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Bipartite Density of Generalized Petersen Graphs

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BIPARTITE DENSITY OF GENERALIZED PETERSEN GRAPHS

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ABSTRACT

The bipartite density $b(G)$ of a graph G with m edges is the maximum ratio $\frac{m_0}{m}$ where m_0 is the number of edges in a bipartite subgraph of G. In this study we determine the bipartite density of several classes of Generalized Petersen Graphs. These graphs are denoted by $P(n, k)$, where $n \geq 3$ and $1 \leq k < n$ with $n \neq 2k$. The Generalized Petersen Graph $P(n, k)$ has vertices $\{v_i\}_{i=0}^{n-1} \cup \{w_i\}_{i=0}^{n-1}$ and edges $\{v_i w_i\}_{i=0}^{n-1} \cup \{v_i v_{i+1}\}_{i=0}^{n-1} \cup \{w_i w_{i+k}\}_{i=0}^{n-1}$ where subscript addition is modulo n. We define subgraphs $P'(n, k)$ of $P(n, k)$ by deleting the edge $v_{n-1}v_0$ and the edges $w_i w_{i+k}$ for $n-k \leq i \leq n-1$. For $P'(n, k)$ and many classes of $P(n, k)$, we determine the exact number of edges which must be removed from $P(n, k)$ to reduce it to a bipartite subgraph. In many classes of Generalized Petersen Graphs the exact bipartite density is derived. For example: $b(P(n, k)) = 1$ for n even, k odd; $b(P(n, k)) = 1 - \frac{k+1}{3n}$ for n and k odd and $n > k^2$; $b(P(n, k))$ is asymptotically $1 - \frac{1}{3k}$ for n odd, k even.

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CHAPTER 1

Introduction

1. Background

This thesis is a study of maximal bipartite subgraphs in Generalized Petersen Graphs. The study of "bipartite density" can be traced back at least to 1964 when Erdős [**1**] proved that every graph G has a bipartite subgraph containing at least half the edges of G. In a subsequent article [**2**] Erdős proposed what has become perhaps the most vexing problem related to bipartite subgraphs:

Conjecture 1.1. *If G is a triangle-free graph with n vertices, then G may be reduced to a bipartite subgraph by the removal of no more than $\frac{n^2}{25}$ edges.*

Erdős provided a simple class of examples to motivate the conjecture and show that if true, then it is best possible. Hopkins and Staton [**3**] derived an upper bound of $\frac{n^2}{16}$ for regular graphs. Erdős, Faudree, Pach and Spencer [**4**] improved the bound to $\frac{n^2}{18}$, but the question is probably quite difficult and is seemingly far from solution. Alexander, Hopkins and Staton [**5**] verified the $\frac{n^2}{25}$ conjecture for $n \leq 16$.

The study of bipartite density in more restricted classes of graphs began in the 1980's. Staton [**6**] considered graphs of maximum degree three and proved the following.

Theorem 1.2. *Let G be a connected cubic graph. Then:*

(*i*) $b(G) \geq \frac{2}{3}$;

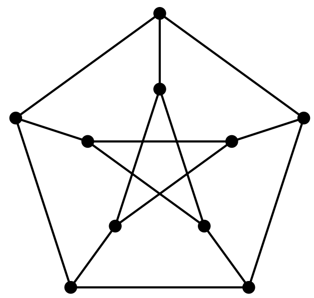

FIGURE 1.1. $P(5,2)$: The Petersen Graph

(*ii*) *If* $G \neq K_4$, *then* $b(G) \geq \frac{7}{9}$;

(*iii*) *If* G *contains no* K_3, *then* $b(G) \geq \frac{11}{14}$.

The bipartite density, $b(G)$, is defined below. Staton noted that (*i*) and (*ii*) are best possible, and conjectured that the lower bound in (*iii*) could be improved to $\frac{4}{5}$, which he noted, is the bipartite density of the dodecahedron and the Petersen Graph, $P(5,2)$.

This conjecture was verified by Hopkins and Staton [**7**], and their result was improved by Bondy and Locke [**8**] who extended the $\frac{4}{5}$ bound to all triangle-free graphs of maximum degree three. Locke [**9**] introduced the term "bipartite density". McColgan [**10**] considered bipartite density of regular graphs of degree four and derived a lower bound of $\frac{17}{24}$ for bipartite density of triangle free 4-regular graphs. McColgan and Staton [**11**] found an exact formula in terms of the degree sequence of a graph G, for the bipartite density of the line graph $L(G)$.

2. Definitions

The reader of this thesis is presumed to be acquainted with the basic definitions of Graph Theory available in a standard text such as [**12**]. In everything that follows graphs will have no loops or multiple edges. A graph is bipartite if its vertices may be partitioned into sets

A and B so that every edge joins a vertex of A and a vertex of B. Among the first theorems of any graph theory course is the following characterization:

Theorem 1.3. *A graph G is bipartite if and only if G contains no odd cycle.*

The bipartite density $b(G)$ of a graph G with m edges is the ratio $\frac{m_0}{m}$ where m_0 is the maximum number of edges among the bipartite subgraphs of G. Generalized Petersen Graphs are defined as follows.

Definition 1.4. *Suppose $n \geq 3$ and $1 \leq k < n$ with $n \neq 2k$. The Generalized Petersen Graph $P(n,k)$ has vertices $\{v_i\}_{i=0}^{n-1} \cup \{w_i\}_{i=0}^{n-1}$ and edges $\{v_i w_i\}_{i=0}^{n-1} \cup \{v_i v_{i+1}\}_{i=0}^{n-1} \cup \{w_i w_{i+k}\}_{i=0}^{n-1}$ where subscript addition is modulo n.*

Following Castagna and Prins [**13**], we call the edges $v_i w_i$ spokes. Edges $v_i v_{i+1}$ will be called outer edges. Edges $w_i w_{i+k}$ will be called inner edges. These conventions are in accordance with the standard visual presentation of $P(n,k)$ (see Figure 1.2). Many authors include the condition that n and k are relatively prime in the definition of $P(n,k)$. This condition has been omitted here because some of the results derived here hold even without this assumption.

It is clear that the graph $P(n,k)$ is cubic with $2n$ vertices and $3n$ edges. It is also clear that $P(n,k)$ is isomorphic to $P(n,n-k)$. Hence in what follows it will be assumed that $1 \leq k < \frac{n}{2}$. In investigating bipartite density of $P(n,k)$ it has been convenient to focus on the edges which are removed rather than those remaining in a largest bipartite subgraph. Hence we define for a graph G, $f(G) = f =$ smallest m such that G may be reduced to a bipartite subgraph by the removal of some set of m edges. From $f(G)$, the bipartite density

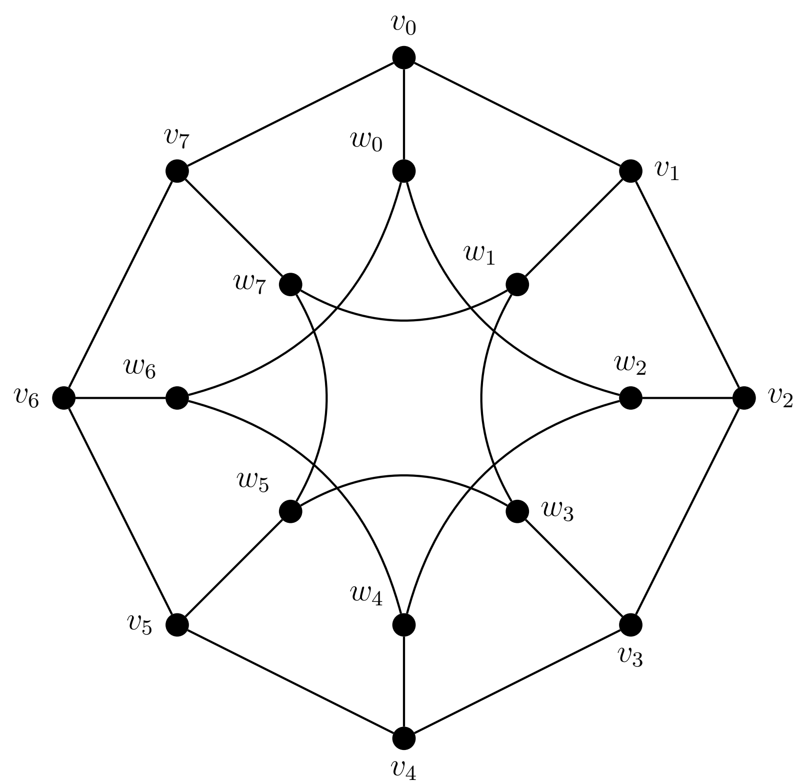

FIGURE 1.2. $P(8,2)$

of Generalized Petersen Graphs is easily found by $b(G) = \frac{3n - f(G)}{3n}$. We also let $f = f_i + f_o + f_s$ where f_i is the number of inner edges removed, f_o is the number of outer edges removed, and f_s is the number of spokes removed.

It is convenient in this study to "linearize" the $P(n,k)$ by considering subgraphs $P'(n,k)$ by modifying the edge set to the set $\{v_i w_i\}_{i=0}^{n-1} \cup \{v_i v_{i+1}\}_{i=0}^{n-2} \cup \{w_i w_{i+k}\}_{i=0}^{n-1-k}$. $P'(n,k)$ is a spanning subgraph of $P(n,k)$ with $3n - 1 - k$ edges. We will refer to a graph $P'(n,k)$ as a Petersen path (see Figure 1.3).

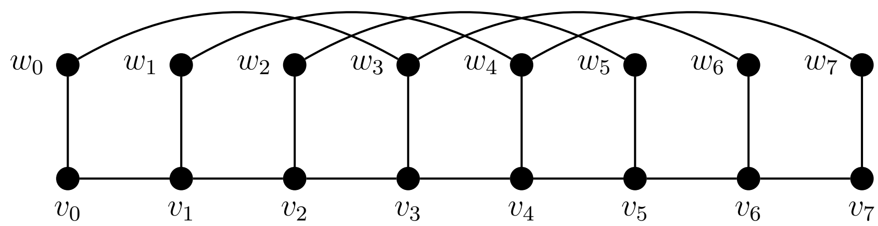

FIGURE 1.3. $P'(8,3)$

CHAPTER 2

Petersen Paths and $f(P(n,k))$ for Small k

1. Petersen Paths

It has proven to be quite helpful to begin the study of $f(P(n,k))$ by computing $f(P'(n,k))$, a far easier task.

THEOREM 2.1. *For k odd, $f(P'(n,k)) = 0$.*

PROOF. Let $A = \{v_i \mid i \text{ even}\} \cup \{w_j \mid j \text{ odd}\}$ and $B = \{v_i \mid i \text{ odd}\} \cup \{w_j \mid j \text{ even}\}$. This is a bipartition. □

THEOREM 2.2. *For k even, $f(P'(n,k)) = \lfloor \frac{n-1}{k} \rfloor$.*

PROOF. Remove $v_{i-1}v_i$ for $i \equiv 0 \bmod k$. Exactly $\lfloor \frac{n-1}{k} \rfloor$ edges have been removed. For every i, let L_i be the least nonnegative residue of i mod $2k$ and color as follows:

(1) w_i blue and v_i red if $i \equiv 0 \bmod 2$ and $L_i < k$ or if $i \equiv 1 \bmod 2$ and $L_i \geq k$;

(2) w_i red and v_i blue if $i \equiv 1 \bmod 2$ and $L_i < k$ or if $i \equiv 0 \bmod 2$ and $L_i \geq k$.

It is clear that $v_i w_i$ join blue to red. The $v_{i-1}v_i$ edges join opposite colors only when $i \equiv 0 \bmod k$, but these are precisely the edges that have been removed. Finally, the $w_i w_{i+k}$ edges join vertices with subscripts of the same parity, but $L_i \neq L_{i+k}$, so w_i and w_{i+k} have opposite colors. So $f(P'(n,k)) \leq \lfloor \frac{n-1}{k} \rfloor$.

To see that $f(P'(n,k)) \geq \lfloor \frac{n-1}{k} \rfloor$, note that there exist $n-k$ cycles of the type $v_i v_{i+1} \ldots v_{i+k} w_{i+k} w_i v_i$ where $0 \leq i \leq n-k-1$, each of odd length $k+3$. Each contains k outer edges, two spokes and one inner edge. Hence, in order to eliminate all odd cycles, it is necessary to remove at least $\lceil \frac{n-k}{k} \rceil$ edges. Let $n = qk + r$ with $0 \le r < k$. Then $\lceil \frac{n-k}{k} \rceil = \lceil \frac{qk+r-k}{k} \rceil = \lceil \frac{(q-1)k+r}{k} \rceil =$

$$q - 1 + \lceil \tfrac{r}{k} \rceil = \begin{cases} q-1 & \text{if } r = 0 \\ \\ q & \text{if } r \ge 1. \end{cases}$$

This is the same as $\lfloor \frac{n-1}{k} \rfloor = \lfloor \frac{qk+r-1}{k} \rfloor = q - 1 + \lfloor \frac{k+r-1}{k} \rfloor = \begin{cases} q-1 & \text{if } r = 0 \\ \\ q & \text{if } r \ge 1. \end{cases}$

We have shown $f(P'(n,k)) = \lfloor \frac{n-1}{k} \rfloor$. □

The dependence on the parity of k seen in these theorems will have consequences for the study of $f(P(n,k))$. From the values of $f(P'(n,k))$, we deduce the following inequalities for $f(P(n,k))$:

Corollary 2.3.

(i) *If k is odd, then $f(P(n,k)) \le k + 1$.*

(ii) *If k is even, then $f(P(n,k)) \le \lfloor \frac{n-1}{k} \rfloor + k + 1$.*

(iii) *If k is even, $k|n$, and $\frac{n}{k}$ is odd, then $f(P(n,k)) \le \lfloor \frac{n-1}{k} \rfloor + k$.*

Proof. From $P(n,k)$ one may remove first the $k+1$ edges in $P(n,k)$ not in $P'(n,k)$ and then the necessary edges to reduce $P'(n,k)$ to a bipartite graph. Finally if k is even, $k|n$, and $\frac{n}{k}$ is odd, note that v_0 and v_{n-1} have opposite colors in the 2-coloring of Theorem 2.2, so it is unnecessary to remove the edge $v_0 v_{n-1}$. Hence $f(P(n,k)) \le \lfloor \frac{n-1}{k} \rfloor + k$. □

2. $f(P(n,2))$

The next cases to be considered will be the graphs $P(n,2)$ where $n \geq 5$. These will be divided into four subcases according to congruence classes (mod 4).

PROPOSITION 2.4. *If $n \geq 5$ and $n \equiv 0 \bmod 4$, then $f(P(n,2)) = \frac{n}{2}$.*

PROOF. Remove from $f(P(n,2))$ the edges v_iv_{i+1} where $i \equiv 0 \bmod 2$. Color the vertices in two colors red and blue according to Table 1.

TABLE 1. For $n \equiv 0 \bmod 4$, $i \equiv 0 \bmod 4$

Red	**Blue**
v_i	v_{i+2}
v_{i+1}	v_{i+3}
w_{i+2}	w_i
w_{i+3}	w_{i+1}

Now, inner edges join w vertices with subscripts differing by two, and such vertices never have the same color. Spokes join v_i to w_i, and in every case these have opposite colors. If i is even, the edge v_iv_{i+1} was removed. If i is odd, v_i and v_{i+1} have opposite colors. It follows that $P(n,2)$ with the $\frac{n}{2}$ edges removed is bipartite, so $f(P(n,2)) \leq \frac{n}{2}$. Now, in $f(P(n,2))$ there are n 5-cycles of the form $v_iv_{i+1}v_{i+2}w_{i+2}w_iv_i$, each containing two outer edges, two spokes and one inner edge. So, removal of an edge eliminates either one or two of these cycles. Hence at least $\frac{n}{2}$ edges must be removed, so $f(P(n,2)) \geq \frac{n}{2}$. □

PROPOSITION 2.5. *If $n \geq 5$ and $n \equiv 1 \bmod 4$, then $f(P(n,2)) = \frac{n+1}{2}$.*

PROOF. As in the proof of Proposition 2.4, at least $\frac{n}{2}$ edges must be removed, hence $f(P(n,2)) \geq \frac{n+1}{2}$. Now remove the following edges: v_0w_0, w_1w_{n-1}, $v_{2i}v_{2i+1}$ for $1 \leq i \leq \frac{n-3}{2}$. Color the vertices as in Table 2.

TABLE 2. For $n \equiv 1 \bmod 4$, $i \equiv 0 \bmod 4$

Red	**Blue**
v_0	
v_{i+2}	v_i, $i \neq 0$
v_{i+3}	v_{i+1}
w_i	w_{i+2}
w_{i+1}	w_{i+3}

Note that the end vertices of each remaining edge receive opposite colors. □

PROPOSITION 2.6. *If* $n \geq 5$ *and* $n \equiv 2 \bmod 4$, *then* $f(P(n,2)) = \frac{n+2}{2}$.

PROOF. As in the proofs of the previous two propositions, there are n 5-cycles, each involving two outer edges, two spokes and one inner edge. Additionally, there are two disjoint $\frac{n}{2}$-cycles consisting entirely of inner edges. Since $n \equiv 2 \bmod 4$, these are odd cycles and must be destroyed. This requires removal of two inner edges. This eliminates two of the n 5-cycles, leaving $n-2$, which can be eliminated only by removing $\frac{n-2}{2}$ additional edges. Since $2 + \frac{n-2}{2} = \frac{n+2}{2}$, it follows that $f(P(n,2)) \geq \frac{n+2}{2}$. Now remove the following edges: w_0w_2, w_1w_{n-1}, $v_{2i}v_{2i+1}$ for $1 \leq i \leq \frac{n-2}{2}$. Color the vertices as in Table 3.

It is routine to check that each remaining edge joins vertices of opposite colors. Hence $f(P(n,2)) = \frac{n+2}{2}$. □

Table 3. For $n \equiv 2 \bmod 4$, $i \equiv 0 \bmod 4$

Red	**Blue**
v_0	w_0
v_{i+2}	v_i, $i \neq 0$
v_{i+3}	v_{i+1}
w_i, $i \neq 0$	w_{i+2}
w_{i+1}	w_{i+3}

Proposition 2.7. *If $n \geq 5$ and $n \equiv 3 \bmod 4$, then $f(P(n,2)) = \frac{n+1}{2}$.*

Proof. As in the proof of Proposition 2.5, at least $\frac{n+1}{2}$ edges must be removed. Now remove edges $w_1 w_{n-1}$, $v_i v_{i+1}$ for i odd. Color the vertices as in Table 4.

Table 4. For $n \equiv 3 \bmod 4$, $i \equiv 0 \bmod 4$

Red	**Blue**
v_{i+1}	v_i
v_{i+2}	v_{i+3}
w_i	w_{i+1}
w_{i+3}	w_{i+2}

Again, it is routine to check the coloring. $\square$

Using the results from this section, we obtain the following theorem:

Theorem 2.8. *If $n \geq 5$, then*
$$f(P(n,2)) = \begin{cases} \frac{n}{2} & \text{if } n \equiv 0 \bmod 4 \\ \frac{n+2}{2} & \text{if } n \equiv 2 \bmod 4 \\ \frac{n+1}{2} & \text{if } n \text{ is odd.} \end{cases}$$

3. $f(P(n,3))$

The next cases to be considered will be the graphs $P(n,3)$ where $n \geq 7$.

THEOREM 2.9. *If $n \geq 7$, then* $f(P(n,3)) = \begin{cases} 0 & \textit{if } n \textit{ is even} \\ 4 & \textit{if } n \textit{ is odd.} \end{cases}$

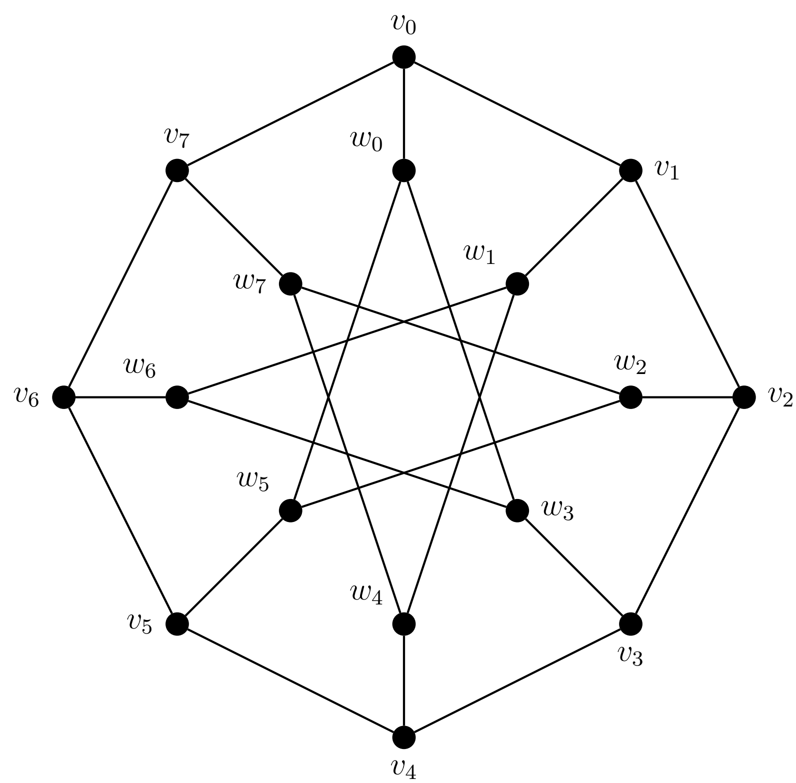

FIGURE 2.1. $P(8,3)$

PROOF. If n is even the bipartition of Theorem 2.1 works. If $n \equiv 3 \bmod 6$, then the inner edges form three disjoint odd cycles. The outer edges form another. Four edges are clearly necessary. But the graph $P'(n,3)$ is bipartite, and lacks exactly four of the edges of $P(n,3)$. So $f(P(n,3)) = 4$.

If $n \equiv \pm 1 \bmod 6$, the same argument shows $f(P(n,3)) \leq 4$. If $n \equiv 1 \bmod 6$ there are n cycles of length $\frac{n+8}{3}$, each of the form $v_i w_i w_{i+3} w_{i+6} \ldots w_{i-1} v_{i-1} v_i$. Since $n \equiv 1 \bmod 6$, these are odd cycles and each has one outer edge, two spokes, and $\frac{n-1}{3}$ inner edges. At least $\frac{n}{\frac{n-1}{3}}$

edges must be removed in order to eliminate these odd cycles. Since $\frac{n}{\frac{n-1}{3}} > 3$, it follows that $f(P(n,3)) \geq 4$, so in fact $f(P(n,3)) = 4$.

If $n \equiv -1 \bmod 6$, there are n cycles of length $\frac{n+10}{3}$, each of the form $v_i v_{i+1} w_{i+1} w_{i+4} w_{i+7} \ldots w_{i-1} v_{i-1} v_i$. Since $n \equiv -1 \bmod 6$, these are odd cycles, each containing two outer edges, two spokes and $\frac{n-2}{3}$ inner edges. As above, at least $\frac{n}{\frac{n-2}{3}}$ edges must be removed to eliminate the odd cycles, but this fraction is greater than three. $\square$

As noted above, the odd $k = 3$ situation is very different from the even $k = 2$ situation. This will continue as we move to $k = 4$.

4. $f(P(n,4))$

The f-values of the graphs $P(n,4)$ fall into categories according to the congruences of $n \bmod 8$.

THEOREM 2.10. *Suppose $n \geq 9$. Then*
$$f(P(n,4)) = \begin{cases} \frac{n}{4} & \text{if } n \equiv 0 \bmod 8 \\ \frac{n+11}{4} & \text{if } n \equiv 1 \bmod 8 \\ \frac{n+6}{4} & \text{if } n \equiv 2 \bmod 8 \\ \frac{n+5}{4} & \text{if } n \equiv 3 \bmod 8 \\ \frac{n+12}{4} & \text{if } n \equiv 4 \bmod 8 \\ \frac{n+3}{4} & \text{if } n \equiv 5 \bmod 8 \\ \frac{n+10}{4} & \text{if } n \equiv 6 \bmod 8 \\ \frac{n+9}{4} & \text{if } n \equiv 7 \bmod 8. \end{cases}$$

PROOF. The cases $n \equiv 1$ and $n \equiv 6 \bmod 8$ will be considered here. The remaining cases are similar to the two presented, and in fact the cases $n \equiv 0$ and $n \equiv 4$ are covered in Theorem 3.10 below.

If $n \equiv 1 \bmod 8$, consider the subgraph $P(n,4)$ obtained by removing the following edges: $v_{4i}v_{4i+1}$ for $1 \le i \le \frac{n-5}{4}$, v_0w_0, w_1w_{n-3}, w_2w_{n-2}, w_3w_{n-1}. Note that exactly $\frac{n+11}{4}$ edges have been removed. Color the vertices according to the scheme of Table 5.

TABLE 5. For $n \equiv 1 \bmod 8$

Red	**Blue**
v_0	
v_i for $i \equiv \{2,4,5,7\} \bmod 8$	v_i for $i \equiv \{0,1,3,6\} \bmod 8$, $i \neq 0$
w_i for $i \equiv \{0,1,3,6\} \bmod 8$	w_i for $i \equiv \{2,4,5,7\} \bmod 8$

To see that this is a bipartition, first note that spokes other than v_0w_0, which was removed, make no problem, since for $i \neq 0$, v_i and w_i get opposite colors. The inner edges, other than the three which have been removed, join vertices whose subscripts differ by four, but neither $\{0,1,3,6\}$ nor $\{2,4,5,7\}$ contains a pair differing by four. The outer edges, other than $v_{n-1}v_0$, join vertices whose subscripts differ by one. The only such pairs within the two color classes are v_iv_{i+1}, where $i \equiv 0 \bmod 8$ or $i \equiv 4 \bmod 8$. In the case $i \equiv 0 \bmod 8$, v_0 is red and both v_1 and v_{n-1} are blue. In all other cases, these edges have been removed. It follows that $f(P(n,4)) \le \frac{n+11}{4}$.

For the other inequality, consider two types of odd cycles in $(P(n,4))$ with $n \equiv 1 \bmod 8$.

Type A: $v_iw_iw_{i+4}v_{i+4}v_{i+3}v_{i+2}v_{i+1}v_i$

Type B: $v_iw_iw_{i+4}w_{i+8}...w_{i-1}v_{i-1}v_i$

The type A cycles are 7-cycles. The type B cycles are $\frac{n+11}{4}$-cycles, and since $n \equiv 1 \bmod 8$, $\frac{n+11}{4}$ is odd, so both types of cycles must be eliminated. There are n of each type. Removing an inner edge eliminates one type A cycle and $\frac{n-1}{4}$ type B cycles. Removing an outer edge eliminates four type A cycles and one type B cycle. Removing a spoke eliminates two type A and type B cycles. Let $f(P(n,k)) = f = f_i + f_o + f_s$ where f_i is the number of inner edges removed, f_o is the number of outer edges removed, and f_s is the number of spokes removed.

For convenience, we express these parameters in a way reflecting the expectation that f_o will be approximately $\frac{n-5}{4}$. So, let $f_o = \frac{n-5}{4} - Q$ where Q is an integer. After removal of f_o outer edges, there remain at least $5 + 4Q$ type A cycles and $\frac{3n+5+4Q}{4}$ type B cycles.

These cycles must be eliminated using spokes and inner edges, so, to eliminate the remaining type A cycles, at least $\frac{5+4Q}{2} = \frac{10+8Q}{4}$ edges must be removed. So $f = f_o + f_i + f_s \geq f_o + f_s \geq \frac{n-5}{4} - Q + \frac{10+8Q}{4} = \frac{n+5}{4} + Q$. If $Q \geq 1$ then $f \geq \frac{n+9}{4}$, so $f \geq \lceil \frac{n+9}{4} \rceil = \frac{n+11}{4}$.

Note that $f_i \geq 1$ since n and 4 are relatively prime, which implies that the inner edges form an n-cycle. Hence $f \geq f_i + f_o \geq \frac{n-1}{4} - Q$, and it follows that if $Q \leq -3$, $f \geq \frac{n+11}{4}$.

It remains to determine the inequality for $-2 \leq Q \leq 0$. With $\frac{n-5}{4} - Q$ outer edges removed and one inner edge removed, $\frac{3n+5+4Q}{4} - \frac{n-1}{4} = \frac{2n+6+4Q}{4}$ type B cycles remain to be eliminated. For any value of Q, $-2 \leq Q \leq 0$, $\frac{\frac{2n+6+4Q}{4}}{\frac{n-1}{4}} = \frac{2n+6+4Q}{n-1} > 2$. Thus the edges removed in these cases are at least $\frac{n-5}{4} - Q + 1 + 3 \geq \frac{n-11}{4}$. Hence, $f(P(n,4)) \geq \frac{n-11}{4}$ for $n \equiv 1 \bmod 8$.

In the case $n \equiv 6 \bmod 8$, remove the edges $v_{4i}v_{4i+1}$ for $0 \leq i \leq \frac{n-2}{4}$, $w_{n-2}w_2$, and $w_{n-3}w_1$. Note that exactly $\frac{n+10}{4}$ edges have been removed. Now color the vertices according to Table 6.

TABLE 6. For $n \equiv 6 \bmod 8$

Red	**Blue**
v_i for $i \equiv \{0, 1, 3, 6\} \bmod 8$	v_i for $i \equiv \{2, 4, 5, 7\} \bmod 8$
w_i for $i \equiv \{2, 4, 5, 7\} \bmod 8$	w_i for $i \equiv \{0, 1, 3, 6\} \bmod 8$

Clearly the spokes join vertices of opposite colors. The only adjacent v_i's of the same color are the pairs v_iv_{i+1} where $i \equiv 0 \bmod 8$ or $i \equiv 4 \bmod 8$, corresponding exactly to the edges removed, so remaining outer edges join vertices of opposite color. Finally, note that subscripts of inner vertices of the same color never differ by exactly four. Hence, the only adjacent pairs with the same color are $w_{n-2}w_2$ and $w_{n-3}w_1$, again corresponding to edges which have been removed. It follows that $f(P(n,4)) \le \frac{n+10}{4}$.

In $P(n,4)$ there are n 7-cycles of type A as in the case with $n \equiv 1 \bmod 8$. Additionally with $n \equiv 6 \bmod 8$, we define type C cycles to be those of the type $v_0w_0w_4w_8...w_{n-2}v_{n-2}v_{n-1}v_0$ and translations of this cycle by shifting subscripts. There are exactly n of these, each of length $\frac{n+14}{4}$, which is odd since $n \equiv 6 \bmod 8$. Each contains two outer edges, two spokes and $\frac{n-2}{4}$ inner edges. The inner edges form two disjoint $\frac{n}{2}$-cycles, and again $\frac{n}{2}$ is odd. If we set $f_o = \frac{n-2}{4} - Q$ and note that $f_i \ge 2$, then we have removed $\frac{n+6}{4} - Q$ edges, and there remain at least $4Q$ type A cycles and $2Q + 2$ type C cycles.

If $Q < 0$, then $f \ge f_o + f_i \ge \frac{n+6}{4} + 1 = \frac{n+10}{4}$. If $Q = 0$, then there remain two type C cycles which require at least one more edge, so again $f \ge \frac{n+10}{4}$. If $Q > 0$, the remaining $4Q$ type A cycles require removal of at least $2Q$ additional edges, yielding $f \ge \frac{n+6}{4} - Q + 2Q \ge \frac{n+10}{4}$. $\square$

CHAPTER 3

More General Cases of $f(P(n,k))$

We now turn to more general cases of $P(n,k)$. The considerations which follow fall naturally into four cases according to the parities of k and n. The case n and k even is the least interesting because in this case n is not relatively prime with k, and we have only partial results in this case. In the other cases, we have very satisfactory results.

1. n even, k odd

THEOREM 3.1. *For n even, k odd, $f(P(n,k)) = 0$.*

PROOF. Color vertices v_i red and w_i blue for $i \equiv 0 \bmod 2$ and v_i blue and w_i red for $i \equiv 1 \bmod 2$. The vertices of each spoke have opposite colors, as do the vertices v_i and v_{i+1}. Consider the edges $w_i w_{i+k}$. Subscripts i and $i+k$ have opposite parity, thus w_i and w_{i+k} have opposite colors. $\square$

2. n odd, k odd

In the following sections we often use the division algorithm to divide n by k. When we write $n = qk + r$ we mean $0 \leq r < k$. In deriving lower bounds for f, it will be necessary to find odd cycles. We begin this process with the following Lemma.

LEMMA 3.2. *If $n = qk + r$ and $r > 0$, then $P(n,k)$ contains n distinct cycles of length $q + 2 + r$, each of which contains q inner edges, two spokes and r outer edges.*

PROOF. Consider the cycle $v_0w_0w_kw_{2k}...w_{n-r}v_{n-r}v_{n-r+1}...v_{n-1}v_0$. This cycle has the required statistics, and one may begin such a cycle at any of the vertices v_i. $\square$

LEMMA 3.3. *If $n = qk + r$ with n and k odd and $n > k^2$, then $f(P(n,k)) \geq k + 1$.*

PROOF. First, if $r > 0$ the $q + 2 + r$ cycles described in Lemma 3.2 are odd cycles since q and r have opposite parity. Removing an inner edge eliminates q such cycles, a spoke eliminates two such cycles, and an outer edge eliminates r of these cycles. Since $q > r$ and $q > 2$, at least $\frac{n}{\frac{n-r}{k}}$ edges must be removed. Since $\frac{kn}{n-r} > k$, at least $k + 1$ edges must be removed. If $r = 0$, there exist k disjoint odd cycles consisting of q inner edges. In addition, the outer edges form an odd cycle of length n. To eliminate all of these odd cycles, at least $k + 1$ edges must be removed. $\square$

THEOREM 3.4. *If n and k are odd and $n > k^2$, then $f(P(n,k)) = k + 1$.*

PROOF. By Lemma 3.3, $f(P(n,k)) \geq k + 1$. Since $f(P'(n,k)) = 0$ for k odd, and only $k+1$ edges are added to the path to create $P(n,k)$, it suffices to remove these $k+1$ edges. $\square$

We now pause to present an example to illustrate the necessity of the hypothesis $n > k^2$ in Lemma 3.3. Consider the graph $P(45, 19)$. Since $19^2 \equiv 1 \bmod 45$, 19 is the only value of $k < \frac{45}{2}$ so that $P(45, 19) \cong P(45, k)$ [**14**]. We claim that $f(P(45, 19)) = 10$. To see this, remove edges $v_{9i-1}v_{9i}$ and $w_{9i-1}w_{9i+18}$ for each $1 \leq i \leq 5$. We color the vertices with colors -1 and 1 according to the scheme: $v_i \longrightarrow (-1)^{(\lfloor \frac{10i}{9} \rfloor)}$ and $w_i \longrightarrow (-1)^{(\lfloor \frac{10i+9}{9} \rfloor)}$.

It is clear that spokes join vertices of opposite color. For inner and outer edges the potentially problematic edges are $v_{9i-1}v_{9i}$ and $w_{9i-1}w_{9i+18}$, precisely those which have been removed. It follows that $f(P(45,19)) \leq 10$, so $f(P(45,19)) \neq 20 = k+1$. But in fact we can show $f(P(45,19)) = 10$ as follows.

Since there are 45 cycles which are translates of $v_0w_0w_{19}w_{38}v_{38}v_{39}...v_{44}v_0$ and 45 cycles which are translates of $v_0w_0w_{19}...w_{43}v_{43}v_{44}v_0$, we have $7f_o+2f_s+2f_i \geq 45$ and $2f_o+2f_s+7f_i \geq 45$. Adding yields $9f_o+2f_s+9f_i \geq 90$, so $f(P(45,19)) \geq 10$.

3. n odd, k even

Lemma 3.5. *Let k be even. Then $P(n,k)$ has n odd cycles of length $k+3$, each of which contains k outer edges, two spokes and one inner edge.*

Proof. Consider the cycle $v_0w_0w_kv_kv_{k-1}...v_1v_0$. This cycle has the required statistics, and one may begin such a cycle at any of the vertices v_i. □

Let the cycles described in Lemma 3.5 be called type A cycles.

Theorem 3.6. *Let $n = qk+r$ with n and q odd, k even, and $n > k^2$. Then $f(P(n,k)) = q+r$.*

Proof. Refer to the coloring specified in the proof of Theorem 2.2 and the resulting Corollary 2.3, where it is shown that $f(P(n,k)) \leq q+k+1$ with the given conditions. We will show that $k-r+1$ of the edges removed there need not have been removed. First consider the edge v_0v_{n-1}. Since $L_0 < k$ and since q is odd, $L_{n-1} > k$, it follows that v_0 and v_{n-1} have opposite colors. Consider now the edges $w_{(q-1)k+r+i}w_i$ for $0 \leq i \leq k-r-1$. Note that both $L_{(q-1)k+r+i} < k$ and $L_i < k$ for i in this range, and since n is odd, r is odd, so $(q-1)k+r+i$ and i have opposite parity. It follows that these $k-r$ edges need not be removed. Hence $f(P(n,k)) \leq q+k+1-(k-r+1) = q+r$.

To see this many edges are necessary, consider two types of odd cycles of which there are n of each: type A cycles and type B cycles of the type $v_0w_0w_kw_{2k}...w_{qk}w_{k-r}v_{k-r}v_{k-r-1}...v_1v_0$ and translations of this cycle. Type B cycles are $(q+k-r+3)$-cycles consisting of $k-r$ outer edges, two spokes, and $q+1$ inner edges. Since n is odd and k is even, there is at least one odd cycle consisting entirely of inner edges. Thus $f_i \geq 1$ and let $f_o = q - Q$. With these edges removed, at least $n - (k(q-Q)+1) = r + kQ - 1$ type A cycles remain. To eliminate these remaining cycles requires at least $\frac{r+kQ-1}{2}$ edges; thus the removal of at least $q - Q + 1 + \frac{r+kQ-1}{2} = q + \frac{r}{2} + \frac{1}{2} + Q(\frac{k-2}{2})$ edges is required. Since $k \geq r+1$, this is at least $q + \frac{r}{2} + \frac{1}{2} + Q(\frac{r-1}{2})$. Thus if $Q \geq 1$, we have $f(P(n,k)) \geq q + r$.

For $Q \leq 0$, consider the type B cycles remaining after the $q - Q + 1$ edges have been removed. We have at least $n - [(k-r)(q-Q)+q+1] = (r-1)(q+1) + Q(k-r)$ type B cycles remaining, and to eliminate these cycles requires at least $\frac{(r-1)(q+1)+Q(k-r)}{q+1}$ edges removed. Thus the number of edges removed is at least $q + r - Q + Q(\frac{k-r}{q+1})$. Since $n > k^2$, $k - r < q + 1$, so $f(P(n,k)) \geq q + r$. $\square$

THEOREM 3.7. *Let $n = qk + r$ with n odd, k and q even, and $n > k^2$. Then $f(P(n,k)) \leq q + 1 + k - r$.*

PROOF. Remove edges $\begin{cases} v_{ki-1}v_{ki} & \text{for } 0 \leq i \leq \frac{n-r}{k} \\ w_{qk-1-i}w_{k-r-1-i} & \text{for } 0 \leq i \leq k-r-1. \end{cases}$

We will refer to the L-parameter and associated coloring used in Theorem 2.2 and its Corollary 2.3. The edges removed here are some but not all of the edges removed in Corollary 2.3. Here, only $q+1+k-r$ edges have been removed. We must show the r edges removed there but not here present no problem. These are the edges $w_{n-r+i}w_{k-r+i}$ for $0 \leq i \leq r-1$. Since

$L_{n-r+i} < k$ and $L_{k-r+i} < k$ for i in this range, and the difference $(n-r+i)-(k-r+i) = n-k$ is odd, these r edges will join vertices of opposite colors and need not be removed. Thus $f(P(n,k)) \leq q+1+k-r$. $\square$

THEOREM 3.8. *Let $n = qk+1$ with n odd, k and q even, and $n > k^2$. Then $f(P(n,k)) = q+k-1$.*

PROOF. The cases for $k = 2$ and $k = 4$ follow from Proposition 2.5 and Theorem 2.10. Now we will consider all other cases with $k \geq 6$.

$$\text{Remove edges} \begin{cases} v_{ki-1}v_{ki} & \text{for } 0 \leq i \leq \frac{n-1}{k} - 1 \\ v_{n-1}w_{n-1} & \\ w_{qk-1-i}w_{k-2-i} & \text{for } 0 \leq i \leq k-2. \end{cases}$$

Note that exactly $q+k-1$ edges have been removed. As in the proof of Theorem 2.2, let L_i be the least nonnegative residue of i mod $2k$ and for $0 \leq i \leq n-2$ color as follows:

(1) w_i blue and v_i red if $i \equiv 0 \bmod 2$ and $L_i < k$ or if $i \equiv 1 \bmod 2$ and $L_i \geq k$;

(2) w_i red and v_i blue if $i \equiv 1 \bmod 2$ and $L_i < k$ or if $i \equiv 0 \bmod 2$ and $L_i \geq k$;

(3) v_{n-1} and w_{n-1} blue.

To see that this is a bipartition, first note that spokes other than $v_{n-1}w_{n-1}$, which was removed, make no problem, since all other v_i and w_i get opposite colors. The inner edges join vertices whose subscripts differ by k, but other than the $k-1$ edges which have been removed, w_i and w_{i+k} have opposite parity and thus have opposite colors. Other than $v_{n-1}v_0$ which joins vertices of opposite colors, the outer edges join vertices whose subscripts differ by one. The only such pairs within the two color classes are $v_{i-1}v_i$, where $i \equiv 0 \bmod k$, and for all but $v_{n-2}v_{n-1}$, these edges have been removed. Since q is even and $n \equiv 1 \bmod k$, v_{n-2} is red, so $v_{n-2}v_{n-1}$ need not be removed. It follows that $f(P(n,k)) \leq q+k-1$.

To see this many edges are required, consider two types of odd cycles; type A cycles and type B cycles which are $(q+3)$-cycles of the form $v_i w_i w_{i+k} w_{i+2k} \ldots w_{i-1} v_{i-1} v_i$. There are n cycles of each type. Removing an outer edge eliminates k type A cycles and one type B cycle; removing a spoke eliminates two of each type of cycle; and removing an inner edge eliminates one type A cycle and q type B cycles.

Since n is odd and k is even, there is also at least one odd cycle of inner edges, and thus $f_i \geq 1$. Begin with $f_i \geq 1$ and let $f_o = q - Q$. With these edges removed, at least $n - (k(q-Q)+1)) = kQ$ type A cycles remain, and $n - ((q-Q)+q) = qk + 1 - 2q + Q = q(k-2) + 1 + Q$ type B cycles remain.

To eliminate the remaining type B cycles requires at least $\frac{q(k-2)+1+Q}{q}$ edges, that is at least $k-2+\lceil \frac{1+Q}{q} \rceil$ edges. Thus far the number of edges removed is at least $q-Q+1+k-2+\lceil \frac{1+Q}{q} \rceil = q+k-Q-1+\lceil \frac{1+Q}{q} \rceil$.

If $Q \leq 0$, $f(P(n,k)) \geq q+k-1$, thus we will consider the case $Q > 0$.

For $Q > 0$, we must consider the remaining type A cycles. With the additional $k-2+\lceil \frac{1+Q}{q} \rceil$ inner edges removed, there remain $kQ - (k-2+\lceil \frac{1+Q}{q} \rceil) = k(Q-1)+2-\lceil \frac{1+Q}{q} \rceil$ type A cycles. To eliminate these requires $\frac{k(Q-1)+2-\lceil \frac{1+Q}{q} \rceil}{2}$ edges at a minimum. The number of edges required to eliminate all type A and type B cycles is now at least $q+k-Q+\frac{1}{2}(k(Q-1)+\lceil \frac{1+Q}{q} \rceil) \geq q+k-Q+\frac{1}{2}(k(Q-1)+1) \geq q+k+\frac{k}{2}(Q-1)-Q+\frac{1}{2}$. Since $k \geq 3$, this number is at least $q+k+3(Q-1)-Q+\frac{1}{2} = q+k+2Q-\frac{5}{2} \geq q+k-1$. Hence $f(P(n,k)) \geq q+k-1$. $\square$

THEOREM 3.9. *Let $n = qk + r$ with n odd, k and q even and $r > 1$. If $n > k^2$ then $f(P(n,k)) = q + k - r + 1$.*

PROOF. From Theorem 3.7, we have seen that $f(P(n,k)) \leq q+k-r+1$. Now consider two types of odd cycles; type A cycles and type B cycles of the type described in Lemma 3.2. Again $f_i \geq 1$ and let $f_o = q-Q$. Removal of these edges eliminates at most $k(q-Q)+1$ type A cycles and $r(q-Q)+q$ type B cycles, and with n cycles of each type, this leaves at least $r+kQ-1$ type A cycles and $q(k-r-1)+r(1+Q)$ type B cycles. Since $q > r$, to eliminate the remaining type B cycles requires at least $\frac{q(k-r-1)+r(1+Q)}{q} = k-r-1+r(\frac{1+Q}{q})$. The number of edges removed is now at least $q-Q+1+k-r-1+r(\frac{1+Q}{q}) = q+k-r+Q(\frac{r}{q}-1)+\frac{r}{q}$. If $Q \leq 0$, we have $f(P(n,k)) \geq q+k-r+1$.

For $Q > 0$ we must look at the remaining type A cycles. With the additional inner edges deleted, there now remain at least $r+kQ-1-(k-r-1+r(\frac{1+Q}{q})) = 2r+k(Q-1)-r(\frac{1+Q}{q})$ type A cycles. At most, $\frac{2r+k(Q-1)-r(\frac{1+Q}{q})}{2}$ edges can eliminate these remaining cycles, and so $f \geq q+k+\frac{r(1+Q)}{2q} - Q + \frac{k}{2}(Q-1)$. Again, we can consider $k \geq 6$, resulting in $f \geq q+k+2Q-3+\frac{r(1+Q)}{2q}$. With $r > 1$, $f \geq q+k-r+1$. $\square$

4. n even, k even, and $k|n$

THEOREM 3.10. *Suppose n and k are even and $k|n$. Then*

$$f(P(n,k)) = \begin{cases} \frac{n}{k} & \text{if } \frac{n}{k} \text{ is even} \\ \frac{n}{k}+k-1 & \text{if } \frac{n}{k} \text{ is odd.} \end{cases}$$

Proof. The graph $P(n,k)$ has one even outer cycle, and if $\frac{n}{k}$ is even, has only even inner cycles of length $\frac{n}{k}$. There exist n odd cycles of type A. These cycles must be eliminated, and so by Lemma 3.5, $f(P(n,k)) \geq \frac{n}{k}$.

Remove $v_{i-1}v_i$ for $i \equiv 0 \bmod k$. For every i, let L_i be the least nonnegative residue of i mod $2k$ and color as follows:

(1) w_i blue and v_i red if $i \equiv 0 \bmod 2$ and $L_i < k$ or if $i \equiv 1 \bmod 2$ and $L_i \geq k$;

(2) w_i red and v_i blue if $i \equiv 1 \bmod 2$ and $L_i < k$ or if $i \equiv 0 \bmod 2$ and $L_i \geq k$.

It is clear that the edges v_iw_i join blue to red. The $v_{i-1}v_i$ edges join opposite colors only when $i \equiv 0 \bmod k$, but these are precisely the edges that have been removed. Finally, the w_iw_{i+k} edges join vertices with subscripts of the same parity, but $L_i \neq L_{i+k}$, so w_i and w_{i+k} have opposite colors. So $f(P(n,k)) = \frac{n}{k}$.

If $\frac{n}{k}$ is odd, then by Corollary 2.3, $f(P(n,k)) \leq \lfloor \frac{n-1}{k} \rfloor + k = (\frac{n}{k} - 1) + k$. To see that this many edges are necessary, note that the inner edges form k disjoint $\frac{n}{k}$-cycles, requiring removal of k inner edges. The remaining $n-k$ inner edges are contained in $n-k$ type A cycles. These odd cycles require removal of at least $\frac{n-k}{k} = \frac{n}{k} - 1$ additional edges. Hence $f(P(n,k)) \geq \lfloor \frac{n-1}{k} \rfloor + k = (\frac{n}{k} - 1) + k$. $\square$

CHAPTER 4

Bipartite Density

1. Conclusions

Having derived bounds and many exact values for $f(P(n,k))$ we next give the consequences for the bipartite densities of these graphs. As a consequence of Corollary 2.3 we obtain:

THEOREM 4.1. *If $1 \leq k < \frac{n}{2}$ then*

(i) $b(P(n,k)) \geq \frac{3n - \lfloor \frac{n-1}{k} \rfloor - k - 1}{3n}$.

(ii) *If additionally k is odd then* $b(P(n,k)) \geq \frac{3n-k-1}{3n}$.

(iii) *If $n > k^2$ (k even or odd) then* $b(P(n,k)) \geq \frac{5}{6} - \frac{1}{3\sqrt{n}} - \frac{1}{3n}$.

Hence the $P(n,k)$, which, for $n > 3$ and n and k are relatively prime, are triangle-free cubic graphs, almost always have higher bipartite density than the $\frac{4}{5}$ bound derived in [**8**] for this class of graphs.

For small values of k we have the following:

THEOREM 4.2. (i) *For $n \geq 3$,* $b(P(n,1)) = \begin{cases} 1 & n \text{ even} \\ 1 - \frac{2}{3n} & n \text{ odd.} \end{cases}$

$$(ii)\ \textit{For } n \geq 5,\ b(P(n,2)) = \begin{cases} \frac{5}{6} & \textit{if } n \equiv 0 \bmod 4 \\ \frac{5}{6} - \frac{1}{3n} & \textit{if } n \equiv 2 \bmod 4 \\ \frac{5}{6} - \frac{1}{6n} & n \textit{ odd.} \end{cases}$$

$$(iii)\ \textit{For } n \geq 7,\ b(P(n,3)) = \begin{cases} 1 & n \textit{ even} \\ 1 - \frac{4}{3n} & n \textit{ odd.} \end{cases}$$

$$(iii)\ \textit{For } n \geq 9,\ b(P(n,4)) = \begin{cases} \frac{11}{12} & \textit{if } n \equiv 0 \bmod 8 \\ \frac{11}{12} - \frac{11}{12n} & \textit{if } n \equiv 1 \bmod 8 \\ \frac{11}{12} - \frac{1}{2n} & \textit{if } n \equiv 2 \bmod 8 \\ \frac{11}{12} - \frac{5}{12n} & \textit{if } n \equiv 3 \bmod 8 \\ \frac{11}{12} - \frac{1}{n} & \textit{if } n \equiv 4 \bmod 8 \\ \frac{11}{12} - \frac{1}{4n} & \textit{if } n \equiv 5 \bmod 8 \\ \frac{11}{12} - \frac{5}{6n} & \textit{if } n \equiv 6 \bmod 8 \\ \frac{11}{12} - \frac{3}{4n} & \textit{if } n \equiv 7 \bmod 8. \end{cases}$$

It is striking but not surprising that the small odd values of k give rise to bipartite densities of one or perhaps asymptotically one, whereas even k give constants less than one or asymptotic approximations to constants less than one. We turn now to more general values of k.

THEOREM 4.3. *If n and k are odd and $n > k^2$, then* $b(P(n,k)) = 1 - \frac{k+1}{3n} > 1 - \frac{1+\sqrt{n}}{3n} = 1 - \frac{1}{3\sqrt{n}} - \frac{1}{3n}$.

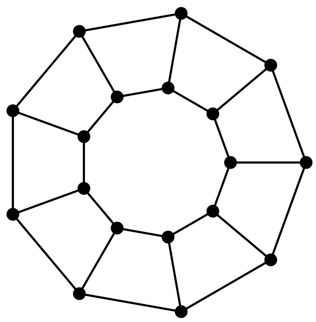

FIGURE 4.1. P(8,1): A Cylinder

THEOREM 4.4. *Let $n = qk + r$ with $0 < r < k$ and n odd, k even and $n > k^2$. Then*

(i) *If q is odd, $b(P(n,k)) = \frac{3n-q-r}{3n} = \frac{3n-\frac{n-r}{k}-r}{3n} = (1-\frac{1}{3k}) + (\frac{r}{3nk} - \frac{r}{3n})$.*

(ii) *If q is even and $r > 1$, $b(P(n,k)) = (1-\frac{1}{3k}) + \frac{r}{3kn} + \frac{r-1-k}{3n}$.*

(iii) *If q is even and $r = 1$, $b(P(n,k)) = (1-\frac{1}{3k}) + \frac{1}{3kn} + \frac{1-k}{3n}$.*

Again, for odd k and $n > k^2$ we have densities which are asymptotically one. For even k, the densities are asymptotically $1 - \frac{1}{3k}$. It should be noted that the formulas derived for the various cases where $n > k^2$ are not valid for all cases. In the example of $P(45, 19)$ presented following the proof of Theorem 3.4, we have $b(P(45, 19)) = \frac{125}{135} = \frac{25}{27}$. This number is considerably larger than the value $\frac{23}{27}$ given in Theorem 3.4.

Finally, some of the graphs $P(n,k)$ where $k^2 > n$ can be handled with our theorems with the help of the following theorem from Steimle and Staton [**14**]:

THEOREM 4.5. *Let $1 \le k < \frac{n}{2}$ with k and n relatively prime, and let $1 \le l \le n-1$. Then $P(n,k) \cong P(n,l)$ if and only if one of the following holds:*

(i) $l = k$

(ii) $l = n - k$

(iii) $kl \equiv 1 \bmod k$

(iiii) $kl \equiv -1 \bmod k$.

Hence, the bipartite density of $P(65, 11)$ may be determined by noting that $P(65, 11) \cong P(65, 6)$. Since $6^2 < 65$, our Theorem 3.6 applies. This procedure does not settle all cases however. Confronted with $P(101, 47)$ for example, one calculates that $P(101, 47)$ is isomorphic to $P(101, 54)$, $P(101, 58)$ and $P(101, 43)$. None of these is helpful. And, as already noted, confronted with $P(45, 19)$, one finds that $P(45, 19)$ is isomorphic only to $P(45, 26)$. The four cases collapse into two because $19^2 \equiv 1 \bmod 45$.

2. Future Research

For a fixed value of k, we have determined exact formulas for all $b(P(n, k))$ where $n > k^2$ and n and k relatively prime, that is, for all but finitely many values of n. We have seen with an example that these formulas are not valid for all small n. The $P(n, k)$ with $n < k^2$ are an attractive possibility for further investigation. I believe the case $k^2 \equiv 1$ modulo n, generalizing the $P(45, 19)$ example may be particularly interesting. Cases such as $P(101, 43)$, where all the l values for which $P(n, k) \cong P(n, l)$ are close to $\frac{n}{2}$ seem to be quite subtle, with odd cycles quite "tangled up." The problem of bipartite density is difficult and certainly attractive in many classes of graphs.

BIBLIOGRAPHY

Bibliography

[1] P. Erdős, *On even subgraphs of graphs (in Hungarian)*, Mat. Lapok. **18**, (1964), 283–388.

[2] P. Erdős, *Some new applications of the probability methods in combinatorial analysis and graph theory, in:*, Proc. Fifth S.E. Conf. on Comb., Graph Th. and Computing, Utilitas Math., Winnipeg, (1974), 39–51.

[3] G. Hopkins and W. Staton, *Maximal bipartite subgraphs*, ARS Combinatoria, Vol. **13**, (1982), 223–226.

[4] P. Erdős, R. Faudree, J. Pach and J. Spencer, *How to make a graph bipartite*, J. Combin. Theory Ser. B. **45**, (1988) no. 1, 86–98.

[5] C. Alexander, G. Hopkins and W. Staton, *Maximal bipartite subgraphs in triangle-free graphs*, Congressus Numerantium, **35**, (1982), 33–46.

[6] W. Staton, *Edge deletions and the chromatic number*, ARS Combinatoria, Vol. **10**, (1980), 103–106.

[7] G. Hopkins and W. Staton, *Extremal bipartite subgraphs of cubic triangle-free graphs*, J. Graph Theory **6**, (1982), 115–121.

[8] J. A. Bondy and S. C. Locke, *Largest bipartite subgraphs in triangle-free graphs with maximum degree three*, J. Graph Theory **10**, (1986), 477–504.

[9] S. C. Locke, *Bipartite density and the independence ratio*, J. Graph Theory **10**, (1986), 47–53.

[10] T. McColgan, *Bipartite density of 4-regular graphs*, Doctoral Dissertation, University of Mississippi, (1992).

[11] T. McColgan and W. Staton, *Bipartite subgraphs of line graphs*, Congressus Numerantium, **104**, (1994), 117-120.

[12] D. B. West, *Introduction to graph theory*, Prentice Hall Inc., Upper Saddle River, NJ, 1996. MR 1367739 (96i:05001)

[13] F. Castagna and G. Prins, *Every generalized petersen graph has a tait coloring*, Pacific J. Math. **40** (1) (1972) 53-58.

[14] A. Steimle and W. Staton, *The isomorphism classes of the generalized petersen graphs*, Discrete Mathematics, (2008), doi:10.1016/j.disc.2007.12.074.

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