2019

On a Generalization of Lucas Numbers

Skylyn Olyvia Irby

University of Mississippi

ON A GENERALIZATION OF LUCAS NUMBERS

by
Skylyn Irby

A thesis submitted to the faculty of The University of Mississippi in partial fulfillment of the requirements of the Sally McDonnell Barksdale Honors College.

Oxford
May 2019

Approved by

Advisor: Dr. Sandra Spiroff

Reader: Dr. Donald Cole

Reader: Dr. Laura Sheppardson

ABSTRACT

In this paper we consider a generalization of Lucas numbers. Recall that Lucas numbers are the sequence of integers defined by the recurrence relation: $L_n = L_{n-1} + L_{n-2}$ with the initial conditions $L_1 = 1$ and $L_2 = 3$ (or $L_0 = 1$ and $L_1 = 3$ if the first subscript is zero). That is, the classical Lucas number sequence is $1, 3, 4, 7, 11, 18, ...$ The goal of the present paper is to study properties of certain generalizations of the Lucas sequence. In particular, we consider the following generalizations of the sequence: $\ell_n = \begin{cases} a\ell_{n-1} + \ell_{n-2} & \text{if } n \text{ is even;} \\ b\ell_{n-1} + \ell_{n-2} & \text{if } n \text{ is odd,} \end{cases}$
for $n = 3, 4, 5, ...,$ where a and b are any nonzero real numbers, with the initial conditions $\ell_0 = 1$ and $\ell_1 = 3$ (see Section 2.0.1) and $l_n = (-1)^n l_{n-1} + l_{n-2}$ for $n = 3, 4, 5, ...$ with the initial conditions $l_1 = 1$ and $l_2 = 3$ (see Section 3.1.2). More precisely, we will determine the generating function and a Binet-like formula for $\{\ell_n\}_{n=0}^{\infty}$ and demonstrate numerical simulations for $\{l_n\}_{n=1}^{\infty}$, proving some relations using Principle of Mathematical Induction.

ACKNOWLEDGEMENTS

I would like to thank Dr. Donald Cole, Ms. Demetria Hereford, and The Ronald E. McNair Post-Baccalaureate Achievement Program at The University of Mississippi for allowing me to begin this research as a Ronald E. McNair Scholar. Thank you to my professor and mentor, Dr. Maksym Derevyagin, former visiting professor in the University of Mississippi Department of Mathematics and now Assistant Professor in the Department of Mathematics at the University of Connecticut for proposing this project and providing guidance throughout this research. Additionally, I would like to thank the Sally McDonald Barksdale Honors College for allowing me to continue this research. I would also like to thank my research advisor, Dr. Sandra Spiroff, Associate Professor of Mathematics for being willing serve has my Honors Thesis Advisor to help continue this research.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

CHAPTER 1

BACKGROUND AND INTRODUCTION

1.1 Background

This research began, with the study of a particular generalization of the Lucas numbers, denoted $\{l_n\}_{n=1}^{\infty}$, during the summer of 2016 under the direction of Visiting Assistant Professor Dr. Maksym Derevyagin. Throughout the summer, we worked on coming up with some conjectures regarding the generalized Lucas number sequence; these are introduced in this paper. We determined five conjectures with the use of *Mathematica* and *Java*, and given the time constraints, we proved three. This project evolved to the study of a more generalized sequence, denoted $\{\ell_n\}_{n=0}^{\infty}$, under the direction of Dr. Sandra Spiroff after the departure of Dr. Derevyagin, who accepted a position at the University of Connecticut. In the re-imagined project, we look more into the mathematics behind these generalized Lucas number sequences.

1.2 Introduction

One of the most well-known sequences is the Fibonacci sequence. Recall that the classical Fibonacci sequence is 1,1,2,3,5,... (or 0,1,1,2,..., if the initial conditions are $F_0 = 0$ and $F_1 = 1$). It is defined by the recurrence relation $F_n = F_{n-1} + F_{n-2}$, for $n \geq 3$, with the initial conditions $F_1 = 1$ and $F_2 = 1$. Interestingly enough, the amazing Fibonacci numbers occur in quite unexpected places. From [5], one learns that Fibonacci numbers can be found in plants, fruit, music, bumble bees, chemistry, and physics. For example, in music, there are $13 = F_7$ notes, in the span of any note through its octaves. A scale is composed of $8(=F_6)$ notes, of which the $5^{th}(=F_5)$ and $3^{rd}(=F_4)$ notes, which create the basic foundation of all chords. This foundation is a combination of $2(=F_3)$ steps and $1(=F_2)$ step from the root tone, which is the 1st note of the scale[5]. Also, there are some poets who use the Fibonacci number sequence to determine the number of words, syllables, or beats in each line or stanza [5]. To see an occurrence of the Fibonacci sequence in nature, let us observe that the scales on pineapples are nearly hexagonal in shape. Since hexagons tessellate a plane perfectly, the scales form three different spiral patterns. The spirals evoke the Fibonacci numbers 8, 13, 21. This example and many more can be found in [5]. The research of the topics related to Fibonacci numbers has been active for centuries and it still produces beautiful results.

The Lucas number sequence is obtained from the Fibonacci recurrence relation when we take initial conditions $L_1 = 1, L_2 = 3$ and set $L_n = L_{n-1} + L_{n-2}$ for $n = 3, 4, 5, ...$ The resulting sequence, $1, 3, 4, 7, 11, ...$, is called the Lucas sequence, named after Edouard Lucas, who was an outstanding French Mathematician. He was a professor of mathematics at the Lycee Saint-Louis and Lycee Charlemagne in Paris. Lucas enjoyed computing. He actually had hopes of developing a computer, but it never materialized. In addition to his contributions to number theory, Lucas is known for his four-volume classic on recreational mathematics. Mainly, Lucas is known for his study of the Fibonacci Sequence.

In [2], the authors have recently introduced a new generalization of the Fibonacci and Lucas sequences. Motivated by this generalization, we decided to study a similar sequence. More specifically, the paper is concerned with the following generalization of the Lucas sequence:

$$l_n = (-1)^n l_{n-1} + l_{n-2}$$

for $n = 3, 4, 5...$ with the initial conditions of $l_1 = 1$ and $l_2 = 3$. One can easily verify that the first few numbers of the sequence are $1, 3, -2, 1, -3, -2, -1, \ldots$.

One can clearly notice that the alternating sequence is obtained from the generalization, l_n, has ties to conditionally defined sequences. More specifically, let $a, b, c,$ and d be any nonzero real numbers. In 2009, Edson and Yayenie [2] introduced the following the definition of a generalized Fibonacci sequence: $q_0 = 0; q_1 = 1$; and for $m \geq 2$, and $a, b \in \mathbb{R} \setminus \{0\}$

$$q_m = \begin{cases} aq_{m-1} + q_{m-2}, & \text{if m is even} \\ bq_{m-1} + q_{m-2} & \text{if m is odd.} \end{cases}$$

In 2012, Yayenie [6] defined the following generalized Lucas sequence:

$V_0 = 2; V_1 = ab$; and for $m \geq 2$, and $a, b \in \mathbb{R} \setminus \{0\}$

$$V_m = \begin{cases} aV_{m-1} + V_{m-2}, & \text{if m is even} \\ bV_{m-1} + V_{m-2} & \text{if m is odd.} \end{cases}$$

Similarly, in 2014, Bilgici [1] defined the the following generalized Lucas sequence:

$\ell_0 = 2; \ell_1 = ab$; and for $m \geq 2$, and $a, b \in \mathbb{R} \setminus \{0\}$

$$\ell_m = \begin{cases} b\ell_{m-1} + \ell_{m-2}, & \text{if m is even} \\ a\ell_{m-1} + \ell_{m-2} & \text{if m is odd.} \end{cases}$$

Therefore, one notices that our generalized definition $\{\ell_n\}_{n=0}^{\infty}$ generalizes all of these sequences.

CHAPTER 2

GENERALIZING CONDITIONAL FIBONACCI AND LUCAS SEQUENCES

The generating function for the Fibonacci sequence is:

$$F(x) = \frac{x}{1 - x - x^2}.$$

The Binet Formula for the Fibonacci sequence is

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1+\sqrt{5}}{2} \right)^n - \left(\frac{1-\sqrt{5}}{2} \right)^n \right],$$

where $\frac{1 \pm \sqrt{5}}{2}$ are the roots of the characteristic equation $x^2 - x - 1 = 0$.

It is a standard question to ask about the generating function and a Binet-like formula for any generalization of the Fibonacci sequence. For each of the three definitions above, the authors determined the generating function and a Binet-like formula.

2.0.1 Generating Function for our Generalized Lucas Numbers

We present our definition of $\{\ell_n\}_{n=0}^{\infty}$.

Definition 2.0.1. For any real non zero numbers a and b, the generalized Lucas sequence $\{\ell_n\}_{n=0}^{\infty}$ is defined recursively by ℓ_0, ℓ_1, and for $m \geq 2$

$$\ell_m = \begin{cases} a\ell_{m-1} + \ell_{m-2}, & \text{if m is even} \\ b\ell_{m-1} + \ell_{m-2} & \text{if m is odd.} \end{cases}$$

If we take $a = b = 1$, we have the classical Lucas sequence. With the help of [4, Lemma 3.1], we have the following identities.

Lemma 2.0.2. *The sequence* $\{\ell_n\}_{n=0}^{\infty}$ *satisfies the following properties for* $n \geq 2$

$$\ell_{2n} = (ab+2)\ell_{2n-2} - \ell_{2n-4},$$

$$\ell_{2n+1} = (ab+2)\ell_{2n-1} - \ell_{2n-3}.$$

Proof. Let $n \geq 2$. Note that

$$\ell_{2n} = a\ell_{2n-1} + \ell_{2n-2}$$

$$= a(b\ell_{2n-2} + \ell_{2n-3}) + \ell_{2n-2}$$

$$= (ab+1)\ell_{2n-2} + a\ell_{2n-3}$$

$$= (ab+1)\ell_{2n-2} + (\ell_{2n-2} - \ell_{2n-4})$$

$$= (ab+2)\ell_{2n-2} - \ell_{2n-4}.$$

Thus,

$$\ell_{2n} = (ab+2)\ell_{2n-2} - \ell_{2n-4}.$$

Now, let us take

$$\ell_{2n+1} = b\ell_{2n} + \ell_{2n-1}$$

$$= b(a\ell_{2n-1} + \ell_{2n-2}) + \ell_{2n-1}$$

$$= (ab+1)\ell_{2n-1} + b\ell_{2n-2}$$

$$= (ab+1)\ell_{2n-1} + (\ell_{2n-1} - \ell_{2n-3})$$

$$= (ab+2)\ell_{2n-1} - \ell_{2n-3}.$$

Thus,

$$\ell_{2n+1} = (ab+2)\ell_{2n-1} - \ell_{2n-3}.$$

□

Theorem 2.0.3. *The generating function of the sequence* $\{\ell_m\}_{m=0}^{\infty}$ *is*

$$L(x) = \frac{\ell_0 + \ell_1 x + (a\ell_1 - (ab+1)\ell_0)x^2 + (b\ell_0 - \ell_1)x^3}{1 - (ab+2)x^2 + x^4}.$$

Proof. We define

$$L_0(x) = \sum_{m=0}^{\infty} \ell_{2m} x^{2m}$$

and

$$L_1(x) = \sum_{m=0}^{\infty} \ell_{2m+1} x^{2m+1}.$$

Note that,

$$L_0(x) = \ell_0 + (a\ell_1 + \ell_0)x^2 + \sum_{m=2}^{\infty} \ell_{2m} x^{2m},$$

$$(ab+2)x^2 L_0(x) = (ab+2)\ell_0 x^2 + \sum_{m=2}^{\infty} (ab+2)\ell_{2m-2} x^{2m},$$

and

$$x^4 L_0(x) = \sum_{m=2}^{\infty} \ell_{2m-4} x^{2m}.$$

Now, let us consider the following:

$$(1 - (ab+2)x^2 + x^4)L_0(x)$$

$$= \ell_0 + (a\ell_1 + \ell_0)x^2 - (ab+2)\ell_0 x^2 + \sum_{m=2}^{\infty} (\ell_{2m} - (ab+2)\ell_{2m-2} + \ell_{2m-4})x^{2m}.$$

From Lemma 2.0.2, we know that

$$\sum_{m=2}^{\infty}(\ell_{2m}-(ab+2)\ell_{2m-2}+\ell_{2m-4})x^{2m}=0.$$

Thus,

$$(1-(ab+2)x^2+x^4)L_0(x)$$

$$=\ell_0+a\ell_1x^2+\ell_0x^2-ab\ell_0x^2-2\ell_0x^2+0$$

$$=\ell_0+(a\ell_1-\ell_0-ab\ell_0)x^2.$$

From the first equation, we have

$$[1-(ab+2)x^2+x^4]L_0(x)=\ell_0+(a\ell_1-(ab+1)\ell_0)x^2,$$

and we get

$$L_0(x)=\frac{\ell_0+(a\ell_1-(ab+1)\ell_0)x^2}{1-(ab+2)x^2+x^4}.$$

Similarly, note that

$$L_1(x)=\ell_1x+((ab+1)\ell_1+b\ell_0)x^3+\sum_{m=2}^{\infty}\ell_{2m+1}x^{2m+1},$$

$$(ab+2)x^2L_1(x)=(ab+2)\ell_1x^3+\sum_{m=2}^{\infty}(ab+2)\ell_{2m-1}x^{2m+1},$$

and

$$x^4L_1(x)=\sum_{m=2}^{\infty}\ell_{2m-3}x^{2m+1}.$$

Now, let us consider the following:

$$(1-(ab+2)x^2+x^4)L_1(x)$$

$$= \ell_1 x + ((ab+1)\ell_1 + b\ell_0)x^3 - (ab+2)\ell_1 x^3 +$$

$$\sum_{m=2}^{\infty} (\ell_{2m+1} - (ab+2)\ell_{2m-1} + \ell_{2m-3)} x^{2m+1}.$$

From Lemma 2.0.2, we know that

$$\sum_{m=2}^{\infty} (\ell_{2m+1} - (ab+2)\ell_{2m-1} + \ell_{2m-3)} x^{2m+1} = 0.$$

Thus,

$$(1 - (ab+2)x^2 + x^4)L_1(x)$$

$$= \ell_1 x + ((ab+1)\ell_1 + b\ell_0)x^3 - (ab+2)\ell_1 x^3 + 0$$

$$= \ell_1 x + ab\ell_1 x^3 + \ell_1 x^3 + b\ell_0 x^3 - ab\ell_1 x^3 - 2\ell_1 x^3$$

$$= \ell_1 x + (b\ell_0 - \ell_1)x^3.$$

From the second equation, we have

$$[1 - (ab+2)x^2 + x^4]L_1(x) = \ell_1 x + (b\ell_0 - \ell_1)x^3,$$

and we get

$$L_1(x) = \frac{\ell_1 x + (b\ell_0 - \ell_1)x^3}{1 - (ab+2)x^2 + x^4}.$$

Since

$$L(x) = L_0(x) + L_1(x),$$

we obtain the desired result of

$$L(x) = \frac{\ell_0 + \ell_1 x + (a\ell_1 - (ab+1)\ell_0)x^2 + (b\ell_0 - \ell_1)x^3}{1 - (ab+2)x^2 + x^4}.$$

□

This rational function produces an infinite series $\ell_0 + \ell_1 x + \ell_2 x^2 + \cdots$; i. e., the coefficients of the series are the generalized sequence $\{\ell_n\}_{n=0}^{\infty}$.

2.0.2 Binet Formula for our Generalized Lucas Sequence

As calculated for other generalizations of these types of sequences, we determine a Binet-like formula for $\{\ell_m\}_{m=0}^{\infty}$.

Theorem 2.0.4. *The n^{th} term of a generalized Lucas sequence $\{\ell_n\}_{n=0}^{\infty}$ is given by*

$$\ell_n = \frac{1}{a^{\lfloor \frac{n+1}{2} \rfloor}} \frac{1}{b^{\lfloor \frac{n}{2} \rfloor}} \left[\left(a\ell_1 - \frac{1}{2} ab\ell_0 \right) \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{1}{2} \ell_0 (\alpha^n + \beta^n) \right],$$

where $\alpha = \dfrac{ab + \sqrt{(ab)^2 + 4ab}}{2}$ *and* $\beta = \dfrac{ab - \sqrt{(ab)^2 + 4ab}}{2}$.

Proof. We note that α and β are roots of the characteristic equation $x^2 - abx - ab = 0$ and

$$\xi(n) = \begin{cases} 0, & \text{if n is even} \\ 1, & \text{if n is odd} \end{cases}$$

is the parity function. Throughout the proof, we need the following properties given in [2]: $(i)(\alpha+1)(\beta+1) = 1, (ii)\alpha + \beta = ab, (iii)\alpha.\beta = -ab, (iv)\alpha + 1 = \dfrac{\alpha^2}{ab}, (v) - \beta + 1 = \dfrac{\beta^2}{ab}, (vi) - \beta(\alpha+1) = \alpha, (vii) - \alpha(\beta+1) = \beta.$

Again, if we let

$$L_0(x) = \sum_{m=0}^{\infty} \ell_{2m} x^{2m} \quad \text{and} \quad L_1(x) = \sum_{m=0}^{\infty} \ell_{2m+1} x^{2m+1},$$

then $L(x) = L_0(x) + L_1(x)$. We will use the seven identities mentioned above and the Maclaurin series expansion:

$$\frac{Ax + B}{x^2 - C} = -\sum_{n=0}^{\infty} AC^{-n-1} x^{2n+1} - \sum_{n=0}^{\infty} BC^{-n-1} x^{2n} = -\sum_{n=0}^{\infty} \frac{A}{C^{n+1}} x^{2n+1} - \sum_{n=0}^{\infty} \frac{B}{C^{n+1}} x^{2n}.$$

We first simplify the even part :

$$L_0(x) = \frac{\ell_0 + (a\ell_1 - (ab+1)\ell_0)x^2}{1-(ab+2)x^2+x^4}.$$

Setting the denominator equal to zero and solving, we get

$$x^2 = \frac{ab \pm \sqrt{(ab)^2+4ab}}{2} + 1.$$

We will set $x^2 = \alpha+1, \beta+1$, where α and β are the roots of $x^2 - abx - ab$. Thus, $1-(ab+2)x^2+x^4 = (x^2-(\alpha+1))(x^2-(\beta+1))$.

□

We will now apply the Maclaurin Series Expansion:

$$\frac{\ell_0 + (a\ell_1 - (ab+1)\ell_0)x^2}{1-(ab+2)x^2+x^4} = \frac{A_1x+B_1}{(x^2-(\alpha+1))} + \frac{A_2x+B_2}{(x^2-(\beta+1))}.$$

We multiply both sides of the equation by $(x^2-(\alpha+1))(x^2-(\beta+1))$. Then we have,

$$\ell_0 + (a\ell_1 - (ab+1)\ell_0)x^2 = (A_1x+B_1)(x^2-(\beta+1)) + (A_2x+B_2)(x^2-(\alpha+1)).$$

Since the coefficients B_i are associated with the even terms in the Maclaurin series expansion, we need only find the values of B_1 and B_2; in fact $A_1 = A_2 = 0$. Let us evaluate the equation for when $x=0$, $x=1$, $x=-1$.

When $x=0$:

$$\ell_0 = -B_1(\beta+1) - B_2(\alpha+1).$$

When $x=1$:

$$a(\ell_1 - b\ell_0) = (A_1+B_1)(-\beta) + (A_2+B_2)(-\alpha).$$

When $x=-1$:

$$a(\ell_1 - b\ell_0) = (-A_1 + B_1)(-\beta) + (-A_2 + B_2)(-\alpha).$$

Let us add the two equations for when $x=1$ and $x=-1$ to get the A values to cancel, leaving us with B values. Thus, we have

$$-a(\ell_1 - b\ell_0) = B_1\beta + B_2\alpha.$$

We solve the following system:

$$-a(\ell_1 - b\ell_0) = B_1\beta + B_2\alpha$$

$$\ell_0 = -B_1(\beta+1) - B_2(\alpha+1).$$

We find that

$$B_1 = -\frac{(-a\ell_1 + ab\ell_0 + \ell_0)\alpha + a(b\ell_0 - \ell_1)}{\alpha - \beta}$$

and

$$B_2 = \frac{(-a\ell_1 + ab\ell_0 + \ell_0)\beta + a(b\ell_0 - \ell_1)}{\alpha - \beta}.$$

Thus, we can simplify $L_0(x)$ as follows:

$$\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\left[\frac{(-a\ell_1 + ab\ell_0 + \ell_0)\alpha + a(b\ell_0 - \ell_1)}{\alpha - \beta} - \frac{(-a\ell_1 + ab\ell_0 + \ell_0)\beta + a(b\ell_0 - \ell_1)}{\alpha - \beta}\right] x^{2n}.$$

Using the seven properties mentioned previously [2], we can further simplify the above expression.

$$L_0(x) = \frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\Bigg[(\ell_0 + ab\ell_0 - a\ell_1)\left(\frac{\alpha}{(\alpha+1)^{n+1}} - \frac{\beta}{(\beta+1)^{n+1}}\right)$$

$$+a(b\ell_0-\ell_1)\left(\frac{1}{(\alpha+1)^{n+1}}-\frac{1}{(\beta+1)^{n+1}}\right)\Big]x^{2n}.$$

We know that $(\alpha+1)(\beta+1)=1$, therefore $L_0(x)$

$$=\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}[(\ell_0+ab\ell_0-a\ell_1)(\alpha(\beta+1)^{n+1}-\beta(\alpha+1)^{n+1})$$

$$+a(b\ell_0-\ell_1)((\beta+1)^{n+1}-(\alpha+1)^{n+1})]^{2n}.$$

Similarly, we know that $\alpha+1=\frac{\alpha^2}{ab}$, hence $L_0(x)$

$$=\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\Big[(\ell_0+ab\ell_0-a\ell_1)\Big(\alpha\Big(\frac{\beta^2}{ab}\Big)^{n+1}-\beta\Big(\frac{\alpha^2}{ab}\Big)^{n+1}\Big)$$

$$+a(b\ell_0-\ell_1)\Big(\Big(\frac{\beta^2}{ab}\Big)^{n+1}-\Big(\frac{\alpha^2}{ab}\Big)^{n+1}\Big)\Big]x^{2n}$$

$$=\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\left[(\ell_0+ab\ell_0-a\ell_1)\Big(\frac{\alpha\beta^{2n+2}-\beta\alpha^{2n+2}}{(ab)^{n+1}}\Big)+a(b\ell_0-\ell_1)\Big(\frac{\beta^{2n+2}-\alpha^{2n+2}}{(ab)^{n+1}}\Big)\right]x^{2n}$$

$$=\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\left[(\ell_0+ab\ell_0-a\ell_1)\Big(\frac{\alpha^{2n+1}-\beta^{2n+1}}{(ab)^{n}}\Big)+a(b\ell_0-\ell_1)\Big(\frac{\beta^{2n+2}-\alpha^{2n+2}}{(ab)^{n+1}}\Big)\right]x^{2n}.$$

After further simplification we have that,

$$L_0(x)=\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\frac{1}{(ab)^n}[(a\ell_1-\beta\ell_0)\alpha^{2n}-(a\ell_1-\alpha\ell_0)\beta^{2n}]x^{2n}.$$

We will now simplify the odd part:

$$L_1(x)=\frac{\ell_1x+(b\ell_0-\ell_1)x^3}{1-(ab+2)x^2+x^4}.$$

Once again, we use $1-(ab+2)x^2+x^4=(x^2-(\alpha+1))(x^2-(\beta+1))$.

We will now apply the Maclaurin Series Expansion:

$$\frac{\ell_1x+(b\ell_0-\ell_1)x^3}{1-(ab+2)x^2+x^4}=\frac{A_1x+B_1}{(x^2-(\alpha+1))}+\frac{A_2x+B_2}{(x^2-(\beta+1))}$$

Let us multiply both sides of the equation by $(x^2-(\alpha+1))(x^2-(\beta+1))$. Then we have,

$$\ell_1 x + (b\ell_0 - \ell_1)x^3 = (A_1x+B_1)(x^2-(\beta+1)) + (A_2x+B_2)(x^2-(\alpha+1)).$$

Since the coefficients A_i are associated with the odd terms in the Maclaurin series expansion, we want to find the values of A_1 and A_2. In fact, $B_1 = B_2 = 0$. So, let us evaluate the equation for when $x = 1$ and $x = 2$. When $x = 1:$

$$b\ell_0 = -A_2\beta - A_2\alpha.$$

When $x = 2:$

$$4b\ell_0 - 3\ell_1 = A_1(3-\beta) + A_2(3-\alpha).$$

When we solve for the following system:

$$b\ell_0 = -A_2\beta - A_2\alpha$$

$$4b\ell_0 - 3\ell_1 = A_1(3-\beta) + A_2(3-\alpha),$$

we find that

$$A_1 = -\frac{(\ell_1 - b\ell_0)\alpha - b\ell_0}{\alpha-\beta}.$$

and

$$A_2 = \frac{(\ell_1 - b\ell_0)\beta - b\ell_0}{\alpha-\beta}.$$

Thus, we can simplify $L_1(x)$ as follows:

$$\frac{1}{\alpha-\beta}\sum_{n=0}^{\infty}\left[\frac{(\ell_1 - b\ell_0)\alpha - b\ell_0)}{(\alpha+1)^{n+1}} - \frac{(\ell_1 - b\ell_0)\beta - b\ell_0)}{(\beta+1)^{n+1}}\right]x^{2n+1}$$

$$= \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left[(\ell_1 - b\ell_0) \frac{\alpha^{2n+1} - \beta^{2n+1}}{(ab)^n} + b\ell_0 \frac{\alpha^{2n+1} - \beta^{2n+2}}{(ab)^{n+1}} \right] x^{2n+1}.$$

After using the seven identities and further simplification, we have that

$$L_1(x) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \left[(\ell_1 - b\ell_0) \Big(\frac{\alpha^{2n+1} - \beta^{2n+1}}{(ab)^n} \Big) - b\ell_0 \Big(\frac{\beta^{2n+2} - \alpha^{2n+2}}{(ab)^{n+1}} \Big) \right] x^{2n+1}.$$

Hence,

$$L_1(x) = \frac{b}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{1}{(ab)^{n+1}} \left[(a\ell_1 - \beta\ell_0)\alpha^{2n+1} - (a\ell_1 - \alpha\ell_0)\beta^{2n+1} \right] x^{2n+1}.$$

In summary, we know that $L(x) = L_0(x) + L_1(x)$. Thus,

$$L(x) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{b^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \left[(a\ell_1 - \beta\ell_0)\alpha^n - (a\ell_1 - \alpha\ell_0)\beta^n \right] x^n.$$

Next, recall that $\alpha = \dfrac{ab + \sqrt{(ab)^2 + 4ab}}{2}$, $\beta = \dfrac{ab - \sqrt{(ab)^2 + 4ab}}{2}$, and hence, $\alpha - \beta = \sqrt{(ab)^2 + 4ab}$. Therefore,

$$L(x) = \frac{1}{\alpha - \beta} \sum_{n=0}^{\infty} \frac{b^{\xi(n)}}{(ab)^{\lfloor \frac{n+1}{2} \rfloor}} \Big[(a\ell_1 - \Big(\frac{ab - \sqrt{(ab)^2 + 4ab}}{2} \Big) \ell_0 \Big) \alpha^n$$

$$- \Big(a\ell_1 - \Big(\frac{ab + \sqrt{(ab)^2 + 4ab}}{2} \Big) \ell_0 \Big) \beta^n \Big] x^n$$

and

$$L(x) = \sum_{n=0}^{\infty} \frac{1}{a^{\lfloor \frac{n+1}{2} \rfloor}} \frac{1}{b^{\lfloor \frac{n}{2} \rfloor}} \left[\Big(a\ell_1 - \frac{1}{2} ab\ell_0 \Big) \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{1}{2} \ell_0 (\alpha^n + \beta^n) \right] x^n.$$

CHAPTER 3

A GENERALIZATION OF THE LUCAS NUMBERS

3.1 Previous Work: Computing, Analyzing, and Forming Conjectures for our Generalized Lucas Numbers

Initially, in 2016, to determine a larger number of the Lucas number sequence, we used Wolfram Mathematica on a MacBook Air laptop computer that uses the macOS Sierra operating system. We simply used the appropriate syntax to generate a table for Lucas numbers. A good feature of the software is that we did not have to create a syntax for the equation for Lucas numbers because *Mathematica* already has the Lucas number function embedded into the program. The syntax used to produce the Classical Lucas numbers is as follows:

```
Table[LucasL[n], {n, 10}]
{1, 3, 4, 7, 11, 18, 29, 47, 76, 123}.
```

To generate the generalized Lucas numbers, we created the function based on the existing commands of the Wolfram Language. The following code returns the first 10 generalized Lucas numbers. There are several ways to approach computing terms of a sequence given by a recurrence relation. For instance, the command RSolve is one of the options, and we will use this option.

```
RSolve[{a[n] == (−1)^n*a[n − 1] + a[n − 2], a[1] == 1,
a[2] == 3}, a[n], n];
Table[FullSimplify[a[n] /. First[\%]], {n, 10}]
{1, 3, −2, 1, −3, −2, −1, −3, 2, −1}
```

While computing terms of the classical and generalized Lucas sequences, we noticed that *Mathematica* requires much time to compute the terms with larger indices, such as fifty or one-hundred. This suggested that to avoid time consumption, we should use an alternative method to compute the numbers. In our case, the alternative method was to write a program code in *Java*, a computer programming language. The following code allows us to compute any number of the Classical Lucas numbers in *Java*:

```
public class Lucas{
    public static void main(String[] args){

    int L2 = 2;
    int L1 = 1;
    System.out.print(L1);
    for( int i = 0; i <= 10; i++ ) {
        int L = L1 + L2;
        System.out.print("\n" + L);
        L2 = L1;
        L1 = L;
     }
        }
    }
```

Figure 3.1: *Java* Code Computing the Classical Lucas numbers

The following modification of the previous code allows us to compute the Generalized Lucas numbers:

```
public class LucasGeneralized{
    public static void main(String[] args){

    int L2 = 3;
    int L1 = 1;
    int N = -1;
    System.out.println(""+L1 + "\n" + L2);
    for( int i = 0; i <= 10; i++ ) {
        int L = ( N * L2 ) + (L1);
        N = - N;
        System.out.print("\n" + L);
        L1 = L2;
        L2 = L;

     }
        }
    }
```

Figure 3.2: *Java* Code Computing the Generalized Lucas numbers

Both Figure 3.1 and Figure 3.2 are codes that were created to see if *Java* would be more efficient in computing a larger number the term of the classical and generalized Lucas sequence.

We ran the same computations using *Mathematica*. To see the exact time elapsed for *Mathematica* to compute the numbers, we can use the following code:

```
RSolve[{a[n] == (−1)^n*a[n − 1] + a[n − 2], a[1] == 1,
a[2] == 3}, a[n], n];
Timing[Table[FullSimplify[a[n] /. First[\%]], {n, 50}]].
```

To see the exact amount of time that it takes *Java* to compute the numbers, we used the following code:

```
public class LucasGeneralized{
    public static void main(String[] args){

    int L2 = 3;
    int L1 = 1;
    int N = -1;

    long startTime = System.nanoTime();
    System.out.println(""+L1 + "\n" + L2);
    for( int i = 0; i <= 48; i++ ) {
        int L = ( N * L2 ) + (L1);
        N = - N;
        System.out.print("\n" + L);
        L1 = L2;
        L2 = L;

     }
     long endTime   = System.nanoTime();
        long totalTime = endTime - startTime;
        System.out.println("");
        System.out.println(totalTime);
        double seconds = totalTime/ 1000000000.0;
        System.out.println(seconds);
        }
    }
```

Figure 3.3: *Java* Code for Time Elapsed

Table 3.1, shows the exact values of the amount of time in seconds that it took each program to execute and compute the numbers.

Generalized Lucas Numbers		
n	Mathematica(seconds)	Java (seconds)
10	1.0811	.000707475
20	2.04639	.000577721
30	6.84106	.000840512
40	19.6633	.001007388
50	64.2744	.001209008

Table 3.1: Time Elapsed for Generalzied Lucas Numbers

Although *Mathematica* was not the ideal method for computing a larger number of the terms of the sequences, it was beneficial in offering methods to produce graphs and tables. We used specific syntax to generate detailed graphs and tables in *Mathematica*. These graphs and tables were helpful for comparing the behavior of the classical and generalized Lucas sequences. The following code generates Table 3.2:

```
TableForm[Table[{LucasL[n],FullSimplify[a[n] /.
First[\%]]}, {n, 10}],
TableHeadings -> {None,{"Lucas Numbers",
"Generalized Lucas Numbers"}}
```

Lucas Numbers	Generalized Lucas Numbers
$L_1 = 1$	$l_1 = 1$
$L_2 = 3$	$l_2 = 3$
$L_3 = 4$	$l_3 = -2$
$L_4 = 7$	$l_4 = 1$
$L_5 = 11$	$l_5 = -3$
$L_6 = 18$	$l_6 = -2$
$L_7 = 29$	$l_7 = -1$
$L_8 = 47$	$l_8 = -3$
$L_9 = 76$	$l_9 = 2$
$L_{10} = 123$	$l_{10} = -1$

Table 3.2: Classical and Generalized Lucas Numbers

Table 3.2 shows the classical and generalized Lucas numbers up to $n = 10$. The following code generates Figure 3.4:

```
RSolve[{a[n] == (−1)^n∗a[n − 1] + a[n − 2], a[1] == 1,
a[2] == 3}, a[n], n];
ListPlot[{Table[LucasL[n], {n, 10}],
Table[Simplify[a[n] /.First[\%]],{n, 10}]},
PlotLegends -> {"Classical Lucas Numbers",
"Generalized Lucas Numbers"}]
```

Mathematica offers many different forms of syntax to produce different tables and graphs. The ListPlot command that we used, allows us to see a dot graph, showing the exact values of each point on the graph.

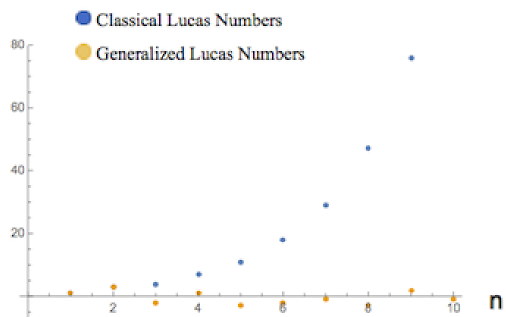

Figure 3.4: Comparing Classical and Generalized Lucas Numbers

Figure 3.4 compares the classical Lucas numbers and the generalized Lucas numbers. After studying this graph carefully, one can conclude that classical Lucas numbers and Generalized Lucas numbers behave somewhat differently. Namely, as one can see from the graph, the classical Lucas numbers grow exponentially while the generalized Lucas numbers demonstrate a fluctuating behavior. The difference in behavior suggests considering the ratio of the generalized and classical

Lucas sequences: $\frac{GeneralizedLucasNumbers}{ClassicalLucasNumbers}$.

We were in the position to use Wolfram *Mathematica* again to understand the behavior of the ratios. The following code allows us to create a graph of the ratio $\frac{l_n}{L_n}$:

```
RSolve[{a[n] == (-1)^n*a[n - 1] + a[n - 2], a[1] == 1,
a[2] == 3}, a[n], n];
Ratios[{Table[LucasL[n], {n, 10}],
Table[FullSimplify[a[n] /. First[\%]], {n, 10}]}]
ListLinePlot[{1, 1, -(1/2), 1/7, -(3/11),
-(1/9), -(1/29), -(3/47), 1/38, -(1/123)}].
```

This code gives us the following figure.

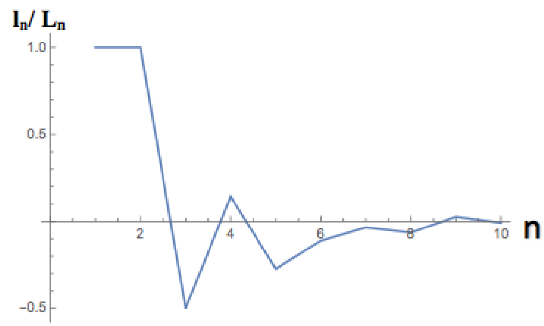

Figure 3.5: Graphing the Ratio of Generalized/Classical Lucas Numbers

Figure 3.5 demonstrates the ratios for the range from $n = 1$ to $n = 10$. This gives us an idea of what the graph may look like as n increases. Just from observing Figure 3.5 and knowing that the increasing denominator will cause the ratio to eventually become zero, one can decide that the graph will eventually resort to zero.

Conjecture 3.1.1. *We have that*

$$\lim_{n\to\infty} \frac{l_n}{L_n}.$$

To conclude this section, we make some observations about our use of *Mathematica* and *Java*. In particular, *Mathematica* is beneficial in creating graphs and tables to help describe data. It is important to note that we used *Mathematica* to calculate all terms sequentially, using the recurrence relation. On the other hand, we used *Java* to calculate individual terms of the sequence. Throughout our methods of computing the Lucas numbers and creating different graphs, we make use of features of both programming languages.

3.1.1 Conjectures about Classical and Generalized Lucas Numbers

To get some conjectures that might work for the generalized Lucas sequence, we used identities from [5]. Let us start with Cassini's formula for the Lucas numbers [5, Theorem 5.9]:

$$L_{n-1} * L_{n+1} - L_n{}^2 = 5(-1)^{n-1}, n \geq 2.$$

To see if this formula can be adapted to the Generalized Lucas numbers, we can set up a table to analyze the relationship between different terms of the sequence.

n	$l_{n-1} * l_{n+1}$	l_n^2	(+)
2	-2	9	$9 - 2 = 7$
3	3	4	$3 + 4 = 7$
4	6	1	$6 + 1 = 7$

Table 3.3: Adapting Cassini's Formula

One should simply substitute the different values of n to compute the second and third columns and study the relationship amongst all the numbers. In analyzing, notice that when the terms of columns 2 and 3 are added, they equal seven for $n = 2, 3, 4$. So, one should alter the original formula to reflect how the Generalized Lucas numbers behave in relation to Cassini's formula. In other words, the subtraction sign is replaced with an addition sign and equated to 7. Based on this observation, we can conjecture the following:

Conjecture 3.1.2. *For every natural number $n \geq 2$, we have that $l_{n-1} * l_{n+1} + l_n{}^2 = 7$.*

Since we only checked Conjecture 3.1.2 for a few values of n. We now want to be sure the formula holds true for a larger value of n. So, we can use Wolfram *Mathematica* to check if the formula works for larger values of n. The code

```
f[n_] := (-1)^n*f[n - 1] + f[n - 2]; f[1] = 1; f[2] = 3
Table[f[n - 1]*f[n + 1] + f[n]^2, {n, 2, 30}]
{7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7,7, 7, 7,
7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7, 7}
```

shows that the values of the expression that appear on the left of Conjecture 3.1.2 are all seven, for n from 2 to 30, which assures us even more about the validity of Conjecture 3.1.2.

We note that when trying to adapt Cassini's Formula $F_{n-1} * F_{n+1} - F_n{}^2 = (-1)^n$ for the Generalized Fibonacci number sequence, $f_n = f_{n-1} + (-1)^n f_{n-2}$, the author of [3] noticed that Cassini's formula did not work for the generalized Fibonacci numbers. As a result, in [3] another generalization of Cassini's formula was proposed. To see if this generalization also works for the generalized Lucas numbers, let us look at the following terms: $L_{n-1} * L_{n+2}$ and $L_n * L_n + 1$. Once again, we can create a table to analyze the relationships amongst the different parts that we chose to test.

n	$l_{n-1} * l_{n+2}$	$l_n * l_{n+1}$	(-)
2	1	-6	1-(-6) = 7
3	-9	-2	-9-(-2) = -7
4	4	-3	4-(-3) = 7
5	-1	6	-1- 6 = -7

Table 3.4: Adapting to Cassini's Formula

Additionally, we can substitute the different values of n to compute the term and study the relationship amongst all the numbers. In analyzing, notice that when the terms of columns 2 and 3 are subtracted, the result equals either seven or negative seven for $n = 2, 3, 4,$ and 5. Having seen how Generalized Lucas numbers behave in Table 3.4 We can make our hypothesis about the corresponding formula.

Conjecture 3.1.3. *For every natural number $n \geq 2$, we have that* $l_{n-1} * l_{n+2} - l_n * l_{n+1} = 7(-1)^n$.

Since we only checked Conjecture 3.1.3 for four values of n, we wanted to be sure the formula works for larger number values of n. To do that, we used Wolfram *Mathematica* to check if the formula works for larger values of n. Executing the code:

```
f[n_] := (-1)^n*f[n - 1] + f[n - 2]; f[1] = 1; f[2] = 3
Table[f[n - 1]*f[n + 2] - f[n]*f[n + 1], {n, 2, 30}]
{7, -7, 7, -7, 7, -7, 7, -7, 7, 7, -7, 7, -7, 7,
-7, 7, -7, 7, 7, -7, 7, -7, 7, -7, 7, -7, 7, -7, 7}.
```

We see that the values of the expression on the left of Conjecture 3.1.3 are all sevens with alternating signs and the first value being a positive seven just as hypothesized in Figure 1.4. This shows that the conjecture holds true for all values of n from 2 to 30.

The following will aid in proving Conjecture 3.1.3:

Remark 3.1.4. For every natural number $n \geq 5$, we have that

$$l_n = l_{n-2} - l_{n-4}.$$

This follows from Lemma 2.0.2 with $a = 1$ and $b = -1$.

Proposition 3.1.5. *For every natural number $n \geq 4$, we have that*

$$l_n = (-1)^n l_{n-3}.$$

Proof. Let us prove the statement using Principle Mathematical Induction. First, we need to check that the formula holds true for $n = 4$. That is:

$$l_4 = (-1)^4 l_1,$$

$$1 = 1.$$

This is true.

Now, let us assume the formula holds true for $n = k$. That is, for some natural number k we have that:

$$l_k = (-1)^k l_{k-3}.$$

Now let us see if true for $n = k + 1$.

That is,

$$l_{k+1} = l_{k-1} - l_{k-3}$$

$$= [(-1)^{k-1} l_{k-2} + l_{k-3}] - l_{k-3}$$

$$= (-1)^{k-1} l_{k-2}$$

$$= (-1)^{k+1} l_{k-2}.$$

Thus, proved. □

In fact, Conjecture 3.1.3 came from something completely new, and therefore, we must check to see if the conjecture is also true for the classical Lucas numbers. So, we set up a similar table for the classical Lucas sequence.

n	$L_{n-1} * L_{n+2}$	$L_n * L_{n+1}$	(-)
2	7	12	-5
3	33	28	5
4	72	77	-5

Table 3.5: Conjecture 3.1.3 for the Classical Lucas Numbers

After substituting different values of n to find each term, we studied the relationship amongst all the numbers. Thus, we can see that when the terms of columns 2 and 3 are subtracted, the result equals either five or negative five for $n = 2, 3$, and 4. Hence, we altered the formula to reflect how the classical Lucas numbers behave in Table 1.5. We added a subtraction sign between the two parts and equate the formula to $5(-1)^{n+1}$ to account for the sign change behavior because when subtracted, the equations all have positive or negative five in common. Since Table 1.5 shows that the sequence begins with a negative five, we can make the exponent $n+1$ for $5(-1)$ so that when starting at two, the first exponent will be odd, forcing the first value to be negative five. To this end, we arrived at the following conjecture:

Conjecture 3.1.6. *For every natural number $n \geq 2$, we have that*

$$L_{n-1} * L_{n+2} - L_n * L_{n+1} = 5(-1)^{n+1}.$$

Since we only checked the conjecture for a few values of n, we wanted sure the formula would hold true for a larger value of n. We used Wolfram Mathematica to check if the equation works for larger values of n. The code is as follows:

```
Table[LucasL[n − 1]∗LucasL[n + 2] −
LucasL[n]∗LucasL[n + 1], {n, 2, 30}]
{−5, 5, −5, 5, −5, 5, −5, 5, −5, 5, −5, 5, −5, 5,
−5, 5, −5, 5, −5, 5, −5, 5, −5, 5, −5, 5, −5, 5, −5}.
```

When executed, the values are all fives with alternating signs and the first value being a negative five just as hypothesized in Table 1.5. This concluded that the conjecture works for all values of n from 2 to 30.

Finally, the last conjecture is on how we adapted the following formula [5, Theorem 5.6]

$$\sum_{i=1}^{n} L_i = L_{n+2} - 3$$

to the generalization under consideration. This process is not as simple as analyzing the relationship between values of a table. To come up with a possible statement, we must use a different strategy. So, let us try a few steps. Based on the definition of the generalized Lucas numbers we have:

$l_3 = -l_2 + l_1$, or $l_1 = l_3 + l_2$, or $l_1 = l_3 + 3$.

Next, let us see what happens when we have two terms. That is, we start with the definition once again, and consider: $l_1 = l_3 + l_2$ and $l_2 = l_4 - l_3$ So, the sum can be found as follows: $l_1 + l_2 = l_2 + l_4 = 3 + l_4$. Then, let us see what happens when we have three terms. Consider: $l_1 + l_2 = l_2 + l_4$ and $l_3 = l_5 + l_4$. So, the sum can be found as follows: $l_1 + l_2 - l_3 = l_2 - l_5 = 3 - l_5$. From this process, we can come up with the following conjecture for the generalized Lucas numbers:

Conjecture 3.1.7. *We have that*

$$\sum_{i=1}^{n} (-1)^{\frac{i(i+1)}{2}+1} * l_i = 3 + (-1)^{\frac{n(n+1)}{2}+1} * l_{n+2}$$

for every natural number n.

We only checked Conjecture 3.1.7 for a few values of n, and we want to be sure the formula holds true for a larger value of n. So, we can use Wolfram Mathematica to check if the equation works for larger values of n. The code is as follows:

```
Table[Sum[(-1)^(i*(i + 1)/2 + 1)*f[i], {i, 1, n}]
- 3 - (-1)^(n*(n + 1)/2 + 1)*f[n + 2], {n, 1, 30}]
{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,
0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}.
```

When executed, the values of the difference of the left and right sides of Conjecture 3.1.7 are all zeros, which shows that the conjecture works for all values of n from 2 to 30.

3.1.2 Proofs of Conjectures

In this section, we will prove some of the conjectures using the Principle of Mathematical Induction. Thus, we will make them theorems.

Theorem 3.1.8. *For every natural number $n \geq 2$, we have that*

$$l_{n-1} * l_{n+1} + {l_n}^2 = 7.$$

Proof. Let us prove the statement using Principle Mathematical Induction. First, we need to check that the formula holds true for $n = 2$. That is:

$$l_{2-1} * l_{2+1} + l_2^2 = 7,$$

$$l_1 * l_3 + l_2^2 = 7,$$

$$1 * (-2) + 3^2 = 7,$$

$$-2 + 9 = 7,$$

$$7 = 7.$$

This is true.

Now, let us assume the formula holds true for $n = k$ That is, for some natural number k we have that:

$$l_{k-1} * l_{k+1} + l_k^2 = 7$$

So, we need to prove the relation holds true for $n = k + 1$. That is,

$$l_{k+1-1} * l_{k+1+1} + l_{k+1}^2 = 7,$$

$$l_k * l_{k+2} + l_{k+1}^2 = 7.$$

To this end, consider the sum:

$$l_k * l_{k+2} + l_{k+1}^2 = (-1)^{k+1}[(l_{k+1} - l_{k-1}) * ((-1)^{k+2} l_{k+1} - l_k)] + l_{k+1}^2$$

$$= -l_{k+2}^2 - (-1)^{k+1} l_{k+1} l_k - (-1)^{k+1}(-1)^{k+2} l_{k-1} l_{k+1} + (-1)^{k+1} l_k l_{k-1} + l_{k+1}^2$$

$$= -(-1)^{k+1} l_{k+1} l_k + l_{k-1} l_{k+1} + (-1)^{k+1} l_k l_{k-1}$$

$$= -(-1)^{k+1} l_{k+1} l_k + 7 + l_k^2 + (-1)^{k+1} l_k l_{k-1}$$

$$= -(-1)^{k+1} l_{k+1} l_k + 7 + (-1)^{k+1} l_k l_{k+1}$$

$$= 7.$$

This shows that the relation holds true for $n = k + 1$. Therefore, by the Principle Mathematical Induction, the relation holds true for all natural numbers n. This completes the proof. □

Theorem 3.1.9. *For every natural number $n \geq 2$, we have that*

$$l_{n-1} l_{n+2} - l_n l_{n+1} = 7(-1)^n.$$

Proof. Let us prove the statement using Principle Mathematical Induction. First, we need to check that the formula holds true for $n = 2$. That is:

$$l_1 l_4 - l_2 l_3,$$

$$(1)(1) - 3(-2) = 1 + 6 = 7.$$

This is true.

Now, let us assume the formula holds true for $n = k$.

That is, for some natural number k we have that:

$$l_{k-1} l_{k+2} - l_k l_{k+1} = 7(-1)^k.$$

Now, let us see if true for $n = k + 1$. That is,

$$l_k l_{k+3} - l_{k+1} l_{k+2} = 7(-1)^{k+1}$$

$$= ((-1)^k l_{k-3})((-1)^{k+3} l_k) - ((-1)^{k+1} l_{k-2})((-1)^{k+2} l_{k-1})$$

$$= (-1)^{2k+3} l_{k-3} l_k - (-1)^{2k+3} l_{k-2} l_{k-1}$$

$$= -l_{k-3} l_k + l_{k-2} l_{k-1}$$

$$= l_{k-2} l_{k-1} - l_{k-3} l_k$$

$$7(-1)^{k-3} = 7(-1)^{k+1}.$$

This shows that the relation holds true for $n = k + 1$. Therefore, by the Principle Mathematical Induction, the relation holds true for all natural numbers n. This completes the proof.

□

Theorem 3.1.10. *For every natural number $n \geq 2$, we have that*

$$L_{n-1} L_{n+2} - L_n L_{n+1} = 5(-1)^{n+1}.$$

Proof. Let us prove the statement using Principle Mathematical Induction. First, we need to check that the formula holds true for $n = 2$. That is:

$$L_1 L_4 - L_2 L_3,$$

$$= (1)(7) - (3)(4)$$

$$= 7 - 12,$$

$$= -5$$

$$= 5(-1)^{2+1}.$$

This is true.

Now, let us assume the formula holds true for $n = k$. That is, for some natural number k we have that:

$$L_{k-1}L_{k+2} - L_kL_{k+1} = 5(-1)^{k+1}.$$

Now, let us check to see if true for $n = k+1$. That is,

$$L_kL_{k+3} - L_{k+1}L_{k+2} = 5(-1)^{k+2}$$

$$= (L_{k-1} + L_{k-2})(L_{k+2} + L_{k+1}) - (L_k + L_{k-1})(L_{k+1} + L_k)$$

$$= L_{k-1}L_{k+2} + L_{k-1}L_{k+1} + L_{k-2}L_{k+2} + L_{k-2}L_{k+1} - L_kL_{k+1} - L_k^2 - L_kL_{k+1} - L_{k-1}L_k$$

$$= 5(-1)^{k+1} + 5(-1)^k + L_{k-2}L_{k+2} - L_k^2$$

$$= L_{k-2}(L_{k+1} + L_k) - L_k^2$$

$$= 5(-1)^k + L_{k-1}L_k + L_{k-2}L_k - L_k^2$$

$$= 5(-1)^k + L_k(L_{k-1} + L_{k-2}) - L_k^2$$

$$= 5(-1)^k + L_k(L_k) - L_k^2$$

$$5(-1)^{k+2} = 5(-1)^k.$$

This shows that the relation holds true for $n = k+1$. Therefore, by the Principle Mathematical Induction, the relation holds true for all natural numbers n. This completes the proof.

$\square$

Theorem 3.1.11. *For any natural number n we have that*

$$\sum_{i=1}^{n}(-1)^{\frac{i(i+1)}{2}+1} * l_i = 3 + (-1)^{\frac{n(n+1)}{2}+1} * l_{n+2}.$$

Proof. Let us use the Principle Mathematical Induction. So, we need to check that the formula holds true for $n = 1$. That is:

$$(-1)^{\frac{1(1+1)}{2}+1} * l_1 = 3 + (-1)^{\frac{1(1+1)}{2}+1} * l_{1+2},$$

$$1 * 1 = 3 + 1 * (-2),$$

$$1 = 1.$$

This is true.

Now let us assume the formula holds true for $n = k$. That is, we have

$$\sum_{i=1}^{k}(-1)^{\frac{i(i+1)}{2}+1} * l_i = 3 + (-1)^{\frac{k(k+1)}{2}+1} * l_{k+2}$$

for some natural number k. So, we need to prove the relation holds true for $n = k + 1$. To this end, consider the sum

$$\sum_{i=1}^{k+1}(-1)^{\frac{i(i+1)}{2}+1} * l_i = 3 + (-1)^{\frac{(k+1)((k+1)+1)}{2}+1} * l_{(k+1)+2}.$$

This can be written as follows:

$$\sum_{i=1}^{k+1}(-1)^{\frac{i(i+1)}{2}+1} * l_i = \sum_{i=1}^{k}(-1)^{\frac{i(i+1)}{2}+1} * l_i + (-1)^{\frac{(k+1)(k+1+1)}{2}+1} * l_{k+1}$$

$$= 3 + (-1)^{\frac{k(k+1)}{2}+1} * l_{k+2} + (-1)^{\frac{(k+1)(k+2)}{2}+1} * l_{k+1}$$

$$= 3 + (-1)^{\frac{(k+1)(k+2)}{2}+1} * l_{k+3}.$$

To see that the last two steps are true, note that

$$l_{k+3} = (-1)^{k+3} l_{k+2} + l_{k+1}.$$

Therefore,

$$l_{k+3} = (-1)^{\frac{(k+1)(k+2)}{2}+1} * (-1)^{\frac{k(k+1)}{2}+1} * l_{k+2}$$

$$+(-1)^{\frac{(k+1)(k+2)}{2}+1} * (-1)^{\frac{(k+1)(k+2)}{2}+1} * l_{k+1}.$$

We want to make sure $(-1)^{\frac{(k+1)(k+2)}{2}+1} * (-1)^{\frac{k(k+1)}{2}+1} = (-1)^{k+3}$.

$$(-1)^{\frac{(k+1)(k+2)}{2}+1} * (-1)^{\frac{k(k+1)}{2}+1} = (-1)^{\frac{(k+1)k+2(k+1)}{2}+2+\frac{k(k+1)}{2}},$$

$$= (-1)^{2+\frac{(k+1)k)}{2}+2\frac{2(k+1)}{2}+\frac{k(k+1)}{2}},$$

$$= (-1)^{2+k(k+1)+(k+1)},$$

$$= (-1)^{k+3+k(k+1)} = (-1)^{k+3}.$$

Consequently, we have $l_{k+3} = (-1)^{k+3} l_{k+2} + l_{k+1}$, which is true. So, by the Principle Mathematical Induction, we see that the relation holds true for all natural numbers n. This completes the proof.

□

CHAPTER 4

CONCLUSION

Previously, studying this generalization of the Lucas numbers showed us how to analyze and manipulate different formulas to come up with similar conjectures and theorems that work for our specific generalization of the Lucas number sequence. This process showed that many formulas can be generalized, but not always in a simple, straightforward manner. Sometimes we must adapt and dedicate extra time and skill to certain formulas to come up with conjectures and to then prove them.

We can conclude that the generating function of our sequence $\{\ell_m\}_{m=0}^{\infty}$ is

$$L(x) = \frac{\ell_0 + \ell_1 x + (a\ell_1 - (ab+1)\ell_0)x^2 + (b\ell_0 - \ell_1)x^3}{1 - (ab+2)x^2 + x^4}.$$

In particular, for the specialized sequence $l_n = (-1)^n l_{n-1} + l_{n-2}$, we can immediately find the generating function by taking $a = 1$ and $b = -1$; i.e.,

$$L(x) = \frac{\ell_0 + \ell_1 x + \ell_1 x^2 - (\ell_0 + \ell_1)x^3}{1 - x^2 + x^4}.$$

In addition, we have successfully found the Binet-like formula for the sequence $\{\ell_m\}_{m=0}^{\infty}$.

Binet-like formula:

$$\ell_n = \frac{1}{a^{\lfloor \frac{n+1}{2} \rfloor}} \frac{1}{b^{\lfloor \frac{n}{2} \rfloor}} \left[\left(a\ell_1 - \frac{1}{2}ab\ell_0 \right) \frac{\alpha^n - \beta^n}{\alpha - \beta} + \frac{1}{2}\ell_0(\alpha^n + \beta^n) \right].$$

Since were able to define a more general formula, our formula encompasses all that the previous authors found in their results on conditionally defined Fibonacci and Lucas sequences.

Bibliography

[1] G. Bilgici. Two generalizations of lucas sequence. *Applied Mathematics and Computation, Elsevier*, (245):526–538, 2014.

[2] M.Edson and O.Yayenie. A new generalization of fibonacci sequence and extended binets formula. *Integers*, a(6), 2009.

[3] V. Robinson. On a generalizatioin of the fibonacci sequence. *University of Mississippi McNair Research Journal*, 2017.

[4] M. Sahin. The gelin-cesaro identity in some conditional sequences. *Hacet.J. Math.*, 40(6):5603–5611, 2011.

[5] T.Koshy. Fibonacci and lucas numbers with applications. 2001.

[6] O. Yayenie. New identities for generalized fibonacci sequences and new generalization of lucas sequences. *Southeast Asian Bulletin of Mathematics*, 36:739–752, 2012.