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A Complex-Envelope FDTD Formulation Using Real-Valued Field-Variables

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A COMPLEX-ENVELOPE FDTD FORMULATION USING REAL-VALUED FIELD-VARIABLES

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ABSTRACT

Use of the complex-envelope (CE) representation of band-pass limited sources and their resulting fields increases the allowable time-step in finite-difference time-domain (FDTD) simulations. The complex envelope representation transforms band-pass limited fields and sources to complex-valued low-pass limited form and Maxwells equations from real-valued partial differential equations (PDEs) to complex-valued PDEs. Previous CE FDTD schemes have used complex valued difference equations in terms of complex valued field quantities to approximate these complex PDEs. This choice requires the use of complex numbers and complex operations in the computer program implementing the solution. An alternative CE FDTD scheme using only real numbers and operations can be derived from the real valued PDEs obtained by substituting the rectangular form of the complex field and source quantities into the complex PDEs and then separating each resulting complex PDE into its two equivalent real valued PDEs. The formulation of the CE FDTD using real values is demonstrated for a two-dimensional geometry where the electric field has only a z component. This implicit formulation requires only the solution of tridiagonal matrices. Results are presented for a 2D cavity problem with an electric current source. A reference solution for this problem is obtained by first solving the problem in the frequency domain and then transform it to the time-domain using the inverse fast Fourier transform (IFFT). Comparison of the two solutions demonstrates the accuracy of the new formulation.

DEDICATION

To Qin Yao, my lovely wife, for her patience and encouragement. Without her long term support, I would have given up long time ago.

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I would like to express my sincere gratitude to my advisor, Dr. Paul M. Goggans, a respectable, resourceful and responsible scholar. His idea of applying Complex-Envelope technique with finite-difference time-domain formulation and splitting complex-valued partial differential equations into real ones by using real field-variables is the origin of this thesis. His patient guidance and long term support is what encouraged me to keep working during my years in the University of Mississippi and it is deeply appreciated.

I would also like to thank the faculty of Department of Electrical Engineering, who helped me by providing academic and financial support. In addition, I would like to express my appreciation to the instructors of the classes I attended from whom I learned a tremendous amount of knowledge benefiting me deeply.

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CHAPTER 1

INTRODUCTION

In the past decade the finite-difference time-domain (FDTD) method has been widely used in solving electromagnetic problems. As a tool to solve Maxwell's equations, it is based on approximations that simplifies differential equations into difference equations. There are many different schemes that can be applied in the simplification process. It is well known that implicit methods are better than explicit methods in terms of accuracy and freedom from dispersion. However, implicit methods can involve solving dense matrices at each time step which is time-consuming. Usually further approximations to the difference equations will be made so that the dense matrices will be transformed in to some other form, for example tridiagonal matrices, and consequently a solution can be obtained by simplified calculations. However, because of the additional approximations, additional errors will be introduced and as a result the new schemes may not be stable.

According to the sampling theorem, in the time domain, the sampling frequency f_s has to be larger than $2f_{max}$, where f_{max} is the maximum frequency of a band-pass limited signal. The maximum frequency is a function of the center frequency f_o and bandwidth B. In the time domain, while sampling, the time step (the sampling period) is a function the f_{max}, so that, as f_{max} becomes larger, the time step becomes very small. Consequently due to the sampling theorem requirement, the usual FDTD methods suffer from the disadvantage that the time step is a function of the maximum frequency of the source. Particularly, for high center frequency signals, the time step can be very small which means many time steps will be needed to compute the fields for some fixed length of time. Because errors in each step of calculation accumulate, needing a very large number of time steps to obtain the solution is problematic.

Nevertheless benefitting from introducing complex-envelope (CE) formulations, this problem can be solved by shifting the center frequency of band-pass limited signals to zero resulting in a much smaller f_{max}, especially for signals of narrow bandwidth because now f_{max} is equal to half the bandwidth. By doing so, the time steps could be several orders of magnitude larger than before so that many fewer time steps will be needed. In other words, solution can be obtained much faster than before, even with the extra calculation needed because all fields are complex-valued.

Recently, CE-FDTD formulations are applied to different areas and many alternative forms have been presented. In [1], the advantageous properties of CE-FDTD technique were discussed through its application in transmission line (TL) problems. After using such technique together with the original expression of electric fields from the well-known telegrapher's equations in the TL problem presented, the number of temporal cells has been reduced significantly. By CE formulation, the center frequency was shifted to be zero which means the temporal discretization step only depends on the bandwidth rather than the center frequency, as the sampling theorem implies. In other words, by reducing the maximum frequency of the signal through CE formulations, the time step in FDTD formulation becomes several orders larger than the conventional one. From the result of a 1D example, several advantages of CE-FDTD were demonstrated, such as less memory and processing time and usefulness in the investigation of the immunity effectiveness of a great variety of devices containing TL's against modern spread spectrum communication applications. In [2], unconditionally stable CE split-step perfectly matched layer (PML) algorithm is presented for modeling open region FDTD problems. Based on the CE-FDTD formulations [3], the time step was split into six pieces equally, together with the modification on spatial derivatives in absorbing boundary conditions (ABCs), open region problems were modeled. From the results of a 2D example, a great improvement has been made by the proposed algorithm reducing the maximum reflection error for different Courant-Friedrichs-Lewy (CFL) numbers and for different cell sizes per wavelength. In [4], for modelling PML-ABCs, the alternating direction implicit (ADI) scheme was incorporated into the CE-FDTD implementations of the scalar wave-equation derived in the PML region at the domain boundaries. Through the given 2D example, the proposed technique possesses higher accuracy compared with the classical ADI scalar wave equation PML formulations in modelling band limited electromagnetic applications. In [5], the stability condition for the CE-FDTD with an explicit formulation is examined and explicit formulation of the CE-FDTD are shown, to give up their most valuable property, increased time step, because of stability requirement.

In this thesis, the CE-FDTD formulation is applied to Maxwell's curl equations. Instead of computing complex-valued fields, Maxwell's equations in CE form are split into an equivalent set of real-valued PDEs by grouping the *real* and *imaginary* terms separately. Except for an extra term, these equations possess the same form as the equations derived from Maxwell' equations with real-valued variables. As will be explained in *Section 3* of *Chapter 2*, the extra terms can be treated as part of the known current source. As a result, any implicit scheme, which can be applied to the real-valued equations, can also be applied to the real and imaginary parts of CE forms. In *Chapter 2*, a 2D example is given and solved by applying an approximate Crank-Nicolson scheme. A reference solution is given in *Chapter 3*. In the reference solution, the CE fields are calculated in the frequency domain and then transformed to the time domain. Finally in *Chapter 4*, results from Matlab programs based on those two methods are presented. The results show that the CE FDTD formulation using real-valued field-variables is accurate and stable.

CHAPTER 2

COMPLEX-ENVELOPE FDTD

In this chapter the complex-envelope (CE) finite-difference time-domain (FDTD) using real-valued field-variables method is presented.

In *Section 2.1*, the CE formulation is introduced using a sinusoidal function example. After that we will discuss *pre-envelope* and *Hilbert Transform*, from which the relation between of the Fourier-transform results of one function and its complex envelope will be illustrated.

In *Section 2.2*, the finite-difference (FD) approximation will be discussed. *Central-difference* approximations for the first and second derivatives will be given. Using these approximations partial differential equations (PDEs) can be transformed into difference equations.

In *Section 2.3*, the general 3D formulation of the proposed numerical method will be given. Compared with the original real wave equations, it will be demonstrated that the proposed method possesses the potential to be implemented with any kind of implicit FDTD scheme; however the solutions based upon different schemes may not all be stable.

In *Section 2.4*, an example of a 2D TM_z cavity problem with perfect electric conductor (PEC) boundary conditions will be given to demonstrate the proposed method in practice.

Maxwell's equations of complex fields will be expanded in Cartesian coordinate system and then separated into groups according to the directions of terms. After this, complex fields will be divided into the combination of real parts and imaginary parts, all real and imaginary parts will be organized separately to form two groups of equations.

In *Section 2.5* and *Section 2.6*, both the real group and imaginary group will be discussed in detail, though they are quite similar. Based upon the FD approximation, those PDEs will be transformed into difference equations sampled spatially and temporally by Δx, Δy and Δt. After making some substitution and reorganization, the in-phase part and quadrature part of the electric fields will be obtained respectively by solving only tridiagonal matrices.

2.1 Complex-Envelope Representation

A real band-pass limited signal with bandwidth B can be represented as

$$x(t) = A(t)\cos[2\pi f_o t + \phi(t)] \tag{2.1}$$

where f_o is the center frequency, $A(t)$ is the amplitude and $\phi(t)$ is the phase. Using Euler's equation, (2.1) can be written as

$$x(t) = \Re\Big\{A(t)\exp\left[j\phi(t)\right]\exp\left[j2\pi f_o t\right]\Big\}. \tag{2.2}$$

The complex-envelope representation of $x(t)$ is

$$\tilde{x}(t) = A(t)\exp\left[j\phi(t)\right] = x_p(t) + jx_q(t), \tag{2.3}$$

where $x_p(t)$ and $x_q(t)$ are the in-phase and quadrature portions of $x(t)$. Both $x_p(t)$ and $x_q(t)$ are low-pass limited with bandwidth $B/2$ With (2.4), we can rewrite (2.2) as

$$x(t) = \Re\Big\{\tilde{x}(t)\exp\left[j2\pi f_o t\right]\Big\}, \tag{2.4}$$

or

$$x(t) = \Re\Big\{[x_p(t) + jx_q(t)][\cos(2\pi f_o t) + j\sin(2\pi f_o t)]\Big\}. \tag{2.5}$$

Then the bandpass-limited signal $x(t)$ can be expressed as [6] [3]

$$x(t) = x_p(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t). \tag{2.6}$$

The bandwidth and center frequency of $x(t)$ can be expressed as

$$B = f_{max} - f_{min} \tag{2.7}$$

and

$$f_o = \frac{f_{max} + f_{min}}{2} \tag{2.8}$$

where f_{max} and f_{min} are the maximum and minimum frequencies in $x(t)$.

To understand the relationship between the Fourier transform of $x(t)$ and the Fourier transform of $\tilde{x}(t)$, it is useful to introduce the pre-envelope, $x_+(t)$, and Hilbert transform, $\hat{x}(t)$, of $x(t)$. These three time function are related by th expression

$$x_+(t) = x(t) + j\hat{x}(t). \tag{2.9}$$

Consider the Hilbert Transform pairs

$$H\{a(t)\cos(2\pi f_o t)\} = a(t)\sin(2\pi f_o t) \tag{2.10}$$

and

$$H\{a(t)\sin(2\pi f_o t)\} = -a(t)\cos(2\pi f_o t), \tag{2.11}$$

where $H\{\}$ is the Hilbert Transform operation. After substituting (2.6) into (2.9) we have

$$\begin{aligned}
x_+(t) &= \Big\{x_p(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t)\Big\} + jH\Big\{x_p(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t)\Big\} \\
&= \Big\{x_p(t)\cos(2\pi f_o t) - x_q(t)\sin(2\pi f_o t)\Big\} + j\Big\{x_p(t)\sin(2\pi f_o t) + x_q(t)\cos(2\pi f_o t)\Big\} \\
&= \Big\{x_p(t) + jx_q(t)\Big\}\Big\{\cos(2\pi f_o t) + j\sin(2\pi f_o t)\Big\} \\
&= \tilde{x}(t)\mathrm{e}^{j2\pi f_o t}.
\end{aligned} \tag{2.12}$$

Here, the standard engineering definition of Fourier Transform is used so that

$$X(f) = \int_{-\infty}^{\infty} x(t)\mathrm{e}^{-j2\pi f t} dt \tag{2.13}$$

and

$$x(t) = \int_{-\infty}^{\infty} X(f) \mathrm{e}^{j2\pi ft} dt. \tag{2.14}$$

We obtain the Fourier Transform of the complex envelope signal as

$$\begin{aligned}
\tilde{X}(f) &= F\{\tilde{x}(t)\} \\
&= F\{x_+(t)\mathrm{e}^{-j2\pi f_o t}\} \\
&= F\{x(t)\mathrm{e}^{-j2\pi f_o t} + j\hat{x}(t)\mathrm{e}^{-j2\pi f_o t}\} \\
&= X(f+f_o) + jF\{[x(t) \otimes \frac{1}{\pi t}]\mathrm{e}^{-j2\pi f_o t}\} \\
&= X(f+f_o) + j\{X(f+f_o)[-j\,\mathrm{sgn}(f+f_o)]\} \\
&= X(f+f_o) + \{X(f+f_o)\,\mathrm{sgn}(f+f_o)\},
\end{aligned} \tag{2.15}$$

where

$$\mathrm{sgn}(f) = \begin{cases} 1, & \text{if } f > 0; \\ 0, & \text{if } f = 0; \\ -1, & \text{if } f < 0. \end{cases} \tag{2.16}$$

Using (2.15) and (2.16), the relationship between $X(f)$ and $\tilde{X}(f)$ can be written as

$$\tilde{X}(f) = \begin{cases} 2X(f+f_o) & \text{if } f+f_o > 0 \\ 0 & \text{if } f+f_o \leq 0. \end{cases} \tag{2.17}$$

Thus in frequency domain, $\tilde{X}(f)$ is the positive part of $X(f)$ with double magnitude after shifting down to the baseband with a center frequency of zero. This process is illustrated in Figure 2.1 and Figure 2.2.

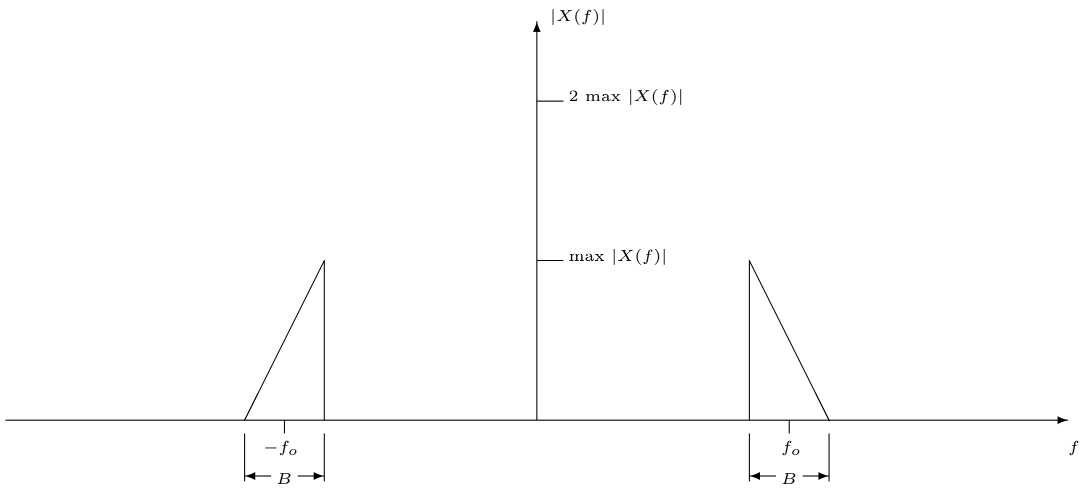


Figure 2.1. The frequency spectrum of a real bandpass-limited signal $X(f)$.

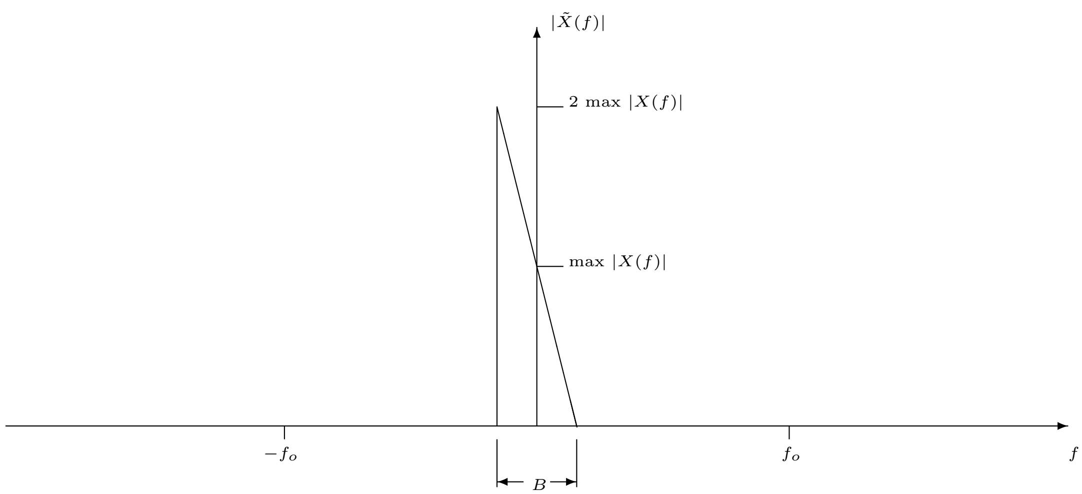


Figure 2.2. The frequency spectrum of the complex envelope of the real bandpass-limited signal $\tilde{X}(f)$.

2.2 Finite-Difference approximations

An arbitrary continuous function can be sampled at discrete points and the accuracy is determined by the sampling rate. After sampling, the discrete function we obtained is used to represent the original continuous one with acceptable error. This technique can be used to approximate the partial differential equations in electromagnetic problems of interest [7].

Consider the an arbitrary continuous function $f(x)$. From the Taylor series we have

$$f(x+\Delta x) = f(x) + \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) + \frac{(\Delta x)^3}{3!} f'''(x) + \frac{(\Delta x)^4}{4!} f''''(x) + ... \tag{2.18}$$

and

$$f(x-\Delta x) = f(x) - \Delta x f'(x) + \frac{(\Delta x)^2}{2!} f''(x) - \frac{(\Delta x)^3}{3!} f'''(x) + \frac{(\Delta x)^4}{4!} f''''(x) + ... \tag{2.19}$$

where $f'(x)$ is the first derivative of $f(x)$ respect to x and $f''(x)$ is the second derivative of $f(x)$ respect to x and so forth. Subtracting the two equations above yields

$$f(x+\Delta x) - f(x-\Delta x) = 2\Delta x f'(x) + 2\frac{(\Delta x)^3}{3!} f'''(x) + ... \tag{2.20}$$

so that

$$\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = f'(x) + \frac{(\Delta x)^2}{3!} f'''(x) + ... \quad . \tag{2.21}$$

Thus

$$\begin{aligned} f'(x) &= \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} - \frac{(\Delta x)^2}{3!} f'''(x) + ... \\ &= \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} + O((\Delta x)^2), \end{aligned} \tag{2.22}$$

where $O((\Delta x)^2)$ is the order of the error meaning that the error is proportional to Δx^2.

Neglecting the error term yields the second-order accurate central-difference approximation to the first derivative

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
&\approx \frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x}.
\end{aligned} \tag{2.23}
$$

Adding (2.18) and (2.19) gives

$$
f(x+\Delta x) + f(x-\Delta x) = 2f(x) + 2\frac{(\Delta x)^2}{2!}f''(x) + 2\frac{(\Delta x)^4}{4!}f''''(x) + ..., \tag{2.24}
$$

so that

$$
\begin{aligned}
f''(x) &= \frac{f(x+\Delta x) + f(x-\Delta x) - 2f(x) - 2\frac{(\Delta x)^4}{4!}f''''(x) + ...}{(\Delta x)^2} \\
&= \frac{f(x+\Delta x) + f(x-\Delta x) - 2f(x)}{(\Delta x)^2} - 2\frac{(\Delta x)^2}{4!}f''''(x) + ... \\
&= \frac{f(x+\Delta x) + f(x-\Delta x) - 2f(x)}{(\Delta x)^2} + O((\Delta x)^2).
\end{aligned} \tag{2.25}
$$

Neglecting the error term yields the second-order accurate approximation to the second-order derivative:

$$
f''(x) = \frac{f(x+\Delta x) + f(x-\Delta x) - 2f(x)}{(\Delta x)^2}. \tag{2.26}
$$

In the following sections the second-order accurate approximation for the first and second derivatives will be used to approximate both temporal and spatial derivatives.

2.3 Three-Dimensional Complex-Envelope FDTD Formulation

The proposed complex-envelope formulation possesses the potential to be used with any type of implicit FDTD scheme based on the Crank-Nicolson approximation [8], although the results from different schemes may not be all stable. In order to explain why, both the *complex-envelope* and wide band *real* wave equations will be presented and compared.

2.3.1 Wave Equations with Complex-Valued Field-Variables

In free space Maxwell's curl equations for real-valued field-variables are

$$\nabla \times \mathbf{H} = \varepsilon_o \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \tag{2.27}$$

and

$$\nabla \times \mathbf{E} = -\mu_o \frac{\partial \mathbf{H}}{\partial t} - \mathbf{M} \tag{2.28}$$

where ε_o and μ_o are the permittivity and permeability of free space. Substituting CE field-variables into the expressions above yields

$$\nabla \times \Re\left\{\tilde{\mathbf{H}} \cdot \mathrm{e}^{j2\pi f_o t}\right\} = \varepsilon_o \frac{\partial \Re\left\{\tilde{\mathbf{E}}\mathrm{e}^{j2\pi f_o t}\right\}}{\partial t} + \Re\left\{\tilde{\mathbf{J}}\mathrm{e}^{j2\pi f_o t}\right\} \tag{2.29}$$

and

$$\nabla \times \Re\left\{\tilde{\mathbf{E}}\mathrm{e}^{j2\pi f_o t}\right\} = -\mu_o \frac{\partial \Re\left\{\tilde{\mathbf{H}}\mathrm{e}^{j2\pi f_o t}\right\}}{\partial t} - \Re\left\{\tilde{\mathbf{M}}\mathrm{e}^{j2\pi f_o t}\right\}. \tag{2.30}$$

where $\tilde{\mathbf{E}}$ is complex envelope of the electric field intensity, $\tilde{\mathbf{H}}$ is complex envelope of the magnetic field intensity, $\tilde{\mathbf{J}}$ is complex envelope of the electric current density, and $\tilde{\mathbf{M}}$ is complex envelope of the magnetic current density. Removing the real operator and dividing through by $\exp(j2\pi f_o t)$ yields the complex envelope version of Maxwell's curl equations

$$\nabla \times \tilde{\mathbf{H}} = \varepsilon_o \frac{\partial \tilde{\mathbf{E}}}{\partial t} + j2\pi f_o \tilde{\mathbf{E}} + \tilde{\mathbf{J}} \tag{2.31}$$

and

$$\nabla \times \tilde{\mathbf{E}} = -\mu_o \frac{\partial \tilde{\mathbf{H}}}{\partial t} - j2\pi f_o \tilde{\mathbf{H}} - \tilde{\mathbf{M}}. \tag{2.32}$$

After expanding the curl operators, (2.31) and (2.32) can be written as

$$\begin{aligned}
&\frac{1}{\varepsilon_o}\Big[\hat{\mathbf{a}}_x\big(\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z}\big) + \hat{\mathbf{a}}_y\big(\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x}\big) + \hat{\mathbf{a}}_z\big(\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y}\big)\Big] \\
&\quad = \hat{\mathbf{a}}_x\big[\frac{\partial \tilde{E}_x}{\partial t} + j2\pi f_o \tilde{E}_x\big] + \hat{\mathbf{a}}_y\big[\frac{\partial \tilde{E}_y}{\partial t} + j2\pi f_o \tilde{E}_y\big] + \hat{\mathbf{a}}_z\big[\frac{\partial \tilde{E}_z}{\partial t} + j2\pi f_o \tilde{E}_z\big] \\
&\qquad + \hat{\mathbf{a}}_x \frac{1}{\varepsilon_o}\tilde{J}_x + \hat{\mathbf{a}}_y \frac{1}{\varepsilon_o}\tilde{J}_y + \hat{\mathbf{a}}_z \frac{1}{\varepsilon_o}\tilde{J}_z
\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}
&\frac{1}{\mu_o}\Big[\hat{\mathbf{a}}_x\big(\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}\big) + \hat{\mathbf{a}}_y\big(\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x}\big) + \hat{\mathbf{a}}_z\big(\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}\big)\Big] \\
&\quad = -\hat{\mathbf{a}}_x\big[\frac{\partial \tilde{H}_x}{\partial t} + j2\pi f_o \tilde{H}_x\big] - \hat{\mathbf{a}}_y\big[\frac{\partial \tilde{H}_y}{\partial t} + j2\pi f_o \tilde{H}_y\big] - \hat{\mathbf{a}}_z\big[\frac{\partial \tilde{H}_z}{\partial t} + j2\pi f_o \tilde{H}_z\big] \\
&\qquad - \hat{\mathbf{a}}_x \frac{1}{\mu_o}\tilde{M}_x - \hat{\mathbf{a}}_y \frac{1}{\mu_o}\tilde{M}_y - \hat{\mathbf{a}}_z \frac{1}{\mu_o}\tilde{M}_z.
\end{aligned} \tag{2.34}$$

Equating the x, y and z component of (2.33) yields

$$\frac{1}{\varepsilon_o}\big(\frac{\partial \tilde{H}_z}{\partial y} - \frac{\partial \tilde{H}_y}{\partial z}\big) = \big[\frac{\partial \tilde{E}_x}{\partial t} + j2\pi f_o \tilde{E}_x\big] + \frac{1}{\varepsilon_o}\tilde{J}_x, \tag{2.35}$$

$$\frac{1}{\varepsilon_o}\big(\frac{\partial \tilde{H}_x}{\partial z} - \frac{\partial \tilde{H}_z}{\partial x}\big) = \big[\frac{\partial \tilde{E}_y}{\partial t} + j2\pi f_o \tilde{E}_y\big] + \frac{1}{\varepsilon_o}\tilde{J}_y \tag{2.36}$$

and

$$\frac{1}{\varepsilon_o}\big(\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y}\big) = \big[\frac{\partial \tilde{E}_z}{\partial t} + j2\pi f_o \tilde{E}_z\big] + \frac{1}{\varepsilon_o}\tilde{J}_z. \tag{2.37}$$

Equating the x, y and z component of (2.34) yields

$$\frac{1}{\mu_o}(\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}) = -[\frac{\partial \tilde{H}_x}{\partial t} + j2\pi f_o \tilde{H}_x] - \frac{1}{\mu_o}\tilde{M}_x, \tag{2.38}$$

$$\frac{1}{\mu_o}(\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x}) = -[\frac{\partial \tilde{H}_y}{\partial t} + j2\pi f_o \tilde{H}_y] - \frac{1}{\mu_o}\tilde{M}_y \tag{2.39}$$

and

$$\frac{1}{\mu_o}(\frac{\partial \tilde{E}_y}{\partial x} - \frac{\partial \tilde{E}_x}{\partial y}) = -[\frac{\partial \tilde{H}_z}{\partial t} + j2\pi f_o \tilde{H}_z] - \frac{1}{\mu_o}\tilde{M}_z. \tag{2.40}$$

After expanding the complex-envelope terms into in-phase and quadrature parts the equations above can be written as

$$\begin{aligned}\frac{1}{\varepsilon_o}[\frac{\partial(H_z^p + jH_z^q)}{\partial y} - \frac{\partial(H_y^p + jH_y^q)}{\partial z}] &= [\frac{\partial(E_x^p + jE_x^q)}{\partial t} + j2\pi f_o\big(E_x^p + jE_x^q\big)] \\ &\quad + \frac{1}{\varepsilon_o}(J_x^p + jJ_x^q),\end{aligned} \tag{2.41}$$

$$\begin{aligned}\frac{1}{\varepsilon_o}[\frac{\partial(H_x^p + jH_x^q)}{\partial z} - \frac{\partial(H_z^p + jH_z^q)}{\partial x}]\mathrm{e} &= [\frac{\partial(E_y^p + jE_y^q)}{\partial t} + j2\pi f_o\big(E_y^p + jE_y^q\big)] \\ &\quad + \frac{1}{\varepsilon_o}(J_y^p + jJ_y^q),\end{aligned} \tag{2.42}$$

$$\begin{aligned}\frac{1}{\varepsilon_o}[\frac{\partial(H_y^p + jH_y^q)}{\partial x} - \frac{\partial(H_x^p + jH_x^q)}{\partial y}] &= [\frac{\partial(E_z^p + jE_z^q)}{\partial t} + j2\pi f_o\big(E_z^p + jE_z^q\big)] \\ &\quad + \frac{1}{\varepsilon_o}(J_z^p + jJ_z^q),\end{aligned} \tag{2.43}$$

$$\begin{aligned}\frac{1}{\mu_o}[\frac{\partial(E_z^p + jE_z^q)}{\partial y} - \frac{\partial(E_y^p + jE_y^q)}{\partial z}] &= -[\frac{\partial(H_x^p + jH_x^q)}{\partial t} + j2\pi f_o\big(H_x^p + jH_x^q\big)] \\ &\quad - \frac{1}{\mu_o}(M_x^p + jM_x^q),\end{aligned} \tag{2.44}$$

$$\begin{aligned}\frac{1}{\mu_o}[\frac{\partial(E_x^p + jE_x^q)}{\partial z} - \frac{\partial(E_z^p + jE_z^q)}{\partial x}] &= -[\frac{\partial(H_y^p + jH_y^q)}{\partial t} + j2\pi f_o\big(H_y^p + jH_y^q\big)] \\ &\quad - \frac{1}{\mu_o}(M_y^p + jM_y^q)\end{aligned} \tag{2.45}$$

and

$$\frac{1}{\mu_o}\Big[\frac{\partial(E_y^p+jE_y^q)}{\partial x}-\frac{\partial(E_x^p+jE_x^q)}{\partial y}\Big]=-\Big[\frac{\partial(H_z^p+jH_z^q)}{\partial t}+j2\pi f_o\big(H_z^p+jH_z^q\big)\Big] \\ -\frac{1}{\mu_o}(M_z^p+jM_z^q). \tag{2.46}$$

Equations (2.41) to (2.46) are similar respect to the structure, so here we take just (2.41) for example. Carrying through the derivative operator in (2.41) yields

$$\begin{aligned}&\frac{1}{\varepsilon_o}\Big[\Big(\frac{\partial H_z^p}{\partial y}+j\frac{\partial H_z^q}{\partial y}\Big)-\Big(\frac{\partial H_y^p}{\partial z}+j\frac{\partial H_y^q}{\partial z}\Big)\Big]\\ &\quad=\frac{\partial E_x^p}{\partial t}+j\frac{\partial E_x^q}{\partial t}+j2\pi f_o\big(E_x^p+jE_x^q\big)+\frac{1}{\varepsilon_o}(J_x^p+jJ_x^q)\mathrm{e}^{j2\pi f_o t}\\ &\quad=\left\{\frac{\partial E_x^p}{\partial t}-2\pi f_o E_x^q+\frac{1}{\varepsilon_o}J_x^p\right\}+j\left\{\frac{\partial E_x^q}{\partial t}+2\pi f_o E_x^p+\frac{1}{\varepsilon_o}J_x^q\right\}.\end{aligned} \tag{2.47}$$

Equating the real and imaginary parts of (2.47) gives

$$\frac{1}{\varepsilon_o}\Big[\frac{\partial H_z^p}{\partial y}-\frac{\partial H_y^p}{\partial z}\Big]=\frac{\partial E_x^p}{\partial t}+\frac{1}{\varepsilon_o}J_x^p-2\pi f_o E_x^q \tag{2.48}$$

and

$$\frac{1}{\varepsilon_o}\Big[\frac{\partial H_z^q}{\partial y}-\frac{\partial H_y^p}{\partial z}\Big]=\frac{\partial E_x^q}{\partial t}+\frac{1}{\varepsilon_o}J_x^q+2\pi f_o E_x^p. \tag{2.49}$$

The CE equations above are entirely real and in terms of real-valued field-variables. The ten equations corresponding to equations (2.42) through (2.46) are similar to the two equations above.

2.3.2 Real Wave Equations

Starting from Maxwell's curl equations with real-valued fields,

$$\nabla\times\mathbf{H}=\frac{\partial\mathbf{D}}{\partial t}+\mathbf{J} \tag{2.50}$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} - \mathbf{M}, \tag{2.51}$$

After expanding the curl operators and equating the x, y and z components we have

$$\frac{1}{\varepsilon_o}(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}) = \frac{\partial E_x}{\partial t} + \hat{\mathbf{a}}_x \frac{1}{\varepsilon_o} J_x, \tag{2.52}$$

$$\frac{1}{\varepsilon_o}(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}) = \frac{\partial E_y}{\partial t} + \hat{\mathbf{a}}_y \frac{1}{\varepsilon_o} J_y, \tag{2.53}$$

$$\frac{1}{\varepsilon_o}(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}) = \frac{\partial E_z}{\partial t} + \frac{1}{\varepsilon_o} J_z, \tag{2.54}$$

$$\frac{1}{\mu_o}(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}) = -\frac{\partial H_x}{\partial t} - \hat{\mathbf{a}}_x \frac{1}{\mu_o} M_x, \tag{2.55}$$

$$\frac{1}{\mu_o}(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}) = -\frac{\partial H_y}{\partial t} - \hat{\mathbf{a}}_y \frac{1}{\mu_o} M_y \tag{2.56}$$

and

$$\frac{1}{\mu_o}(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}) = -\frac{\partial H_z}{\partial t} - \frac{1}{\mu_o} M_z. \tag{2.57}$$

2.3.3 Comparison of The Real Wave Equations and CE Wave Equations with Real Field-Variables

In order to make a comparison, (2.52), (2.48) and (2.49) are repeated here

$$\frac{1}{\varepsilon_o}(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}) = \frac{\partial E_x}{\partial t} + \frac{1}{\varepsilon_o} J_x, \tag{2.52}$$

$$\frac{1}{\varepsilon_o}[\frac{\partial H_z^p}{\partial y} - \frac{\partial H_y^p}{\partial z}] = \frac{\partial E_x^p}{\partial t} + \frac{1}{\varepsilon_o} J_x^p - 2\pi f_o E_x^q \tag{2.48}$$

and

$$\frac{1}{\varepsilon_o}[\frac{\partial H_z^q}{\partial y} - \frac{\partial H_y^p}{\partial z}] = \frac{\partial E_x^q}{\partial t} + \frac{1}{\varepsilon_o} J_x^q + 2\pi f_o E_x^p. \tag{2.49}$$

When computing the real-valued field-variables using the complex-envelope equations, (2.48) and (2.49) are employed alternatively at times offset by half a time step. For example, while processing (2.48), all the terms are unknowns except the last two on the right. There are J_x^p, the source which is known for all time, and E_x^q, which is known by use of (2.49) at the previous half time step. The same situation exists for (2.49) in which J_x^q is known and E_x^p is known by use of (2.48) at the previous half time step. In this way, the time varying fields can be calculated by employing (2.48) and (2.49) alternately at half time steps.

Now compare (2.48) with (2.52). All the terms are exactly the same except the notation p and the term E_x^q. Since E_x^q is known when we use (2.48), it can be treated as part of the source and after doing that all the unknowns are positioned the same as they are in (2.52) which means any implicit FDTD scheme that can be used for the *real* wave equations is potentially appropriate for the *complex-envelope* equations, although the stability of the new method is not certain. However after considering the high similarity of (2.41) to (2.46) one can make the conclusion that the *complex-envelope* equations possess the potential application with any type of implicit FDTD scheme.

2.4 Two-Dimensional Cavity Problem with TM_z Polarized Source

To illustrate the CE-FDTD method with real-valued field-variables, a two-dimensional cavity problem, infinite in the z-direction, was chosen as an example. In the example the source current results in fields that are transverse-magnetic to z (TM_z) so that $\tilde{E}_x = \tilde{E}_y = \tilde{H}_z = 0$, $\tilde{\mathbf{J}} = \hat{\mathbf{a}}_z \tilde{J}_z$ and $\tilde{\mathbf{M}} = \hat{\mathbf{a}}_x \tilde{M}_x + \hat{\mathbf{a}}_y \tilde{M}_y$.

Because the fields are independent of z, all fields in (2.35), (2.36) and (2.40) are zero; however (2.37), (2.38) and (2.39) can be written as

$$\frac{1}{\varepsilon_o}(\frac{\partial \tilde{H}_y}{\partial x} - \frac{\partial \tilde{H}_x}{\partial y}) = \frac{\partial \tilde{E}_z}{\partial t} + j2\pi f_o \tilde{E}_z + \frac{1}{\varepsilon_o}\tilde{J}_z, \tag{2.58}$$

$$\frac{1}{\mu_o}(\frac{\partial \tilde{E}_z}{\partial y} - \frac{\partial \tilde{E}_y}{\partial z}) = -[\frac{\partial \tilde{H}_x}{\partial t} + j2\pi f_o \tilde{H}_x] - \frac{1}{\mu_o}\tilde{M}_x \tag{2.59}$$

and

$$\frac{1}{\mu_o}(\frac{\partial \tilde{E}_x}{\partial z} - \frac{\partial \tilde{E}_z}{\partial x}) = -[\frac{\partial \tilde{H}_y}{\partial t} + j2\pi f_o \tilde{H}_y] - \frac{1}{\mu_o}\tilde{M}_y. \tag{2.60}$$

Here using real part terms and imaginary terms respectively to build new equations, each of the three equations above can be split into two groups as *Real Terms Equations*

$$\frac{1}{\varepsilon_o}[\frac{\partial H_y^p}{\partial x} - \frac{\partial H_x^p}{\partial y}] = [\frac{\partial E_z^p}{\partial t} - 2\pi f_o E_z^q + \frac{1}{\varepsilon_o} J_z^p], \tag{2.61}$$

$$\frac{1}{\mu_o}\frac{\partial E_z^p}{\partial y} = -[\frac{\partial H_x^p}{\partial t} - 2\pi f_o H_x^q + \frac{1}{\mu_o} M_x^p] \tag{2.62}$$

and

$$-\frac{1}{\mu_o}\frac{\partial E_z^p}{\partial x} = -[\frac{\partial H_y^p}{\partial t} - 2\pi f_o H_y^q + \frac{1}{\mu_o} M_y^p], \tag{2.63}$$

and *Imaginary Terms Equations*

$$j\frac{1}{\varepsilon_o}\Big[\frac{\partial H_y^q}{\partial x}-\frac{\partial H_x^q}{\partial y}\Big]=j\Big[\frac{E_z^q}{\partial t}+2\pi f_oE_z^p+\frac{1}{\varepsilon_o}J_z^q\Big], \tag{2.64}$$

$$j\frac{1}{\mu_o}\frac{\partial E_z^q}{\partial y}=-j\Big[\frac{H_x^q}{\partial t}+2\pi f_oH_x^p+\frac{1}{\mu_o}M_x^q\Big] \tag{2.65}$$

and

$$-j\frac{1}{\mu_o}\frac{\partial E_z^q}{\partial x}=-j\Big[\frac{H_y^q}{\partial t}+2\pi f_oH_y^p+\frac{1}{\mu_o}M_y^q\Big]. \tag{2.66}$$

Finally after rearranging the equations, we obtain *real part equations* as

$$\frac{\partial H_x^p}{\partial t}=-\frac{1}{\mu_o}M_x^p-\frac{1}{\mu_o}\frac{\partial E_z^p}{\partial y}+2\pi f_oH_x^q, \tag{2.67a}$$

$$\frac{\partial H_y^p}{\partial t}=-\frac{1}{\mu_o}M_y^p+\frac{1}{\mu_o}\frac{\partial E_z^p}{\partial x}+2\pi f_oH_y^q, \tag{2.67b}$$

$$\frac{\partial E_z^p}{\partial t}=\frac{1}{\varepsilon_o}\Big[\frac{\partial H_y^p}{\partial x}-\frac{\partial H_x^p}{\partial y}-J_z^p\Big]+2\pi f_oE_z^q, \tag{2.67c}$$

and *imaginary part equations* as

$$\frac{\partial H_x^q}{\partial t}=-\frac{1}{\mu_o}M_x^q-\frac{1}{\mu_o}\frac{\partial E_z^q}{\partial y}-2\pi f_oH_x^p, \tag{2.68a}$$

$$\frac{\partial H_y^q}{\partial t}=-\frac{1}{\mu_o}M_y^q+\frac{1}{\mu_o}\frac{\partial E_z^q}{\partial x}-2\pi f_oH_y^p, \tag{2.68b}$$

$$\frac{\partial E_z^q}{\partial t}=\frac{1}{\varepsilon_o}\Big[\frac{\partial H_y^q}{\partial x}-\frac{H_x^q}{\partial y}-J_z^q\Big]-2\pi f_oE_z^p. \tag{2.68c}$$

2.5 Real Part of Curl Equations

Now consider the equations group (2.67). Using the *central-difference* approximation centered at time point $n+\frac{1}{2}$, (2.67) can be approximated as

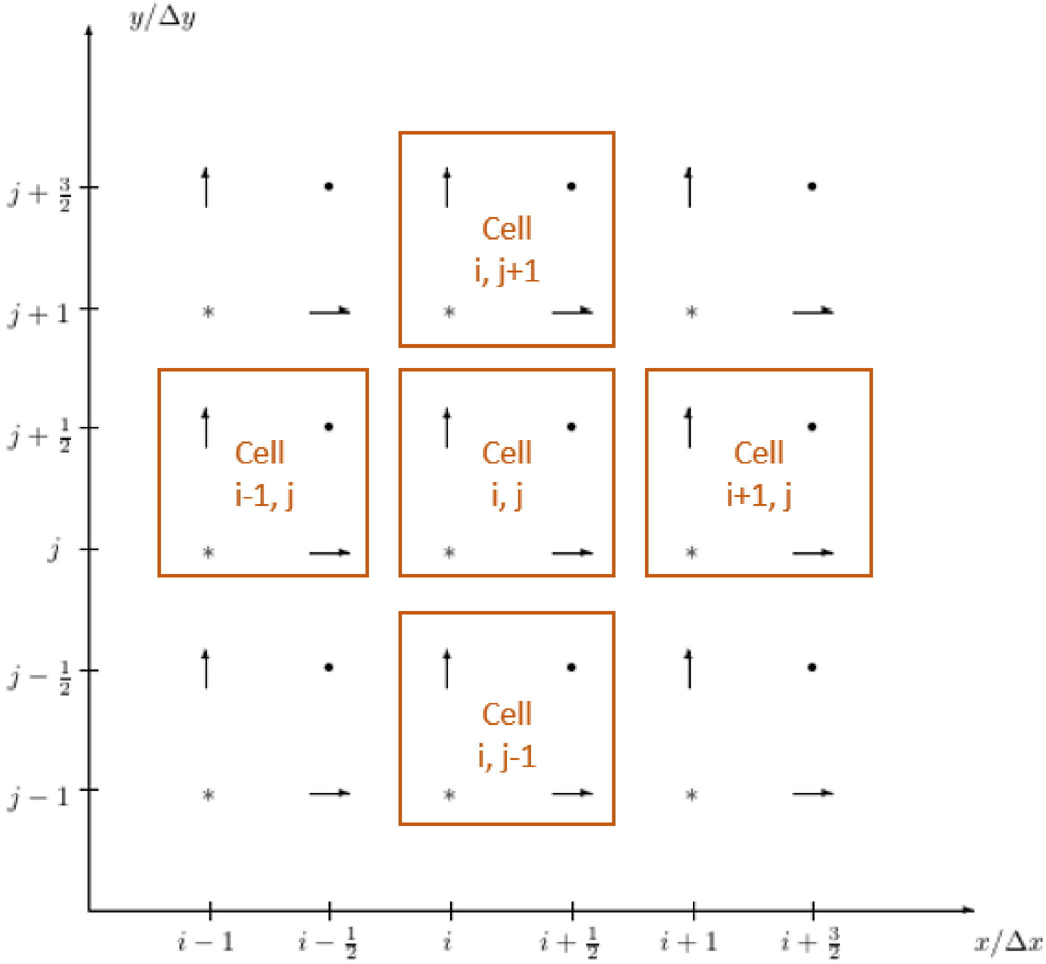
$$\frac{E_z^p|^{n+1} - E_z^p|^n}{\Delta t} = \frac{1}{\varepsilon_o}\Big[\mathrm{D}_x(\frac{H_y^p|^{n+1} + H_y^p|^n}{2}) - \mathrm{D}_y(\frac{H_x^p|^{n+1} + H_x^p|^n}{2}) - (\frac{J_z^p|^{n+1} + J_z^p|^n}{2})\Big] + 2\pi f_o E_z^q|^{n+\frac{1}{2}}, \tag{2.69}$$

$$\frac{H_x^p|^{n+1} - H_x^p|^n}{\Delta t} = -\frac{1}{\mu_o}\mathrm{D}_y(\frac{E_z^p|^{n+1} + E_z^p|^n}{2}) - \frac{1}{2\mu_o}\Big[M_x^p|^{n+1} + M_x^p|^n\Big] + 2\pi f_o H_x^q|^{n+\frac{1}{2}} \tag{2.70}$$

and

$$\frac{H_y^p|^{n+1} - H_y^p|^n}{\Delta t} = \frac{1}{\mu_o}\mathrm{D}_x(\frac{E_z^p|^{n+1} + E_z^p|^n}{2}) - \frac{1}{2\mu_o}\Big[M_y^p|^{n+1} + M_y^p|^n\Big] + 2\pi f_o H_y^q|^{n+\frac{1}{2}}. \tag{2.71}$$

Here the operators D_x and D_y are the *central difference* approximations of the derivatives on x and y. In order to explain how the fields in the equations above are sampled, the Yee grid is illustrated in Figure 2.3.



Symbols for cell i, j:

$*$ Cell reference point at $x = i\Delta x$, $y = j\Delta y$.

$\bullet$ $E_z|_{i,j} = E_z\big(x = [i + 1/2]\Delta x, y = [j + 1/2]\Delta y\big)$

$\rightarrow$ $H_x|_{i,j} = H_x\big(x = [i + 1/2]\Delta x, y = j\Delta y)\big)$

$\uparrow$ $H_y|_{i,j} = H_y\big(x = i\Delta x, y = [j + 1/2]\Delta y)\big)$

Figure 2.3. The Yee Cell — after [9].

The integer i and j are the spatial indices of the fields on the $x-y$ mesh grid. Expanding the D_x and D_y operators centered at the position of E_z in cell i,j gives

$$\begin{aligned}\frac{E_z^p|_{i,j}^{n+1}-E_z^p|_{i,j}^{n}}{\Delta t} &= \frac{1}{\varepsilon_o}\frac{(H_y^p|_{i+1,j}^{n+1}-H_y^p|_{i,j}^{n+1})+(H_y^p|_{i+1,j}^{n}-H_y^p|_{i,j}^{n})}{2\Delta x}\\ &-\frac{1}{\varepsilon_o}\frac{(H_x^p|_{i,j+1}^{n+1}-H_x^p|_{i,j}^{n+1})+(H_x^p|_{i,j+1}^{n}-H_x^p|_{i,j}^{n})}{2\Delta y}\\ &-\frac{1}{\varepsilon_o}(\frac{J_z^p|_{i,j}^{n+1}+J_z^p|_{i,j}^{n}}{2})\\ &+2\pi f_o E_z^q|_{i,j}^{n+\frac{1}{2}}, \qquad (2.72)\\ \frac{H_x^p|_{i,j}^{n+1}-H_x^p|_{i,j}^{n}}{\Delta t} &= -\frac{1}{\mu_o}\frac{(E_z^p|_{i,j}^{n+1}-E_z^p|_{i,j-1}^{n+1})+(E_z^p|_{i,j}^{n}-E_z^p|_{i,j-1}^{n})}{2\Delta y}\\ &-\frac{1}{2\mu_o}\left[M_x^p|_{i,j}^{n+1}+M_x^p|_{i,j}^{n}\right]+2\pi f_o H_x^q|_{i,j}^{n+\frac{1}{2}}, \qquad (2.73)\end{aligned}$$

and

$$\begin{aligned}\frac{H_y^p|_{i,j}^{n+1}-H_y^p|_{i,j}^{n}}{\Delta t} &= \frac{1}{\mu_o}\frac{(E_z^p|_{i,j}^{n+1}-E_z^p|_{i-1,j}^{n+1})+(E_z^p|_{i,j}^{n}-E_z^p|_{i-1,j}^{n})}{2\Delta x}\\ &-\frac{1}{2\mu_o}\left[M_y^p|_{i,j}^{n+1}+M_y^p|_{i,j}^{n}\right]+2\pi f_o H_y^q|_{i,j}^{n+\frac{1}{2}}. \qquad (2.74)\end{aligned}$$

After rearrangement, we have

$$\begin{aligned}E_z^p|_{i,j}^{n+1}-E_z^p|_{i,j}^{n} &= \frac{\Delta t}{2\Delta x\varepsilon_o}\left[(H_y^p|_{i+1,j}^{n+1}-H_y^p|_{i,j}^{n+1})+(H_y^p|_{i+1,j}^{n}-H_y^p|_{i,j}^{n})\right]\\ &-\frac{\Delta t}{2\Delta y\varepsilon_o}\left[(H_x^p|_{i,j+1}^{n+1}-H_x^p|_{i,j}^{n+1})+(H_x^p|_{i,j+1}^{n}-H_x^p|_{i,j}^{n})\right]\\ &-\frac{\Delta t}{2\varepsilon_o}\left[J_z^p|_{i,j}^{n+1}+J_z^p|_{i,j}^{n}\right]\\ &+\Delta t 2\pi f_o E_z^q|_{i,j}^{n+\frac{1}{2}}, \qquad (2.75)\\ H_x^p|_{i,j}^{n+1}-H_x^p|_{i.j}^{n} &= -\frac{\Delta t}{2\Delta y\mu_o}\left[(E_z^p|_{i,j}^{n+1}-E_z^p|_{i,j-1}^{n+1})+(E_z^p|_{i,j}^{n}-E_z^p|_{i,j-1}^{n})\right]\\ &-\frac{\Delta t}{2\mu_o}\left[M_x^p|_{i,j}^{n+1}+M_x^p|_{i,j}^{n}\right]+\Delta t 2\pi f_o H_x^q|_{i,j}^{n+\frac{1}{2}} \qquad (2.76)\end{aligned}$$

and

$$H_y^p\Big|_{i,j}^{n+1} - H_y^p\Big|_{i,j}^{n} = \frac{\Delta t}{2\Delta x \mu_o}\Big[(E_z^p\Big|_{i,j}^{n+1} - E_z^p\Big|_{i-1,j}^{n+1}) + (E_z^p\Big|_{i,j}^{n} - E_z^p\Big|_{i-1,j}^{n})\Big]$$
$$- \frac{\Delta t}{2\mu_o}\Big[M_y^p\Big|_{i,j}^{n+1} + M_y^p\Big|_{i,j}^{n}\Big] + \Delta t 2\pi f_o H_y^q\Big|_{i,j}^{n+\frac{1}{2}}. \tag{2.77}$$

Now in order to simplify the equations, new constants are introduced as follows:

$$\frac{\Delta t}{2\Delta x \varepsilon_o} = \frac{\eta_o}{2}\alpha_x, \tag{2.78}$$

$$\frac{\Delta t}{2\Delta x \mu_o} = \frac{1}{2\eta_o}\alpha_x, \tag{2.79}$$

$$\frac{\Delta t}{2\Delta y \varepsilon_o} = \frac{\eta_o}{2}\alpha_y \tag{2.80}$$

and

$$\frac{\Delta t}{2\Delta y \mu_o} = \frac{1}{2\eta_o}\alpha_y \tag{2.81}$$

where

$$\alpha_x = \frac{\Delta t}{\sqrt{\mu_o \varepsilon_o}\Delta x} = \frac{c\Delta t}{\Delta x} \tag{2.82}$$

and

$$\alpha_y = \frac{\Delta t}{\sqrt{\mu_o \varepsilon_o}\Delta y} = \frac{c\Delta t}{\Delta y}. \tag{2.83}$$

Here c is the speed of light and η_o is the characteristic impedance in free space. If we define the new source terms as

$$S_z^p\Big|_{i,j}^{n} = -\frac{\Delta t}{2\varepsilon_o}\Big[J_z^p\Big|_{i,j}^{n+1} + J_z^p\Big|_{i,j}^{n}\Big], \tag{2.84}$$

$$T_x^p\Big|_{i,j}^{n} = -\frac{\Delta t}{2\mu_o}\Big[M_x^p\Big|_{i,j}^{n+1} + M_x^p\Big|_{i,j}^{n}\Big], \tag{2.85}$$

and

$$R_y^p\Big|_{i,j}^{n} = -\frac{\Delta t}{2\mu_o}\Big[M_y^p\Big|_{i,j}^{n+1} + M_y^p\Big|_{i,j}^{n}\Big], \tag{2.86}$$

then (2.75) to (2.77) can be rewritten as

$$
\begin{aligned}
E_z^p\big|_{i,j}^{n+1} = E_z^p\big|_{i,j}^{n} &+ \frac{\eta}{2}\alpha_x\big[(H_y^p\big|_{i+1,j}^{n+1} - H_y^p\big|_{i,j}^{n+1}) + (H_y^p\big|_{i+1,j}^{n} - H_y^p\big|_{i,j}^{n})\big] \\
&- \frac{\eta}{2}\alpha_y\big[(H_x^p\big|_{i,j+1}^{n+1} - H_x^p\big|_{i,j}^{n+1}) + (H_x^p\big|_{i,j+1}^{n} - H_x^p\big|_{i,j}^{n})\big] \\
&+ S_z^p\big|_{i,j}^{n} \\
&+ \Delta t 2\pi f_o E_z^q\big|_{i,j}^{n+\frac{1}{2}}, \qquad (2.87) \\
H_x^p\big|_{i,j}^{n+1} = H_x^p\big|_{i,j}^{n} &- \frac{1}{2\eta}\alpha_y\big[(E_z^p\big|_{i,j}^{n+1} - E_z^p\big|_{i,j-1}^{n+1}) + (E_z^p\big|_{i,j}^{n} - E_z^p\big|_{i,j-1}^{n})\big] \\
&+ T_x^p\big|_{i,j}^{n} + \Delta t 2\pi f_o H_x^q\big|_{i,j}^{n+\frac{1}{2}} \qquad (2.88)
\end{aligned}
$$

and

$$
\begin{aligned}
H_y^p\big|_{i,j}^{n+1} = H_y^p\big|_{i,j}^{n} &+ \frac{1}{2\eta}\alpha_x\big[(E_z^p\big|_{i,j}^{n+1} - E_z^p\big|_{i-1,j}^{n+1}) + (E_z^p\big|_{i,j}^{n} - E_z^p\big|_{i-1,j}^{n})\big] \\
&+ R_y^p\big|_{i,j}^{n} + \Delta t 2\pi f_o H_y^q\big|_{i,j}^{n+\frac{1}{2}}. \qquad (2.89)
\end{aligned}
$$

Consider (2.87) can also be written as

$$
\begin{aligned}
E_z^p\big|_{i,j}^{n+1} &= E_z^p\big|_{i,j}^{n} \\
&+ \frac{\alpha_x}{2}\big[\eta_o H_y^p\big|_{i+1,j}^{n+1}\big] \\
&- \frac{\alpha_x}{2}\big[\eta_o H_y^p\big|_{i,j}^{n+1}\big] \\
&+ \frac{\alpha_x}{2}\big[\eta_o H_y^p\big|_{i+1,j}^{n} - H_y^p\big|_{i,j}^{n}\big] \\
&- \frac{\alpha_y}{2}\big[\eta_o H_x^p\big|_{i,j+1}^{n+1}\big] \\
&+ \frac{\alpha_y}{2}\big[\eta_o H_x^p\big|_{i,j}^{n+1}\big] \\
&- \frac{\alpha_y}{2}\big[\eta_o H_x^p\big|_{i,j+1}^{n} - \eta H_x^p\big|_{i,j}^{n}\big] \\
&+ S_z^p\big|_{i,j}^{n} + \Delta t 2\pi f_o E_z^q\big|_{i,j}^{n+\frac{1}{2}}. \qquad (2.90)
\end{aligned}
$$

Now after substituting (2.88) and (2.89) into (2.90), we have

$$
\begin{aligned}
E_z^p\big|_{i,j}^{n+1} = E_z^p\big|_{i,j}^{n} \\
&+ \frac{\alpha_x}{2}\Big\{\eta_o H_y^p\big|_{i+1,j}^{n} + \frac{\alpha_x}{2}\big[(E_z^p\big|_{i+1,j}^{n+1} - E_z^p\big|_{i,j}^{n+1}) + (E_z^p\big|_{i+1,j}^{n} - E_z^p\big|_{i,j}^{n})\big] \\
&\qquad + \eta_o R_y^p\big|_{i+1,j}^{n} + \eta_o \Delta t 2\pi f_o H_y^q\big|_{i+1,j}^{n+\frac{1}{2}}\Big\} \\
&- \frac{\alpha_x}{2}\Big\{\eta_o H_y^p\big|_{i,j}^{n} + \frac{\alpha_x}{2}\big[(E_z^p\big|_{i,j}^{n+1} - E_z^p\big|_{i-1,j}^{n+1}) + (E_z^p\big|_{i,j}^{n} - E_z^p\big|_{i-1,j}^{n})\big] \\
&\qquad + \eta_o R_y^p\big|_{i,j}^{n} + \eta_o \Delta t 2\pi f_o H_y^q\big|_{i,j}^{n+\frac{1}{2}}\Big\} \\
&+ \frac{\alpha_x}{2}\big[\eta_o H_y^p\big|_{i+1,j} - \eta_o H_y^p\big|_{i,j}^{n}\big] \\
&- \frac{\alpha_y}{2}\Big\{\eta_o H_x^p\big|_{i,j+1}^{n} - \frac{\alpha_y}{2}\big[(E_z^p\big|_{i,j+1}^{n+1} - E_z^p\big|_{i,j}^{n+1}) + (E_z^p\big|_{i,j+1}^{n} - E_z^p\big|_{i,j}^{n})\big] \\
&\qquad + \eta_o T_x^p\big|_{i,j+1}^{n} + \eta_o \Delta t 2\pi f_o H_x^q\big|_{i,j+1}^{n+\frac{1}{2}}\Big\} \\
&+ \frac{\alpha_y}{2}\Big\{\eta_o H_x^p\big|_{i,j}^{n} - \frac{\alpha_y}{2}\big[(E_z^p\big|_{i,j}^{n+1} - E_z^p\big|_{i,j-1}^{n+1}) + (E_z^p\big|_{i,j}^{n} - E_z^p\big|_{i,j-1}^{n})\big] \\
&\qquad + \eta_o T_x^p\big|_{i,j}^{n} + \eta_o \Delta t 2\pi f_o H_x^q\big|_{i,j}^{n+\frac{1}{2}}\Big\} \\
&- \frac{\alpha_y}{2}\big[\eta_o H_x^p\big|_{i,j+1}^{n} - \eta_o H_x^p\big|_{i,j}^{n}\big] \\
&+ S_z^p\big|_{i,j}^{n} + \Delta t 2\pi f_o E_z^q\big|_{i,j}^{n+\frac{1}{2}}. \qquad (2.91)
\end{aligned}
$$

Now if we expand those brackets

$$
\begin{aligned}
E_z^p|_{i,j}^{n+1} = E_z^p|_{i,j}^{n} \\
&+ \Big\{ \frac{\alpha_x}{2}\eta H_y^p|_{i+1,j}^{n} + \frac{\alpha_x^2}{4}(E_z^p|_{i+1,j}^{n+1} - E_z^p|_{i,j}^{n+1}) + \frac{\alpha_x^2}{4}(E_z^p|_{i+1,j}^{n} - E_z^p|_{i,j}^{n}) \\
&\qquad + \eta_o R_y^p|_{i+1,j}^{n} + \frac{\alpha_x}{2}\eta \Delta t 2\pi f_o H_y^q|_{i+1,j}^{n+\frac{1}{2}} \Big\} \\
&- \Big\{ \frac{\alpha_x}{2}\eta H_y^p|_{i,j}^{n} + \frac{\alpha_x^2}{4}(E_z^p|_{i,j}^{n+1} - E_z^p|_{i-1,j}^{n+1}) + \frac{\alpha_x^2}{4}(E_z^p|_{i,j}^{n} - E_z^p|_{i-1,j}^{n}) \\
&\qquad + \eta_o R_y^p|_{i,j}^{n} + \frac{\alpha_x}{2}\eta \Delta t 2\pi f_o H_y^q|_{i,j}^{n+\frac{1}{2}} \Big\} \\
&+ \frac{\alpha_x}{2}\big[\eta H_y^p|_{i+1,j} - H_y^p|_{i,j}^{n}\big] \\
&- \Big\{ \frac{\alpha_y}{2}\eta H_x^p|_{i,j+1}^{n} - \frac{\alpha_y^2}{4}(E_z^p|_{i,j+1}^{n+1} - E_z^p|_{i,j}^{n+1}) - \frac{\alpha_y^2}{4}(E_z^p|_{i,j+1}^{n} - E_z^p|_{i,j}^{n}) \\
&\qquad + \eta_o T_x^p|_{i,j+1}^{n} + \frac{\alpha_x}{2}\eta \Delta t 2\pi f_o H_x^q|_{i,j+1}^{n+\frac{1}{2}} \Big\} \\
&+ \Big\{ \frac{\alpha_y}{2} H_x^p|_{i,j}^{n} - \frac{\alpha_y^2}{4}(E_z^p|_{i,j}^{n+1} - E_z^p|_{i,j-1}^{n+1}) - \frac{\alpha_y^2}{4}(E_z^p|_{i,j}^{n} - E_z^p|_{i,j-1}^{n})\big] \\
&\qquad + \eta_o T_x^p|_{i,j}^{n} + \frac{\alpha_y}{2}\eta \Delta t 2\pi f_o H_x^q|_{i,j}^{n+\frac{1}{2}} \Big\} \\
&- \frac{\alpha_y}{2}\big[\eta H_x^p|_{i,j+1}^{n} - \eta H_x^p|_{i,j}^{n}\big] \\
&+ S_z^p|_{i,j}^{n} + \Delta t 2\pi f_o E_z^q|_{i,j}^{n+\frac{1}{2}},
\end{aligned} \tag{2.92}
$$

that is

$$
\begin{aligned}
&\big(1 + \frac{\alpha_x^2}{2} + \frac{\alpha_y^2}{2}\big) E_z^p|_{i,j}^{n+1} - \frac{\alpha_x^2}{4}\big(E_z^p|_{i+1,j}^{n+1} + E_z^p|_{i-1,j}^{n+1}\big) - \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^{n+1} + E_z^p|_{i,j-1}^{n+1}\big) \\
&\quad = \big(1 + \frac{\alpha_x^2}{2} + \frac{\alpha_y^2}{2}\big) E_z^p|_{i,j}^{n} + \frac{\alpha_y^2}{4}\big(E_z^p|_{i+1,j}^{n} + E_z^p|_{i-1,j}^{n}\big) + \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^{n} + E_z^p|_{i,j-1}^{n}\big) \\
&\quad + \alpha_x\big[\eta H_y^p|_{i+1,j}^{n} - \eta H_y^p|_{i,j}^{n}\big] + \frac{2\pi f_o \Delta t \alpha_x}{2}\big[\eta H_y^q|_{i+1,j}^{n+\frac{1}{2}} - \eta H_y^q|_{i,j}^{n+\frac{1}{2}}\big] \\
&\quad - \alpha_y\big[\eta H_x^p|_{i,j+1}^{n} - \eta H_x^p|_{i,j}^{n}\big] - \frac{2\pi f_o \Delta t \alpha_y}{2}\big[\eta H_x^q|_{i,j+1}^{n+\frac{1}{2}} - \eta H_x^q|_{i,j}^{n+\frac{1}{2}}\big] + 2\pi f_o \Delta t E_z^q|_{i,j}^{n+\frac{1}{2}} \\
&\quad + S_z^p|_{i,j}^{n} + \frac{\alpha_x}{2}\big[\eta_o R_y^p|_{i+1,j}^{n} - \eta_o R_y^p|_{i,j}^{n}\big] - \frac{\alpha_y}{2}\big[\eta_o T_x^p|_{i,j+1}^{n} - \eta_o T_x^p|_{i,j}^{n}\big].
\end{aligned} \tag{2.93}
$$

Here for simplicity we define

$$F_R^n = \alpha_x\big[\eta H_y^p|_{i+1,j}^n - \eta H_y^p|_{i,j}^n\big] + \frac{2\pi f_o\Delta t\alpha_x}{2}\big[\eta H_y^q|_{i+1,j}^{n+\frac{1}{2}} - \eta H_y^q|_{i,j}^{n+\frac{1}{2}}\big]$$
$$- \alpha_y\big[\eta H_x^p|_{i,j+1}^n - \eta H_x^p|_{i,j}^n\big] - \frac{2\pi f_o\Delta t\alpha_y}{2}\big[\eta H_x^q|_{i,j+1}^{n+\frac{1}{2}} - \eta H_x^q|_{i,j}^{n+\frac{1}{2}}\big] + 2\pi f_o\Delta t E_z^q|_{i,j}^{n+\frac{1}{2}} \quad (2.94)$$

and

$$W^p|_{i,j}^n = S_z^p|_{i,j}^n + \frac{\alpha_x}{2}\big[R_y^p|_{i+1,j}^n - R_y^p|_{i,j}^n\big] - \frac{\alpha_y}{2}\big[T_x^p|_{i,j+1}^n - T_x^p|_{i,j}^n\big], \quad (2.95)$$

then after reorganizing the terms, (2.93) can be written as

$$E_z^p|_{i,j}^{n+1} - \frac{\alpha_x^2}{4}\big(E_z^p|_{i+1,j}^{n+1} - 2E_z^p|_{i,j}^{n+1} + E_z^p|_{i-1,j}^{n+1}\big) - \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^{n+1} - 2E_z^p|_{i,j}^{n+1} + E_z^p|_{i,j-1}^{n+1}\big)$$
$$= E_z^p|_{i,j}^n + \frac{\alpha_y^2}{4}\big(E_z^p|_{i+1,j}^n - 2E_z^p|_{i,j}^n + E_z^p|_{i-1,j}^n\big) + \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^n - 2E_z^p|_{i,j}^n + E_z^p|_{i,j-1}^n\big)$$
$$+ F_R^n + W^p|_{i,j}^n \quad (2.96)$$

Then after expanding α_x and α_y, we have

$$E_z^p|_{i,j}^{n+1} - \left(\frac{c\Delta t}{2}\right)^2\left[\frac{E_z^p|_{i+1,j}^{n+1} - 2E_z^p|_{i,j}^{n+1} + E_z^p|_{i-1,j}^{n+1}}{(\Delta x)^2}\right] - \left(\frac{c\Delta t}{2}\right)^2\left[\frac{E_z^p|_{i,j+1}^{n+1} - 2E_z^p|_{i,j}^{n+1} + E_z^p|_{i,j-1}^{n+1}}{(\Delta y)^2}\right]$$
$$= E_z^p|_{i,j}^n + \left(\frac{c\Delta t}{2}\right)^2\left[\frac{E_z^p|_{i+1,j}^n - 2E_z^p|_{i,j}^n + E_z^p|_{i-1,j}^n}{(\Delta x)^2}\right]$$
$$+ \left(\frac{c\Delta t}{2}\right)^2\left[\frac{E_z^p|_{i,j+1}^n - 2E_z^p|_{i,j}^n + E_z^p|_{i,j-1}^n}{(\Delta y)^2}\right] + F_R^n + W^p|_{i,j}^n. \quad (2.97)$$

Here if we define

$$b = \frac{c\Delta t}{2} \quad (2.98)$$

and consider(2.97), then

$$(1 - b^2D_{2x} - b^2D_{2y})E_z^p|_{i,j}^{n+1} = (1 + b^2D_{2x} + b^2D_{2y})E_z^p|_{i,j}^n + F_R^n + W^p|_{i,j}^n \quad (2.99)$$

Here D_{2x} and D_{2y} are the *second-order central-difference* approximations with respect to x

and y. From [10], in this case one type of approximation has been proposed so that (2.99) can be written as

$$(1-b^2D_{2x})(1-b^2D_{2y})E_z^p|_{i,j}^{n+1} = (1+b^2D_{2x})(1+b^2D_{2y})E_z^p|_{i,j}^n + F_R^n + W^p|_{i,j}^n \tag{2.100}$$

Now if we define

$$E_{ss}^p|_{i,j} = \big(1-b^2D_{2y}\big)E_z^p|_{i,j}^{n+1} \tag{2.101}$$

then the left hand side of (2.100) can be written as

$$\begin{aligned}\big(1-b^2D_{2x}\big)E_{ss}^p|_{i,j} &= E_{ss}^p|_{i,j} - b^2D_{2x}E_{ss}^q|_{i,j}\\ &= E_{ss}^p|_{i,j} - (\frac{c\Delta t}{2})^2\frac{1}{\Delta x^2}\big[E_{ss}^p|_{i+1,j} - 2E_{ss}^p|_{i,j} + E_{ss}^p|_{i-1,j}\big]\\ &= \big(1+\frac{\alpha_x^2}{2}\big)E_{ss}^p|_{i,j} - \frac{\alpha_x^2}{4}\big(E_{ss}^p|_{i+1,j} + E_{ss}^p|_{i-1,j}\big).\end{aligned} \tag{2.102}$$

Now consider $E_{ss}^p|_{i,j}$, similarly we have

$$\big(1-b^2D_{2y}\big)E_z^p|_{i,j}^{n+1} = \big(1+\frac{\alpha_y^2}{2}\big)E_z^p|_{i,j}^{n+1} - \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^{n+1} + E_z^p|_{i,j-1}^{n+1}\big). \tag{2.103}$$

Consider the right hand side of (2.100). Similarly we have

$$\big(1+b^2D_{2y}\big)E_z^p|_{i,j}^n = \big(1-\frac{\alpha_y^2}{2}\big)E_z^p|_{i,j}^n + \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^n + E_z^p|_{i,j-1}^n\big), \tag{2.104}$$

then

$$\begin{aligned}\big(1+b^2D_{2x}\big)\big(1+b^2D_{2y}\big)E_z^p|_{i,j}^n &= \big(1+b^2D_{2y}\big)E_z^p|_{i,j}^n + b^2D_{2x}\Big[\big(1+b^2D_{2y}\big)E_z^p|_{i,j}^n\Big]\\ &= \big(1-\frac{\alpha_y^2}{2}\big)E_z^p|_{i,j}^n + \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^n + E_z^p|_{i,j-1}^n\big)\\ &+ b^2D_{2x}\Big[\big(1-\frac{\alpha_y^2}{2}\big)E_z^p|_{i,j}^n + \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^n + E_z^p|_{i,j-1}^n\big)\Big],\end{aligned} \tag{2.105}$$

that is

$$
\begin{aligned}
\big(1+b^2D_{2x}\big)\big(1+b^2D_{2y}\big)E_z^p|_{i,j}^n = \big(1-\frac{\alpha_y^2}{2}\big)E_z^p|_{i,j}^n + \frac{\alpha_y^2}{4}\big(E_z^p|_{i,j+1}^n + E_z^p|_{i,j-1}^n\big) \\
+ (\frac{c\Delta t}{2})^2\frac{1}{\Delta x^2}\Big[\big(1-\frac{\alpha_y^2}{2}\big)\big(E_z^p|_{i+1,j}^n - 2E_z^p|_{i,j}^n + E_z^p|_{i-1,j}^n\big) \\
+ \frac{\alpha_y^2}{4}\big(E_z^p|_{i+1,j+1}^n - 2E_z^p|_{i,j+1}^n + E_z^p|_{i-1,j+1}^n\big) \\
+ \frac{\alpha_y^2}{4}\big(E_z^p|_{i+1,j-1}^n - 2E_z^p|_{i,j-1}^n + E_z^p|_{i-1,j-1}^n\big)\Big]. \qquad (2.106)
\end{aligned}
$$

After reorganizing, we have

$$
\begin{aligned}
\big(1+b^2D_{2x}\big)\big(1+b^2D_{2y}\big)E_z^p|_{i,j}^n = \big(1-\frac{\alpha_x^2}{2}\big)\big(1-\frac{\alpha_y^2}{2}\big)E_z^p|_{i,j}^n \\
+ \frac{\alpha_y^2}{4}\big(1-\frac{\alpha_x^2}{2}\big)\big[E_z^p|_{i,j+1}^n + E_z^p|_{i,j-1}^n\big] \\
+ \frac{\alpha_x^2}{4}\big(1-\frac{\alpha_y^2}{2}\big)\big[E_z^p|_{i+1,j}^n + E_z^p|_{i-1,j}^n\big] \\
+ \frac{\alpha_x^2\alpha_y^2}{16}\big[E_z^p|_{i+1,j+1}^n + E_z^p|_{i-1,j+1}^n \\
+ E_z^p|_{i+1,j-1}^n + E_z^p|_{i-1,j-1}^n\big]. \qquad (2.107)
\end{aligned}
$$

When solving (2.100), the right hand side, equivalent to right hand side of (2.107) plus terms F_R^n and $W^p|_{i,j}^n$, are all knowns and for the left hand side, (2.101) and (2.102) can be used. In this way, at each time step, $E_{ss}^p|_{i,j}$ can be obtained by solving a tridiagonal matrix based on $(1-b^2D_{2x})$ in (2.102) and knowns on the right of (2.100). After that, $E_{ss}^p|_{i,j}$ is a known and by (2.101) and (2.103), similarly $E_z^p|_{i,j}^{n+1}$ can be obtained by solving a tridiagonal matrix.

2.6 Imaginary Part of Curl Equations

Now consider the equations group (2.68). Using the *central-difference* approximation centered at time point n, (2.68) can be approximated as

$$\begin{aligned}\frac{E_z^q|^{n+\frac{1}{2}} - E_z^q|^{n-\frac{1}{2}}}{\Delta t} &= \frac{1}{2\varepsilon_o} D_x\big(H_y^q|^{n+\frac{1}{2}} + H_y^q|^{n-\frac{1}{2}}\big) - \frac{1}{2\varepsilon_o} D_y\big(H_x^q|^{n+\frac{1}{2}} + H_x^q|^{n-\frac{1}{2}}\big) \\ &- \frac{1}{2\varepsilon_o}\big(J_z^q|^{n+\frac{1}{2}} + J_z^q|^{n-\frac{1}{2}}\big) - 2\pi f_o E_z^p|^n, \qquad (2.108) \\ \frac{H_x^q|^{n+\frac{1}{2}} - H_x^q|^{n-\frac{1}{2}}}{\Delta t} &= -\frac{1}{2\mu_o} D_y\big(E_z^q|^{n+\frac{1}{2}} + E_z^q|^{n-\frac{1}{2}}\big) - \frac{1}{2\mu_o}\big(M_x^q|^{n+\frac{1}{2}} + M_x^q|^{n-\frac{1}{2}}\big) \\ &- 2\pi f_o H_x^p|^n \qquad (2.109)\end{aligned}$$

and

$$\begin{aligned}\frac{H_y^q|^{n+\frac{1}{2}} - H_y^q|^{n-\frac{1}{2}}}{\Delta t} &= \frac{1}{2\mu_o} D_x\big(E_z^q|^{n+\frac{1}{2}} + E_z^q|^{n-\frac{1}{2}}\big) - \frac{1}{2\mu_o}\big(M_y^q|^{n+\frac{1}{2}} + M_y^q|^{n-\frac{1}{2}}\big) \\ &- 2\pi f_o H_y^p|^n. \qquad (2.110)\end{aligned}$$

Now similar as what has been done in the previous section, after expanding those equations by approximating them on the spatial scheme, we will obtain

$$\begin{aligned}\frac{E_z^q\big|_{i,j}^{n+\frac{1}{2}} - E_z^q\big|_{i,j}^{n-\frac{1}{2}}}{\Delta t} &= \frac{1}{2\varepsilon_o\Delta x}\Big(H_y^q\big|_{i+1,j}^{n+\frac{1}{2}} - H_y^q\big|_{i,j}^{n+\frac{1}{2}} + H_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - H_y^q\big|_{i+1,j}^{n-\frac{1}{2}}\Big) \\ &- \frac{1}{2\varepsilon_o\Delta y}\Big(H_x^q\big|_{i,j+1}^{n+\frac{1}{2}} - H_x^q\big|_{i,j}^{n+\frac{1}{2}} + H_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - H_x^q\big|_{i,j}^{n-\frac{1}{2}}\Big) \\ &- \frac{1}{2\varepsilon_o}\Big(J_z^q\big|_{i,j}^{n+\frac{1}{2}} + J_z^q\big|_{i,j}^{n-\frac{1}{2}}\Big) - 2\pi f_o E_z^p\big|_{i,j}^n, \qquad (2.111) \\ \frac{H_x^q\big|_{i,j}^{n+\frac{1}{2}} - H_x^q\big|_{i,j}^{n-\frac{1}{2}}}{\Delta t} &= -\frac{1}{2\mu_o\Delta y}\Big(E_z^q\big|_{i,j}^{n+\frac{1}{2}} - E_z^q\big|_{i,j-1}^{n+\frac{1}{2}} + E_z^q\big|_{i,j}^{n-\frac{1}{2}} - E_z^q\big|_{i,j-1}^{n-\frac{1}{2}}\Big) \\ &- \frac{1}{2\mu_o}\Big(M_x^q\big|_{i,j}^{n+\frac{1}{2}} + M_x^q\big|_{i,j}^{n-\frac{1}{2}}\Big) \\ &- 2\pi f_o H_x^p\big|_{i,j}^n, \qquad (2.112)\end{aligned}$$

and

$$\frac{H_y^q\big|_{i,j}^{n+\frac{1}{2}} - H_y^q\big|_{i,j}^{n-\frac{1}{2}}}{\Delta t} = \frac{1}{2\mu_o \Delta x}\big(E_z^q\big|_{i,j}^{n+\frac{1}{2}} - E_z^q\big|_{i-1,j}^{n+\frac{1}{2}} + E_z^q\big|_{i,j}^{n-\frac{1}{2}} - E_z^q\big|_{i-1,j}^{n-\frac{1}{2}}\big)$$
$$- \frac{1}{2\mu_o}\big(M_y^q\big|_{i,j}^{n+\frac{1}{2}} + M_y^q\big|_{i,j}^{n-\frac{1}{2}}\big)$$
$$- 2\pi f_o H_y^p\big|_{i,j}^{n}. \tag{2.113}$$

By introducing the simplified expression (2.78) to (2.81) and source expressions as

$$\frac{\Delta t}{2\varepsilon_o \Delta x} = \frac{\eta_o}{2}\alpha_x, \tag{2.78}$$

$$\frac{\Delta t}{2\varepsilon_o \Delta y} = \frac{\eta_o}{2}\alpha_y, \tag{2.79}$$

$$\frac{\Delta t}{2\mu_o \Delta x} = \frac{1}{2\eta_o}\alpha_x, \tag{2.80}$$

$$\frac{\Delta t}{2\mu_o \Delta y} = \frac{1}{2\eta_o}\alpha_y, \tag{2.81}$$

$$S_z^q\big|_{i,j}^{n-\frac{1}{2}} = -\frac{\Delta t}{2\varepsilon_o}\big[J_z^q\big|_{i,j}^{n+\frac{1}{2}} + J_z^q\big|_{i,j}^{n-\frac{1}{2}}\big], \tag{2.114}$$

$$T_x^q\big|_{i,j}^{n-\frac{1}{2}} = -\frac{\Delta t}{2\mu_o}\big[M_x^q\big|_{i,j}^{n+\frac{1}{2}} + M_x^q\big|_{i,j}^{n-\frac{1}{2}}\big] \tag{2.115}$$

and

$$R_y^q\big|_{i,j}^{n-\frac{1}{2}} = -\frac{\Delta t}{2\mu_o}\big[M_y^q\big|_{i,j}^{n+\frac{1}{2}} + M_y^q\big|_{i,j}^{n-\frac{1}{2}}\big], \tag{2.116}$$

and rearranging (2.111) to (2.113), we will have

$$
\begin{aligned}
E_z^q\big|_{i,j}^{n+\frac{1}{2}} = E_z^q\big|_{i,j}^{n-\frac{1}{2}} &+ \frac{\alpha_x}{2}\big(\eta_o H_y^q\big|_{i+1,j}^{n+\frac{1}{2}} - \eta_o H_y^q\big|_{i,j}^{n+\frac{1}{2}} + \eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - \eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}}\big) \\
&- \frac{\alpha_y}{2}\big(\eta_o H_x^q\big|_{i,j+1}^{n+\frac{1}{2}} - \eta_o H_x^q\big|_{i,j}^{n+\frac{1}{2}} + \eta_o H_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \eta_o H_x^q\big|_{i,j}^{n-\frac{1}{2}}\big) \\
&+ S_z^q\big|_{i,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o E_z^p\big|_{i,j}^{n}, \qquad (2.117) \\
H_x^q\big|_{i,j}^{n+\frac{1}{2}} = H_x^q\big|_{i,j}^{n-\frac{1}{2}} &- \frac{\alpha_y}{2}\big(\frac{1}{\eta_o} E_z^q\big|_{i,j}^{n+\frac{1}{2}} - \frac{1}{\eta_o} E_z^q\big|_{i,j-1}^{n+\frac{1}{2}} + \frac{1}{\eta_o} E_z^q\big|_{i,j}^{n-\frac{1}{2}} - \frac{1}{\eta_o} E_z^q\big|_{i,j-1}^{n-\frac{1}{2}}\big) \\
&+ T_x^q\big|_{i,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o H_x^p\big|_{i,j}^{n} \qquad (2.118)
\end{aligned}
$$

and

$$
\begin{aligned}
H_y^q\big|_{i,j}^{n+\frac{1}{2}} = H_y^q\big|_{i,j}^{n-\frac{1}{2}} &+ \frac{\alpha_x}{2}\big(\frac{1}{\eta_o} E_z^q\big|_{i,j}^{n+\frac{1}{2}} - \frac{1}{\eta_o} E_z^q\big|_{i-1,j}^{n+\frac{1}{2}} + \frac{1}{\eta_o} E_z^q\big|_{i,j}^{n-\frac{1}{2}} - \frac{1}{\eta_o} E_z^q\big|_{i-1,j}^{n-\frac{1}{2}}\big) \\
&+ R_y^q\big|_{i,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o H_y^p\big|_{i,j}^{n}. \qquad (2.119)
\end{aligned}
$$

In order to avoid confusion in the equation transform, (2.117) can be written as

$$
\begin{aligned}
E_z^q\big|_{i,j}^{n+\frac{1}{2}} &= E_z^q\big|_{i,j}^{n-\frac{1}{2}} \\
&+ \frac{\alpha_x}{2}\eta_o H_y^q\big|_{i+1,j}^{n+\frac{1}{2}} \\
&- \frac{\alpha_x}{2}\eta_o H_y^q\big|_{i,j}^{n+\frac{1}{2}} \\
&+ \frac{\alpha_x}{2}\big(\eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - \eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}}\big) \\
&- \frac{\alpha_y}{2}\eta_o H_x^q\big|_{i,j+1}^{n+\frac{1}{2}} \\
&+ \frac{\alpha_y}{2}\eta_o H_x^q\big|_{i,j}^{n+\frac{1}{2}} \\
&- \frac{\alpha_y}{2}\big(\eta_o H_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \eta_o H_x^q\big|_{i,j}^{n-\frac{1}{2}}\big) \\
&+ S_z^q\big|_{i,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o E_z^p\big|_{i,j}^{n}. \qquad (2.120)
\end{aligned}
$$

After substituting (2.118) and (2.119) into (2.120), we have

$$
\begin{aligned}
E_z^q\big|_{i,j}^{n+\frac{1}{2}} &= E_z^q\big|_{i,j}^{n-\frac{1}{2}} \\
&+ \frac{\alpha_x}{2}\eta_o\Big[H_y^q\big|_{i+1,j}^{n-\frac{1}{2}} + \frac{\alpha_x}{2}\big(\frac{1}{\eta_o}E_z^q\big|_{i+1,j}^{n+\frac{1}{2}} - \frac{1}{\eta_o}E_z^q\big|_{i,j}^{n+\frac{1}{2}} + \frac{1}{\eta_o}E_z^q\big|_{i+1,j}^{n-\frac{1}{2}} - \frac{1}{\eta_o}E_z^q\big|_{i,j}^{n-\frac{1}{2}}\big) \\
&\quad + R_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o H_y^p\big|_{i+1,j}^{n}\Big] \\
&- \frac{\alpha_x}{2}\eta_o\Big[H_y^q\big|_{i,j}^{n-\frac{1}{2}} + \frac{\alpha_x}{2}\big(\frac{1}{\eta_o}E_z^q\big|_{i,j}^{n+\frac{1}{2}} - \frac{1}{\eta_o}E_z^q\big|_{i-1,j}^{n+\frac{1}{2}} + \frac{1}{\eta_o}E_z^q\big|_{i,j}^{n-\frac{1}{2}} - \frac{1}{\eta_o}E_z^q\big|_{i-1,j}^{n-\frac{1}{2}}\big) \\
&\quad + R_y^q\big|_{i,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o H_y^p\big|_{i,j}^{n}\Big] \\
&+ \frac{\alpha_x}{2}\big(\eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - \eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}}\big) \\
&- \frac{\alpha_y}{2}\eta_o\Big[H_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \frac{\alpha_y}{2}\big(\frac{1}{\eta_o}E_z^q\big|_{i,j+1}^{n+\frac{1}{2}} - \frac{1}{\eta_o}E_z^q\big|_{i,j}^{n+\frac{1}{2}} + \frac{1}{\eta_o}E_z^q\big|_{i,j+1}^{n-\frac{1}{2}} - \frac{1}{\eta_o}E_z^q\big|_{i,j}^{n-\frac{1}{2}}\big) \\
&\quad + T_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \Delta t 2\pi f_o H_x^p\big|_{i,j+1}^{n}\Big] \\
&+ \frac{\alpha_y}{2}\eta_o\Big[H_x^q\big|_{i,j}^{n-\frac{1}{2}} - \frac{\alpha_y}{2}\big(\frac{1}{\eta_o}E_z^q\big|_{i,j}^{n+\frac{1}{2}} - \frac{1}{\eta_o}E_z^q\big|_{i,j-1}^{n+\frac{1}{2}} + \frac{1}{\eta_o}E_z^q\big|_{i,j}^{n-\frac{1}{2}} - \frac{1}{\eta_o}E_z^q\big|_{i,j-1}^{n-\frac{1}{2}}\big) \\
&\quad + T_x^q\big|_{i,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o H_x^p\big|_{i,j}^{n}\Big] \\
&- \frac{\alpha_y}{2}\big(\eta_o H_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \eta_o H_x^q\big|_{i,j}^{n-\frac{1}{2}}\big) \\
&+ S_z^q\big|_{i,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o E_z^p\big|_{i,j}^{n}.
\end{aligned}
\tag{2.121}
$$

That is equivalent to

$$
\begin{aligned}
E_z^q\big|_{i,j}^{n+\frac{1}{2}} = E_z^q\big|_{i,j}^{n-\frac{1}{2}} & \\
+ \frac{\alpha_x}{2}\Big[&\eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}} + \frac{\alpha_x}{2}\big(E_z^q\big|_{i+1,j}^{n+\frac{1}{2}} - E_z^q\big|_{i,j}^{n+\frac{1}{2}} + E_z^q\big|_{i+1,j}^{n-\frac{1}{2}} - E_z^q\big|_{i,j}^{n-\frac{1}{2}}\big) \\
&+ \eta_o R_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - \eta_o \Delta t 2\pi f_o H_y^p\big|_{i+1,j}^{n}\Big] \\
- \frac{\alpha_x}{2}\Big[&\eta_o H_y^q\big|_{i,j}^{n-\frac{1}{2}} + \frac{\alpha_x}{2}\big(E_z^q\big|_{i,j}^{n+\frac{1}{2}} - E_z^q\big|_{i-1,j}^{n+\frac{1}{2}} + E_z^q\big|_{i,j}^{n-\frac{1}{2}} - E_z^q\big|_{i-1,j}^{n-\frac{1}{2}}\big) \\
&+ \eta_o R_y^q\big|_{i,j}^{n-\frac{1}{2}} - \eta_o \Delta t 2\pi f_o H_y^p\big|_{i,j}^{n}\Big] \\
- \frac{\alpha_y}{2}\Big[&\eta_o H_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \frac{\alpha_y}{2}\big(E_z^q\big|_{i,j+1}^{n+\frac{1}{2}} - E_z^q\big|_{i,j}^{n+\frac{1}{2}} + E_z^q\big|_{i,j+1}^{n-\frac{1}{2}} - E_z^q\big|_{i,j}^{n-\frac{1}{2}}\big) \\
&+ \eta_o T_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \eta_o \Delta t 2\pi f_o H_x^p\big|_{i,j+1}^{n}\Big] \\
+ \frac{\alpha_y}{2}\Big[&H_x^q\big|_{i,j}^{n-\frac{1}{2}} - \frac{\alpha_y}{2}\big(E_z^q\big|_{i,j}^{n+\frac{1}{2}} - E_z^q\big|_{i,j-1}^{n+\frac{1}{2}} + E_z^q\big|_{i,j}^{n-\frac{1}{2}} - E_z^q\big|_{i,j-1}^{n-\frac{1}{2}}\big) \\
&+ \eta_o T_x^q\big|_{i,j}^{n-\frac{1}{2}} - \eta_o \Delta t 2\pi f_o H_x^p\big|_{i,j}^{n}\Big] \\
+ \frac{\alpha_x}{2}&\big(\eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - \eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}}\big) - \frac{\alpha_y}{2}\big(\eta_o H_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \eta_o H_x^q\big|_{i,j}^{n-\frac{1}{2}}\big) \\
+ S_z^q&\big|_{i,j}^{n-\frac{1}{2}} - \Delta t 2\pi f_o E_z^p\big|_{i,j}^{n}.
\end{aligned} \tag{2.122}
$$

After grouping same terms together, it can be written as

$$
\begin{aligned}
&\big(1 + \frac{\alpha_x^2}{2} + \frac{\alpha_y^2}{2}\big)E_z^q\big|_{i,j}^{n+\frac{1}{2}} - \frac{\alpha_x^2}{4}\big(E_z^q\big|_{i+1,j}^{n+\frac{1}{2}} - E_z^q\big|_{i-1,j}^{n+\frac{1}{2}}\big) - \frac{\alpha_y^2}{4}\big(E_z^q\big|_{i,j+1}^{n+\frac{1}{2}} - E_z^q\big|_{i,j-1}^{n+\frac{1}{2}}\big) \\
&\quad = \big(1 - \frac{\alpha_x^2}{2} - \frac{\alpha_y^2}{2}\big)E_z^q\big|_{i,j}^{n-\frac{1}{2}} + \frac{\alpha_x^2}{4}\big(E_z^q\big|_{i+1,j}^{n-\frac{1}{2}} - E_z^q\big|_{i-1,j}^{n-\frac{1}{2}}\big) + \frac{\alpha_y^2}{4}\big(E_z^q\big|_{i,j+1}^{n-\frac{1}{2}} - E_z^q\big|_{i,j-1}^{n-\frac{1}{2}}\big) \\
&\quad + \alpha_x\big[\eta_o H_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - \eta_o H_y^q\big|^{n-\frac{1}{2}}\big]_{i,j} - \frac{\alpha_x 2\pi f_o \Delta t}{2}\big[\eta_o H_y^p\big|_{i+1,j}^{n} - \eta_o H_y^p\big|_{i,j}^{n}\big] \\
&\quad - \alpha_y\big[\eta_o H_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - \eta_o H_x^q\big|^{n-\frac{1}{2}}\big]_{i,j} + \frac{\alpha_y 2\pi f_o \Delta t}{2}\big[\eta_o H_x^p\big|_{i,j+1}^{n} - \eta_o H_x^p\big|_{i,j}^{n}\big] - \Delta t 2\pi f_o E_z^p\big|_{i,j}^{n} \\
&\quad + S_z^q\big|_{i,j}^{n-\frac{1}{2}} + \frac{\alpha_x}{2}\big[R_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - R_y^q\big|_{i,j}^{n-\frac{1}{2}}\big] - \frac{\alpha_y}{2}\big[T_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - T_x^q\big|_{i,j}^{n-\frac{1}{2}}\big].
\end{aligned} \tag{2.123}
$$

Now if we define

$$
W^q\big|_{i,j}^{n-\frac{1}{2}} = S_z^q\big|_{i,j}^{n-\frac{1}{2}} + \frac{\alpha_x}{2}\big[R_y^q\big|_{i+1,j}^{n-\frac{1}{2}} - R_y^q\big|_{i,j}^{n-\frac{1}{2}}\big] - \frac{\alpha_y}{2}\big[T_x^q\big|_{i,j+1}^{n-\frac{1}{2}} - T_x^q\big|_{i,j}^{n-\frac{1}{2}}\big] \tag{2.124}
$$

and

$$F_I^{n-\frac{1}{2}} = \alpha_x\big[\eta_o H_y^q|_{i+1,j}^{n-\frac{1}{2}} - \eta_o H_y^q|^{n-\frac{1}{2}}\big]_{i,j} - \frac{\alpha_x 2\pi f_o \Delta t}{2}\big[\eta_o H_y^p|_{i+1,j}^n - \eta_o H_y^p|_{i,j}^n\big]$$
$$- \alpha_y\big[\eta_o H_x^q|_{i,j+1}^{n-\frac{1}{2}} - \eta_o H_x^q|^{n-\frac{1}{2}}\big]_{i,j} + \frac{\alpha_y 2\pi f_o \Delta t}{2}\big[\eta_o H_x^p|_{i,j+1}^n - \eta_o H_x^p|_{i,j}^n\big] - \Delta t 2\pi f_o E_z^p|_{i,j}^n, \tag{2.125}$$

then the equation above can be simplified as

$$\begin{aligned} E_z^q|_{i,j}^{n+\frac{1}{2}} &- (\frac{c\Delta t}{2})^2\big[\frac{E_z^q|_{i+1,j}^{n+\frac{1}{2}} - 2E_z^q|_{i,j}^{n+\frac{1}{2}} + E_z^q|_{i-1,j}^{n+\frac{1}{2}}}{\Delta x^2}\big] \\ &- (\frac{c\Delta t}{2})^2\big[\frac{E_z^q|_{i,j+1}^{n+\frac{1}{2}} - 2E_z^q|_{i,j}^{n+\frac{1}{2}} + E_z^q|_{i,j-1}^{n+\frac{1}{2}}}{\Delta y^2}\big] \\ &= E_z^q|_{i,j}^{n-\frac{1}{2}} + (\frac{c\Delta t}{2})^2\big[\frac{E_z^q|_{i+1,j}^{n-\frac{1}{2}} - 2E_z^q|_{i,j}^{n-\frac{1}{2}} + E_z^q|_{i-1,j}^{n-\frac{1}{2}}}{\Delta x^2}\big] \\ &\qquad - (\frac{c\Delta t}{2})^2\big[\frac{E_z^q|_{i,j+1}^{n-\frac{1}{2}} - 2E_z^q|_{i,j}^{n-\frac{1}{2}} + E_z^q|_{i,j-1}^{n-\frac{1}{2}}}{\Delta y^2}\big] \\ &\qquad + F_I^{n-\frac{1}{2}} + W^q|_{i,j}^{n-\frac{1}{2}}. \end{aligned} \tag{2.126}$$

Here again if we define $b = \frac{c\Delta t}{2}$, then

$$\big(1 - b^2 D_{2x} - b^2 D_{2y}\big)E_z^q|_{i,j}^{n+\frac{1}{2}} = \big(1 + b^2 D_{2x} + b^2 D_{2y}\big)E_z^q|^{n-\frac{1}{2}} + F_I^{n-\frac{1}{2}} + W^q|_{i,j}^{n-\frac{1}{2}}. \tag{2.127}$$

Here similar as what has been done in the previous section, from [10] an approximation has been proposed so that the equation above can be written as

$$\big(1 - b^2 D_{2x}\big)\big(1 - b^2 D_{2y}\big)E_z^q|_{i,j}^{n+\frac{1}{2}} = \big(1 + b^2 D_{2x}\big)\big(1 + b^2 D_{2y}\big)E_z^q|^{n-\frac{1}{2}} + F_I^{n-\frac{1}{2}} + W^q|_{i,j}^{n-\frac{1}{2}}. \tag{2.128}$$

Now if we define

$$E_{ss}^q|_{i,j} = \big(1 - b^2 D_{2y}\big)E_z^q|_{i,j}^{n+\frac{1}{2}} \tag{2.129}$$

then the left hand side of (2.128) can be written as

$$\begin{aligned}
\big(1-b^2D_{2x}\big)E^q_{ss}|_{i,j} &= E^q_{ss}|_{i,j} - b^2D_{2x}E^q_{ss}|_{i,j} \\
&= E^q_{ss}|_{i,j} - (\frac{c\Delta t}{2})^2\frac{1}{\Delta x^2}\big[E^q_{ss}|_{i+1,j} - 2E^q_{ss}|_{i,j} + E^q_{ss}|_{i-1,j}\big] \\
&= \big(1+\frac{\alpha_x^2}{2}\big)E^q_{ss}|_{i,j} - \frac{\alpha_x^2}{4}\big(E^q_{ss}|_{i+1,j} + E^q_{ss}|_{i-1,j}\big). \qquad (2.130)
\end{aligned}$$

Now consider $E^q_{ss}|_{i,j}$, similarly we have

$$\big(1-b^2D_{2y}\big)E^q_z|^{n+\frac12}_{i,j} = \big(1+\frac{\alpha_y^2}{2}\big)E^q_z|^{n+\frac12}_{i,j} - \frac{\alpha_y^2}{4}\big(E^q_z|^{n+\frac12}_{i,j+1} + E^q_z|^{n+\frac12}_{i,j-1}\big). \qquad (2.131)$$

Consider the right hand side of (2.128). Similarly we have

$$\big(1+b^2D_{2y}\big)E^q_z|^{n-\frac12}_{i,j} = \big(1-\frac{\alpha_y^2}{2}\big)E^q_z|^{n-\frac12}_{i,j} + \frac{\alpha_y^2}{4}\big(E^q_z|^{n-\frac12}_{i,j+1} + E^q_z|^{n-\frac12}_{i,j-1}\big). \qquad (2.132)$$

Then we will obtain

$$\begin{aligned}
\big(1+b^2D_{2x}\big)\big(1+b^2D_{2y}\big)E^q_z|^{n-\frac12}_{i,j} &= \big(1+b^2D_{2y}\big)E^q_z|^{n-\frac12}_{i,j} + b^2D_{2x}\Big[\big(1+b^2D_{2y}\big)E^q_z|^{n-\frac12}_{i,j}\Big] \\
&= \big(1-\frac{\alpha_y^2}{2}\big)E^q_z|^{n-\frac12}_{i,j} + \frac{\alpha_y^2}{4}\big(E^q_z|^{n-\frac12}_{i,j+1} + E^q_z|^{n-\frac12}_{i,j-1}\big) \\
&\quad + b^2D_{2x}\Big[\big(1-\frac{\alpha_y^2}{2}\big)E^q_z|^{n-\frac12}_{i,j} + \frac{\alpha_y^2}{4}\big(E^q_z|^{n-\frac12}_{i,j+1} + E^q_z|^{n-\frac12}_{i,j-1}\big)\Big],
\end{aligned} \qquad (2.133)$$

that is

$$\begin{aligned}
\big(1+b^2D_{2x}\big)\big(1+b^2D_{2y}\big)E^q_z|^{n-\frac12}_{i,j} &= \big(1-\frac{\alpha_y^2}{2}\big)E^q_z|^{n-\frac12}_{i,j} + \frac{\alpha_y^2}{4}\big(E^q_z|^{n-\frac12}_{i,j+1} + E^q_z|^{n-\frac12}_{i,j-1}\big) \\
&\quad + (\frac{c\Delta t}{2})^2\frac{1}{\Delta x^2}\Big[\big(1-\frac{\alpha_y^2}{2}\big)\big(E^q_z|^{n-\frac12}_{i+1,j} - 2E^q_z|^{n-\frac12}_{i,j} + E^q_z|^{n-\frac12}_{i-1,j}\big) \\
&\qquad + \frac{\alpha_y^2}{4}\big(E^q_z|^{n-\frac12}_{i+1,j+1} - 2E^q_z|^{n-\frac12}_{i,j+1} + E^q_z|^{n-\frac12}_{i-1,j+1}\big) \\
&\qquad + \frac{\alpha_y^2}{4}\big(E^q_z|^{n-\frac12}_{i+1,j-1} - 2E^q_z|^{n-\frac12}_{i,j-1} + E^q_z|^{n-\frac12}_{i-1,j-1}\big)\Big]. \qquad (2.134)
\end{aligned}$$

After reorganizing, we have

$$
\begin{aligned}
\big(1+b^2D_{2x}\big)\big(1+b^2D_{2y}\big)E_z^q\big|_{i,j}^{n-\frac{1}{2}} = {}& \big(1-\frac{\alpha_x^2}{2}\big)\big(1-\frac{\alpha_y^2}{2}\big)E_z^q\big|_{i,j}^{n-\frac{1}{2}} \\
&+\frac{\alpha_y^2}{4}\big(1-\frac{\alpha_x^2}{2}\big)\big[E_z^q\big|_{i,j+1}^{n-\frac{1}{2}}+E_z^q\big|_{i,j-1}^{n-\frac{1}{2}}\big] \\
&+\frac{\alpha_x^2}{4}\big(1-\frac{\alpha_y^2}{2}\big)\big[E_z^q\big|_{i+1,j}^{n-\frac{1}{2}}+E_z^q\big|_{i-1,j}^{n-\frac{1}{2}}\big] \\
&+\frac{\alpha_x^2\alpha_y^2}{16}\big[E_z^q\big|_{i+1,j+1}^{n-\frac{1}{2}}+E_z^q\big|_{i-1,j+1}^{n-\frac{1}{2}} \\
&\quad +E_z^q\big|_{i+1,j-1}^{n-\frac{1}{2}}+E_z^q\big|_{i-1,j-1}^{n-\frac{1}{2}}\big]. \qquad (2.135)
\end{aligned}
$$

Similar as what has been done in previous section, the right hand side of (2.128), equivalent to right hand side of (2.135) plus $F_I^{n-\frac{1}{2}}$ and $W_q\big|_{i,j}^{n-\frac{1}{2}}$, are all knowns and for the left hand side, (2.129) and (2.130) can be used. At each time step, $E_{ss}^q|_{i,j}$ can be obtained by solving a tridiagonal matrix based on $(1-b^2D_{2x})$ in (2.130) and knowns on the right hand side of (2.128). After that $E_{ss}^q|_{i,j}$ is a known, and by (2.129) and (2.131), similarly $E_z^q\big|_{i,j}^{n+\frac{1}{2}}$ can be obtained by solving a tridiagonal matrix.

CHAPTER 3

ANALYTICAL SOLUTION USING THE FREQUENCY DOMAIN

In this chapter, a reference solution for the 2D cavity problem will be presented to compare with the proposed FDTD solution given in *Chapter 2*. A completely different method, the reference solution is first obtained in frequency domain and then transformed to time domain. Image theory is used to account for the reflections from the cavity walls.

In FDTD method, a single sampling point is used to represent the source region, which means in reality we have a squared surface source region, with area equal to $\Delta x \times \Delta y$. However, in reference solution with that source current, a surface integral of the Hankel's function of second kind is required which is very complicated. So a compressed line source, original source current multiplied by the area $\Delta x \times \Delta y$ in the origin, is used for approximation instead the original squared surface which simplifies the problem. However, in the region close to the source, the size of the source can not be neglected, which means a line source approximation is not appropriate. In addition, due to the singularity problem of the Hankel's function of the second kind at the origin, no solution can be obtained at the origin in this way. Consequently the reference solution can only be used to compared with the FDTD solution at a limited number of spatial points which is relatively far from the source. With source region of small size and large cavity, most of points can be covered in the reference solution.

In *Section 3.1*, a general analysis based upon an electric surface source, with only a z component, is presented. Starting from the magnetic field in terms of an electric source and Green's function, the electric field is given by the product of some constants and the surface integral of the electric source current and the Hankel's function of the second kind.

In *Section 3.2*, image theory will be introduced in order to formulate the perfect electric conductor (PEC) boundary conditions. Two figures are used to illustrate the idea of generating reflection waves by building a limited number of image sources.

3.1 Electric Field in Free Space

Our source current is the Gaussian function of time t as

$$\mathbf{J}(\boldsymbol{\rho'}, t) = \hat{\mathbf{a}}_z \exp\left[-(t-t_d)^2/2\sigma^2\right]\cos(2\pi f_o t)\delta(\rho'), \tag{3.1}$$

where ρ' is the magnitude of $\boldsymbol{\rho'}$. Then after using Fourier transform, in the frequency domain

$$\begin{aligned}\mathbf{J}(\boldsymbol{\rho'}, f) = {} & \hat{\mathbf{a}}_z \sigma\sqrt{\pi/2}\exp\left\{-2\left[(f-f_o)\pi\sigma\right]^2\right\}\exp\left\{-j2\pi t_d(f-f_o)\right\}\delta(\rho') \\ & + \hat{\mathbf{a}}_z \sigma\sqrt{\pi/2}\exp\left\{-2\left[(f+f_o)\pi\sigma\right]^2\right\}\exp\left\{-j2\pi t_d(f+f_o)\right\}\delta(\rho'). \end{aligned} \tag{3.2}$$

Using the equations in [11], the electric field outside of the source region due to the z-directed electric current source can be written as

$$E_z(\boldsymbol{\rho}, f) = -j\eta_o k \int J_z(\boldsymbol{\rho'}, f) g(\boldsymbol{\rho}, \boldsymbol{\rho'}) dS'. \tag{3.3}$$

where

$$g(\boldsymbol{\rho}, \boldsymbol{\rho'}) = \frac{1}{4j} H_o^{(2)}(k\Delta\rho) \tag{3.4}$$

and

$$\Delta\rho = |\boldsymbol{\rho} - \boldsymbol{\rho'}| = \sqrt{(x-x')^2 + (y-y')^2}. \tag{3.5}$$

Here x and y represent the position of the observation point in Cartesian coordinate system while x' and y' represent the position of the source.

Substituting the Green's function into (3.3) yields

$$\begin{aligned}
E_z(\boldsymbol{\rho}, f) &= -\frac{\eta_o k}{4} \int J_z(\boldsymbol{\rho'}, f) H_o^{(2)}(k\Delta\rho) dS' \\
&= -\frac{\eta_o k}{4} \sigma \sqrt{\pi/2} \exp\left\{-2\big[(f-f_o)\pi\sigma\big]^2\right\} \exp\left\{-j2\pi t_d(f-f_o)\right\} H_o^{(2)}(k\rho) \\
&\quad - \frac{\eta_o k}{4} \sigma \sqrt{\pi/2} \exp\left\{-2\big[(f+f_o)\pi\sigma\big]^2\right\} \exp\left\{-j2\pi t_d(f+f_o)\right\} H_o^{(2)}(k\rho) \\
&= -\frac{\mu_o f \sigma \pi}{2} \sqrt{\pi/2} \exp\left\{-2\big[(f-f_o)\pi\sigma\big]^2\right\} \exp\left\{-j2\pi t_d(f-f_o)\right\} H_o^{(2)}\left[\frac{2\pi f \rho}{f_o \lambda_o}\right] \\
&\quad - \frac{\mu_o f \sigma \pi}{2} \sqrt{\pi/2} \exp\left\{-2\big[(f+f_o)\pi\sigma\big]^2\right\} \exp\left\{-j2\pi t_d(f+f_o)\right\} H_o^{(2)}\left[\frac{2\pi f \rho}{f_o \lambda_o}\right],
\end{aligned} \tag{3.6}$$

where ρ is the magnitude of $\boldsymbol{\rho}$. By applying the CE transformation via (2.17) to 3.6, the terms around $(f + f_o)$ are removed, then the remaining terms are translated from f_0 to 0 and the magnitude is doubled so that

$$\tilde{E}_z(\boldsymbol{\rho}, f) = -\mu_o f \sigma \pi \sqrt{\pi/2} \exp\left\{-2\big[(f-f_o)\pi\sigma\big]^2\right\} \exp\left\{-j2\pi t_d(f-f_o)\right\} H_o^{(2)}\big[\frac{2\pi f \rho}{f_o \lambda_o}\big]. \tag{3.7}$$

Finally the solution in the time domain is given by

$$\tilde{E}_z(\boldsymbol{\rho}, t) = \mathbf{F}^{-1}\left\{\tilde{E}_z(\boldsymbol{\rho}, f)\right\}. \tag{3.8}$$

$\mathbf{F}^{-1}$ is the inverse Fourier transform operator and in Matlab it can be performed by

$$\tilde{E}_z(\boldsymbol{\rho}, t) = N\Delta f\, \mathbf{ifft}\left\{\tilde{E}_z(\boldsymbol{\rho}, f)\right\}, \tag{3.9}$$

where in the equation above **ifft** is the inverse fast Fourier transform in Matlab, N is the number of points of $\tilde{E}_z(\boldsymbol{\rho}, f)$, and Δf is the frequency step when sampling $\tilde{E}_z(\boldsymbol{\rho}, f)$ in the frequency domain.

3.2 Image Theory

With the PEC (Perfect Electric Conductor) boundaries around the square domain, in order to analyze the behavior of the total electric field, the effect from the reflection waves has to be considered. In [12], if the source is near the PEC boundary, the image source can be determined as illustrated in Figure 3.1.

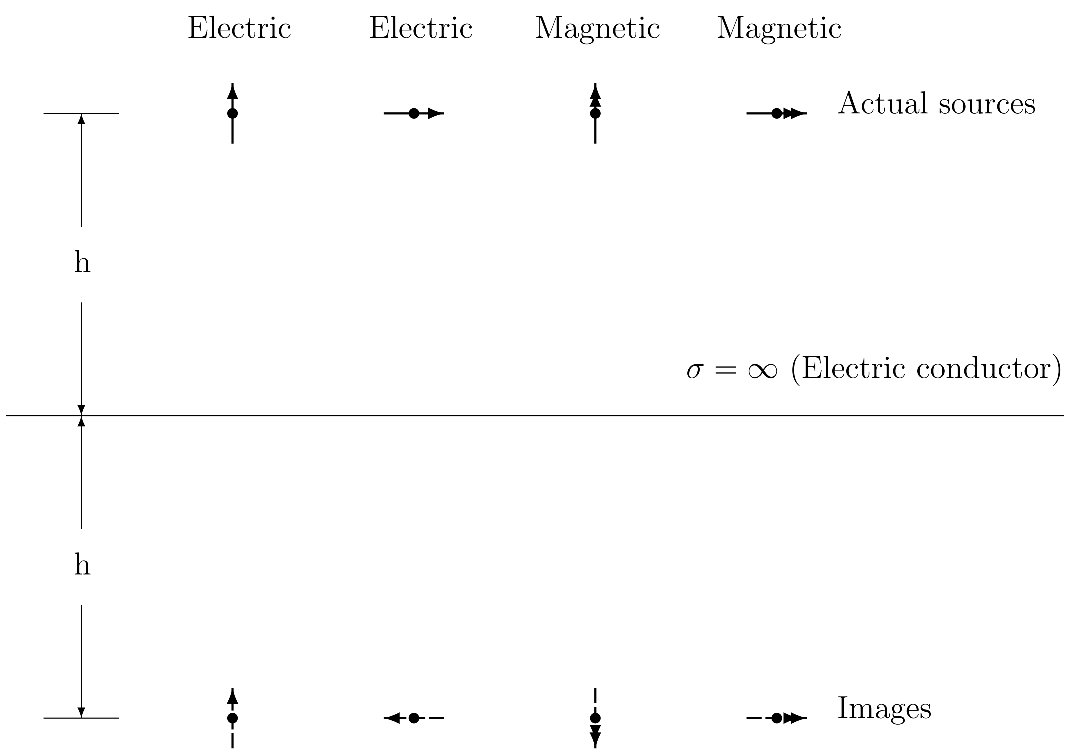


Figure 3.1. Electric and magnetic sources and their images near the electric conductor —after [12].

Through this figure we will be able to approximate the effect of PEC boundary condition by placing one or more images on the other side of the boundaries with correct phases. One image is sufficient for replacing the effect of one boundary exactly; however, with multiple boundaries, due to the multiple reflection paths from different angles, an infinite number of images are needed to model the reflection waves for all time. This seems impossible, but in our problem the end time is set to a fixed value so only the waves traveling from near by image sources will propagate to the field point of interest before the end time. Because of this a finite number of image sources can be used to model the effect of the PEC boundaries of the 2D cavity.

Figure 3.2 illustrates the original and image sources for the cavity problem. In the figure, the red box represents the original domain of the cavity while purple boxes represent image domains with image sources; those sources with filled circles represent a positive source (the same as the actual source), while the unfilled circles represent the negative sources (minus the actual source). By selecting a reasonably small ending time, a limited number of image sources around the real source are required to approximate the PEC boundaries. This is illustrated in examples given in the following chapter.

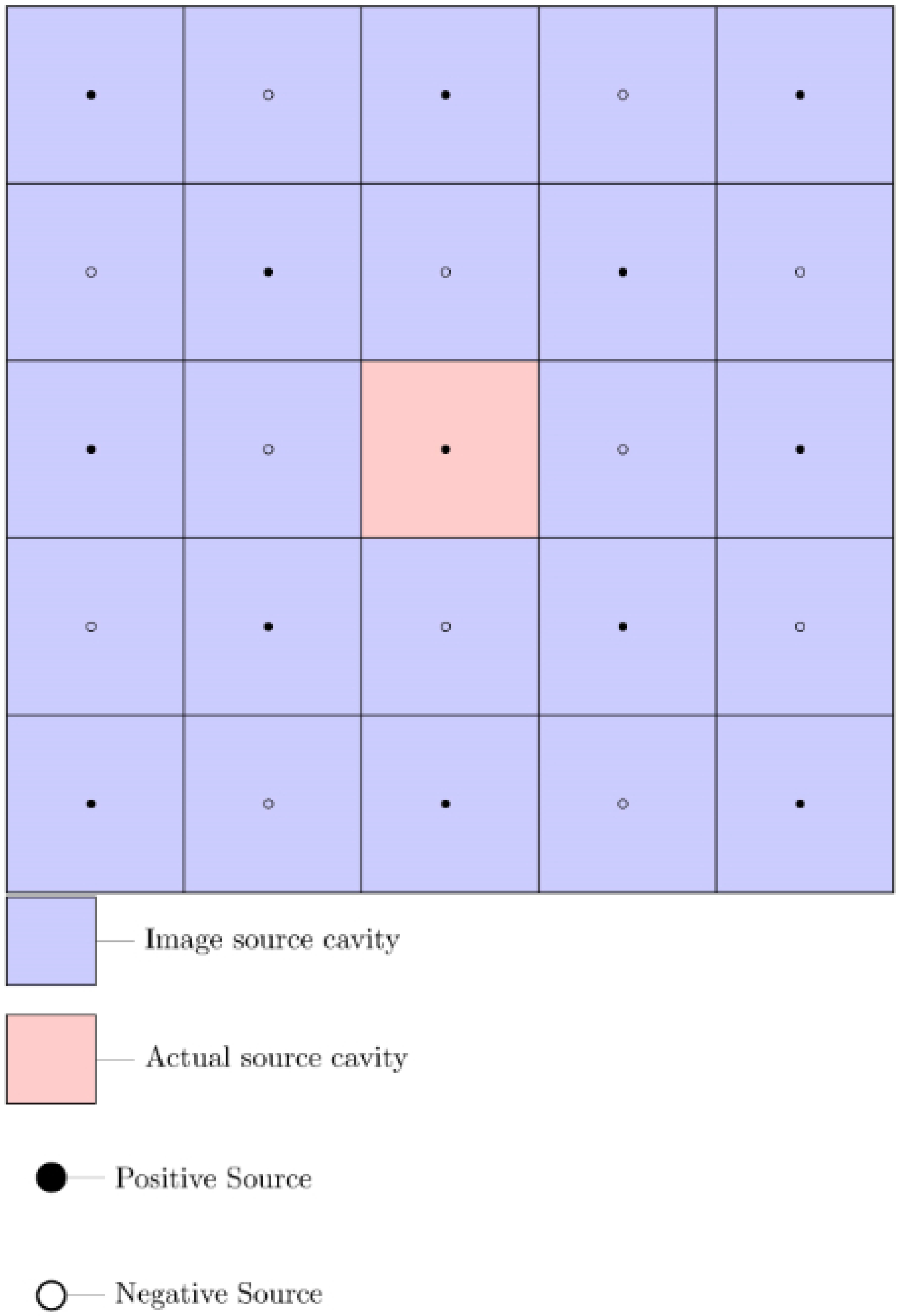


Figure 3.2. Images of a two-dimensional cavity with PEC walls and a transverse-magnetic source —after [13].

CHAPTER 4

RESULT

In this chapter results from the CE FDTD method and the reference IFFT/Image Theory method will be compared. The physical problem considered here is illustrated in Fig 4.1. The physical problem is a square area of free space, $20\lambda_o \times 20\lambda_o$ with PEC walls at the boundaries. Excitation is provided by a line source of electric current placed at the center of the domain.

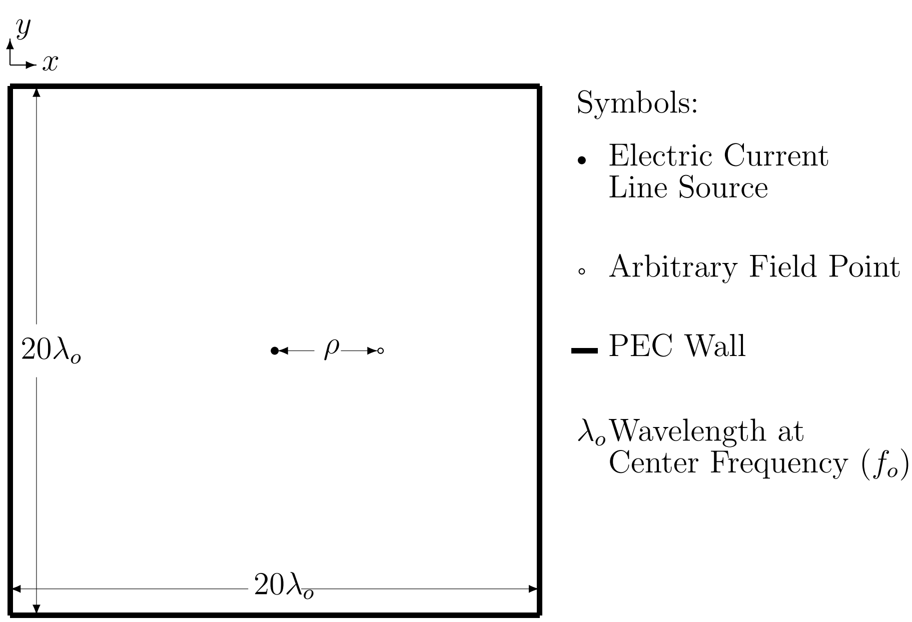


Figure 4.1. Domain of calculation.

In the CE-FDTD method the spatial domain needs to be sampled and then filled with Yee cells. Here the original geometry is square, so it is convenient for the Yee cells to be squares too and their dimension is given by $\Delta x \times \Delta y$ where $\Delta x = \Delta y$. The expression of the source current density implemented on the center Yee cell is

$$\mathbf{J} = \hat{\mathbf{a}}_z \frac{1}{\Delta x \Delta y} \exp\left\{-(t-t_d)^2/2\sigma^2\right\} \cos(2\pi f_o t), \tag{4.1}$$

where t_d is the time delay for the Gaussian pulse function to reach its peak,

$$t_d = 4\sqrt{2}B, \tag{4.2}$$

and the pulse width is determined by

$$\sigma = \frac{4}{\pi B} \tag{4.3}$$

where B is the bandwidth of the source as defined by (2.7). Here

$$f_o = 10 \text{ MHz}, \tag{4.4}$$

$$f_{min} = 8 \text{ MHz}, \tag{4.5}$$

$$f_{max} = 12 \text{ MHz}, \tag{4.6}$$

$$B = 4 \text{ MHz}, \tag{4.7}$$

and the temporal and spatial sampling intervals are

$$\Delta x = \frac{\lambda_{min}}{N_x}, \tag{4.8}$$

$$\Delta y = \frac{\lambda_{min}}{N_y} \tag{4.9}$$

and

$$\Delta t = \frac{2}{N_t B}, \tag{4.10}$$

where

$$\lambda_{min} = \frac{c_o}{f_{min}}, \tag{4.11}$$

$$N_x = 40, \tag{4.12}$$

$$N_y = 40 \tag{4.13}$$

and

$$N_t = 20. \tag{4.14}$$

For CE-FDTD method, the equations in *Chapter 2* are used and the reference results are obtained using the inverse fast Fourier transform (IFFT) solution in *Chapter 3*.

Section 4.1 presents the complex-envelope electric field versus time for three points on the x-axis. For each point, the in-phase part, the quadrature part and the magnitude are given. Figures illustrating the difference between the CE-FDTD method and the IFFT/Image Theory method are also presented. In *Section 4.2*, surface plots showing the spatial distribution of the complex-envelope electric field for a fixed time are given.

4.1 Electric Field over Time

Figures 4.2 through Figure 4.7 plot the electric field versus time for a field point on the x-axis $5\lambda_o$ from the origin for the time interval 0 to $4\mu s$. In $4\mu s$, the electric field can propagate over $40\lambda_o$ from the source at the origin. Hence 24 image sources are sufficient to determine the CE electric field using IFFT/Image Theory method.

In Figures 4.2 to 4.4, for every time the differences between the FDTD and IFFT solutions for the in-phase parts and quadrature parts are quite small, and tend to increase as time increases. However, compared with the peak magnitude of electric field from both methods, differences are still relatively small at the end time of $4\mu s$.

One reason for the error increase with time is likely the accumulation of errors in the FDTD method as the computations progress. On the other hand, it might also result from errors in the IFFT/Image Theory method. As time increases, an increasing number of image sources contribute to the field. At $4\mu s$ the result is a superposition of fields from 25 sources. In order to obtain a correct result, all fields from different sources must possess perfect phase. This is difficult to achieve for a large number of sources.

Figures 4.8 through Fig 4.13 plot the electric field versus time for a fixed point on the x-axis $2\lambda_o$ from the origin. Figures 4.14 through 4.19 plot the electric field versus time for a fixed point on the x-axis $0.5\lambda_o$ from the origin. As expected, errors for the field points at $5\lambda_o$, $2\lambda_o$ and $0.5\lambda_o$ from the origin are similar in magnitude and exhibit the same increasing error with increasing time characteristic.

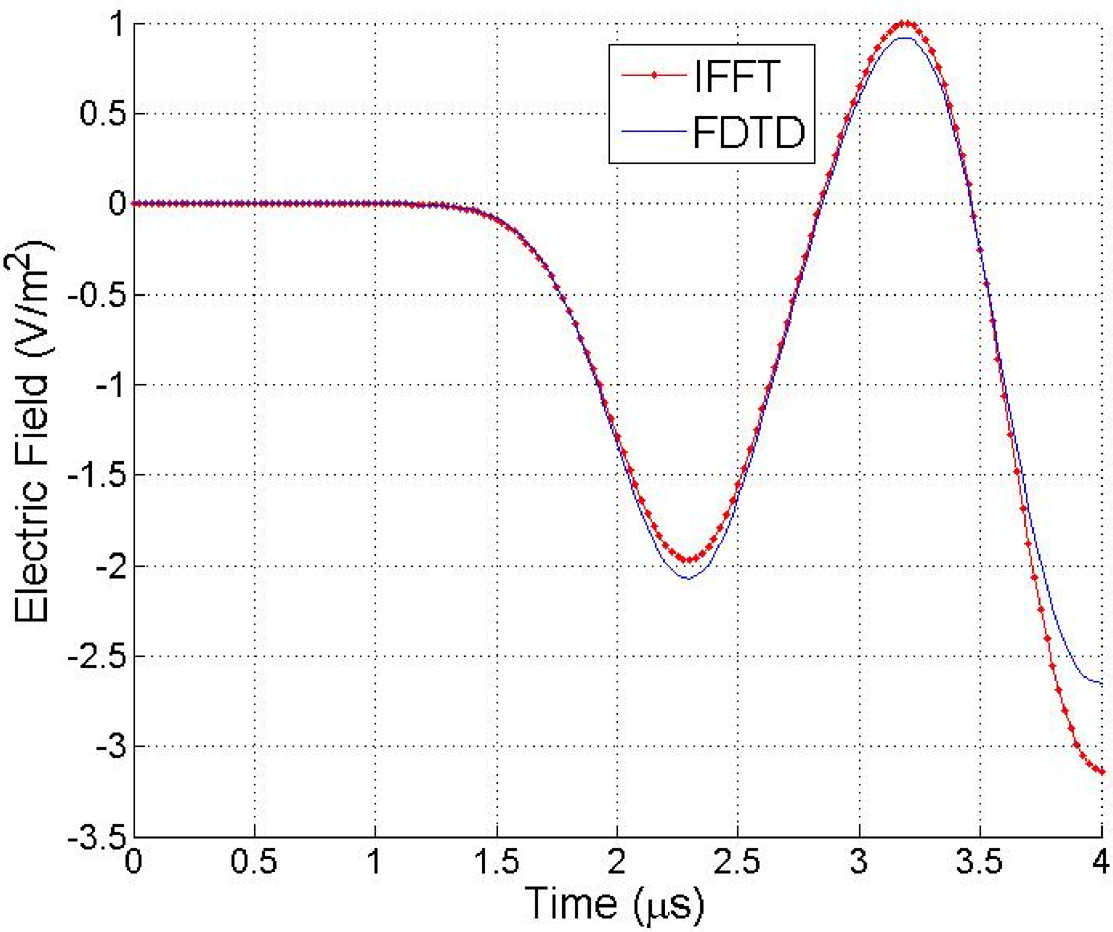


Figure 4.2. In-phase part of the electric field for $\rho = 5\lambda_o$.

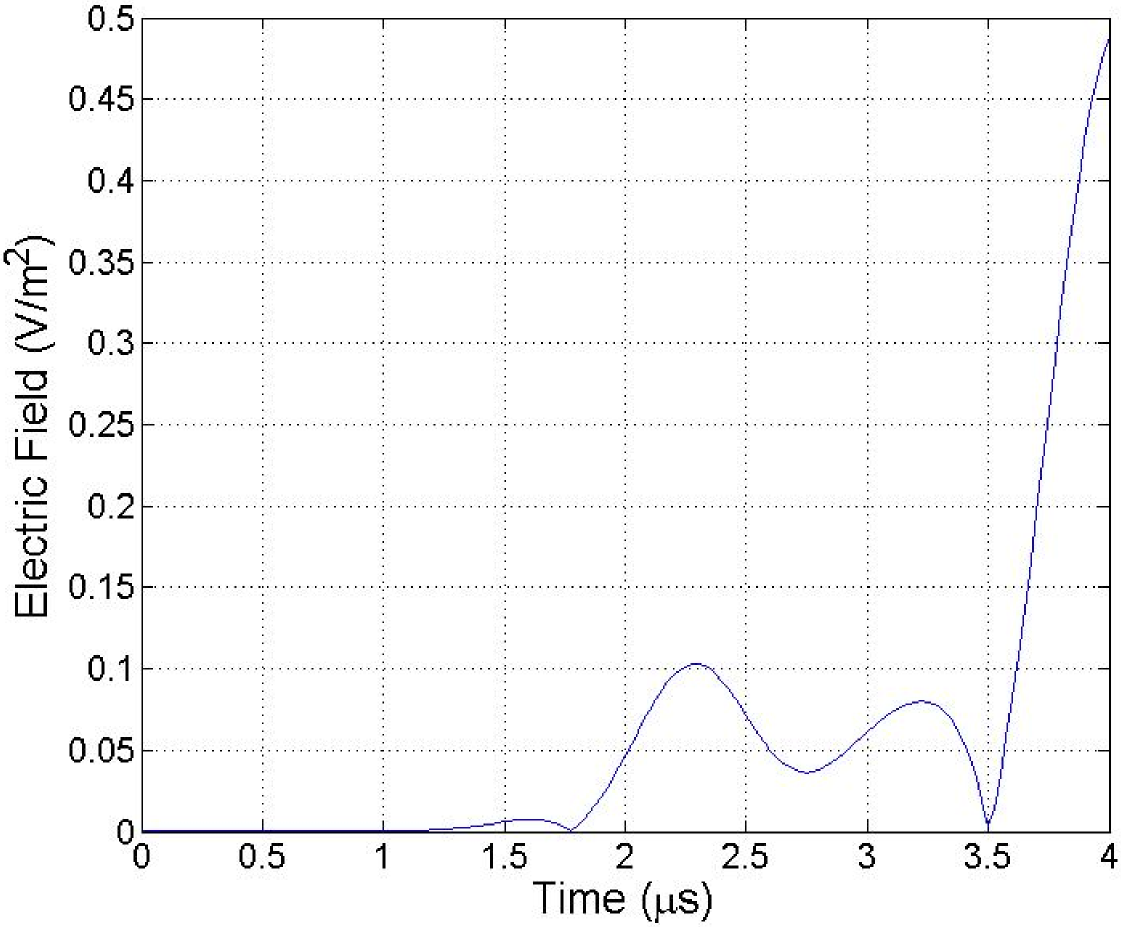


Figure 4.3. Magnitude of the difference between the in-phase parts of the IFFT and FDTD solutions for $\rho = 5\lambda_o$.

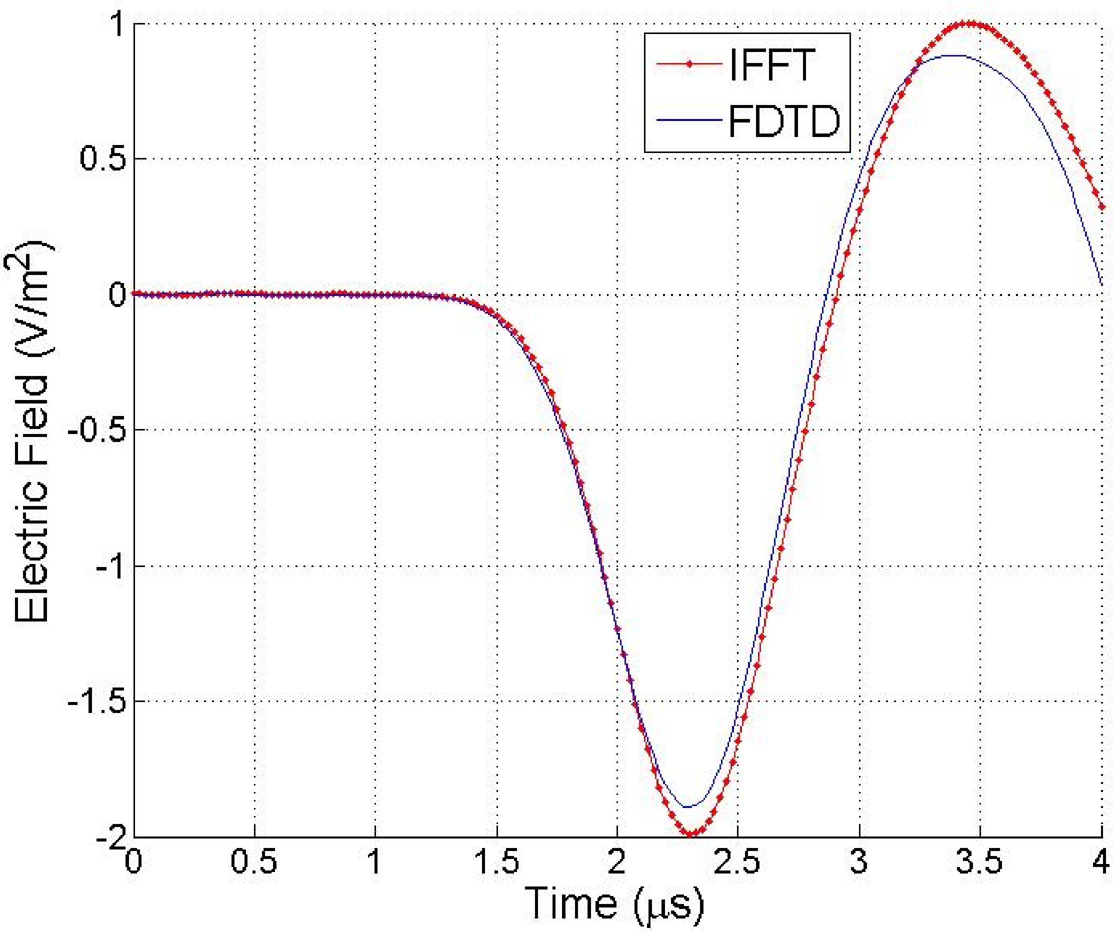


Figure 4.4. Quadrature part of the electric field for $\rho = 5\lambda_o$.

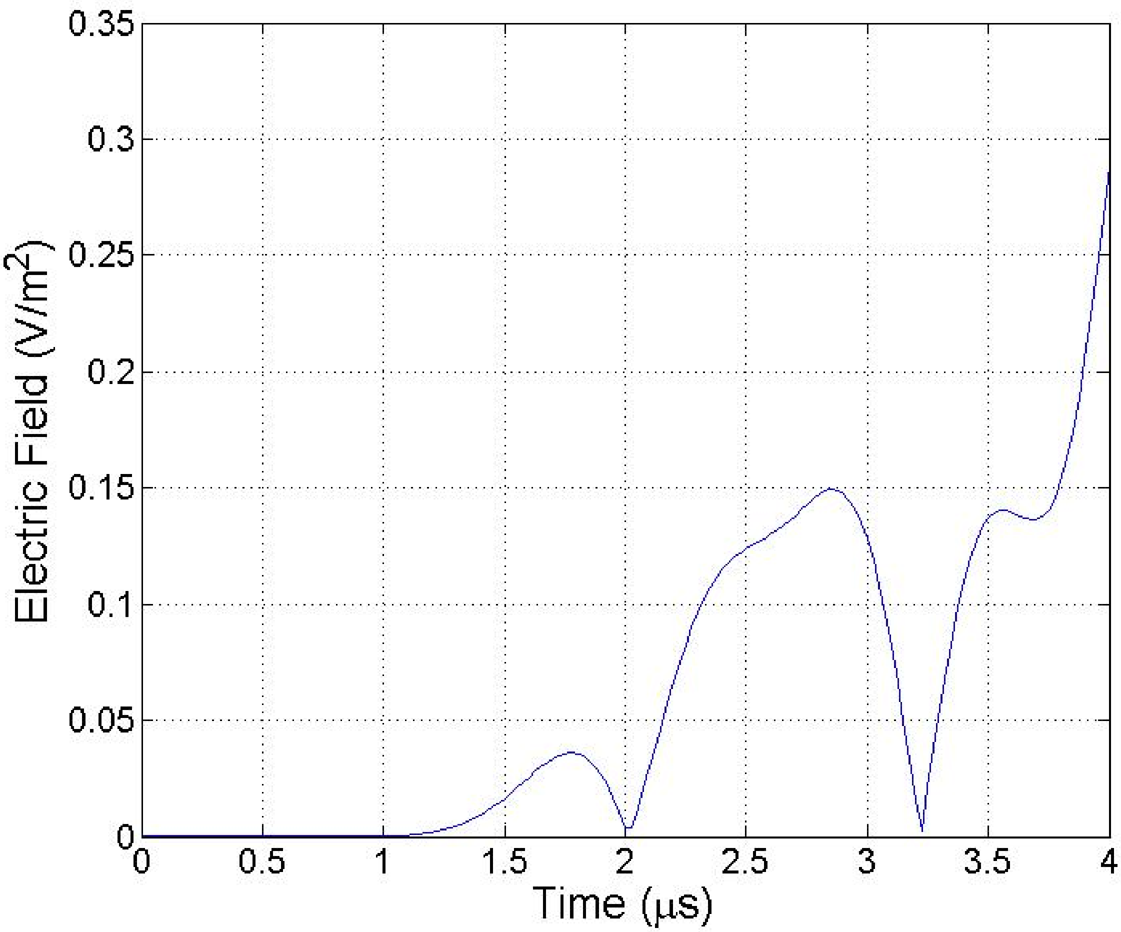


Figure 4.5. Magnitude of the difference between the quadrature parts of the IFFT and FDTD solutions for $\rho = 5\lambda_o$.

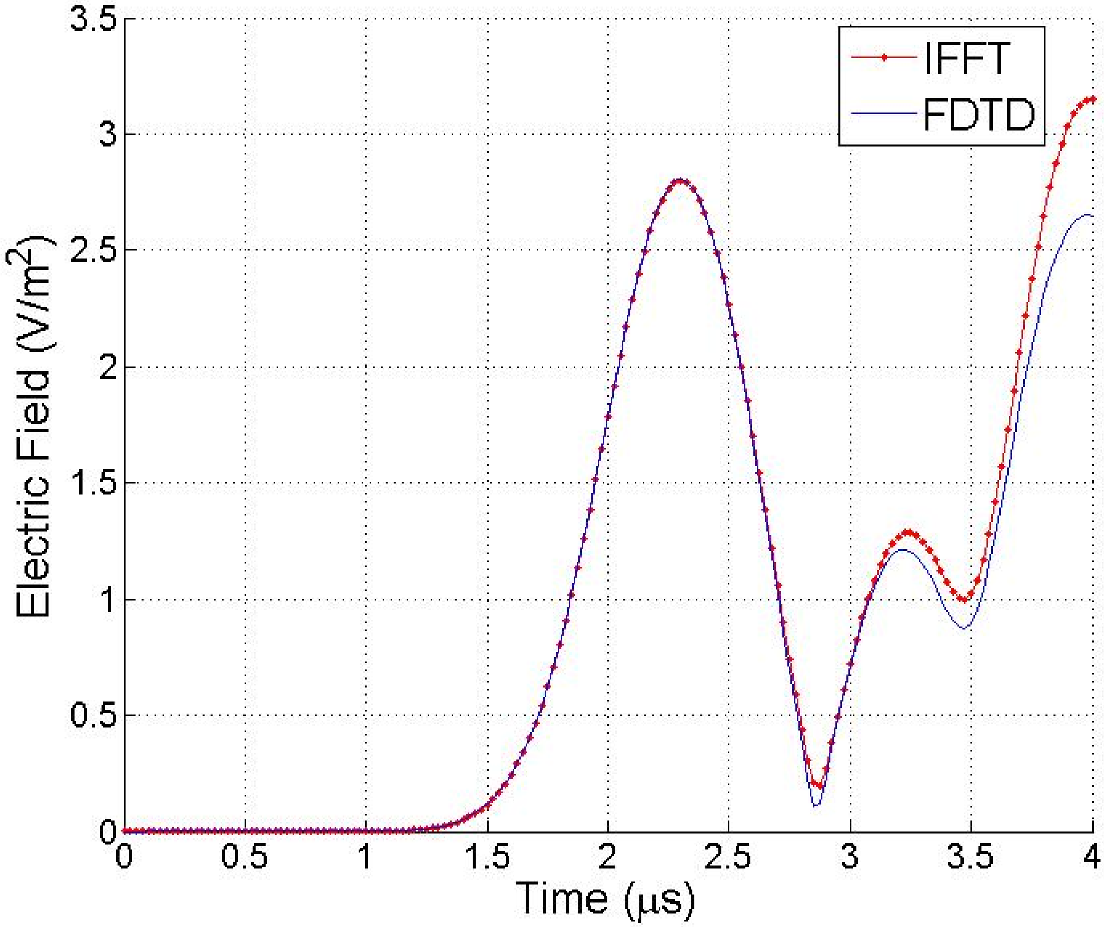


Figure 4.6. Magnitude of the CE electric field for $\rho = 5\lambda_o$.

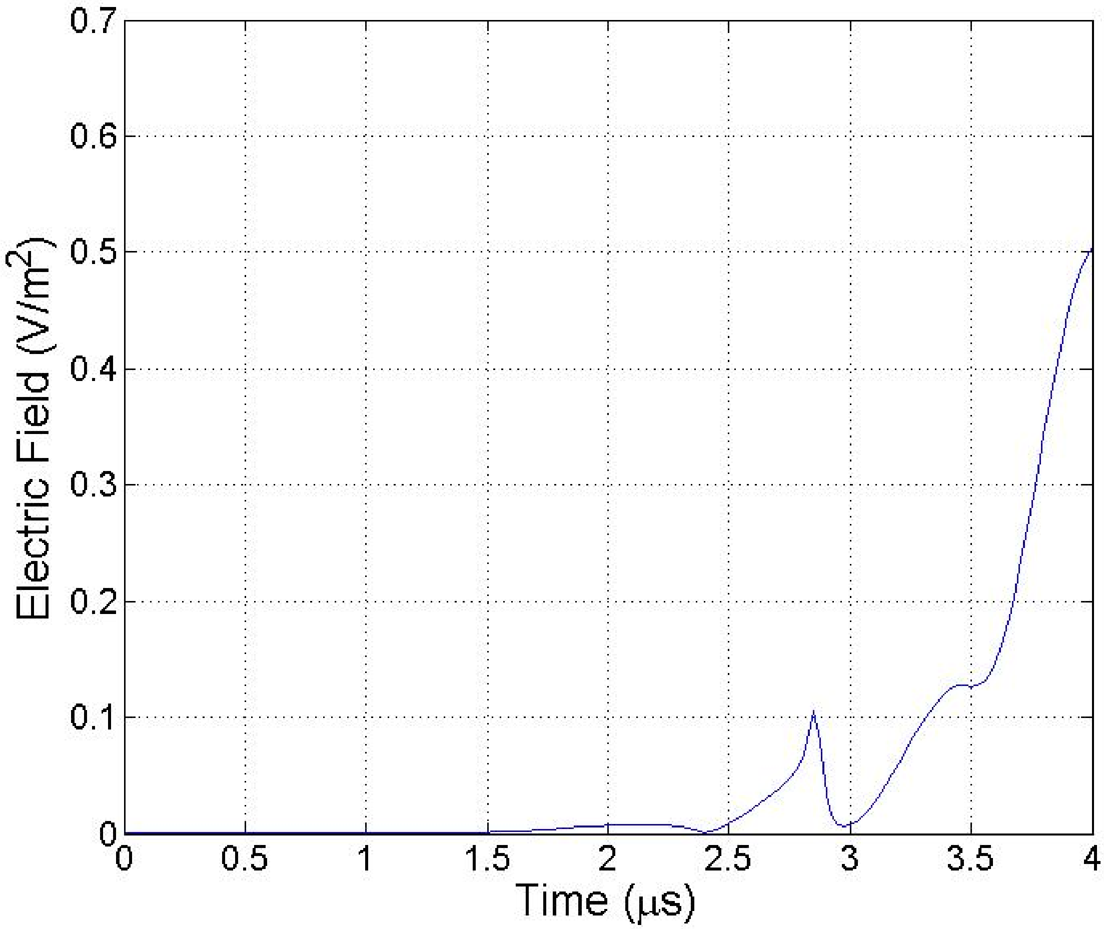


Figure 4.7. Magnitude of difference between the magnitudes of the CE electric field for the IFFT and FDTD solutions for $\rho = 5\lambda_o$.

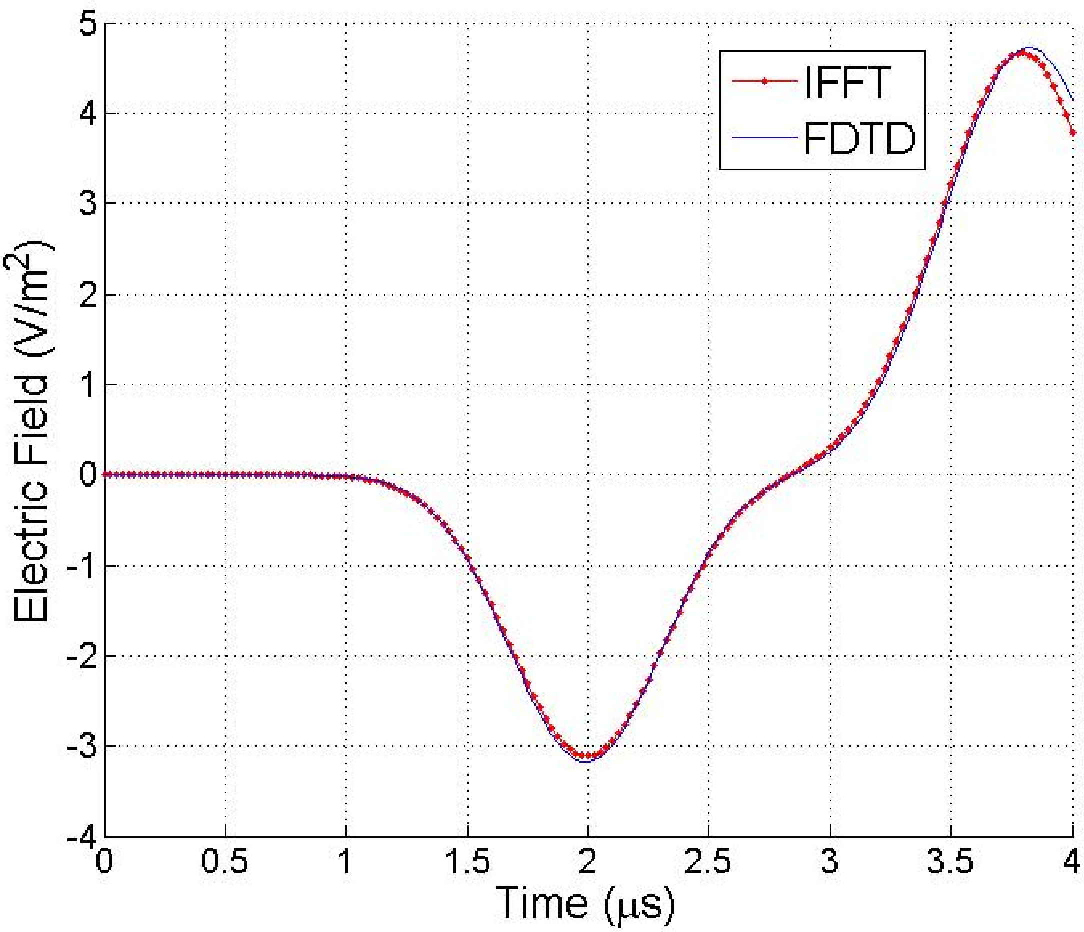


Figure 4.8. In-phase part of the electric field for $\rho = 2\lambda_o$.

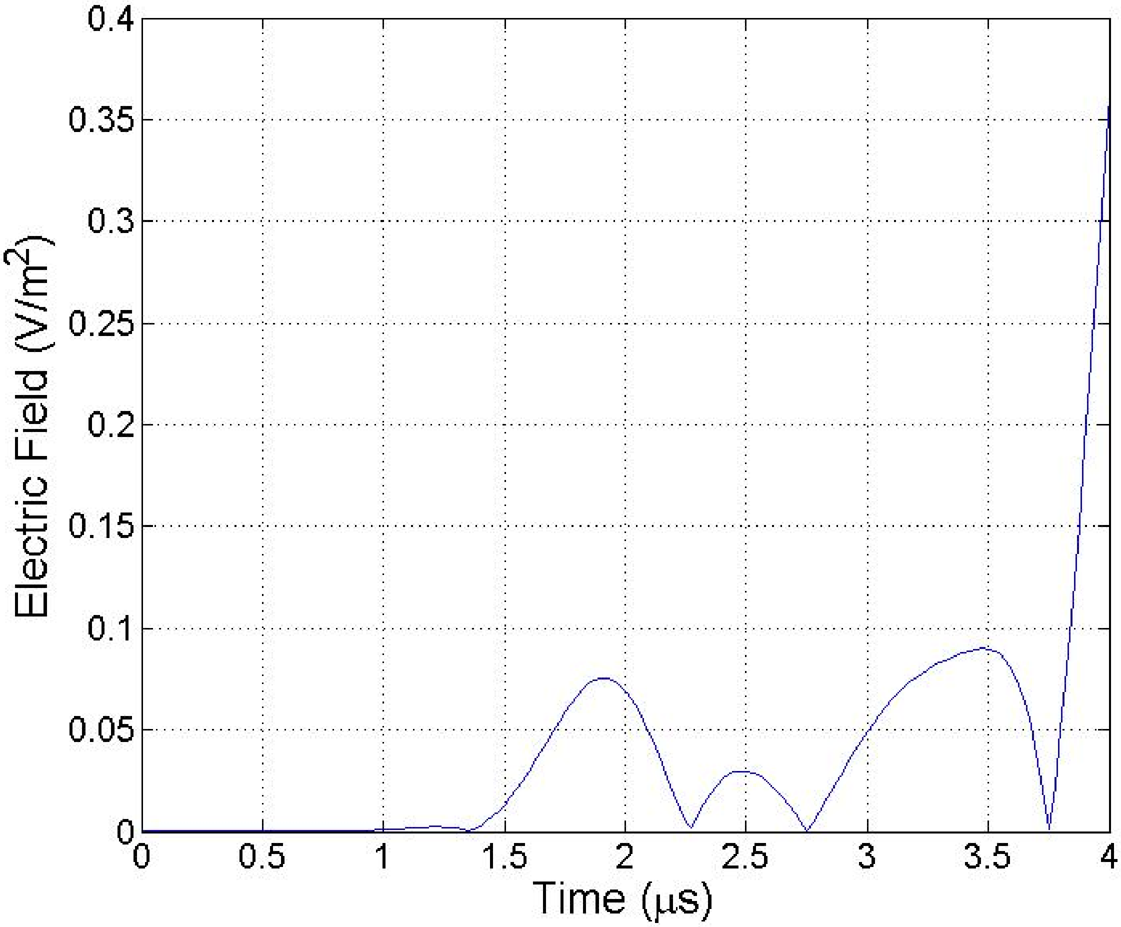


Figure 4.9. Magnitude of the difference between the in-phase parts for the IFFT and FDTD solutions for $\rho = 2\lambda_o$.

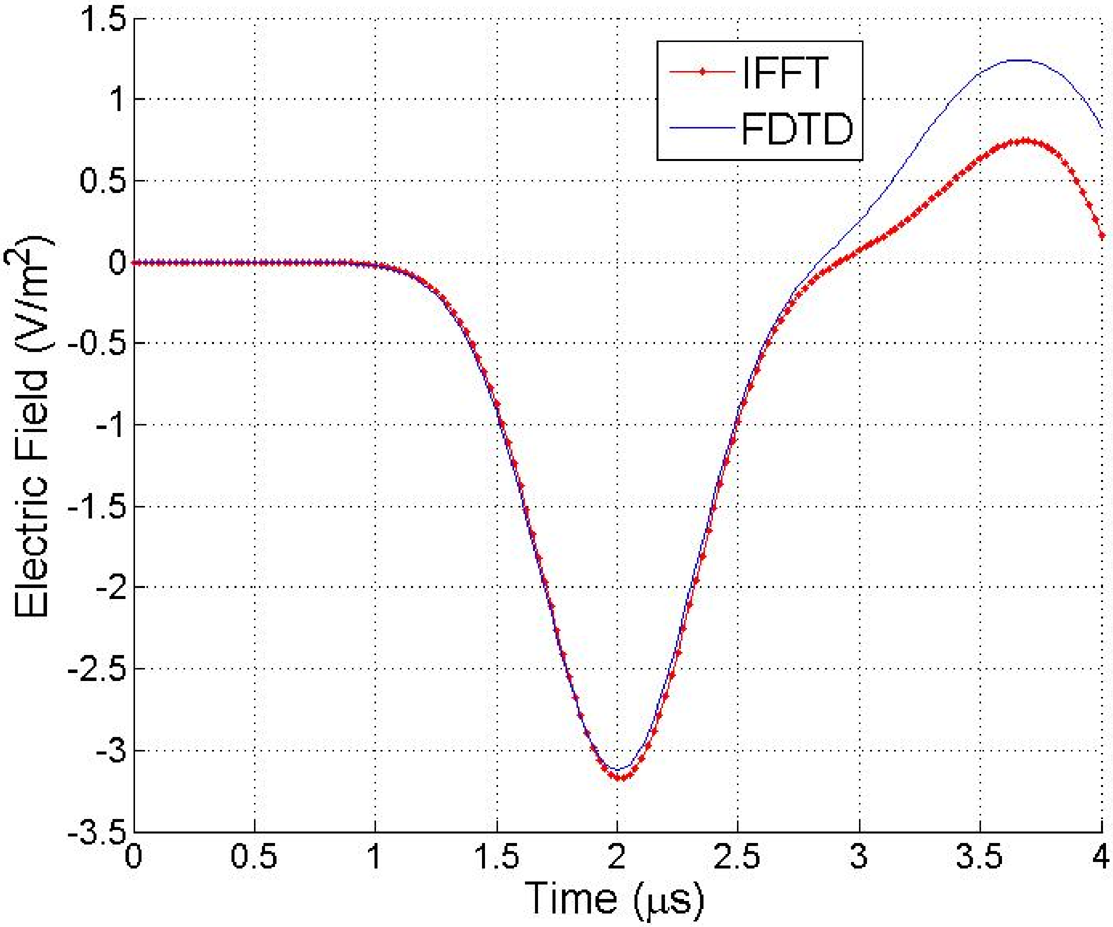


Figure 4.10. Quadrature part of the electric field for $\rho = 2\lambda_o$.

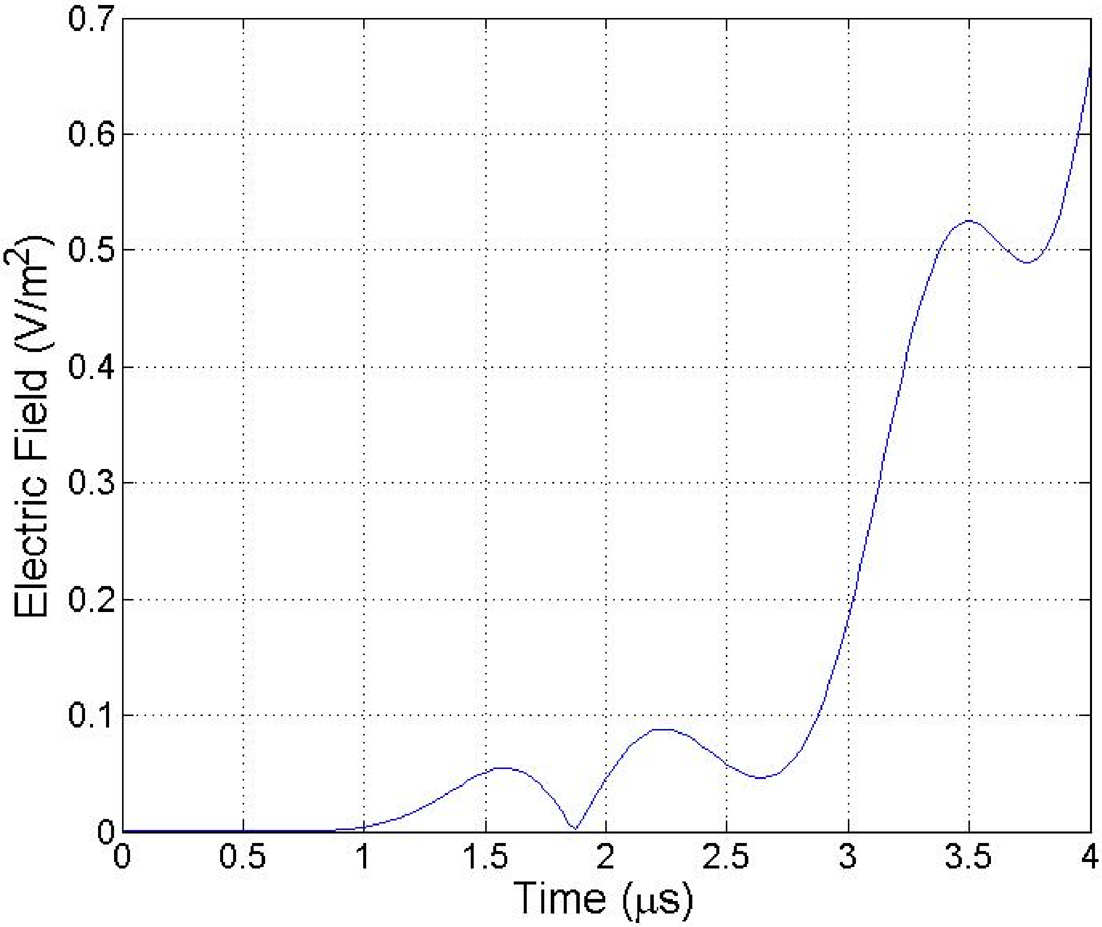


Figure 4.11. Magnitude of the difference between the quadrature parts for the IFFT and FDTD solutions for $\rho = 2\lambda_o$.

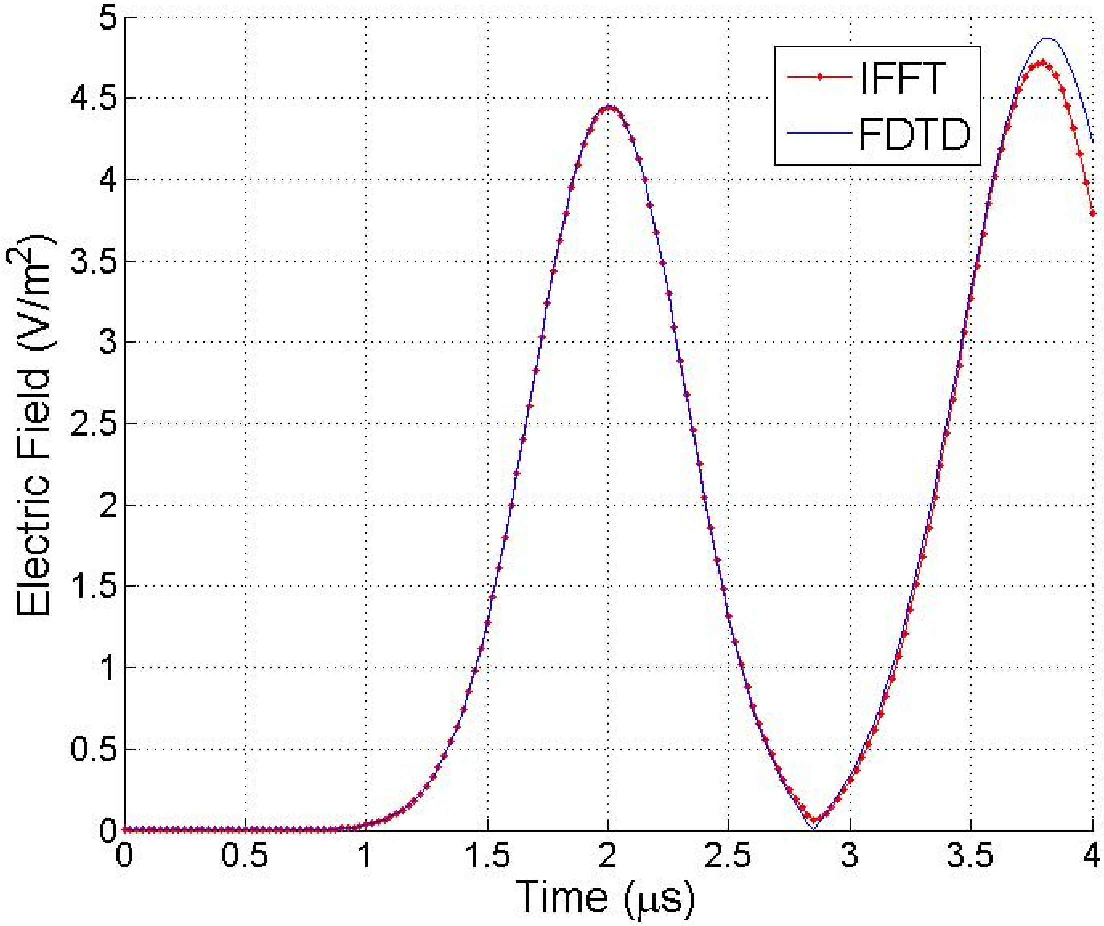


Figure 4.12. Magnitude of the CE electric field for $\rho = 2\lambda_o$.

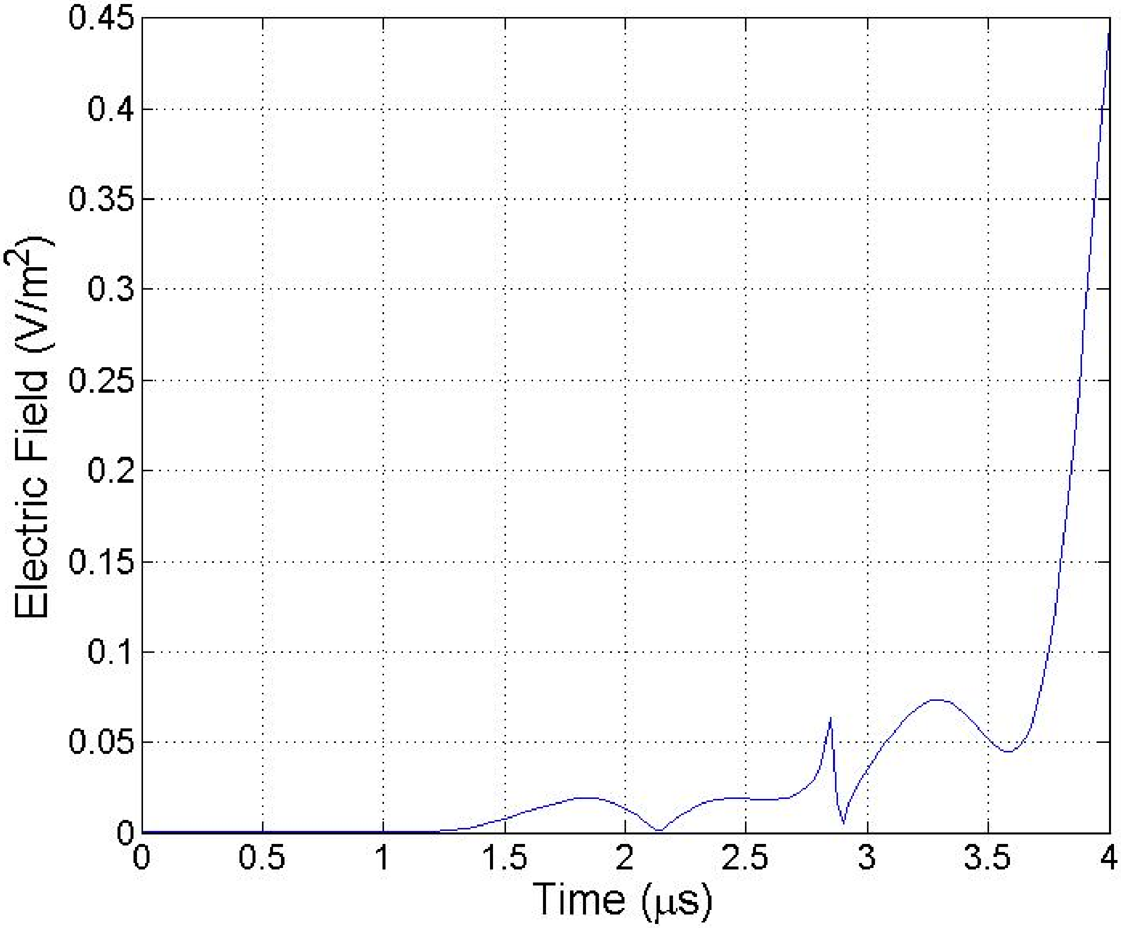


Figure 4.13. Magnitude of difference between the magnitudes of the CE electric field for the IFFT and FDTD solutions for $\rho = 2\lambda_o$.

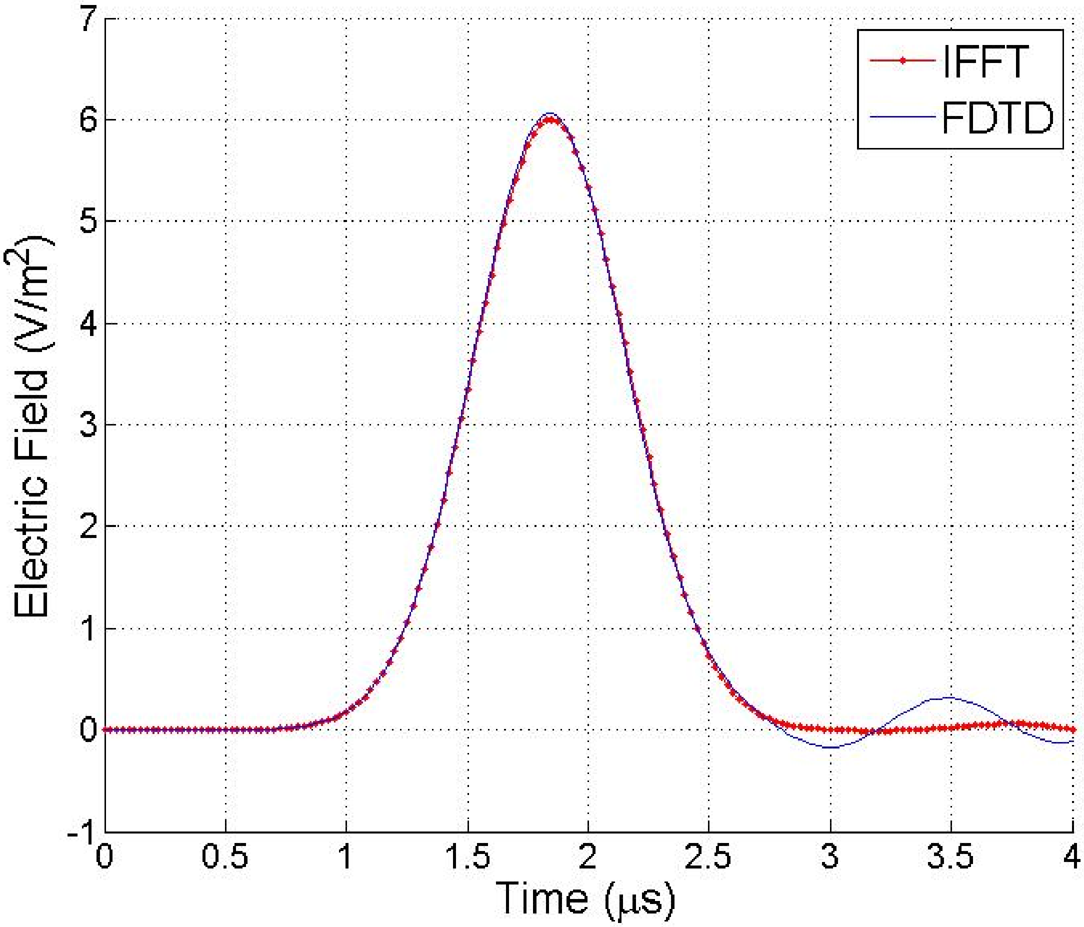


Figure 4.14. In-phase part of the electric field for $\rho = 0.5\lambda_o$.

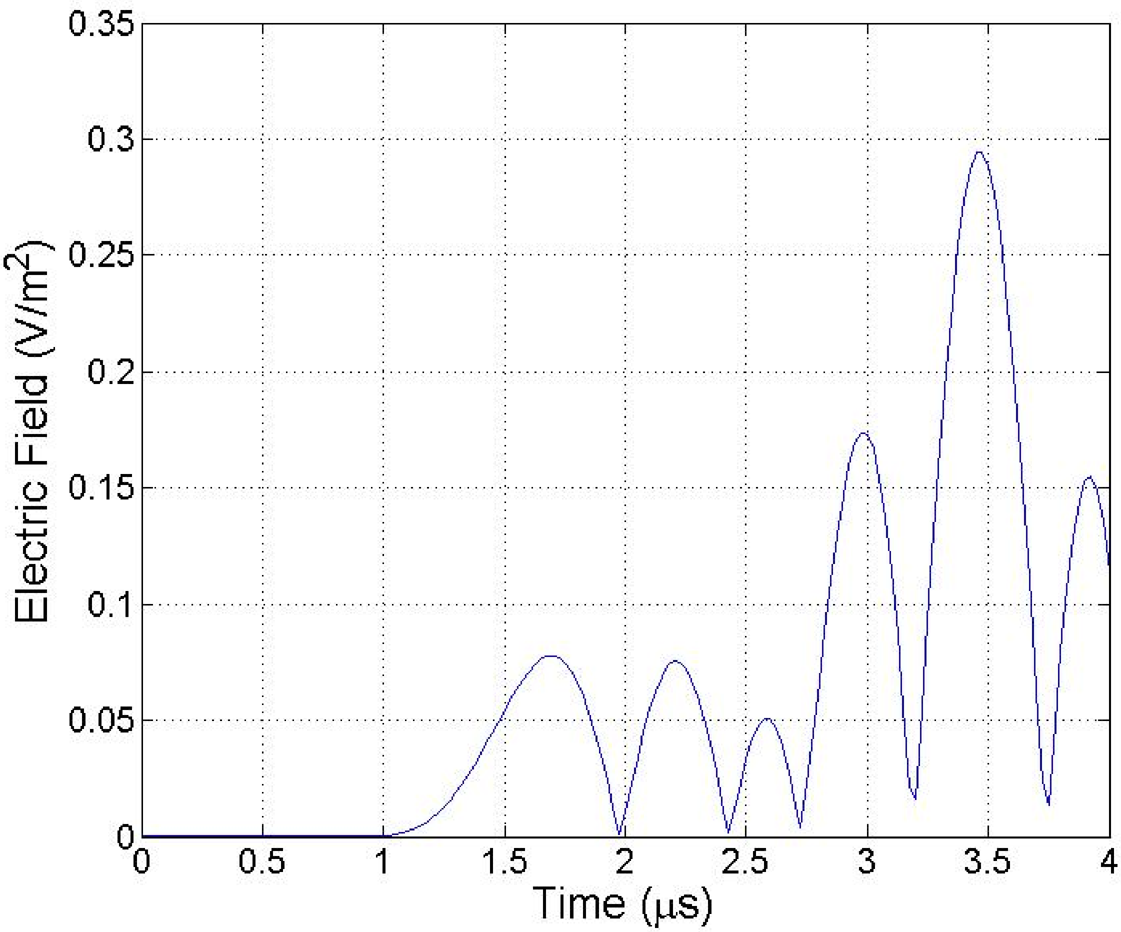


Figure 4.15. Magnitude of difference between the in-phase parts of the electric field for the IFFT and FDTD solutions for $\rho = 0.5\lambda_o$.

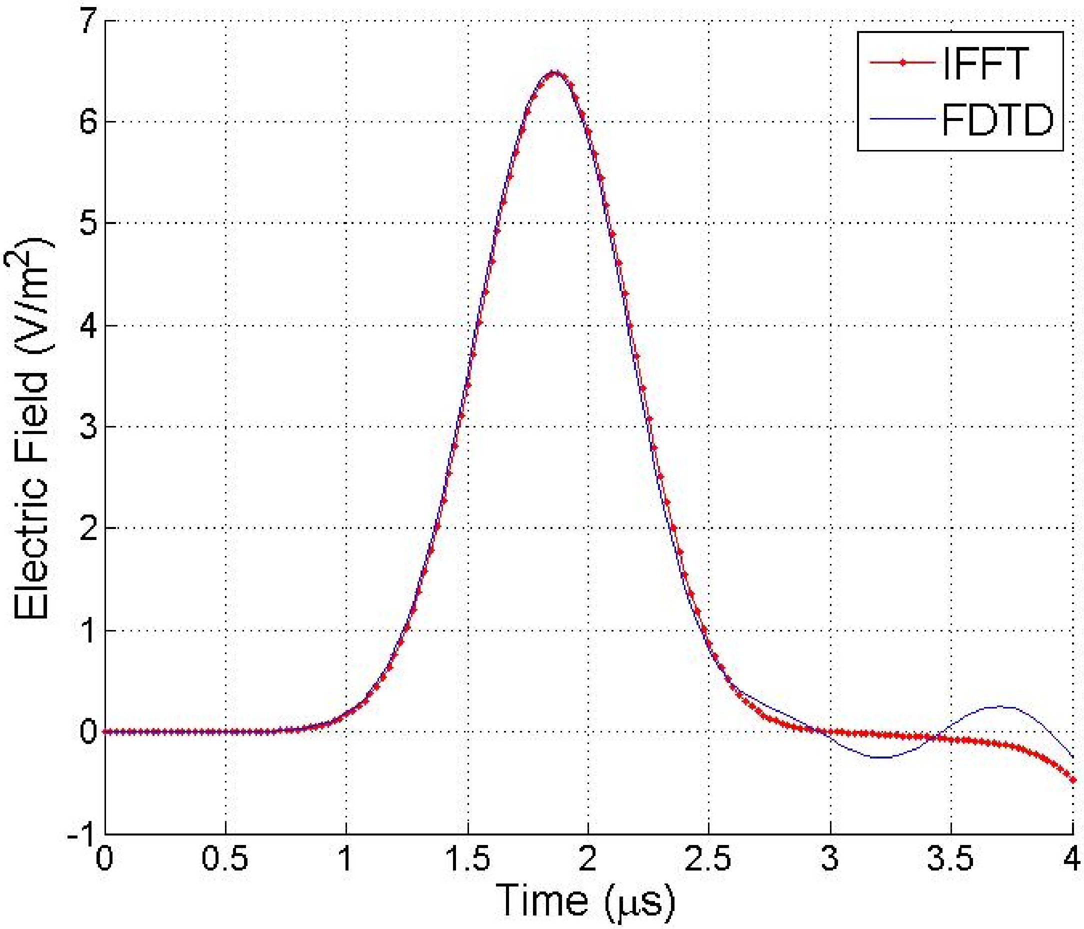


Figure 4.16. Quadrature part of the electric field for $\rho = 0.5\lambda_o$.

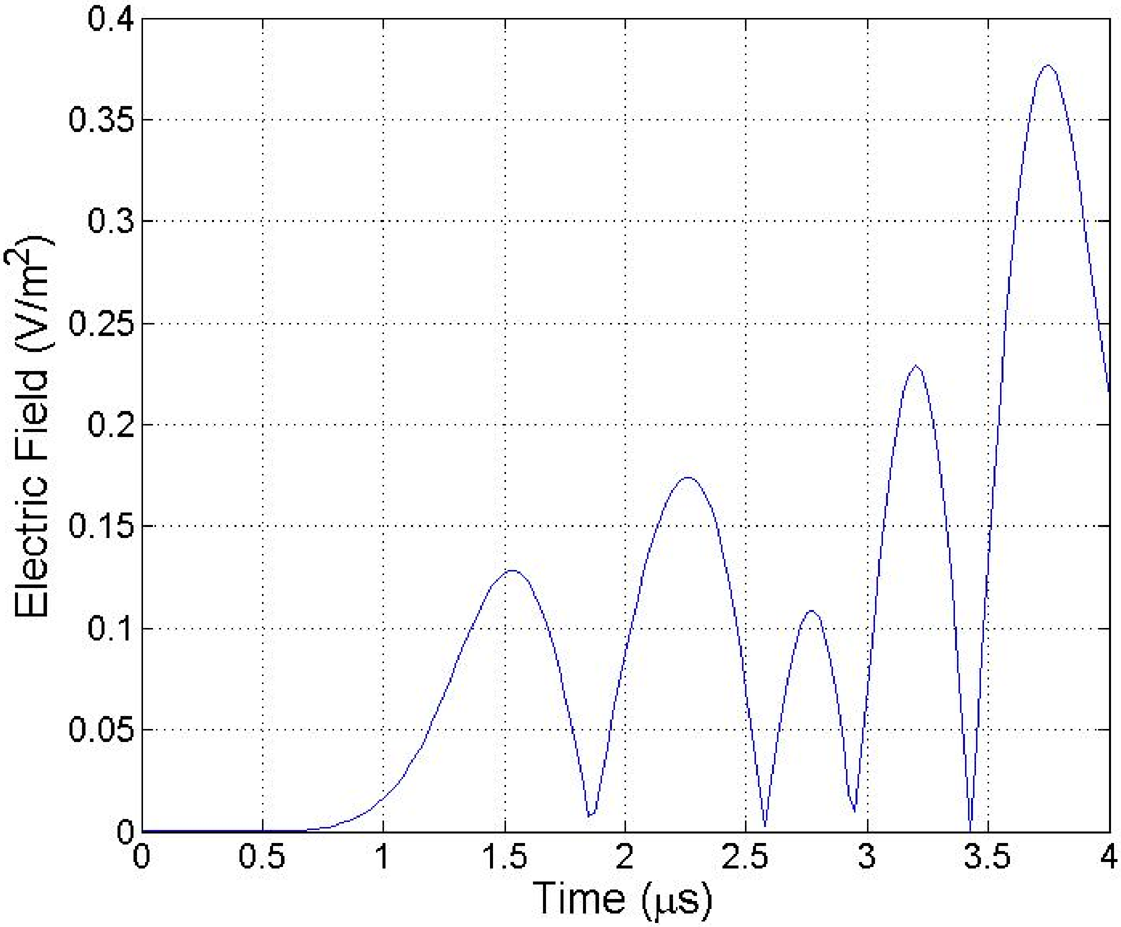


Figure 4.17. Magnitude of difference between the quadrature parts of the IFFT and FDTD solutions for $\rho = 0.5\lambda_o$.

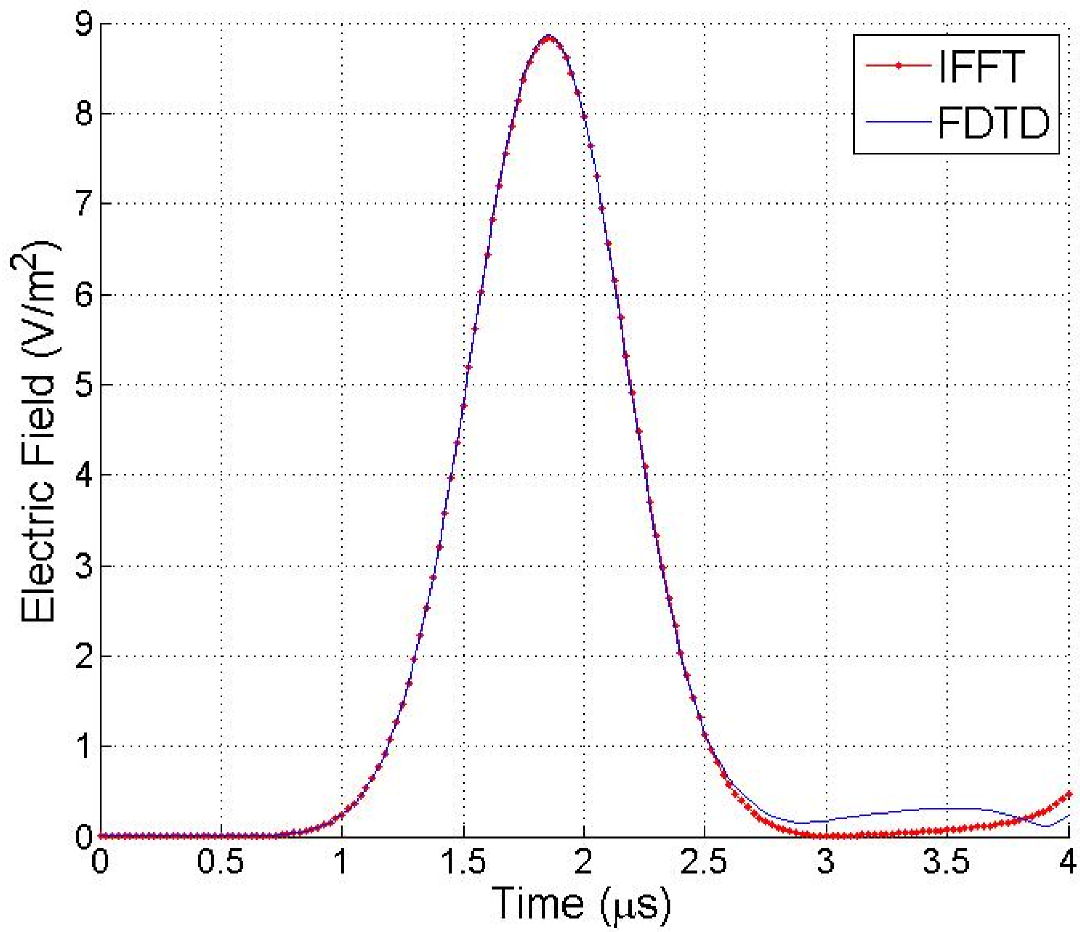


Figure 4.18. Magnitude of CE electric field for $\rho = 0.5\lambda_o$.

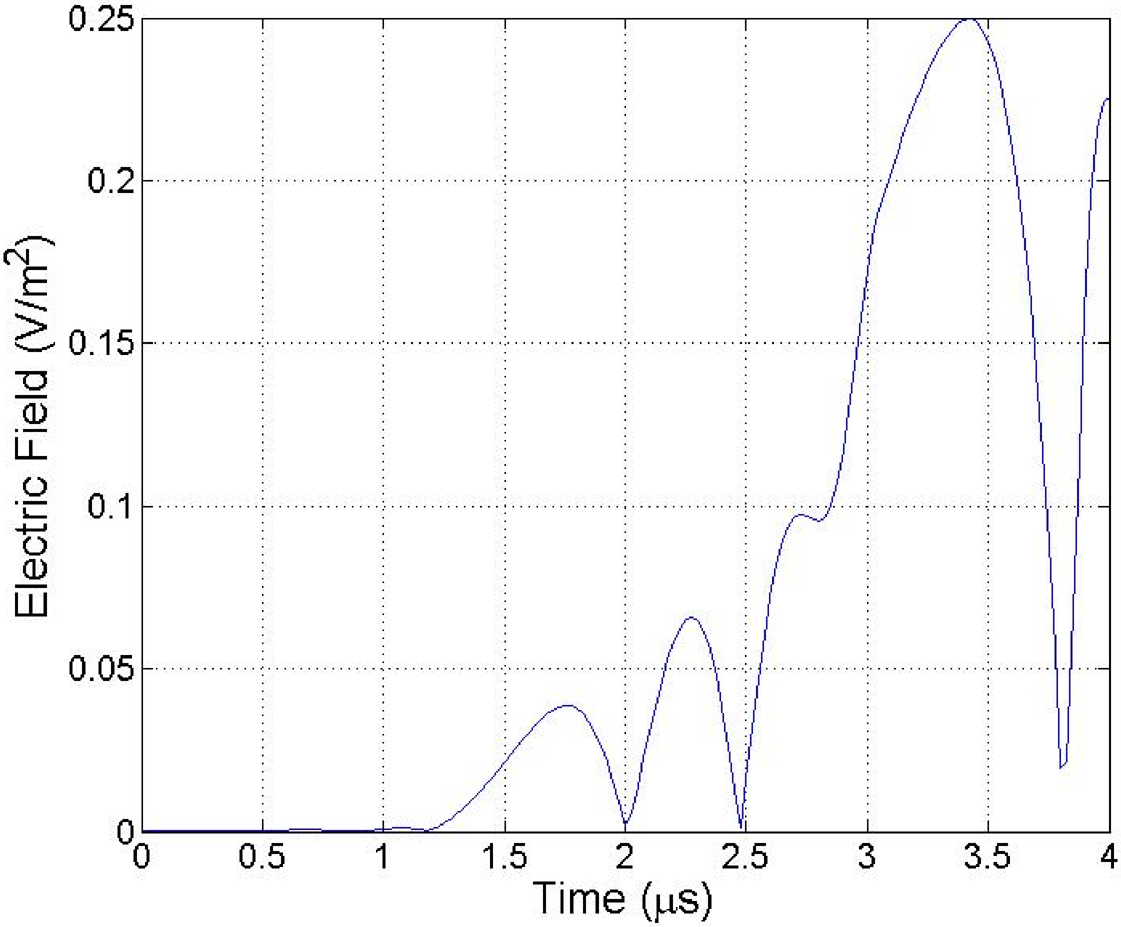


Figure 4.19. Magnitude of difference between the magnitudes of the CE electric field for the IFFT and FDTD solutions for $\rho = 0.5\lambda_o$.

4.2 Electric Field over Space

Figures (4.20) to Figure (4.22) are surface plots of the CE electric field at $2.3\mu s$ for the whole computational domain of the cavity determined using the FDTD method. In all the figures, the fields propagate uniformly away from the source at the origin. In the areas close to the edges, reflections from the PEC boundaries combine with incident waves to generate ripples in the "peak circles".

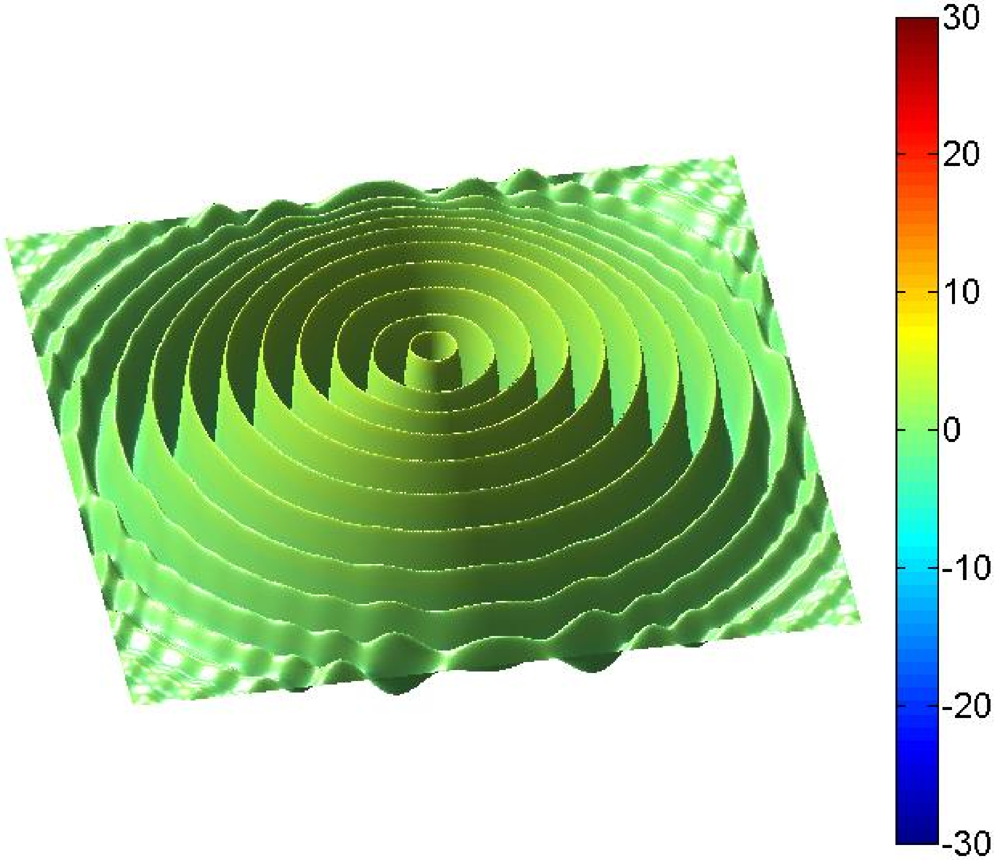


Figure 4.20. In-phase part of the CE electric field versus x and y for $t = t_d + 0.5\mu s$.

In addition, since the ending time, $2.3\mu s$, is small, the Gaussian pulse (source) has just started to decreasing in amplitude. Consequently, the magnitude of electric field near the source region is still very large.

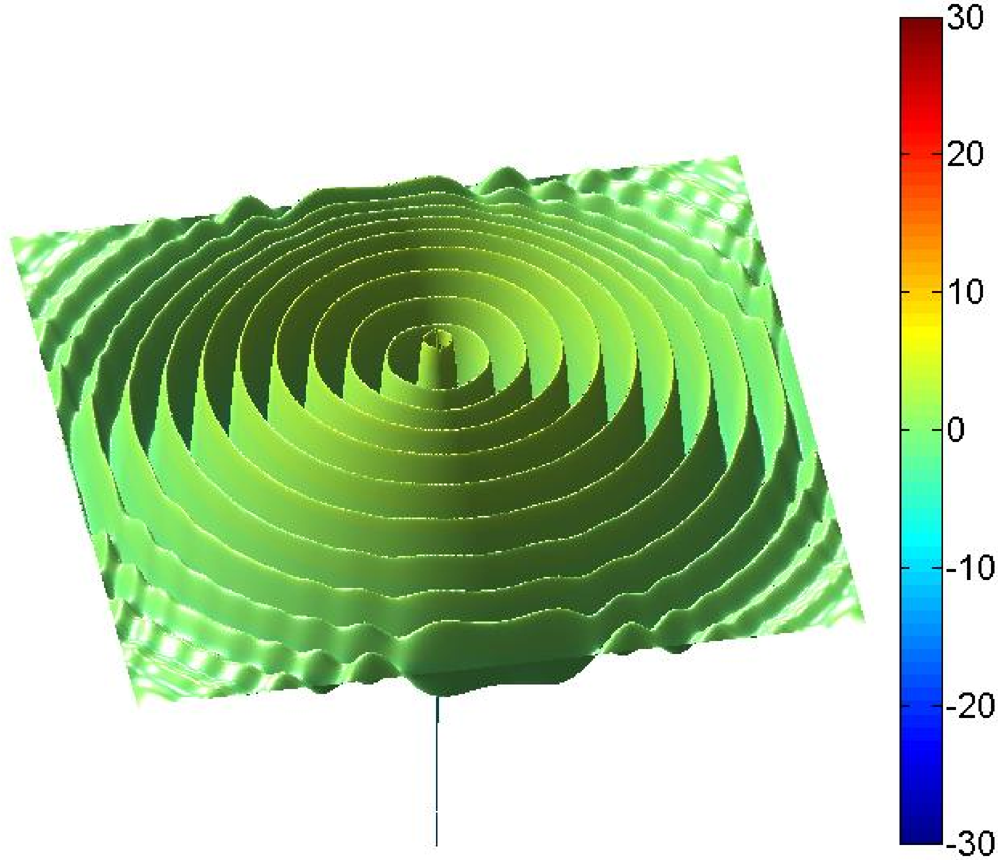


Figure 4.21. Quadrature part of the CE electric field versus x and y for $t = t_d + 0.5\mu s$.

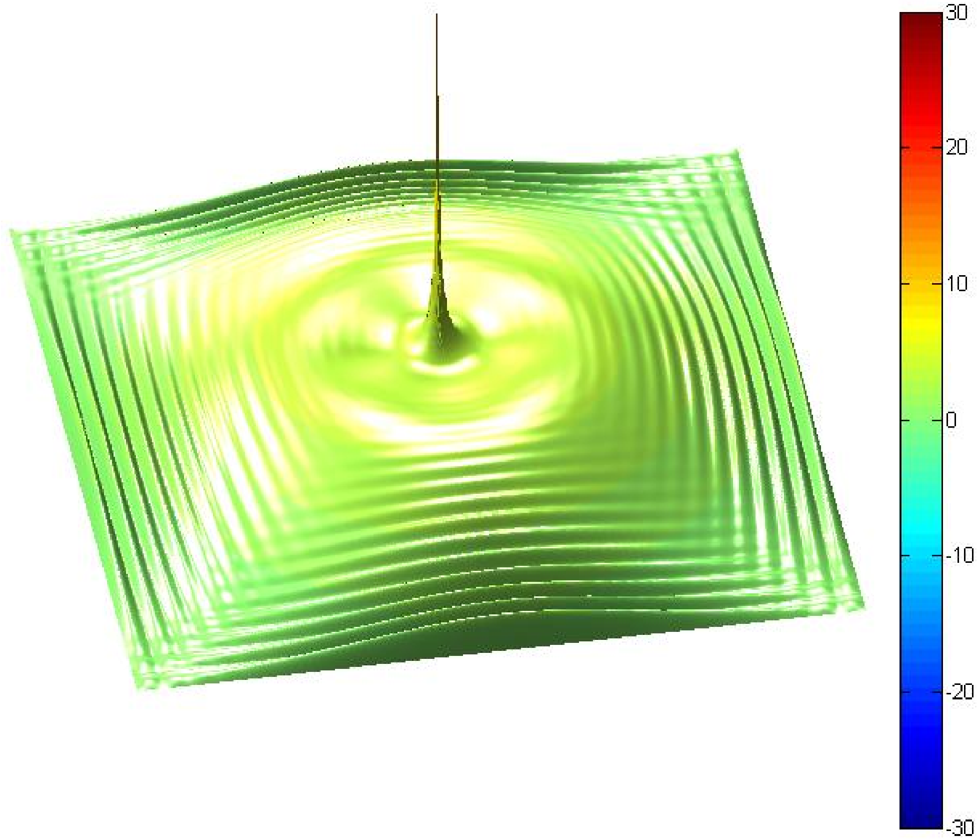


Figure 4.22. Magnitude of the CE electric field versus x and y for $t = t_d + 0.5\mu s$.

CHAPTER 5

CONCLUSION AND FUTURE WORK

The formulation of the CE FDTD method using real-valued field-variables was presented and solutions for a 2D cavity problem from both the proposed numerical method and a reference method were given.

5.1 Conclusion

Using real-valued field-variables, the CE PDEs can be split into two groups of real-valued PDEs that can potentially be used with any implicit FDTD scheme. As a result, band-pass limited electromagnetic problems can be solved in the base band using many existing FDTD formulations.

The primary advantage of the CE FDTD over the standard FDTD is that the time step can be several orders of magnitude larger meaning that fewer time steps are needed for a calculation. A further advantage of the present CE FDTD over existing CE FDTD methods is that existing methods require calculation with complex numbers whereas the present method uses calculation with real numbers alone. This is an advantage because a complex multiplication requires three real multiplications and a complex division requires three real divisions and three real multiplications.

In the plots of electric field versus time for the cavity problem, result from the present method and the reference method are nearly identical until $2\mu s$. In $2\mu s$ the wave can travel 20 free space wavelengths. After $2\mu s$ the error tends to increase, however, it is still relatively small at $4\mu s$. Plots of the electric field over the spatial domain at $2.3\mu s$ shows a stable result with waves that propagate without obvious dispersion.

5.2 Future Work

This thesis presents the central ideas required to formulate a complex-envelope FDTD method with real-valued field-variables and gives an example formulation in 2D based on the Douglas Gunn algorithm given by Sun and Trueman [10]. However, much remains to be done to develop the CE FDTD method with real-valued field-variables into a useful tool for practical electromagnetics problems. First and foremost is the development of a 3D formulation. Also, needed are appropriate absorbing boundaries to terminate the spatial mesh as well as a carefully stability and dispersion analysis.

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VITA

Qi Liu, son of Cuilin Yao and Qingbo Liu, was born on July 21st 1987 in Wuhan. He graduated from Jianghan University in 2009 with Bachelor of Engineering. After one year English study, he came to the United States as an international student pursuing his master degree. In spring 2012, he came to the University of Mississippi as a transfer student majoring in Engineering Science. In the following two and half years in Ole Miss, he studied as a graduate student in the Department of Electrical Engineering while worked as a teaching assistance. In the Spring of 2014, he completed his Master of Engineering Science (electromagnetics) at the University of Mississippi.