an Archimedean real vector lattice. Define $A : \otimes_n E \to \otimes_n E$ by

$$A(\sum_{k=1}^{m} f_{1_k} \otimes \cdots \otimes f_{n_k}) = \sum_{k=1}^{m} \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{p(\sigma)} f_{\sigma(1)_k} \otimes \cdots \otimes f_{\sigma(n)_k}.$$

Let $\sigma \in S_n$. Then $\bar{\otimes}_\sigma : \times_n E \to \bar{\otimes}_n E$ is a vector lattice n-morphism. Therefore, there exists a unique vector lattice homomorphism $\psi_\sigma : \bar{\otimes}_n E \to \bar{\otimes}_n E$ such that $\psi_\sigma \circ \bar{\otimes} = \bar{\otimes}_\sigma$ $(\sigma \in S_n)$. Define $\bar{A} := \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{p(\sigma)} \psi_\sigma$, and denote the vector sublattice $\langle A(\otimes_n E) \rangle$ of $\bar{\otimes}_n E$ by $\bar{\otimes}_{n,a} E$. Following Lemma 4.10, Proposition 4.11, Proposition 4.12, and Corollary 4.13 in [25], one proves that $\bar{A}(\bar{\otimes}_n E) = \bar{\otimes}_{n,a} E$. Set $\bar{\otimes}_a := \bar{A} \circ \bar{\otimes}$. We call $(\bar{\otimes}_{n,a} E, \bar{\otimes}_a)$ the *n-fold antisymmetric Archimedean real vector lattice tensor product* of E.

Analogously to the n-fold symmetric Archimedean vector lattice tensor product, the n-fold antisymmetric Archimedean vector lattice tensor product satisfies a universal property involving antisymmetrized vector lattice n-morphisms. The proof of Theorem 4.21 is similar to the proof of Theorem 4.19. We do not include the proof.

Theorem 4.21. *If E is an Archimedean real vector lattice then the following hold.*

(1) $\bar{\otimes}_{n,a} E$ is an Archimedean real vector lattice.

(2) $\bar{\otimes}_a$ is an antisymmetrized vector lattice n-morphism.

(3) For every Archimedean real vector lattice F and for every antisymmetrized vector lattice n-morphism $T_a : \times_n E \to F$ there exists a uniquely determined vector lattice homomorphism $T_a^{\bar{\otimes}_a} : \bar{\otimes}_{n,a} E \to F$ such that $T_a^{\bar{\otimes}_a} \circ \bar{\otimes}_a = T_a$.

Furthermore, suppose that G is an Archimedean real vector lattice and that $\oslash$ is an antisymmetrized vector lattice n-morphism. If $(G, \oslash)$ satisfies (1)–(3) above then there exists a vector lattice isomorphism $\phi : \bar{\otimes}_{n,a} E \to G$ such that $\oslash = \phi \circ \bar{\otimes}_a$.

We next use the universal property in Theorem 4.21 to define the n-fold antisymmetric Archimedean vector lattice tensor product.

For an Archimedean complex vector lattice E, we call $(\bar{\otimes}_{n,a} E, \bar{\otimes}_a)$ an *n-fold antisymmetric Archimedean complex vector lattice tensor product of* E if the following hold.

(1) $\bar{\otimes}_{n,a} E$ is an Archimedean complex vector lattice.

(2) $\bar{\otimes}_a$ is an antisymmetrized vector lattice $n_{\mathbb{C}}$-morphism.

(3) For every Archimedean complex vector lattice F and for every antisymmetrized vector lattice $n_{\mathbb{C}}$-morphism $T_a : \times_n E \to F$ there exists a unique vector lattice $\mathbb{C}$-homomorphism $T_a^{\bar{\otimes}_a} : \bar{\otimes}_{n,a} E \to F$ such that $T_a^{\bar{\otimes}_a} \circ \bar{\otimes}_a = T_a$.

The proof of the following theorem is similar to the proof of Theorem 4.20.

Theorem 4.22. *If E is an Archimedean complex vector lattice and $n \in \mathbb{N} \setminus \{1\}$ then there exists an n-fold antisymmetric Archimedean complex vector lattice tensor product of E, unique up to vector lattice n-morphism.*

4.4 Powers of Complex Vector Lattices

The Archimedean complex vector lattice tensor product introduced in Section 4.2 is used to prove the existence of s-powers for every $s \in \mathbb{N} \setminus \{1\}$ and every Archimedean complex vector lattice. We prove that the s-power of an Archimedean complex vector lattice E is the vector lattice complexification of the s-power of E_ρ.

Central to the theory of s-powers are orthosymmetric vector lattice n-morphisms. For an Archimedean vector lattice E over $\mathbb{K}$ and for $f, g \in E$, we say that f and g are *disjoint* if $|f| \wedge |g| = 0$. In this case we write $f \perp g$.

Given Archimedean vector lattices E and F over $\mathbb{K}$, a map $T : \times_s E \to F$ is called *orthosymmetric* if $T(f_1, ..., f_s) = 0$ whenever there exist $i, j \in \{1, ..., s\}$ such that $f_i \perp f_j$.

For an Archimedean vector lattice E over $\mathbb{K}$ and for $s \in \mathbb{N} \setminus \{1\}$, we call a pair $(E^{Ⓢ}, Ⓢ)$ an *s-power* of E if the following hold.

(1) $E^{Ⓢ}$ is an Archimedean vector lattice over $\mathbb{K}$.

(2) $Ⓢ : \times_s E \to E^{Ⓢ}$ is an orthosymmetric vector lattice s-morphism.

(3) For every Archimedean vector lattice F over $\mathbb{K}$, and for every orthosymmetric vector lattice s-morphism $T : \times_s E \to F$, there exists a unique vector lattice homomorphism $T^{Ⓢ} : E^{Ⓢ} \to F$ such that $T^{Ⓢ} \circ Ⓢ = T$.

Orthosymmetric $s_{\mathbb{R}}$-linear maps were first introduced for $s = 2$ by Buskes and van Rooij in Definition 1 of [14]. Boulabiar and Buskes extended the notion of orthosymmetric $s_{\mathbb{R}}$-linear maps to general $s \in \mathbb{N} \setminus \{1\}$ in Section 2 of [8]. Similarly, s-powers for Archimedean real vector lattices were introduced for $s = 2$ in Definition 3 of [14] and were later introduced for $s \in \mathbb{N} \setminus \{1\}$ in Definition 3.1 of [8].

We address the existence and uniqueness of s-powers for Archimedean complex vector lattices in Theorem 4.24, which is a complex analogue of Theorem 3.2 in [8] by Boulabiar and Buskes for Archimedean real vector lattices. We need a definition and a few prerequisite results first.

For a nonempty subset A of an Archimedean vector lattice E over $\mathbb{K}$, we denote by $\langle\langle A \rangle\rangle$ the smallest ideal of E that contains A and call $\langle\langle A \rangle\rangle$ the *ideal of E generated by A*.

The following lemma can be found on page 96 of [28] for $\mathbb{K}=\mathbb{R}$. The lemma and its proof also hold for $\mathbb{K}=\mathbb{C}$.

Lemma 4.23. *For a nonempty subset A of an Archimedean vector lattice E over $\mathbb{K}$, we have that*

$$\langle\langle A\rangle\rangle=\{g\in E:|g|\leq|\alpha_1 f_1|+\cdots+|\alpha_n f_n| \text{ for some } \alpha_1,...,\alpha_n\in\mathbb{K} \text{ and } f_1,...,f_n\in E\}.$$

Lemma 4.24. *Let $E_1,...,E_s,F$ be Archimedean real vector latices and let (f_{k_n}) be a sequence in E_k such that $f_{k_n}\overset{ru}{\to}f_k$ $(k\in\{1,...,s\})$. If $T:\times_{k=1}^{s}E_k\to F$ is a positive s-linear map then $T(f_{1_n},...,f_{s_n})\overset{ru}{\to}T(f_1,...,f_s)$.*

Proof. Let $E_1,...,E_s,F$ be Archimedean real vector lattices and let $T:\times_{k=1}^{s}E_k\to F$ be a positive s-linear map. Suppose that for each $k\in\{1,...,s\}$ the sequence (f_{k_n}) converges p_k-uniformly to f_k. By Proposition 1.3(i) in [7], we have

$$|T(g_1,...,g_s)|\leq T(|g_1|,...,|g_s|)\ g_k\in E_k\ (k\in\{1,...,s\}).$$

We thus have for sufficiently large n that

$$\begin{aligned}|T(f_{1_n},...,f_{s_n})-T(f_1,...,f_s)|&=|\sum_{k=1}^{s}T(f_1,...,f_{k-1},f_{k_n}-f_k,f_{(k+1)_n},...,f_{s_n})|\\&\leq\sum_{k=1}^{s}|T(f_1,...,f_{k-1},f_{k_n}-f_k,f_{(k+1)_n},...,f_{s_n})|\\&\leq\sum_{k=1}^{s}|T(|f_1|,...,|f_{k-1}|,|f_{k_n}-f_k|,|f_{(k+1)_n}|,...,|f_{s_n}|)|\\&\leq\sum_{k=1}^{s}|T(|f_1|,...,|f_{k-1}|,\epsilon p_k,|f_{(k+1)_n}|,...,|f_{s_n}|)|.\end{aligned}$$

It is evident that every relatively uniformly convergent sequence is order bounded. For every $k\in\{1,...,s\}$, let $v_k\in E_k^+$ be such that $|f_{k_n}|\leq v_k$ for every $n\in\mathbb{N}$. Then for sufficiently large

$n \in \mathbb{N}$ we have

$$\sum_{k=1}^{s} |T(|f_1|, ..., |f_{k-1}|, \epsilon p_k, |f_{(k+1)_n}|, ..., |f_{s_n}|)| \leq \sum_{k=1}^{s} |T(|f_1|, ..., |f_{k-1}|, \epsilon p_k, v_{k+1}, ..., v_s)|$$
$$= \epsilon \sum_{k=1}^{s} |T(|f_1|, ..., |f_{k-1}|, p_k, v_{k+1}, ..., v_s)|.$$

Therefore, $(T(f_{1_n}, ..., f_{s_n}))$ converges

$$\sum_{k=1}^{s} |T(|f_1|, ..., |f_{k-1}|, p_k, v_{k+1}, ..., v_s)| - \text{uniformly}$$

to $T(f_1, ..., f_s)$. □

Theorem 4.25. *If E is an Archimedean complex vector lattice and $s \in \mathbb{N} \setminus \{1\}$ then there exists an s-power of E, unique up to vector lattice isomorphism.*

Proof. Let E be an Archimedean complex vector lattice, let $s \in \mathbb{N} \setminus \{1\}$, and let I be the smallest uniformly closed ideal of $\bar{\otimes}_s E$ that contains

$$A := \big\{ f_1 \otimes \cdots \otimes f_s : f_1, ..., f_s \in E \text{ and } f_i \perp f_j \text{ for some } i, j \in \{1, ..., s\} \big\}.$$

Using the same notation for pseudo uniform closures in Section 3.1 and Section 4.2, it is straightforward to show that $\langle\langle A \rangle\rangle_\alpha$ is an ideal in E for every ordinal $1 \leq \alpha \leq \omega_1$. Thus $I = \langle\langle A \rangle\rangle_{\omega_1}$, where D_{ω_1} denotes the uniform closure of a nonempty subset D of E (see Proposition 3.1 of Section 3.1). Given $f \in \bar{\otimes}_s E$, we denote the equivalence class of f in $(\bar{\otimes}_s E)/I$ by $[f]$. Note that $(\bar{\otimes}_s E)/I$ is a vector space over $\mathbb{C}$ under the operations

$$[f] + [g] = [f + g] \text{ and } \lambda[f] = [\lambda f] \ ([f], [g] \in (\bar{\otimes}_s E)/I, \lambda \in \mathbb{C}).$$

In particular, we have for $f, g \in E$ that $[f+ig]=[f]+i[g]$. It follows that

$$(\bar{\otimes}_s E)/I = \big((\bar{\otimes}_s E)_\rho / I_\rho\big)_{\mathbb{C}}$$

(also see page 198 of [46]). Moreover, I_ρ is a uniformly closed ideal in $(\bar{\otimes}_s E)_\rho$, and thus $(\bar{\otimes}_s E)_\rho / I_\rho$ is an Archimedean real vector lattice ([28], Theorem 60.2). Define the natural vector lattice homomorphism $p : (\bar{\otimes}_s E)_\rho \to (\bar{\otimes}_s E)_\rho / I_\rho$ by

$$p(f) = [f] \ (f \in (\bar{\otimes}_s E)_\rho).$$

Consider p as a map from $(\bar{\otimes}_s E)_\rho$ to the square mean completion $\big((\bar{\otimes}_s E)_\rho / I_\rho\big)^\mu$. Suppose that $[f],[g] \in (\bar{\otimes}_s E)_\rho / I_\rho$. By Theorem 3.13 in Section 3.3, we have that

$$\mu([f],[g]) = [\mu(f,g)].$$

Moreover, we have that $[\mu(f,g)] \in (\bar{\otimes}_s E)_\rho / I_\rho$ since $(\bar{\otimes}_s E)_\rho$ is square mean complete. Hence $(\bar{\otimes}_s E)_\rho / I_\rho$ is square mean complete, and $(\bar{\otimes}_s E)/I$ is an Archimedean complex vector lattice.

Next let $q : \bar{\otimes}_s E \to (\bar{\otimes}_s E)/I$ be the natural vector lattice homomorphism from $\bar{\otimes}_s E$ to $(\bar{\otimes}_s E)/I$. Then $q \circ \bar{\otimes}$ is a vector lattice s-morphism. If $f_1, ..., f_s \in E$ and $f_i \perp f_j$ for some $i, j \in \{1, ..., s\}$ then $f_1 \otimes \cdots \otimes f_s \in I$. We thus have that $q(f_1 \otimes \cdots \otimes f_s) = 0$. Hence $q \circ \bar{\otimes}$ is orthosymmetric. Let F be an Archimedean complex vector lattice, and let $T : \times_s E \to F$ be an orthosymmetric s-morphism. By Theorem 4.10(1), there exists a unique vector lattice homomorphism $T^{\bar{\otimes}} : \bar{\otimes}_s E \to F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$. Define

$$T^{\circledS}(q(w)) = T^{\bar{\otimes}}(w) \ (w \in \bar{\otimes}_s E).$$

To show that $T^{\circledS}$ is well-defined, suppose that if $q(v) = q(w)$ for some $v, w \in \bar{\otimes}_s E$. Then $q(v-w) = 0$, and thus $v - w \in I$. We use transfinite induction on the pseudo uniform closures of $\langle\langle A\rangle\rangle$ to prove that $T^{\circledS}(q(v)) = T^{\circledS}(q(w))$. First suppose that $v - w \in \langle\langle A\rangle\rangle_1$. By Lemma 4.23, there exist $\alpha_1, ..., \alpha_n \in \mathbb{C}$ and $f_k^1 \otimes \cdots \otimes f_k^s \in A$ $(k \in \{1, ..., n\})$ such that

$$|v - w| \leq |\alpha_1 f_1^1 \otimes \cdots \otimes f_1^s| + \cdots + |\alpha_n f_n^1 \otimes \cdots \otimes f_n^s|.$$

Then we get

$$\begin{aligned}
|T^{\circledS}(q(v)) - T^{\circledS}(q(w))| &= |T^{\bar{\otimes}}(v) - T^{\bar{\otimes}}(w)| \\
&= |T^{\bar{\otimes}}(v - w)| \\
&= T^{\bar{\otimes}}(|v - w|) \\
&\leq T^{\bar{\otimes}}(|\alpha_1 f_1^1 \otimes \cdots \otimes f_1^s| + \cdots + |\alpha_n f_n^1 \otimes \cdots \otimes f_n^s|) \\
&= T^{\bar{\otimes}}(|\alpha_1 f_1^1 \otimes \cdots \otimes f_1^n|) + \cdots + T^{\bar{\otimes}}(|\alpha_n f_n^1 \otimes \cdots \otimes f_n^s|) \\
&= |\alpha_1 T^{\bar{\otimes}}(f_1^1 \otimes \cdots \otimes f_1^s)| + \cdots + |\alpha_n T^{\bar{\otimes}}(f_n^1 \otimes \cdots \otimes f_n^s)| \\
&= |\alpha_1 T(f_1^1, ..., f_1^s)| + \cdots + |\alpha_n T(f_n^1, ..., f_n^s)| \\
&= 0.
\end{aligned}$$

Thus $T^{\circledS}(q(v)) = T^{\circledS}(q(w))$. Next let $\alpha > 1$ be a successor ordinal and suppose that $T^{\circledS}(q(v)) = T^{\circledS}(q(w))$ whenever $v - w \in \langle\langle A\rangle\rangle_{\alpha-1}$. Let $v - w \in \langle\langle A\rangle\rangle_\alpha$. There exists a sequence (x_n) in $\langle\langle A\rangle\rangle_{\alpha-1}$ such that $x_n \overset{ru}{\to} v - w$. It follows from Lemma 4.24 that $T^{\bar{\otimes}}(x_n) \overset{ru}{\to} T^{\bar{\otimes}}(v - w)$. By our inductive hypotheses we get

$$T^{\bar{\otimes}}(x_n) = T^{\circledS}(q(x_n)) = T^{\circledS}(q(0)) = T^{\bar{\otimes}}(0) = 0 \ (n \in \mathbb{N}).$$

Furthermore, there exists $0 < u \in \langle\langle A\rangle\rangle_{\alpha-1}$ such that for every $\epsilon > 0$ and for sufficiently large n we have

$$\begin{aligned}|T^{\circledS}(q(v)) - T^{\circledS}(q(w))| &= |T^{\bar{\otimes}}(v) - T^{\bar{\otimes}}(w)| \\ &= |T^{\bar{\otimes}}(v-w)| \\ &= |T^{\bar{\otimes}}(v-w) - T^{\bar{\otimes}}(x_n)| \\ &< \epsilon u.\end{aligned}$$

Since E is Archimedean, we conclude that $T^{\circledS}(q(v)) = T^{\circledS}(q(w))$. Finally, suppose that α is a limit ordinal and that for every ordinal $\beta < \alpha$ and every $v, w \in \langle\langle A\rangle\rangle_\beta$ we have that $T^{\circledS}(q(v)) = T^{\circledS}(q(w))$. If $v, w \in \langle\langle A\rangle\rangle_\alpha$ then there exists an ordinal $\beta < \alpha$ such that $v, w \in \langle\langle A\rangle\rangle_\beta$. It follows that $T^{\circledS}(q(v)) = T^{\circledS}(q(w))$. Therefore, $T^{\circledS}$ is well-defined. It is straightforward to prove that $T^{\circledS}$ is a vector lattice homomorphism and to prove that $T^{\circledS} \circ (q \circ \bar{\otimes}) = T$. We conclude that $\big((\bar{\otimes}_s E)/I, q \circ \bar{\otimes}\big)$ is an s-power of E. The proof of the uniqueness of s-powers is the same as the real case. □

We next prove that if E is an Archimedean complex vector lattice, $s \in \mathbb{N} \setminus \{1\}$, and $((E_\rho)^{\circledS}, \circledS)$ is the s-power of E_ρ then $(((E_\rho)^{\circledS})_{|\mathbb{C}|}, \circledS_{|\mathbb{C}|})$ is the unique s-power of E. In particular, we will prove that $((E_\rho)^{\circledS})_{|\mathbb{C}|}$ and $E^{\circledS}$ are isomorphic as complex vector lattices. We start with some prerequisite results.

Lemma 4.26. *Let E and F be Archimedean real vector lattices.*

(1) If F is uniformly complete and $T : \times_s E \to F$ is an s-linear, positive, orthosymmetric map then T extends uniquely to an s-linear, positive, orthosymmetric map $T^u : \times_s E^u \to F$.

(2) If F is $\mathcal{D}$-complete for some nonempty $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}^+_{|\ |}(\mathbb{R}^m)$ and $T : \times_s E \to F$ is an orthosymmetric vector lattice s-morphism then T extends uniquely to an orthosymmetric vector lattice s-morphism $T^{\mathcal{D}} : \times_s E^{\mathcal{D}} \to F$.

Proof. (1) We use transfinite induction on the pseudo uniform closures of Archimedean real vector lattices, using the same notation for pseudo uniform closures in Section 3.1. Let $T : \times_s E \to F$ be a positive, s-linear, orthosymmetric map. It was proven in Proposition 3.2(1) that $E^u = E_{\omega_1}$ and that for every ordinal $1 \leq \alpha \leq \omega_1$ there exists a unique extension $T_\alpha : \times_s E_\alpha \to F$ of T that is a positive and s-linear. We only need to verify that each T_α is orthosymmetric.

First note that T_1 is clearly orthosymmetric. Suppose that $\alpha > 1$ is a successor ordinal and that $T_{\alpha-1}$ is orthosymmetric. Let $f_1, ..., f_s \in (E_\alpha)^+$ be such that $f_i \perp f_j$ for some $i, j \in \{1, ..., s\}$. For each $k \in \{1, ..., s\}$ there exists $p_k \in (E_{\alpha-1})^+$ and a sequence (f_{k_n}) in $E_{\alpha-1}$ that converges p_k-uniformly to f_k. In fact, we can choose the sequence (f_{k_n}) so that $|f_{k_n} - f_k| \leq n^{-1} p_k$ for every $n \in \mathbb{N}$. Then we have

$$f_{k_n} - n^{-1} p_k \leq f_k \ (n \in \mathbb{N}, k \in \{1, ..., s\}).$$

We thus have that $0 \leq (f_{k_n} - n^{-1} p_k)^+ \leq f_k^+ = f_k$ for every $n \in \mathbb{N}$ and every $k \in \{1, ..., s\}$. Moreover, the sequence $((f_{k_n} - n^{-1} p_k)^+)$ converges $2p_k$-uniformly to f_k. Noting that T_α is positive, Lemma 4.24 implies that

$$T_\alpha((f_{1_n} - n^{-1} p_1)^+, ..., (f_{s_n} - n^{-1} p_s)^+) \xrightarrow{ru} T_\alpha(f_1, ..., f_s).$$

Furthermore, we get that

$$0 \leq (f_{i_n} - n^{-1} p_i)^+ \wedge (f_{j_n} - n^{-1} p_j)^+ \leq f_i \wedge f_j = 0 \ (n \in \mathbb{N})$$

Therefore, we conclude that $(f_{i_n} - n^{-1} p_i)^+ \perp (f_{j_n} - n^{-1} p_j)^+$ for every $n \in \mathbb{N}$. Since $T_{\alpha-1}$ is orthosymmetric, we obtain

$$T_{\alpha-1}((f_{1_n} - n^{-1} p_1)^+, ..., (f_{s_n} - n^{-1} p_s)^+) = 0 \ (n \in \mathbb{N}).$$

Since T_α extends $T_{\alpha-1}$, we have $T_\alpha(f_1, ..., f_s) = 0$. This proves that T_α is orthosymmetric. Next suppose α is a limit ordinal and that for every $\beta < \alpha$ the map T_β is orthosymmetric. Let $f_1, ..., f_s \in (E_\alpha)^+$ be such that $f_i \perp f_j$ for some $i, j \in \{1, ..., s\}$. There exists an ordinal $\beta < \alpha$ such that $f_1, ..., f_s \in (E_\beta)^+$. Thus we have

$$T_\alpha(f_1, ..., f_s) = T_\beta(f_1, ..., f_s) = 0.$$

We conclude that T_α is orthosymmetric.

(2) Suppose F is $\mathcal{D}$-complete for some nonempty $\mathcal{D} \subseteq \bigcup_{m\in\mathbb{N}} \mathcal{H}^+_{|\ |}(\mathbb{R}^m)$, and suppose that a map $T : \times_s E \to F$ is an orthosymmetric vector lattice s-morphism. By Theorem 3.19 in Section 3.3, the map T extends uniquely to a vector lattice s-morphism $T^{\mathcal{D}} : \times_s E^{\mathcal{D}} \to F$. Consider $T^{\mathcal{D}}$ as a map from $\times_s E^{\mathcal{D}}$ to F^u. It was proven in Theorem 3.19 that

$$T^{\mathcal{D}} = T^u|_{\times_s E^{\mathcal{D}}}.$$

Then $T^{\mathcal{D}}$ is orthosymmetric by part (1) of this lemma. □

Part (3) of the following proposition is evident, and the proof of part (2) is similar to the proof of part (1). We only prove part (1).

Proposition 4.27. *Let E and F be Archimedean real vector lattices.*

(1) If F is square mean complete and $T : \times_s E \to F$ is an orthosymmetric vector lattice $s_{\mathbb{R}}$-morphism then $T_{|\mathbb{C}|}$ is an orthosymmetric vector lattice $s_{\mathbb{C}}$-morphism.

(2) If F is uniformly complete and $T : \times_s E \to F$ is a positive, orthosymmetric, $s_{\mathbb{R}}$-linear map then $T_{|\mathbb{C}|}$ is a positive, orthosymmetric, $s_{\mathbb{C}}$-linear map.

(3) If $T : \times_s E_{|\mathbb{C}|} \to F_{|\mathbb{C}|}$ is a real orthosymmetric map then T_ρ is orthosymmetric.

Proof. (1) Suppose that F is square mean complete and that $T : \times_s E \to F$ is an orthosymmetric vector lattice s-morphism. By Lemma 4.26(2), the map $T^\mu : \times_s E^\mu \to F$ is also an orthosymmetric vector lattice $s_{\mathbb{R}}$-morphism. It follows from Proposition 4.3(1) in Section 4.1 that $T_{|\mathbb{C}|}$ is a vector lattice $s_{\mathbb{C}}$-morphism. Moreover, we have

$$T_{|\mathbb{C}|}(f_0^1 + if_1^1, ..., f_0^s + if_1^s) = \sum_{\epsilon_k \in \{0,1\}} T^{\mu}(f_{\epsilon_1}^1, ..., f_{\epsilon_s}^s) i^{\sum_{k=1}^{s} \epsilon_k}$$

for every $f_0^k + if_1^k \in (E_k)_{|\mathbb{C}|}$ $(k \in \{1, ..., s\})$. Suppose that $(f_0^j + if_1^j) \perp (f_0^k + if_1^k)$ for some $j, k \in \{1, ..., s\}$. Then we have $f_p^j \perp f_q^k$ for every $p, q \in \{0, 1\}$ ([46], page 192). Therefore, we get

$$T^{\mu}(f_{\epsilon_1}^1, ..., f_{\epsilon_s}^s) = 0 \ (\epsilon_k \in \{0, 1\}, k \in \{1, ..., s\})$$

since T^{μ} is orthosymmetric. Then $T_{|\mathbb{C}|}(f_0^1 + if_1^1, ..., f_0^s + if_1^s) = 0$, and thus T is an orthosymmetric $s_{\mathbb{C}}$-morphism. □

We are ready to prove the final result of this section.

Theorem 4.28. *If E is an Archimedean complex vector lattice, $s \in \mathbb{N} \setminus \{1\}$, and $((E_\rho)^{ⓢ}, ⓢ)$ is the s-power of E_ρ then $(((E_\rho)^{ⓢ})_{|\mathbb{C}|}, ⓢ_{|\mathbb{C}|})$ is the s-power of E.*

Proof. Let E be an Archimedean complex vector lattice. Suppose that $s \in \mathbb{N} \setminus \{1\}$ and that $((E_\rho)^{ⓢ}, ⓢ)$ is the s-power of E_ρ. Let F be an Archimedean complex vector lattice, and suppose that $T : \times_s E \to F$ is an orthosymmetric vector lattice $s_{\mathbb{C}}$-morphism. From Theorem 3.2 in [8], the map ⓢ induces a unique vector lattice $\mathbb{R}$-homomorphism $T_\rho^{ⓢ} : (E_\rho)^{ⓢ} \to F$ such that $T_\rho^{ⓢ} \circ ⓢ = T_\rho$. The map $T_\rho^{ⓢ}$ uniquely extends to a vector lattice $\mathbb{R}$-homomorphism $(T_\rho^{ⓢ})^{\mu}$ on $(E_\rho^{ⓢ})^{\mu}$ (Corollary 4.1). Moreover, $ⓢ_{|\mathbb{C}|}$ is an orthosymmetric vector lattice $s_{\mathbb{C}}$-morphism (Proposition 4.27(1)). Define $T^{ⓢ_{|\mathbb{C}|}} := (T_\rho^{ⓢ})_{|\mathbb{C}|}$. Then $T^{ⓢ_{|\mathbb{C}|}}$ is a vector lattice homomorphism by Proposition 4.3(1). That $T^{ⓢ_{|\mathbb{C}|}} \circ ⓢ_{|\mathbb{C}|} = T$ follows from $(T_\rho)^{ⓢ} \circ ⓢ = T_\rho$. Since every vector lattice $\mathbb{C}$-homomorphism is real, the uniqueness of $T^{ⓢ_{|\mathbb{C}|}}$ follows from the fact that $(T_\rho^{ⓢ})^{\mu}$ is the unique vector lattice $\mathbb{R}$-homomorphism from $(E_\rho^{ⓢ})^{\mu}$ to F such that $(T_\rho^{ⓢ})^{\mu} \circ ⓢ = T$. □

4.5 Complex Maps of Order Bounded Variation

The Archimedean complex vector lattice tensor product is be used to obtain results for $n_{\mathbb{C}}$-linear maps of ordered bounded variation in this section.

Let E be an Archimedean complex vector lattice and suppose that $a \in E^+$. A *partition* of a is a finite sequence $\{x_k\}_{k=1}^m$ in E^+ such that $\sum_{k=1}^{m} x_k = a$. As in Section 2 of [14], we denote the set of all partitions of a by $\prod a$ and abbreviate a partition $\{x_k\}_{k=1}^m$ of a by x. This explains the shorthand $x \in \prod a$ that will appear throughout this section.

Let $E_1, ..., E_n, F$ be Archimedean complex vector lattices. We say that an $n_{\mathbb{C}}$-linear map $T : \times_{k=1}^n E_k \to F$ is of *order bounded variation* if for all $a_k \in E_k^+$ $(k \in \{1, ..., n\})$ the set

$$\{\sum_{m_1,...,m_n} |T(x^1_{m_1}, .., x^n_{m_n})| : x^k \in \prod a_k (k \in \{1, ..., n\})\}$$

is order bounded. We denote by $\mathcal{L}_{bv}(E_1, ..., E_n; F)$ the space of all $n_{\mathbb{C}}$-linear maps of order bounded variation from $\times_{k=1}^n E_k$ into F. We note that spaces of multilinear maps of order bounded variation between Archimedean real vector lattices were introduced by Buskes and van Rooij in Section 2 of [14].

Let V be a vector space over $\mathbb{K}$. We call $K \subseteq V$ a *cone* in V if

(1) $K + K \subseteq K$,

(2) $\lambda K \subseteq K$ for every $\lambda \in \mathbb{K}^+$, and

(3) $K \cap (-K) = \{0\}$.

A pair (V, K) is called a *complex ordered vector space* if V is a vector space over $\mathbb{C}$ and K is a cone in V.

If $E_1, ..., E_n, F$ are Archimedean complex vector lattices then $\mathcal{L}_{bv}(E_1, ..., E_n; F)$ is a complex ordered vector space with the set of all positive maps in $\mathcal{L}_{bv}(E_1, ..., E_n; F)$, which we denote by $\mathcal{L}_{bv}^{+}(E_1, ..., E_n; F)$, as a cone.

Let $(V_1, K_1), ..., (V_n, K_n), (W, K)$ be complex ordered vector spaces. We say that a map $T : \times_{k=1}^{n} V_k \to W$ is *positive* if $T(\times_{k=1}^{n} K_k) \subseteq K$. A positive $\mathbb{C}$-linear map T is called an *ordered vector space isomorphism* if T bijective and has a positive inverse. We say that complex ordered vector spaces (V, K) and (W, K') are *isomorphic as ordered vector spaces* if there exists an ordered vector space isomorphism between them.

For the proof of the following lemma, let E be an Archimedean complex vector lattice, let (V, K) be a complex ordered vector space, and suppose that $\phi : E \to V$ is an ordered vector space isomorphism with respect to the cones E^+ and K. It is readily checked that the map $m(v) = \phi(|\phi^{-1}(v)|)$ $(v \in V)$ is an Archimedean modulus on V with $m(V) = K$.

Lemma 4.29. *Let E be an Archimedean complex vector lattice, and let (V, K) be a complex ordered vector space. If $\phi : E \to V$ is an ordered vector space isomorphism with respect to the cones E^+ and K then*

(1) V is an Archimedean complex vector lattice with K as positive cone, and

(2) ϕ is a vector lattice isomorphism.

Let E and F be Archimedean vector lattices over $\mathbb{K}$ such that F is Dedekind complete. We denote by $\mathcal{L}_b(E, F)$ the space of all order bounded linear maps from E into F. If $\mathbb{K} = \mathbb{R}$ then $\mathcal{L}_b(E, F)$ is a Dedekind complete real vector lattice by Theorem 1.18 of [1]. For $\mathbb{K} = \mathbb{C}$ it is proven on pages 201–202 of [46] that $\mathcal{L}_b(E, F)$ and $\mathcal{L}_b(E_\rho, F_\rho)_{\mathbb{C}}$ are isomorphic as complex vector lattices if E_ρ is uniformly complete. The proof on pages 201–202 of [46] holds under the mere assumption that E_ρ is square mean complete. Hence, $\mathcal{L}_b(E, F)$ is a Dedekind complete complex vector lattice.

The following result is a complex analogue of Theorem 3.1 in [14] and extends Proposition 3.2(4) in [19].

Theorem 4.30. *Let $E_1, ..., E_n, F$ be Archimedean complex vector lattices such that F is Dedekind complete.*

(1) For any $n_{\mathbb{C}}$-linear map of order bounded variation $T : \times_{k=1}^n E_k \to F$ there exists a unique order bounded $\mathbb{C}$-linear map $T^{\bar{\otimes}} : \bar{\otimes}_{k=1}^n E_k \to F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.

(2) $\mathcal{L}_{bv}(E_1, ..., E_n; F)$ is a Dedekind complete Archimedean complex vector lattice and the correspondence $T \mapsto T^{\bar{\otimes}}$ is a vector lattice isomorphism from $\mathcal{L}_{bv}(E_1, ..., E_n; F)$ onto $\mathcal{L}_b(\bar{\otimes}_{k=1}^n E_k, F)$.

(3) For $T \in \mathcal{L}_{bv}(E_1, ..., E_n; F)$ and for every $a_k \in E_k^+$ $(k \in \{1, ..., n\})$ we have

$$|T|(a_1, ..., a_n) = \sup\{\sum_{m_1,...,m_n} |T(x^1_{m_1}, ..., x^n_{m_n})| : x^k \in \prod a_k \ (k \in \{1, ..., n)\}.$$

Proof. (1) For the uniqueness, suppose that $T : \times_{k=1}^n E_k \to F$ is an $n_{\mathbb{C}}$-linear map of order bounded variation, and assume that S_1, S_2 are complex order bounded linear maps from $\bar{\otimes}_{k=1}^n E_k$ to F such that $T = S_1 \circ \bar{\otimes}$ and $T = S_2 \circ \bar{\otimes}$. Then $S_1 - S_2 = 0$ identically on $\otimes_{k=1}^n E_k$. By relatively uniform density, $S_1 - S_2 = 0$ on $(\otimes_{k=1}^n E_k)_2$, where $(\otimes_{k=1}^n E_k)_2$ denotes the pseudo uniform closure of $\otimes_{k=1}^n E_k$ in $\bar{\otimes}_{k=1}^n E_k$ (see Section 3.1). By relatively uniform density again, $S_1 - S_2 = 0$ on $(\otimes_{k=1}^n E_k)_3$, where $(\otimes_{k=1}^n E_k)_3$ denotes the pseudo uniform closure of $(\otimes_{k=1}^n E_k)_2$ in $\bar{\otimes}_{k=1}^n E_k$. From Theorem 4.10(3) we have $(\otimes_{k=1}^n E_k)_3 = \bar{\otimes}_{k=1}^n E_k$. We next turn to the existence. To this end, define

$$\bar{T}_+(a_1, ..., a_n) := \sup\{\sum_{m_1,...,m_n} |T(x^1_{m_1}, ..., x^n_{m_n})| : x^k \in \prod a_k(k \in \{1, ..., n\})\}$$

for every $a_k \in E_k^+$ $(k \in \{1, ..., n\})$. Like in the proof of Theorem 3.1 of [14], one infers that $\bar{T}_+$ is additive and positively homogeneous in each variable separately. By routine reasoning,

T_+ uniquely extends to a positive $n_{\mathbb{R}}$-linear map $\bar{T} : \times_{k=1}^n E_{k\rho} \to F_\rho$, and subsequently to a positive $n_{\mathbb{C}}$-linear map $\bar{T}_{\mathbb{C}} : \times_{k=1}^n E_k \to F$. Then $\bar{T}_{\mathbb{C}} - T$ is also a positive $n_{\mathbb{C}}$-linear map. By Theorem 4.16, there exists unique positive linear maps $\bar{T}_{\mathbb{C}}^{\bar{\otimes}}$ and $(\bar{T}_{\mathbb{C}} - T)^{\bar{\otimes}}$ from $\bar{\otimes}_{k=1}^n E_k$ into F with $\bar{T}_{\mathbb{C}} = \bar{T}_{\mathbb{C}}^{\bar{\otimes}} \circ \bar{\otimes}$ and $(\bar{T}_{\mathbb{C}} - T) = (\bar{T}_{\mathbb{C}} - T)^{\bar{\otimes}} \circ \bar{\otimes}$. Then for $T^{\bar{\otimes}} := \bar{T}_{\mathbb{C}}^{\bar{\otimes}} - (\bar{T}_{\mathbb{C}} - T)^{\bar{\otimes}}$, we have $T^{\bar{\otimes}} \circ \bar{\otimes} = T$.

(2) The map $\Phi : \mathcal{L}_{bv}(E_1, ..., E_n; F) \to \mathcal{L}_b(\bar{\otimes}_{k=1}^n E_k, F)$ is evidently a $\mathbb{C}$-linear map. Suppose that $T : \times_{k=1}^n E_k \to F$ is a positive n-linear map, and let $\sum_{j=1}^m f_j^1 \otimes \cdots \otimes f_j^n \in \otimes_{k=1}^n E_k$ be such that $f_j^k \in E_k^+$ for every $j \in \{1, ..., m\}$ $(k \in \{1, ..., n\})$. Then we have that

$$T^{\bar{\otimes}}(\sum_{j=1}^m f_j^1 \otimes \cdots \otimes f_j^n) = \sum_{j=1}^m T(f_j^1, ..., f_j^n) \in F^+.$$

By relatively uniform density, $T^{\bar{\otimes}}$ is positive on $\bar{\otimes}_{k=1}^n E_k$. Therefore, Φ is positive. It follows from Theorem 4.16 that for every positive linear map $S : \bar{\otimes}_{k=1}^n E_k \to F$, there exists a unique positive map $T \in \mathcal{L}_{bv}(E_1, ..., E_n; F)$ such that $T^{\bar{\otimes}} = S$. Therefore, Φ and Φ^{-1} are ordered vector space isomorphisms with respect to the cones $\mathcal{L}_{bv}^+(E_1, ..., E_n; F)$ and $\big(\mathcal{L}_b(\bar{\otimes}_{k=1}^n E_k, F)\big)^+$. It follows from Lemma 4.29 that $\mathcal{L}_{bv}(E_1, ..., E_n; F)$ is an Archimedean complex vector lattice with $\mathcal{L}_{bv}^+(E_1, ..., E_n; F)$ as positive cone. The map Φ is a vector lattice isomorphism by Lemma 4.29, and thus $\mathcal{L}_{bv}(E_1, ..., E_n; F)$ is Dedekind complete.

(3) Let $a_k \in E_k^+$ $(k \in \{1, ..., n\})$, let $\theta \in [0, 2\pi]$, and let $T \in \mathcal{L}_{bv}(E_1, ..., E_n; F)$. Let $\bar{T}$ be as in part (1) of this proof. We have that $\bar{T}(a_1, ..., a_n) \geq \big(Re(e^{-i\theta} T)\big)(a_1, ..., a_n)$ and thus $\bar{T} \geq \sup\{Re(e^{-i\theta} T) : \theta \in [0, 2\pi]\} = |T|$. On the other hand, if $x^k \in \prod a_k$ for every $k \in \{1, ..., n\}$ then we get

$$\sum_{m_1, ..., m_n} |T(x_{m_1}^1, ..., x_{m_n}^n)| \leq \sum_{m_1, ..., m_n} |T|(x_{m_1}^1, ..., x_{m_n}^n) = |T|(a_1, ..., a_n).$$

Therefore, we have $\bar{T} \leq |T|$. □

4.6 The Complex Banach Lattice Tensor Product

We review the complexification of real Banach lattices in this section, and we construct a tensor product for complex Banach lattices (Theorem 4.32).

For a normed vector space E over $\mathbb{R}$ that is also a real vector lattice, we call the norm $|| \ ||$ on E a *vector lattice norm* if $||f|| \leq ||g||$ whenever $|f| \leq |g|$ $(f, g \in E)$. A real vector lattice E is called a *real normed vector lattice* if E is equipped with a vector lattice norm. If a real normed vector lattice E is a Banach space with respect to its vector lattice norm, we call E a *real Banach lattice.*

Let E be a normed real vector lattice with norm $|| \ ||$. It is evident that $||f|| = || \ |f| \ ||$ for every $f \in E$. Moreover, if E is a Banach lattice then E is Archimedean and uniformly complete ([47], page 85 and Theorem 15.3). In particular, if E is a Banach lattice then E is square mean complete. Thus $E_{\mathbb{C}}$ is an Archimedean complex vector lattice. The *vector lattice complexification norm* $|| \ ||_{\mathbb{C}}$ on $E_{\mathbb{C}}$ is defined by

$$||f||_{\mathbb{C}} := || \ |f| \ || \ (f \in E_{\mathbb{C}}).$$

A complex vector lattice E is called a *complex Banach lattice* (under the vector lattice complexification norm) if E_ρ is a real Banach lattice (see [47], Exercise 15.11).

We next review the real Banach lattice tensor product, which was introduced for $n = 2$ by Fremlin in Section 1 of [21].

Let $E_1, ..., E_n$ be real Banach lattices. We call a pair $(\tilde{\otimes}_{k=1}^n E_k, \tilde{\otimes})$ a *real Banach lattice tensor product* of $E_1, ..., E_n$ if the following hold.

(1) $\tilde{\otimes}_{k=1}^n E_k$ is a Banach lattice.

(2) $\tilde{\otimes}$ is a vector lattice n-morphism.

(3) For every Banach lattice F and every positive n-linear map $T : \times_{k=1}^n E_k \to F$ there exists a unique positive linear map $T^{\tilde{\otimes}} : \tilde{\otimes}_{k=1}^n E \to F$ such that $T^{\tilde{\otimes}} \circ \tilde{\otimes} = T$ and $||T^{\tilde{\otimes}}|| = ||T||$.

If $E_1, ..., E_n$ are real Banach lattices then there exists a unique real Banach lattice tensor product $(\tilde{\otimes}_{k=1}^n E_k, \tilde{\otimes})$ of $E_1, ..., E_n$ up to vector lattice isomorphism ([21], 1E and [38], Section 2). If F is a real Banach lattice then a map $T : \times_{k=1}^n E_k \to F$ is a vector lattice n-morphism if and only if $T^{\tilde{\otimes}} : \tilde{\otimes}_{k=1}^n E \to F$ is a vector lattice homomorphism ([21], 1E(iv)).

Most of the following outline of the construction of the real Banach lattice tensor product was given by Schep in Section 2 of [38]. Given real Banach lattices $E_1, ..., E_n$, one defines the *positive projective norm* on $\bar{\otimes}_{k=1}^n E_k$ by

$$||u||_{|\pi|} := \inf\{\sum_{j=1}^m \prod_{k=1}^n ||x_j^k|| : x_j^k \in E_k^+, |u| \leq \sum_{j=1}^m (x_j^1 \otimes \cdots \otimes x_j^n)\} \ (u \in \bar{\otimes}_{k=1}^n E_k).$$

The positive projective norm is a vector lattice norm, and the norm completion $\tilde{\otimes}_{k=1}^n E_k$ of $\bar{\otimes}_{k=1}^n E_k$ with respect to $|| \ ||_{|\pi|}$ is a Banach lattice. We again use $|| \ ||_{|\pi|}$ to denote the completed vector lattice norm on $\tilde{\otimes}_{k=1}^n E_k$. Following Fremlin's proof on page 91 and page 92 of [21], one proves that

$$||u||_{|\pi|} = \inf\{\sum_{j=1}^\infty \prod_{k=1}^n ||x_j^k|| : x_j^k \in E_k^+, |u| \leq \sum_{j=1}^\infty (x_j^1 \otimes \cdots \otimes x_j^n)\} \ (u \in \tilde{\otimes}_{k=1}^n E_k).$$

Finally, setting $\tilde{\otimes} := \bar{\otimes}$ gives us the real Banach lattice tensor product.

Before we prove the existence of the analogous complex Banach lattice tensor product, we need the following complex analogue of Proposition 1.3(i) in [7].

Lemma 4.31. *Let $E_1, ..., E_n, F$ be Archimedean complex vector lattices, and suppose that $T : \times_{k=1}^{n} E_k \to F$ is a real $n_{\mathbb{C}}$-linear map. The map T is positive if and only if*

$$|T(f_1, ..., f_n)| \leq T(|f_1|, ..., |f_n|) \ (f_k \in E_k, k \in \{1, ..., n\}).$$

Proof. We first suppose that T is positive and that $n = 1$. Let $f + ig \in E_1$, and let $\theta_0 \in [0, 2\pi]$. We get that

$$\begin{aligned} T(|f + ig|) &= T_\rho(\sup\{f \cos\theta + g \sin\theta : \theta \in [0, 2\pi]\}) \\ &\geq T_\rho(f \cos\theta_0 + g \sin\theta_0) \\ &= T_\rho(f) \cos\theta_0 + T_\rho(g) \sin\theta_0. \end{aligned}$$

Thus we have

$$T(|f + ig|) \geq \sup\{T_\rho(f) \cos\theta + T(g) \sin\theta : \theta \in [0, 2\pi]\} = |T(f + ig)|.$$

Next we suppose that $n \in \mathbb{N} \setminus \{1\}$. Let $f_k \in E_k$ $(k \in \{1, ...n\})$, and consider T to be a map from $\times_{k=1}^{n} E_k$ to F^u. By Theorem 4.16 there exists a unique positive linear map $T^{\bar{\otimes}} : \bar{\otimes} E_k \to F$ such that $T^{\bar{\otimes}} \circ \bar{\otimes} = T$. It follows that

$$|T(f_1, ..., f_n)| = |T^{\bar{\otimes}}(f_1 \otimes \cdots \otimes f_n)| \leq T^{\bar{\otimes}}(|f_1 \otimes \cdots \otimes f_n|).$$

By Proposition 3.1(2) in [7], we have that

$$|\bar{\otimes}_\rho(g_1, ..., g_n)| = \bar{\otimes}_\rho(|g_1|, ..., |g_n|) \ (g_k \in E_{k\rho}, k \in \{1, ..., n\}).$$

As a consequence, we have from Proposition 6 in [45] that

$$|g_1 \otimes \cdots \otimes g_n| = |g_1| \otimes \cdots \otimes |g_n| \ (g_k \in E_k, k \in \{1, ..., n\}).$$

Hence we have

$$T^{\bar{\otimes}}(|f_1 \otimes \cdots \otimes f_n|) = T^{\bar{\otimes}}(|f_1| \otimes \cdots \otimes |f_n|) = T(|f_1|, ..., |f_n|).$$

On the other hand, if $|T(f_1, ..., f_n)| \leq T(|f_1|, ..., |f_n|)$ for every $f_k \in E_k$ $(k \in \{1, ..., n\})$ then $|T_\rho(f_1, ..., f_n)| \leq T_\rho(|f_1|, ..., |f_n|)$ for every $f_k \in E_{k\rho}$ $(k \in \{1, ..., n\})$. Then T_ρ is positive by Proposition 1.3 in [7]. By Proposition 4.3(2), the map T is positive as well. □

The definition of a complex Banach lattice tensor product is analogous to the definition of the real Banach lattice tensor product. We next prove its existence. The proof for the uniqueness of the complex Banach lattice tensor product is routine.

Theorem 4.32. *Let $E_1, ..., E_n$ be complex Banach lattices, and let $(\tilde{\otimes}_{k=1}^n E_{k\rho}, \tilde{\otimes})$ denote the real Banach lattice tensor product of $E_{1\rho}, ..., E_{n\rho}$. Endow $(\tilde{\otimes}_{k=1}^n E_{k\rho})_\mathbb{C}$ with the Banach lattice complexification norm $||u||_{|\pi|\mathbb{C}} := ||\ |u|\ ||_{|\pi|}$ $(u \in (\tilde{\otimes}_{k=1}^n E_{k\rho})_\mathbb{C})$. The pair $((\tilde{\otimes}_{k=1}^n E_{k\rho})_\mathbb{C}, \tilde{\otimes}_\mathbb{C})$ is the unique complex Banach lattice tensor product of $E_1, ..., E_n$, unique up to vector lattice isomorphism.*

Proof. Let F be a complex Banach lattice, and let $T : \times_{k=1}^n E_k \to F$ be a positive $n_\mathbb{C}$-linear map. Then $T_\rho : \times_{k=1}^n E_{k\rho} \to F_\rho$ is a positive $n_\mathbb{R}$-linear map. Thus, there exists a unique positive $\mathbb{R}$-linear map $T_\rho^{\tilde{\otimes}} : \tilde{\otimes}_{k=1}^n E_{k\rho} \to F_\rho$ such that $T_\rho^{\tilde{\otimes}} \circ \tilde{\otimes} = T_\rho$. Define $T^{\tilde{\otimes}_\mathbb{C}} := (T_\rho^{\tilde{\otimes}})_\mathbb{C}$. Then $T^{\tilde{\otimes}_\mathbb{C}}$ is a positive $\mathbb{C}$-linear map (Proposition 4.3(2)). It follows from $T^{\tilde{\otimes}} \circ \tilde{\otimes} = T$ that $T^{\tilde{\otimes}_\mathbb{C}} \circ \tilde{\otimes}_\mathbb{C} = T$. It only remains to prove that $||T^{\tilde{\otimes}_\mathbb{C}}|| = ||T||$. To this end, let $||\ ||_{E_k}$, respectively $||\ ||_{E_{k\rho}}$ denote the vector lattice norm on E_k, respectively $E_{k\rho}$ $(k \in \{1, ..., n\})$. Also let $||\ ||_F$, respectively $||\ ||_{F_\rho}$, denote the vector lattice norm on F, respectively F_ρ. We have that

$$\begin{aligned}
||T|| &= \sup\{||T(f_1, ..., f_n)||_F : ||f_k||_{E_k} \leq 1 \ (k \in \{1, ..., n\})\} \\
&= \sup\{|| \ |T(f_1, ..., f_n)| \ ||_{F_\rho} : || \ |f_k| \ ||_{E_{k\rho}} \leq 1 \ (k \in \{1, ..., n\})\} \\
&\leq \sup\{||T_\rho(|f_1|, ..., |f_n|)||_{F\rho} : || \ |f_k| \ ||_{E_{k\rho}} \leq 1 \ (k \in \{1, ..., n\})\} \\
&\leq \sup\{||T_\rho(f_1, ..., f_n)||_{F\rho} : ||f_k||_{E_{k\rho}} \leq 1 \ (k \in \{1, ..., n\})\} \\
&= ||T_\rho||,
\end{aligned}$$

where the first inequality follows from Lemma 4.31. On the other hand, we have

$$\begin{aligned}
||T_\rho|| &= \sup\{||T_\rho(f_1, ..., f_n)||_{F_\rho} : ||f_k||_{E_{k\rho}} \leq 1 \ (k \in \{1, ..., n\})\} \\
&= \sup\{|| \ |T(f_1, ..., f_n)| \ ||_F : || \ |f_k| \ ||_{E_k} \leq 1 \ (k \in \{1, ..., n\})\} \\
&\leq \sup\{||T(|f_1|, ..., |f_n|)||_F : || \ |f_k| \ ||_{E_k} \leq 1 \ (k \in \{1, ..., n\})\} \\
&\leq \sup\{||T(f_1, ..., f_n)||_F : ||f_k||_{E_k} \leq 1 \ (k \in \{1, ..., n\})\} \\
&= ||T||,
\end{aligned}$$

where again the first inequality follows from Lemma 4.31. Thus $||T|| = ||T_\rho||$, and similarly $||T^{\tilde{\otimes}_{\mathbb{C}}}|| = ||T_\rho^{\tilde{\otimes}}||$. The desired result now follows from the fact that $||T_\rho|| = ||T^{\tilde{\otimes}}||$ ([38], Section 2(d)). □

We conclude this chapter with a few facts regarding the complex Banach lattice tensor product. The following corollary follows from its real analogue preceding Lemma 4.31.

Corollary 4.33. *Let $E_1, ..., E_n$ be complex Banach lattices, and let $(\tilde{\otimes}_{k=1}^n E_k, \tilde{\otimes})$ be the Banach lattice tensor product of $E_1, ..., E_n$. The following hold.*

(1) For every $u \in \tilde{\otimes}_{k=1}^{n} E_k$*, we have*

$$||u||_{|\pi|\mathbb{C}} = \inf\{\sum_{j=1}^{\infty}\prod_{k=1}^{n}||x_j^k|| : x_j^k \in E_k^+, |u| \leq \sum_{j=1}^{\infty}(x_j^1 \otimes \cdots \otimes x_j^n)\}.$$

(2) Suppose that F *is a Banach lattice and that* $T : \times_{k=1}^{n} E_k \to F$ *is a positive* $n_{\mathbb{C}}$*-linear map. We have that* $T^{\tilde{\otimes}} : \tilde{\otimes}_{k=1}^{n} E_k \to F$ *is a vector lattice* $\mathbb{C}$*-homomorphism if and only if* T *is a vector lattice* $n_{\mathbb{C}}$*-morphism.*

5 THE COMPLEX CAUCHY-SCHWARZ INEQUALITY

We prove the Cauchy-Schwarz Inequality in four settings in this section (Theorems 5.2 and 5.4, Corollaries 5.3 and 5.5). These settings include Archimedean complex almost f-algebras, uniformly complete semiprime Archimedean complex f-algebras, and complexifications of geometric mean complete Archimedean real vector lattices.

An Archimedean vector lattice A over $\mathbb{K}$ is called an Archimedean *f-algebra over* $\mathbb{K}$ if the following hold.

(1) A is equipped with an associative multiplication (denoted by $\cdot$ throughout).

(2) A is a ring under its vector space addition and associative multiplication.

(3) $f \cdot g \in A^+$ for every $f, g \in A^+$.

(4) If $a, b \in A$ and $a \perp b = 0$ then $c \cdot |a| \wedge |b| = |a| \cdot c \wedge |b| = 0$ for every $c \in A^+$.

If an Archimedean vector lattice A over $\mathbb{K}$ satisfies (1), (2), and (3) above and the map $(f, g) \mapsto f \cdot g$ is orthosymmetric, we call A an Archimedean *almost f-algebra over* $\mathbb{K}$. An Archimedean (almost) f-algebra over $\mathbb{R}$ (over $\mathbb{C}$) will also be called a *real (complex) Archimedean (almost) f-algebra*.

We need the following lemma for Theorem 5.2. We remind the reader that given a vector space V over $\mathbb{C}$ and a nonempty subset A of V, we denote by $[A]$ the vector subspace of V generated by A.

Lemma 5.1. *Let X be a compact Hausdorff space, and let E be a relatively uniformly dense complex vector sublattice of $C(X)_{\mathbb{C}}$. Suppose that F is an Archimedean complex vector lattice* *and that $T : \times_n E \to F$ be a positive n-linear orthosymmetric map. Set*

$$E^n := [\{f_1 \cdots f_n : f_k \in E_k \ (k \in \{1, ..., n\})\}].$$

There exists a positive linear map $B : E^n \to F$ such that

$$T(f_1, ..., f_n) = B(f_1 \dots f_n) \ (f_1, ..., f_n \in C(X)_{\mathbb{C}}).$$

Proof. Consider T as a map from $\times_n E$ to F^u. We have that E_ρ is a relatively uniformly dense vector sublattice of $C(X)$. Then $(E_\rho)^u = C(X)$, and thus T_ρ uniquely extends to a positive orthosymmetric n-linear map on $C(X)$ (Lemma 4.26(1)). Hence we can assume that $E = C(X)_{\mathbb{C}}$. Then $(E^n)_\rho$ and $(E_\rho)^n$ are isomorphic as complex vector lattices. If $n = 2$ then there exists a positive linear map $B : (E_\rho)^2 \to F_\rho$ such that $T_\rho(f_1, f_2) = B(f_1 f_2)$ $(f_1, f_2 \in C(X))$ (see [13], Theorem 1). A simple induction argument shows that if $n \in \mathbb{N} \setminus \{1\}$ then there exists a positive linear map $B : (E_\rho)^n \to F_\rho$ such that

$$T_\rho(f_1, ..., f_n) = B(f_1 \dots f_n) \ (f_1, ..., f_n \in C(X)).$$

Then for $f_0^1 + if_1^1, ..., f_0^n + if_1^n \in C_{\mathbb{C}}(X)$ we have

$$\begin{aligned} T(f_0^1 + if_1^1, ..., f_0^n + if_1^n) &= \sum_{\epsilon_k \in \{0,1\}} T_\rho(f_{\epsilon_1}^1, ..., f_{\epsilon_n}^n) i^{\sum_{k=1}^n \epsilon_k} \\ &= \sum_{\epsilon_k \in \{0,1\}} B(f_{\epsilon_1}^1 \dots f_{\epsilon_n}^n) i^{\sum_{k=1}^n \epsilon_k} \\ &= B_{\mathbb{C}}\Big(\sum_{\epsilon_k \in \{0,1\}} f_{\epsilon_1}^1 \dots f_{\epsilon_n}^n i^{\sum_{k=1}^n \epsilon_k} \Big) \\ &= B_{\mathbb{C}}((f_0^1 + if_1^1) \dots (f_0^n + if_1^n)). \end{aligned}$$

Thus $B_{\mathbb{C}}$ is the positive linear map that we seek. □

Let V be a real vector space, and let A be an Archimedean real almost f-algebra. Suppose that $T : V \times V \to A$ is a bilinear map such that

(1) $T(v, v) \geq 0$ for every $v \in V$, and

(2) $T(u, v) = T(v, u)$ for every $u, v \in V$.

From Corollary 4 in [13] we have $T(u, v) \cdot T(u, v) \leq T(u, u) \cdot T(v, v)$ $(u, v \in V)$. We give a complex analogue of this result for sesquilinear maps.

Let U, V, W be vector spaces over $\mathbb{C}$. We call a map $T : U \times V \to W$ *sesquilinear* if

(1) $T(\alpha u + \beta u', v) = \alpha T(u, v) + \beta T(u', v)$ $(u, u' \in U, v \in V, \alpha, \beta \in \mathbb{C})$, and

(2) $T(u, \alpha v + \beta v') = \bar{\alpha} T(u, v) + \bar{\beta} T(u, v')$ $(u \in U, v, v' \in V, \alpha, \beta \in \mathbb{C})$.

We use the following definitions in the proof of our next theorem.

Let E be an Archimedean vector lattice over $\mathbb{K}$. We call $u \in E^+$ an *order unit* in E if for every $f \in E$ there exists $n \in \mathbb{N}$ such that $|f| \leq nu$. For $\mathbb{K} = \mathbb{C}$ and $h = f + ig \in E$, we write $\mathrm{Re}(f) := f$ and $\mathrm{Im}(h) := g$.

The proof of Theorem 5.2 combines techniques found in the proof of Corollary 4 of [13] with techniques from the proof of Theorem 1.4 in [16].

Theorem 5.2. *Let V be a vector space over $\mathbb{C}$, and let A be an Archimedean complex almost f-algebra. Suppose that $T : V \times V \to A$ be a sesquilinear map such that*

(1) $T(v, v) \geq 0$ for every $v \in V$, and

(2) $T(u, v) = \overline{T(v, u)}$ for every $u, v \in V$.

Then we have $|T(u, v)| \cdot |T(u, v)| \leq T(u, u) \cdot T(v, v)$ for every $u, v \in V$.

Proof. Let $u, v \in V$. For all $\alpha \in \mathbb{C}$ we have

$$0 \leq T(u - \alpha v, u - \alpha v) = T(u, u) - \alpha T(v, u) - \bar{\alpha} T(u, v) + |\alpha|^2 T(v, v).$$

Set $f := T(u, u), g := T(v, u)$, and $h := T(v, v)$. Then the inequality above becomes

$$0 \leq f - \alpha g - \bar{\alpha}\bar{g} + |\alpha|^2 h \ (\alpha \in \mathbb{C}).$$

Let I be the ideal of A_ρ generated by $\{\mathrm{Re}(f), \mathrm{Im}(f), \mathrm{Re}(g), \mathrm{Im}(g), \mathrm{Re}(h), \mathrm{Im}(h)\}$. Then $|\mathrm{Re}(f)| + |\mathrm{Im}(f)| + |\mathrm{Re}(g)| + |\mathrm{Im}(g)| + |\mathrm{Re}(h)| + |\mathrm{Im}(h)|$ is an order unit in I. Thus there exists a compact Hausdorff space X and a vector lattice isomorphism $f \mapsto \hat{f}$ from I to a relatively uniformly dense vector sublattice $\hat{I}$ of $C(X)$ (see [17], Theorem 13.11). Since A is an Archimedean complex vector lattice, A_ρ is square mean complete (see the introduction in Section 4.1). Then $\mu(a, b) \in A_\rho$ for every $a, b \in I$, where μ denotes the square mean and $\mu(a, b)$ is defined via functional calculus (see Section 3.2). By Corollary 3.9 we have

$$0 \leq \sqrt{2}\mu(a, b) = \sup\{a \cos\theta + b \sin\theta : \theta \in [0, 2\pi]\} = |a + ib| \leq |a| + |b| \in I.$$

Thus I is square mean complete and $\hat{I} \oplus i\hat{I}$ is a relatively uniformly dense complex vector sublattice of $C(X)_{\mathbb{C}}$. Moreover, $\hat{f}, \hat{g}, \hat{h} \in \hat{I} \oplus i\hat{I}$. Let $x \in X$. We have that

$$0 \leq \hat{f}(x) - \alpha\hat{g}(x) - \bar{\alpha}\bar{\hat{g}}(x) + |\alpha|^2 \hat{h}(x) \ (\alpha \in \mathbb{C}).$$

Set $\hat{g}(x) = be^{i\theta}$ $(b \in \mathbb{R}^+, \theta \in \mathbb{R})$ and let $\alpha = te^{-i\theta}$ $(t \in \mathbb{R})$. Then we get

$$\begin{aligned} 0 &\leq \hat{f}(x) - te^{-i\theta}be^{i\theta} - te^{i\theta}be^{-i\theta} + |te^{-i\theta}|^2 \hat{h}(x) \\ &= \hat{f}(x) - 2bt + t^2 \hat{h}(x). \end{aligned}$$

Moreover, setting $c := \hat{f}(x)$ and $a := \hat{h}(x)$ yields

$$0 \leq c - 2bt + at^2.$$

Since t was chosen arbitrarily, the function $q(t) = at^2 - 2bt + c$ $(t \in \mathbb{R})$ satisfies $q(t) \geq 0$ for all $t \in \mathbb{R}$. Thus $q(t) = 0$ for at most one $t \in \mathbb{R}$. Hence $4b^2 - 4ac \leq 0$ and then $|\hat{g}(x)|^2 \leq \hat{f}(x)\hat{h}(x)$. Since x was chosen arbitrarily, we have $|\hat{g}||\hat{g}| \leq \hat{f}\hat{h}$. There exists a positive linear map $B : (\hat{I} \oplus i\hat{I})^2 \to A$ such that $B(\hat{a}\hat{b}) = \hat{a} \cdot \hat{b}$ for every $\hat{a}, \hat{b} \in \hat{I} \oplus i\hat{I}$ (Lemma 5.1). Since B is positive, we have $|B(\hat{g}\hat{g})| \leq B(|\hat{g}\hat{g}|)$ (Lemma 4.31). Hence we have that

$$|\hat{g} \cdot \hat{g}| = |B(\hat{g}\hat{g})| \leq B(|\hat{g}\hat{g}|) \leq B(\hat{f}\hat{h}) = \hat{f} \cdot \hat{h}.$$

Therefore, we obtain $|T(u, v)| \cdot |T(u, v)| \leq T(u, u) \cdot T(v, v)$ $(u, v \in V)$. □

Corollary 5.3 is a special case of the theorem above that involves the geometric mean, studied in Section 3.2. An Archimedean f-algebra A over $\mathbb{K}$ is called *semiprime* if 0 is the only nilpotent element of A. For an Archimedean f-algebra A over $\mathbb{K}$ and $a \in A^+$, we write $r = \sqrt{a}$ if

(1) $r \in A^+$,

(2) $r \cdot r = a$, and

(3) if $s \in A^+$ and $s \cdot s = a$ then $s = r$.

Corollary 5.3. *Let V be a vector space over $\mathbb{C}$, and assume that A is a uniformly complete Archimedean complex semiprime f-algebra. Suppose that $T : V \times V \to A$ is a sesquilinear map such that*

(1) $T(u, u) \geq 0$ for every $u \in V$, and

(2) $T(u, v) = \overline{T(v, u)}$ for every $u, v \in V$.

We have that $|T(u,v)| \leq \sqrt{T(u,u) \cdot T(v,v)}$ *for every* $u, v \in V$.

Proof. Let $u, v \in V$. Since A_ρ is a uniformly complete semiprime Archimedean real f-algebra, there exists a unique element $r \in A^+$ such that $r \cdot r = T(u,u) \cdot T(v,v)$ (see [4], Theorem 4.2). Therefore, $\sqrt{T(u,u) \cdot T(v,v)}$ is well defined. Note that for $a, b \in A^+$ we have $a \leq b$ if and only if $a^2 \leq b^2$ (see [18], Theorem 3.7(ii)). Then taking the square root of both sides of the inequality $|T(u,v)| \cdot |T(u,v)| \leq T(u,u) \cdot T(v,v)$ yields

$$|T(u,v)| \leq \sqrt{T(u,u) \cdot T(v,v)}.$$

□

Under the assumptions of Corollary 5.3 we have

$$\sqrt{T(u,u) \cdot T(v,v)} = \frac{1}{2} \inf\{\theta T(u,u) + \theta^{-1} T(v,v) : \theta \in (0, \infty)\}$$

(see [2], Theorem 2.21). Let $\gamma \in \mathcal{H}(\mathbb{R}^m)$ be the geometric mean (see Example 3.4). By Corollary 3.10 we get

$$\sqrt{T(u,u) \cdot T(v,v)} = \gamma(T(u,u), T(v,v)),$$

where $\gamma(T(u,u), T(v,v))$ is defined via functional calculus (see Section 3.2). We thus have

$$|T(u,v)| \leq \gamma(T(u,u), T(v,v)).$$

The next theorem states that the inequality above holds in a more general setting.

Theorem 5.4. *Let* V *be a vector space over* $\mathbb{C}$, *and suppose that* F *is an Archimedean complex vector lattice such that* F_ρ *is geometric mean complete. If* $T : V \times V \to F$ *is a sesquilinear map such that*

(1) $T(u,u) \geq 0$ *for every* $u \in V$, *and*

(2) $T(u,v) = \overline{T(v,u)}$ *for every* $u, v \in V$

then we have that $|T(u,v)| \leq \gamma(T(u,u), T(v,v))$ *for every* $u, v \in V$.

Proof. Let $u, v \in V$, and let $\theta \in (0,\infty)$. The identity

$$\mathrm{Re}\big(T(u,v)\big) = \frac{1}{2}(\theta T(u,u) + \theta^{-1}T(v,v)) - \frac{1}{2}T(\theta u - v, u - \theta^{-1}v)$$

follows directly from the sesquilinearity of A. Moreover, we have

$$0 \leq T(u-v, u-v) = T(u,u) - 2\mathrm{Re}\big(T(u,v)\big) + T(v,v).$$

We thus get

$$2\mathrm{Re}\big(T(u,v)\big) \leq T(u,u) + T(v,v).$$

Hence for $\theta \in (0,1]$ we have

$$\begin{aligned} T(\theta u - v, u - \theta^{-1}v) &= \theta T(u,u) - 2\mathrm{Re}\big(T(u,v)\big) + \theta^{-1}T(v,v) \\ &\geq \theta T(u,u) - T(u,u) - T(v,v) + \theta^{-1}T(v,v) \\ &\geq T(u,u) - T(u,u) - T(v,v) + \theta^{-1}T(v,v) \\ &= (\theta^{-1} - 1)T(v,v) \geq T(v,v) \geq 0. \end{aligned}$$

On the other hand, for $\theta \in (1, \infty)$ we get

$$\begin{aligned}
T(\theta u - v, u - \theta^{-1} v) &= \theta T(u,u) - 2\mathrm{Re}\big(T(u,v)\big) + \theta^{-1} T(v,v) \\
&\geq \theta T(u,u) - T(u,u) - T(v,v) + \theta^{-1} T(v,v) \\
&\geq \theta T(u,u) - T(u,u) - T(v,v) + T(v,v) \\
&= (\theta - 1) T(u,u) \geq T(u,u) \geq 0.
\end{aligned}$$

Thus $T(\theta u - v, u - \theta^{-1} v) \geq 0$ for every $\theta \in (0, \infty)$. We thus have

$$\mathrm{Re}\big(T(u,v)\big) \leq \frac{1}{2}(\theta T(u,u) + \theta^{-1} T(v,v))$$

for every $\theta \in (0, \infty)$. Therefore, we obtain

$$\mathrm{Re}\big(T(u,v)\big) \leq \frac{1}{2} \inf\{\theta T(u,u) + \theta^{-1} T(v,v) : \theta \in (0, \infty)\}.$$

Let $\lambda \in \mathbb{C}$ with $|\lambda| = 1$. Since the inequality above does not depend on u or v, we can replace u with λu and obtain

$$\begin{aligned}
\mathrm{Re}\big(T(\lambda u, v)\big) &\leq \frac{1}{2} \inf\{\theta T(\lambda u, \lambda u) + \theta^{-1} T(v,v) : \theta \in (0, \infty)\} \\
&= \frac{1}{2} \inf\{\theta \lambda \overline{\lambda} T(u,u) + \theta^{-1} T(v,v) : \theta \in (0, \infty)\} \\
&= \frac{1}{2} \inf\{\theta |\lambda|^2 T(u,u) + \theta^{-1} T(v,v) : \theta \in (0, \infty)\} \\
&= \frac{1}{2} \inf\{\theta T(u,u) + \theta^{-1} T(v,v) : \theta \in (0, \infty)\}.
\end{aligned}$$

It follows from Corollary 3.10 that

$$\gamma(T(u,u), T(v,v)) = \frac{1}{2} \inf\{\theta T(u,u) + \theta^{-1} T(v,v) : \theta \in (0, \infty)\}.$$

Therefore, $\gamma(T(u,u),T(v,v))$ is an upper bound of the set

$$\{\mathrm{Re}\big(\lambda T(u,v)\big) : \lambda \in \mathbb{C}, |\lambda| = 1\}.$$

Hence we have

$$|T(u,v)| = \sup\{\mathrm{Re}\big(\lambda T(u,v)\big) : \lambda \in \mathbb{C}, |\lambda| = 1\} \leq \gamma(T(u,u),T(v,v)).$$

□

By following the proof of Theorem 5.4 above, one can prove a similar result for bilinear maps from a real vector space to a geometric mean complete Archimedean real vector lattice.

Corollary 5.5. *Let V be a vector space over $\mathbb{R}$, and suppose that F is a geometric mean complete Archimedean real vector lattice. If $T : V \times V \to F$ is a bilinear map such that*

(1) $T(u,u) \geq 0$ for every $u \in V$, and

(2) $T(u,v) = T(v,u)$ for every $u,v \in V$

then $|T(u,v)| \leq \gamma(T(u,u),T(v,v))$ for every $u,v \in V$.

6 INDEX OF TERMS

7 INDEX OF SYMBOLS

$(\otimes_{k=1}^n, E_k, \otimes)$	48 (algebraic tensor product)
$\otimes_n E$	62 (n-fold algebraic tensor product)
$\bar{\otimes}_n E$	62 (n-fold Archimedean vector lattice tensor product)
T_σ	63
T_s	63 (symmetrization of a map)
$(\bar{\otimes}_{n,s} E, \bar{\otimes}_s)$	64 (symmetric Archimedean vector lattice tensor product)
$(T_s)_{\lvert\mathbb{C}\rvert}$	67 (complexification of a symmetrized map)
$p(\sigma)$	68
T_a	68 (antisymmetrization of a map)
$(\bar{\otimes}_{n,a} E, \bar{\otimes}_a)$	69 (antisymmetric Archimedean vector lattice tensor product)
$f \perp g$	71 (disjoint elements)
$(E^{\circledS}, \circledS)$	71 (s-power)
$\langle\langle A \rangle\rangle$	71 (ideal generated by A)
$\prod a$	80 (set of partitions of a)
$\mathcal{L}_{bv}(E_1, ..., E_n; F)$	80 (space of multilinear maps of order bounded variation)
$\mathcal{L}_{bv}^+(E_1, ..., E_n; F)$	81 (cone of positive maps of order bouned variation)
$\mathcal{L}_b(E, F)$	81 (space of order bounded maps)
$\|\|f\|\|_{\mathbb{C}}$	84 (complexification of vector lattice norm)
$(\tilde{\otimes}_{k=1}^n E_k, \tilde{\otimes})$	84 (Banach lattice tensor product)
$\|\|u\|\|_{\lvert\pi\rvert}$	85 (positive projective norm)
E^n	91 ($\{a_1 \cdots a_n : a_1, ..., a_n \in E\}$)

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VITA

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The author is grateful for the many constants throughout his life. One of these constants has been the joy that he feels while learning mathematics.