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ASYMPTOTICS OF CARLEMAN POLYNOMIALS FOR LEVEL CURVES OF THE INVERSE OF A SHIFTED JOUKOWSKY TRANSFORMATION

A Thesis

presented in partial fulfillment of requirements

for the degree of Master of Science

in the Department of Mathematics

The University of Mississippi

By

MICHAEL C. NORTHINGTON V

May 2011

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ABSTRACT

Let L be a level curve of the inverse of the shifted Joukowsky transformation $w \mapsto w - 1 + (w-1)^{-1}$, that is,

$$L := \left\{ w - 1 + (w-1)^{-1} : |w| = R \right\}, \quad R > 2.$$

In this thesis we investigate the asymptotic properties of the sequence of polynomials that are orthonormal over the interior domain of L with respect to the area measure. We establish strong asymptotic formulas describing the behavior of these polynomials (as their degree increases) at every point of the complex plane.

DEDICATION

This thesis is dedicated to everyone who has helped me during my academic career. In particular, I would like to thank my family who has supported me throughout life.

ACKNOWLEDGMENTS

I would like to express my deepest appreciation to my advisor, Dr. Erwin Miña-Díaz, for guiding me in the process of finishing this thesis and for helping me understand his research upon which this work is directly built. Also, I would like to thank the members of my thesis committee, Dr. Gerard Buskes and Dr. Iwo Labuda, and Dr. Peter Dragnev who is referenced several times in the thesis for his work with my advisor.

In addition, I would like to thank all of my fellow graduate students who have made the last two years an exciting and enlightening adventure.

Finally, I acknowledge the support of the Graduate School and the Department of Mathematics at the University of Mississippi for providing the necessary tools to make this educational experience a great one. I was able to finance this education by receiving a Graduate Assistance in Areas of National Need (GAANN) Fellowship.

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1 INTRODUCTION

1.1 Carleman polynomials and asymptotic formula

The present work investigates the asymptotic behavior of polynomials that are orthogonal over a domain in $\mathbb{C}$ that is bounded by an analytic Jordan curve. Such polynomials were first studied by Carleman in [1], where he established an asymptotic formula with convergence rates for the polynomials valid in a certain open set containing the closed exterior of the orthogonality domain. To better explain this result we need first to introduce some notation that will be used throughout.

For $r > 0$, we define

$$\mathbb{T}_r := \{w : |w| = r\}, \quad \Delta_r := \{w : r < |w| \leq \infty\}, \quad \mathbb{D}_r := \{w : |w| < r\}.$$

Let G_1 be a bounded simply-connected domain of $\mathbb{C}$, whose boundary L_1 is an analytic Jordan curve, more precisely, L_1 is the range or trajectory of a path $\gamma(e^{i\theta})$, $0 \leq \theta \leq 2\pi$, with γ analytic and univalent in some neighborhood of the unit circle $\mathbb{T}_1$. The space of functions that are analytic and square integrable over G_1 forms a Hilbert space with the inner product

$$< f, g >:= \frac{1}{\pi} \int_{G_1} f(z)\overline{g(z)} dA(z),$$

where dA denotes the two-dimensional Lebesgue (or area) measure (see e.g. [4, Chap. 1]). Applying the standard Gram-Schmidt orthonormalization process to the linearly independent sequence of power functions

$$1, z, z^2, \ldots, z^n, \ldots,$$

we can construct a unique sequence of polynomials $\{p_n(z)\}_{n=0}^{\infty}$ (the sequence of orthonormal polynomials over G_1) satisfying that p_n is a polynomial of degree n with positive leading coefficient and that

$$\frac{1}{\pi}\int_{G_1} p_n(z)\overline{p_m(z)}dA(z) = \begin{cases} 0, & n \neq m, \\ 1, & n = m. \end{cases} \tag{1}$$

Let Ω_1 be the unbounded component of $\overline{\mathbb{C}} \setminus L_1$, and let $\psi(w)$ be the unique conformal map of Δ_1 onto Ω_1 that satisfies $\psi(\infty) = \infty$ and $\psi'(\infty) > 0$. Let $\rho \geq 0$ be the smallest number such that ψ has an analytic and univalent continuation to Δ_ρ. Since L_1 is an analytic Jordan curve, we are guaranteed that $\rho < 1$. Then, for each $\rho \leq r < \infty$, we define

$$\Omega_r := \psi(\Delta_r), \qquad L_r := \partial\Omega_r, \qquad G_r := \mathbb{C} \setminus \overline{\Omega}_r. \tag{2}$$

Notice that for $r > \rho$, L_r is an analytic Jordan curve. Now, we define

$$\phi(z) : \Omega_\rho \to \Delta_\rho$$

to be the inverse of ψ. We are now ready to state the asymptotic formula of Carleman (see [1, Satz IV] and also [4, Sec. 1]).

Theorem 1.1. *For fixed $r > \rho$, we have that*

$$\frac{p_n(z)}{\sqrt{n+1}[\phi(z)]^n} = \phi'(z) + \begin{cases} O\left(\sqrt{n}\rho^n\right), & r \geq 1, \\ O\left(n^{-1/2}(\rho/r)^n\right), & \rho < r < 1, \end{cases} \tag{3}$$

uniformly on $\overline{\Omega}_r$ as $n \to \infty$.

Carleman also proved that for κ_n, the leading coefficient of p_n, we have

$$\kappa_n = \sqrt{n+1}[\phi'(\infty)]^{n+1}(1 + O(\rho^{2n})) \tag{4}$$

as $n \to \infty$.

Notice that the asymptotic behavior of the polynomials is related to both ψ and ϕ. We will expand on this relationship in the next section. Now, we illustrate Carleman's formula with two examples.

The simplest sequence of Carleman polynomials corresponds to G_1 being an open disk, which without loss of generality can be assumed to be the unit disk $\mathbb{D}_1$. Here ψ and ϕ are the identity maps, $\rho = 0$, $\Omega_\rho = \overline{\mathbb{C}} \setminus \{0\}$, and it is very easy to verify that

$$p_n(z) = \sqrt{n+1}\, z^n, \quad n \geq 0,$$

so that (3) holds trivially.

Probably the second simplest, though already more interesting example, corresponds to G_1 being the interior of an ellipse L_1, which without loss of generality can be assumed to have its foci located at $z = \pm 2$. Any such ellipse can be obtained as the image by the Joukowski Transformation

$$J(w) = w + \frac{1}{w} \tag{5}$$

of a circle $\mathbb{T}_R$ for some value of $R > 1$. Given that $J(w)$ maps Δ_1 conformally onto $\overline{\mathbb{C}} \setminus [-2, 2]$ while mapping both the upper and lower halves of $\mathbb{T}_1$ onto $[-2, 2]$, we see that here

$$\psi(w) = Rw + \frac{1}{Rw}, \quad \rho = 1/R, \quad \Omega_\rho = \overline{\mathbb{C}} \setminus [-2, 2],$$

and

$$\phi(z) = \frac{z + \sqrt{z^2 - 4}}{2R},$$

with the branch of the square root chosen to be positive along $(2, \infty)$. Then, (3) gives us that

$$\lim_{n \to \infty} \frac{(2R)^{n+1} p_n(z)}{\sqrt{n+1}\left[z + \sqrt{z^2-4}\right]^n} = \frac{z + \sqrt{z^2-4}}{\sqrt{z^2-4}}$$

locally uniformly as $n \to \infty$ on $\overline{\mathbb{C}} \setminus [-2, 2]$. As a matter of fact, in this case the polynomials $p_n(z)$ can also be computed explicitly. In [6, pp. 258-259], it is proven that if $T_n(z)$ is the Chebyshev polynomial of degree n, that is,

$$T_n(\cos(\theta)) = \cos(n\theta),$$

or equivalently,

$$T_n(z) = \frac{(z + \sqrt{z^2-1})^n + (z - \sqrt{z^2-1})^n}{2},$$

then

$$p_n(z) = \frac{T'_{n+1}(z/2)}{\sqrt{(n+1)[(4R^2)^{n+1} - (4R^2)^{-n-1}]}}.$$

Hence, it is easy to deduce the behavior of $p_n(z)$ for $z \in [-2, 2]$. In fact,

$$p_n(2\cos\theta) = \frac{\sqrt{n+1}\,\sin((n+1)\theta)}{[(4R^2)^{n+1} - (4R^2)^{-n-1}]\sin\theta}, \quad 0 \le \theta \le \pi.$$

1.2 Extension of Carleman's formula

Carleman's formula, (3), establishes the asymptotic behavior of $p_n(z)$ on a subset of the extended complex plane, namely, on Ω_ρ. It is natural to wonder how the polynomials behave at the remaining points of the complex plane, and in particular, whether Ω_ρ is the largest open set where an asymptotic formula like (3) holds true. This question has been recently clarified in [2], [3], and [5]. In particular, in the paper [3], upon which our work is directly built, the asymptotic formula of Carleman is proven to be valid for points z on a subset of G_1 that is, in general, larger than the band $\Omega_\rho \cap G_1$. Before being able to explain this result we need to introduce some new elements that start playing a role.

Let $\varphi(z)$ be a conformal map of G_1 onto the unit disk $\mathbb{D}_1$. By a well-known theorem of Carathéodory, φ can be continuously and univalently extended to $\overline{G}_1$. Moreover, φ has a meromorphic continuation to $G_{1/\rho}$. In fact, if z^* denotes the Schwarz reflection of z about L_1, then

$$z^* = \psi(1/\overline{\phi(z)}), \quad z \in G_{1/\rho} \cap \Omega_\rho,$$

and the meromorphic continuation of φ to $G_{1/\rho}$ is given by

$$\varphi(z) = \frac{1}{\overline{\varphi(z^*)}}, \quad z \in G_{1/\rho} \setminus \overline{G}_1, \tag{6}$$

so that for the point $z_0 \in G_1$ such that $\varphi(z_0) = 0$, if $z_0 \in \overline{G}_\rho$, φ is analytic on $G_{1/\rho}$, and if $z_0 \in G_1 \cap \Omega_\rho$, then φ has a simple pole at z_0^* and is analytic everywhere else in $G_{1/\rho}$.

We shall see that the properties of the composition $\varphi(\psi(w))$ are intimately related to the asymptotic behavior of the polynomials for values of z in G_1. In fact, in [5] (see also Corollary 2.2 in Section 2 below) Miña-Díaz proved a result roughly stating that for $z \in G_1$, $p_n(z)$ behaves as $n \to \infty$ like the integral

$$\frac{1}{2\pi i} \int_{\mathbb{T}_1} \frac{w^n dw}{\varphi(\psi(w)) - \varphi(z)} \,. \tag{7}$$

Notice that the composition $\varphi(\psi(w))$ is a well-defined meromorphic function in the annulus $\rho <$

$|w| < 1/\rho$, so that the problem of determining the asymptotic behavior of the polynomials in G_1 reduces to determining the properties of a contour integral of a function that is meromorphic in a neighborhood of the contour of integration.

We define $\mu \geq 0$ to be the smallest number such that $\varphi(\psi(w))$ has a meromorphic continuation, $h_\varphi(w)$, to the annulus $\{w : \mu < |w| < 1/\rho\}$. Further, we let Σ be the set of points $z \in G_1$ such that the equation

$$h_\varphi(w) = \varphi(z) \tag{8}$$

has at least one solution in the annulus $\mu < |w| < 1$, and let $\Sigma_0 := G_1 \setminus \Sigma$. Given $z \in \Sigma$, we say that a solution ω of (8) has multiplicity $\alpha \geq 1$ if

$$h_\varphi^{(\alpha)}(\omega) \neq 0, \qquad h_\varphi^{(j)}(\omega) = 0, \qquad 1 \leq j < \alpha. \tag{9}$$

Since $h_\varphi(w)$ is nonconstant on $\mu < |w| < 1/\rho$, for fixed $z \in \Sigma$, of the solutions that the equation (8) has in $\mu < |w| < 1/\rho$, only finitely many have largest modulus, and we denote these solutions of largest modulus by $\omega_{z,1}, \ldots, \omega_{z,s}$. Let $\alpha_{z,k}$ denote the multiplicity of h_φ at $\omega_{z,k}$. Then, for every integer $p \geq 1$, we define $\Sigma_p \subset \Sigma$ by

$$\Sigma_p = \{z \in \Sigma : \alpha_{z,1} + \ldots + \alpha_{z,s} = p\} \tag{10}$$

Then, Σ_1 is the set of points $z \in \Sigma$ such that the equation (8) has one solution of largest modulus, and this solution is simple. Finally, we define the functions $\Phi : \Sigma_1 \to \{w : \mu < |w| < 1\}$ and $r : \mathbb{C} \to [\mu, \infty)$ by

$$\Phi(z) := \omega_{z,1}, \qquad z \in \Sigma_1,$$

and

$$r(z) := \begin{cases} |\phi(z)|, & z \in \overline{\Omega}_1, \\ |\omega_{z,1}|, & z \in \Sigma, \\ \mu, & z \in \Sigma_0. \end{cases} \tag{11}$$

It is easy to see as a simple application of Rouche's theorem (see Lemma 11 and Corollary 12 of [3]) that Σ and Σ_1 are open, that $\Phi(z)$ is analytic and univalent, and that $r(z)$ is continuous. Notice that for $z \in G_1 \cap \Omega_\rho$,

$$h_\varphi(\phi(z)) = \varphi(\psi(\phi(z))) = \varphi(z),$$

so we have

$$\Phi(z) = \phi(z), \quad z \in G_1 \cap \Omega_\rho, \tag{12}$$

and therefore, $\Sigma_1 \supset \Omega_\rho \cap G_1$. Moreover, as examples show, Σ_1 is in general larger that $\Omega_\rho \cap G_1$.

Using this partition of G_1 into Σ_p sets, Dragnev and Miña-Díaz proved in [3] the following extension of Carleman's formula. We say that a sequence of functions $f_n : K \to \mathbb{C}$ decays geometrically fast on K if there exists $0 < \epsilon < 1$ such that $f_n(z) = O(\epsilon^n)$ uniformly on K as $n \to \infty$.

Theorem 1.2. *(a) For every $z \in \Sigma_1$,*

$$p_n(z) = \sqrt{n+1}\Phi'(z)[\Phi(z)]^n[1 + e_n(z)], \tag{13}$$

with $e_n(z)$ decaying geometrically fast on compact subsets of Σ_1.

(b) Also,

$$\limsup_{n\to\infty} |p_n(z)|^{1/n} = r(z), \quad z \in G_1\,. \tag{14}$$

Notice that for $z \in \Sigma_1$, (14) is a direct consequence of (13), thus (14) is rather a (weak) statement about points in $z \in G_1 \setminus \Sigma_1$. It is natural to wonder if we can say more about $p_n(z)$ for these points. If $z \in \Sigma_p$, $p > 1$, then the equation $h_\varphi(w) = \varphi(z)$ has in $\mu < |w| < 1$ finitely many (say s) solutions $\omega_{z,1}, \ldots, \omega_{z,s}$ of largest modulus, whose multiplicities $\alpha_{z,k}$ amount exactly to p. Setting

$$\alpha_z = \max_{1\le k\le s} \alpha_{z,k},$$

we can deduce from the residue theorem when analyzing the integral (7) that

$$p_n(z) = \varphi'(z) \sum_{\alpha_{z,k}=\alpha_z} \frac{\alpha_z!\, n^{\alpha_z-1/2}(\omega_{z,k})^{n-\alpha_z+1}}{h_\varphi^{(\alpha_z)}(\omega_{z,k})} + O(r^n(z) n^{\alpha_z-3/2}),$$

as $n \to \infty$. The problem with this estimate is that, in general, it is just a pointwise estimate, though it can be established uniformly in concrete instances where we count on more information about the structure of the Σ_p sets.

1.3 New results

For points $z \in \Sigma_0$, in general, (14) is the best we can say. This is so because for such values of z, the integrand in (7) remains analytic in $\mu < |w| < 1$ but its behavior on the circle $\mathbb{T}_\mu$ can be as erratic as one might imagine.

Nevertheless, for some more specific domains G_1 such as those considered in [5] for which $\mu = \rho > 0$ and ψ maps $\mathbb{T}_\rho$ into a piecewise analytic curve, it is possible to establish the behavior of p_n everywhere in Σ_0 with a great degree of detail. Such domains have a Σ_0 set with nonempty interior, however their Σ_1 set coincides with $\Omega_\rho \cap G_1$.

Thus, what still remains to be explored is a "full featured" example in which Σ_1 be actually *larger* than $\Omega_\rho \cap G_1$, where the interior of Σ_0 be nonempty as well, and we could establish the asymptotic behavior of $p_n(z)$ for every point $z \in \Sigma_0$. It is the purpose of the present work to provide such an example, for which the domain of orthogonality G_1 is simply taken to be the interior of the image by the Joukowski transformation (5) of a circle C_R centered at -1 of radius $R > 2$ (see Figure 1).

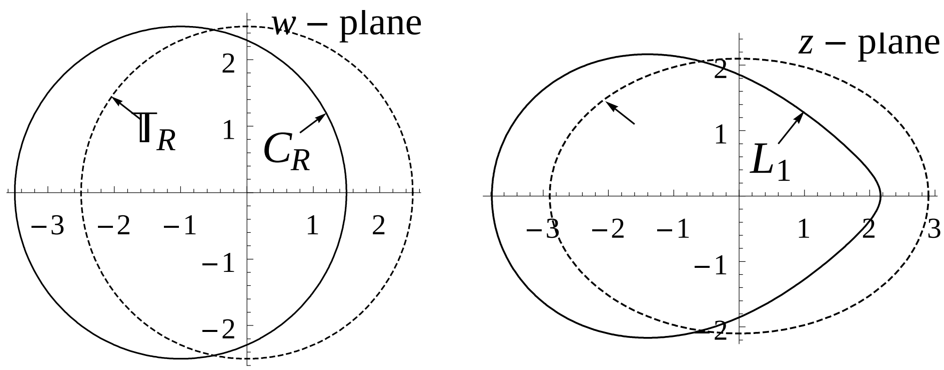

Figure 1: L_1 is the image by the Joukowski transformation of the circle $C_R = \{w : |w-1| = 2.5\}$.

Notice that the boundary, L_1, of G_1 is a level curve of the inverse of the "shifted Joukowski transformation", $w \mapsto w - 1 + (w-1)^{-1}$, that is,

$$L_1 := \left\{ w - 1 + (w-1)^{-1} : |w| = R \right\}.$$

Hence the title of this thesis. The corresponding sets Σ_1 and Σ_0 are illustrated in Figure 2.

It is interesting and somewhat unexpected that this slight variation of the classical example of an ellipse yields orthonormal polynomials $p_n(z)$ with a rich and rather surprising behavior. For instance, we find that after proper normalization and through different subsequences of the positive integers, $p_n(z)$ converges to different functions in the interior of Σ_0. Also, the asymptotic analysis

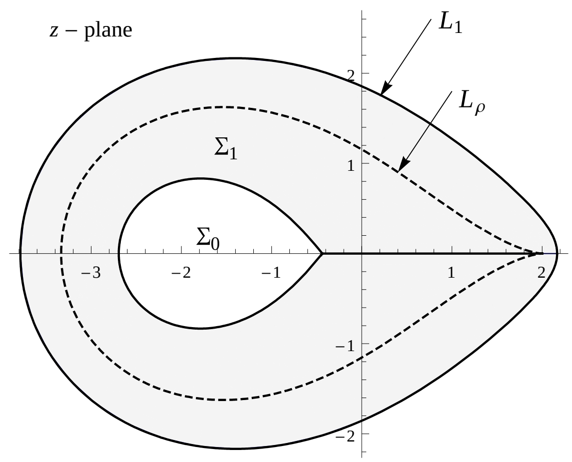

Figure 2: Σ_1 is the greyish region, Σ_0 is the white region and its boundary.

as $n \to \infty$ of the integral (7) is quite challenging and involved. This is all essentially due to the fact that for $z \in \Sigma_0$, the function $[h_\varphi(w) - \varphi(z)]^{-1}$ occurring in (7) encounters in the circle $\mathbb{T}_\mu$ a singularity that attracts infinitely many of its poles.

The remaining portion of this thesis is organized as follows. In Section 2 we briefly outline the asymptotic expansions for p_n that were recently obtained in [2]. We show how these expansions quickly yield the same integral representation for p_n already obtained in [5], but with a different and more advantageous expression for the error term. This allows us to simplify some aspects of the original proof of Theorem 1.2, which we present as a small contribution at the end of Section 2. Our main results are Theorems 3.2 and 3.3, which are stated and proven in Section 3.

2 AUXILIARY RESULTS

2.1 Asymptotic expansion and integral representation for p_n

In this section we quickly outline a series expansion for Carleman polynomials recently obtained in [2] and shown to yield at once Carleman's formula (3). We also show how this expansion automatically produces an integral representation for the polynomials (see Corollary 2.2 below) that will be vital to our study of their asymptotic behavior. Indeed, this integral representation is similar to that previously obtained in [5], but with a more advantageous expression for the error term that allows us to simplify some aspects of the original proof of Theorem 1.2. We present the new simplified proof in Subsection 2.2 below.

We shall denote by P_n the nth monic orthogonal polynomial. Recall that we have denoted the leading coefficient of p_n by κ_n, so that

$$P_n(z) = \kappa_n^{-1} p_n(z)$$

and any asymptotic result for P_n is easily translated to p_n with the help of (4).

Fix $\rho < r < 1$, and a conformal map φ of G_1 onto $\mathbb{D}_1$. Then, for each $n \geq 0$ we recursively define a sequence of functions $\{f_{n,k}\}_{k=0}^{\infty}$ as follows:

$$
\begin{aligned}
f_{n,0}(z) &:= 1, && z \in \overline{\mathbb{C}}, \\
f_{n,2k+1}(z) &:= -\frac{1}{2\pi i}\oint_{L_r} \frac{f_{n,2k}(\zeta)\varphi'(\zeta)[\phi(\zeta)]^{n+1}d\zeta}{\varphi(\zeta)-\varphi(z)}, && z \in G_{1/\rho}, \backslash L_r, \qquad (15)\\
f_{n,2k+2}(z) &:= \frac{1}{2\pi i}\oint_{L_{1/r}} \frac{f_{n,2k+1}(\zeta)\phi'(\zeta)[\phi(\zeta)]^{-n-1}d\zeta}{\phi(\zeta)-\phi(z)}, && z \in \Omega_\rho \backslash L_{1/r}. \qquad (16)
\end{aligned}
$$

In [2, Proof of Theorem 1.1], it is shown that for n large enough, the series $\sum_{k=0}^{\infty} f_{n,2k+2}(z)$ and $\sum_{k=0}^{\infty} f_{n,2k+1}(z)$ converge absolutely and locally uniformly for $z \in \Omega_\rho \backslash L_{1/r}$ and $z \in G_{1/\rho} \backslash L_r$, respectively, and that

$$
\sum_{k=0}^{\infty} f_{n,2k+2}(z) = O(r^{2n}), \quad \sum_{k=0}^{\infty} f_{n,2k+1}(z) = O(r^{n}) \tag{17}
$$

locally uniformly in their respective domains of definition as n tends to ∞.

Also proven in [2] is the following theorem giving us a series expansion for $P_n(z)$.

Theorem 2.1. *For every integer n large enough,*

$$
\begin{aligned}
&(n+1)[\phi'(\infty)]^{n+1}P_n(z) \\
&= \frac{d}{dz}\begin{cases}
[\phi(z)]^{n+1}\sum_{k=0}^{\infty} f_{n,2k}(z), & z \in \Omega_{1/r}, \\
[\phi(z)]^{n+1}\sum_{k=0}^{\infty} f_{n,2k}(z) - \sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in \Omega_r \cap G_{1/r}, \\
-\sum_{k=0}^{\infty} f_{n,2k+1}(z), & z \in G_r.
\end{cases}
\end{aligned}
$$

From Theorem 2.1 and (15) we see that for $z \in G_r$,

$$
(n+1)[\phi'(\infty)]^{n+1}P_n(z) = \frac{\varphi'(z)}{2\pi i}\oint_{L_r} \frac{\left(\sum_{k=0}^{\infty} f_{n,2k}(\zeta)\right)\varphi'(\zeta)\phi(\zeta)^{n+1}d\zeta}{[\varphi(\zeta)-\varphi(z)]^2}.
$$

Deforming the contour of integration L_r into L_1 does not change the value of this last integral for values of $z \in G_r$, and leaves a function that is analytic in G_1. By the uniqueness of the analytic continuation, we must have

$$(n+1)[\phi'(\infty)]^{n+1}P_n(z) = \frac{\varphi'(z)}{2\pi i}\oint_{L_1}\frac{\left(\sum_{k=0}^{\infty} f_{n,2k}(\zeta)\right)\varphi'(\zeta)\phi(\zeta)^{n+1}d\zeta}{[\varphi(\zeta)-\varphi(z)]^2} \tag{18}$$

for all $z \in G_1$.

Now, from the very definition of $f_{n,2k+2}$ in (16), we see that for $k \geq 0$, $f_{n,2k+2}(\psi(w))$ has an analytic continuation $f^*_{n,2k+2}(w)$ to $\mathbb{C} \setminus \mathbb{T}_{1/r}$ given by

$$f^*_{n,2k+2}(w) = \frac{1}{2\pi i}\oint_{\mathbb{T}_{1/r}}\frac{f_{n,2k+1}(\psi(t))t^{-n-1}dt}{t-w}, \qquad |w| \neq 1/r,$$

and if we define

$$F_n(w) := \sum_{k=0}^{\infty} f^*_{n,2k+2}(w), \quad |w| \neq 1/r, \quad n \geq 0,$$

then by (18) and (17),

$$F_n(w) = O(r^{2n}) \quad \text{and} \quad F'_n(w) = O(r^{2n}) \tag{19}$$

locally uniformly as $n \to \infty$ in $|w| \neq 1/r$.

Integration by parts in (18) followed by the change of variables $z = \psi(w)$ yields

$$\begin{aligned}
P_n(z) &= \frac{\varphi'(z)}{(n+1)[\phi'(\infty)]^{n+1}2\pi i}\oint_{L_1}\frac{[1+\sum_{k=0}^{\infty} f_{n,2k+2}(\zeta)\phi(\zeta)^{n+1}]'d\zeta}{\varphi(\zeta)-\varphi(z)} \\
&= \frac{\varphi'(z)}{(n+1)[\phi'(\infty)]^{n+1}2\pi i}\oint_{\mathbb{T}_1}\frac{[(1+F_n(w))w^{n+1}]'dw}{\varphi(\psi(w))-\varphi(z)} \\
&= \frac{\varphi'(z)}{[\phi'(\infty)]^{n+1}2\pi i}\oint_{\mathbb{T}_1}\frac{w^n(1+K^*_n(w))dw}{\varphi(\psi(w))-\varphi(z)}, \qquad z \in G_1,
\end{aligned} \tag{20}$$

with

$$K_n^*(w) = F_n(w) + \frac{wF_n'(w)}{n+1},$$

and so by multiplying (20) by κ_n, letting

$$K_n(z) := \frac{\kappa_n K_n^*(z)}{\sqrt{n+1}[\phi'(\infty)]^{n+1}} + \frac{\kappa_n}{\sqrt{n+1}[\phi'(\infty)]^{n+1}} - 1\,,$$

and using (19) and (4), we arrive at the following

Corollary 2.2. *Let r be any fixed number satisfying that $\rho < r < 1$. Then, for every integer $n \geq 0$,*

$$p_n(z) = \frac{\sqrt{n+1}\varphi'(z)}{2\pi i} \oint_{\mathbb{T}_1} \frac{w^n(1+K_n(w))dw}{\varphi(\psi(w)) - \varphi(z)}, \quad z \in G_1, \tag{21}$$

where $K_n(w)$ is analytic in $|w| < 1$ and $K_n(w) = O(r^{2n})$ locally uniformly as $n \to \infty$ in $|w| < 1$.

Corollary 2.2 will play an essential role in deriving the new results obtained in this thesis, but as a first application we give in the next subsection a proof of Theorem 1.2 that simplifies some of the arguments originally given in [3]. For such a proof as well as for the proof of our main result, Theorem 3.2 of Section 3, we shall need the following lemma, which is stated as Lemma 11 in [3] and proven therein by a simple application of Rouche's Theorem.

Recall from the introduction that we let μ denote the smallest nonnegative number such that the composition $\varphi(\psi(w))$, that is well-defined in $\rho < |w| < 1/\rho$, admits a meromorphic continuation to $\mu < |w| < 1/\rho$. We denote such a continuation by $h_\varphi(w)$. The sets Σ, Σ_p ($p \in \mathbb{N}$) and Σ_0 have been defined in the introduction as well. We shall also employ the notation

$$D_\epsilon(t) := \{z : |z - t| < \epsilon\}, \quad D_\epsilon^*(t) := D_\epsilon(t) \setminus \{t\}.$$

Lemma 2.3. *Suppose $z \in \Sigma_p$ for some integer $p \geq 1$, and that $\omega_{z,1}, \dots, \omega_{z,s}$ are the solutions of largest modulus that the equation (8) has in the annulus $\mu < |w| < 1$, so that the multiplicities $\alpha_{z,k}$ of h_φ at these $\omega_{z,k}$ satisfy that $\alpha_{z,1} + \cdots + \alpha_{z,s} = p$. Let μ' be a number chosen with the properties that*

$$\mu < \mu' < r(z) \; (= |\omega_{z,1}| = \cdots = |\omega_{z,s}|),$$

that h_φ has no poles on $\mathbb{T}_{\mu'}$, and that the only solutions of (8) in $\mu' \leq |w| < 1$ are precisely the $\omega_{z,k}$, $k = 1, \dots, s$. Let $\delta > 0$ be so small that for each $k = 1, \dots, s$, the closed disk $\overline{D_\delta(\omega_{z,k})}$ is contained in $\mu' < |w| < 1$, h_φ has no poles in $\overline{D_\delta(\omega_{z,k})}$, and $h'_\varphi(w) \neq 0$ for $w \in D^*_\delta(\omega_{z,k})$.

Then, there exists $\epsilon > 0$ such that for $0 < |\zeta - z| \leq \epsilon$, the equation $h_\varphi(w) = \varphi(\zeta)$ has exactly p solutions in $\mu' \leq |w| < 1$, which are all simple and with each disk $D_\delta(\omega_{z,k})$ containing $\alpha_{z,k}$ of them.

Clearly this lemma immediately reveals that Σ and Σ_1 are open, while Σ_0 is compact. Notice also that the function $\Phi : z \mapsto \omega_{z,1}$ defined on Σ_1 is continuous and satisfies that

$$(\varphi^{-1} \circ h_\varphi)(\Phi(z)) = z, \quad z \in \Sigma_1.$$

Thus, by the inverse function theorem, Φ is analytic and univalent in Σ_1, and

$$h'_\varphi(\omega_{z,1}) = \frac{\varphi'(z)}{\Phi'(z)}, \quad z \in \Sigma_1\,. \tag{22}$$

2.2 Proof of Theorem 1.2

Fix $z \in \Sigma_1$ and choose corresponding numbers μ' and δ satisfying the hypothesis of Lemma 2.3, which guarantees the existence of an $\epsilon > 0$ such that for every $\zeta \in \overline{D_\epsilon(z)}$, the equation

$h_\varphi(w) = \varphi(\zeta)$ has exactly 1 solutions in $\mu' \le |w| < 1$, which is simple and contained in the open disk $D_\delta(\omega_{z,1})$. Then, for every $\zeta \in \overline{D_\epsilon(z)}$, we obtain from Corollary 2.2, the residue theorem, and equation (22) that

$$\begin{aligned} p_n(\zeta) &= \sqrt{n+1}\,\varphi'(\zeta) \lim_{w\to\omega_{\zeta,1}} w^n(1+K_n(w)) \left[\frac{h_\varphi(w)-\varphi(\zeta)}{w-\omega_{z,1}}\right]^{-1} \\ &\quad + \frac{\sqrt{n+1}\,\varphi'(\zeta)}{2\pi i} \oint_{\mathbb{T}_{\mu'}} \frac{w^n(1+K_n(w))dw}{h_\varphi(w)-\varphi(\zeta)} \\ &= \sqrt{n+1}\,\Phi(\zeta)^n\Phi'(\zeta)\left(1+O(r^{2n})\right) + \frac{\sqrt{n+1}\,\varphi'(\zeta)}{2\pi i} \oint_{\mathbb{T}_{\mu'}} \frac{w^n(1+K_n(w))dw}{h_\varphi(w)-\varphi(\zeta)}\,. \end{aligned}$$

Notice that $\varphi'(\zeta)/[h_\varphi(w)-\varphi(\zeta)]$ is bounded for $(w,\zeta) \in \mathbb{T}_{\mu'} \times \overline{D_\epsilon(z)}$, and since

$$\eta := \min_{|\zeta - z|\le\epsilon} |\Phi(\zeta)| > \mu',$$

we obtain

$$\begin{aligned} p_n(\zeta) &= \sqrt{n+1}\,\Phi(\zeta)^n\Phi'(\zeta)\left(1+O(r^{2n})\right) + O(\sqrt{n}\mu'^n) \\ &= \sqrt{n+1}\,\Phi(\zeta)^n\Phi'(\zeta)\left(1+O(r^{2n}) + O((\mu'/\eta)^n)\right) \\ &= \sqrt{n+1}\,\Phi(\zeta)^n\Phi'(\zeta)(1+O(\varepsilon^n)) \end{aligned}$$

uniformly for $\zeta \in \overline{D_\epsilon(z)}$ as $n \to \infty$, with $\varepsilon := \max\{r^2, \mu'/\eta\} < 1$. This proves Part (a) of Theorem 1.2 for a compact set K that is a sufficiently small closed disk about a point of Σ_1. But then, by the Heine-Borel theorem, it is also true for an arbitrary compact $K \subset \Sigma_1$.

We now prove Part (b) of Theorem 1.2. Let $z \in G_1$ be fixed. We first notice that once again by

Corollary 2.2, if τ is such that $r(z) < \tau < 1$, then

$$p_n(z) = \frac{\sqrt{n+1}\,\varphi'(z)}{2\pi i}\left[\oint_{\mathbb{T}_1}\frac{w^n}{h_\varphi(w)-\varphi(z)}dw + \oint_{\mathbb{T}_\tau}\frac{w^n K_n(w)}{h_\varphi(w)-\varphi(z)}dw\right]$$
$$= \frac{\sqrt{n+1}\,\varphi'(z)}{2\pi i}\oint_{\mathbb{T}_1}\frac{w^n}{h_\varphi(w)-\varphi(z)}dw + O\left(\sqrt{n}(\tau r^2)^n\right) \tag{23}$$

as $n \to \infty$.

Now, the function

$$g_z(w) := \frac{1}{h_\varphi(w)-\varphi(z)}$$

is analytic in $r(z) < |w| < 1/\rho$, and we claim that $g_z(w)$ is not analytic in any annulus $\mu^* < |w| < 1/\rho$ with $\mu^* < r(z)$. This is clear if $r(z) = 0$. If $r(z) > 0$ and $z \in \Sigma$, $g_z(w)$ has at least one pole in the circle $|w| = r(z)$. If $r(z) > 0$ and $z \in \Sigma_0$, and we suppose that there is a $\mu^* < \rho$ such that $g_z(w)$ is analytic in the annulus $\mu^* < |w| < 1/\rho$, then $h_\varphi(w)$ has a meromorphic continuation to $\mu^* < |w| < 1/\rho$ given by,

$$h_\varphi(w) = \varphi(z) + \frac{1}{g_z(w)},$$

which contradicts the very definition of μ. So $g_z(w)$ is analytic in the annulus $r(z) < |w| < 1/\rho$ and $r(z)$ is the smallest number satisfying this condition.

As a consequence, if $g_z(w) = \sum_{n=-\infty}^{\infty} a_n(z)w^j$ for $r(z) < |w| < 1/\rho$, then

$$r(z) = \limsup_{n\to\infty}|a_{-n}(z)|^{1/n} = \limsup_{n\to\infty}\left|\frac{1}{2\pi i}\oint_{\mathbb{T}_1} w^{n-1}g_z(w)dw\right|^{1/n}$$

and we get from (23) that

$$\limsup_{n\to\infty}|p_n(z)|^{1/n} = \limsup_{n\to\infty}\left|\sqrt{n+1}\,\varphi'(z)a_{-(n+1)}(z) + O\left(\sqrt{n}(\tau r^2)^n\right)\right|^{\frac{1}{n}}.$$

It is straightforward to show that for two sequences $\{a_n\}$ and $\{b_n\}$ such that

$$\limsup_{n\to\infty} |a_n|^{1/n} = A > B = \limsup_{n\to\infty} |b_n|^{1/n},$$

we have

$$\limsup_{n\to\infty} |a_n + b_n|^{1/n} = A.$$

Since τ can be taken arbitrarily closed to $r(z)$ and $r < 1$, this implies that

$$\limsup_{n\to\infty} |p_n(z)|^{1/n} = r(z),$$

completing the proof of Theorem 1.2.

3 MAIN RESULTS

In this section we prove two theorems describing the asymptotic behavior of Carleman polynomials corresponding to level curves of the inverse of the ”shifted Joukowsky transformation” $w \mapsto w - 1 + (w-1)^{-1}$. This family of curves was briefly considered in [3] with the purpose of illustrating Theorem 1.2 in a concrete example with the feature that Σ_1 is a set larger that $\Omega_\rho \cap G_1$. For any such level curve, the corresponding set Σ_0 has a nonempty interior, and the main contribution of this thesis consists of establishing the strong asymptotic behavior of the polynomials in all of Σ_0.

3.1 Statement of results

Let $R > 2$ be fixed, set

$$L_1 := \left\{ w - 1 + (w-1)^{-1} : |w| = R \right\} \tag{24}$$

and consider the sequence of polynomials $\{p_n(z)\}_{n=0}^{\infty}$ that are orthonormal over the interior domain G_1 of L_1, that is, satisfying (1).

From very well-known properties of the Joukowsky transformation $w \mapsto w + 1/w$, it follows that L_1 is an analytic Jordan curve, with

$$\psi(w) = Rw - 1 + \frac{1}{Rw-1}, \quad z \in \overline{\mathbb{C}}, \tag{25}$$

mapping Δ_1 conformally onto the exterior Ω_1 of L_1. Moreover, ψ maps both $\{w : |w - 1/R| > 1/R\}$ and $\{w : |w - 1/R| < 1/R\}$ conformally onto $\overline{\mathbb{C}} \setminus [-2, 2]$, while mapping both the closed upper and lower halves of the circle $|w - 1/R| = 1/R$ univalently onto $[-2, 2]$. Hence,

$$\rho = 2/R,$$

and L_ρ is the image by ψ of the circle $\mathbb{T}_{2/R}$ (see Figure 3).

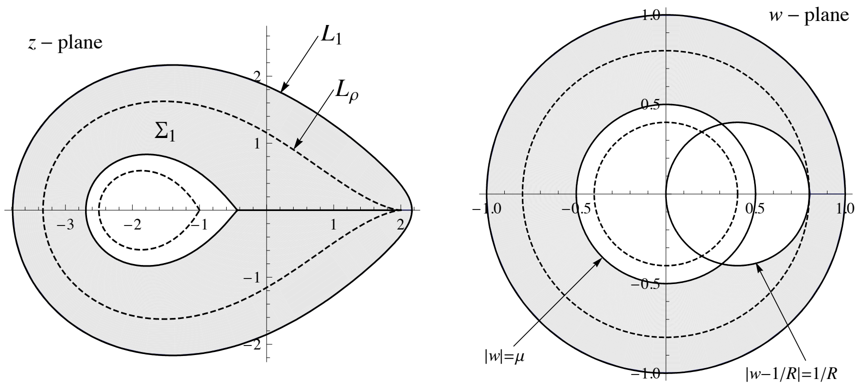

Figure 3: Sets Σ_1, Σ_2 and Σ_0 for the curve L_1 defined in (24) for $R = 2.5$.

Now, for every $z \in \mathbb{C}$, the two solutions of the equation $z = \psi(w)$ are

$$v_{z,1} = \frac{z + 2 + \sqrt{z^2 - 4}}{2R}, \quad v_{z,2} = \frac{z + 2 - \sqrt{z^2 - 4}}{2R}. \tag{26}$$

Here we denote by $\sqrt{z^2 - 4}$ the branch of the square root of $z^2 - 4$ in $\mathbb{C} \setminus [-2, 2]$ that is positive along $(2, \infty)$, extended to $[-2, 2]$ by taking its boundary values from the upper half plane, so that when $z \in \mathbb{C} \setminus [-2, 2]$, $v_{z,1}$ and $v_{z,2}$ lie, respectively, outside and inside the circle $|w - 1/R| = 1/R$, and consequently

$$|v_{z,1}| = \left| R^{-1} + \frac{R^{-2}}{v_{z,2} - R^{-1}} \right| = \left| \frac{R^{-1}}{v_{z,2} - R^{-1}} \right| |v_{z,2}| > |v_{z,2}|,$$

that is, $|v_{z,1}| > |v_{z,2}|$ for every $z \in \mathbb{C} \setminus [-2, 2]$. Of course, if $z \in [-2, 2]$, then $v_{z,1} = \overline{v_{z,2}}$, they lie on the circle $|w - 1/R| = 1/R$, and $v_{z,1} = v_{z,2}$ if and only if $z = \pm 2$. It follows that the inverse of $\psi(w)$ is the function

$$\phi(z) = v_{z,1}, \quad z \in \Omega_\rho,$$

which is indeed analytic and univalent all over $\mathbb{C} \setminus [-2, 2]$.

As for the corresponding Σ_p sets and number μ, the following result was already obtained in [3, Theorem 10]. We shall however give in Subsection 3.3 a new proof of it that is based on finding all the solutions of the equation (8).

Theorem 3.1. *For the domain G_1 bounded by the curve L_1 of (24),*

$$\mu = \frac{R - \sqrt{R^2 - 4}}{2}$$

and Σ consists of those points $z \in G_1$ for which $|v_{z,1}| > \mu$. Furthermore, if $z \in \Sigma$ and ω is one of the solutions of largest modulus that the equation (8) has in $\mu < |w| < 1$, then $\omega \in \{v_{z,1}, v_{z,2}\}$. As a consequence,

$$G_1 = \Sigma_1 \cup \Sigma_2 \cup \Sigma_0,$$

with Σ_1 being the image by ψ of those points of $\mathbb{D}_1$ that lie exterior to both the circle $|w| = \mu$ and the circle $|w - 1/R| = 1/R$, and $\Sigma_2 = (R^2\mu^2, 2]$ (see Figure 3).

Thus Σ_1 is larger than $\Omega_\rho \cap G_1$ and Σ_0 has nonempty interior. Moreover, $\Omega_\rho \cup \Sigma_1$ is a domain (i.e. open and connected) contained in $\mathbb{C} \setminus [-2, 2]$, so that by the uniqueness of the analytic continuation and (12), we must have that $\Phi(z) = v_{z,1}$ for all $z \in \Sigma_1$. Hence, we obtain directly from Theorems

1.1 and 1.2 that

$$\frac{(2R)^{n+1}p_n(z)}{\sqrt{n+1}(z+2+\sqrt{z^2-4})^n}-\frac{z+\sqrt{z^2-4}}{\sqrt{z^2-4}}$$

decays geometrically fast on compact subsets of $\Omega_\rho \cup \Sigma_1$.

The behavior of $p_n(z)$ at the remaining points, especially for $z \in \Sigma_0$, is definitely more challenging to determine, but we are rewarded with really interesting results. We shall denote the fractional part of a number x by $\langle x \rangle$. Here the functions

$$\Psi_n(t) := t^2 \int_0^\infty \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} s e^{-(\mu^{-1}-\mu)ts} ds, \quad \Re(t) > 0, \quad n \geq 1, \tag{27}$$

and the doubly infinite series

$$\chi_q(t) := t \sum_{k=-\infty}^{\infty} \mu^{4(k+1-q)} \exp(-(\mu^{-1}-\mu)\mu^{4(k+1-q)} t), \quad \Re(t) > 0,\ q \in [0,1]$$

play a definitive role. Notice that

$$\Psi_n(\mu^4 t) = \Psi_n(t), \quad \chi_q(\mu^4 t) = \chi_q(t)\,.$$

Theorem 3.2. *(a) The estimate*

$$\frac{(2R)^{n+1}p_n(z)}{\sqrt{n+1}(z+2+\sqrt{z^2-4})^n} = \frac{z+\sqrt{z^2-4}}{\sqrt{z^2-4}} + O(1/n)$$

holds uniformly as $n \to \infty$ on compact subsets of $\Sigma_1 \cup \partial\Sigma_0 \setminus \{R^2\mu^2\}$;

(b) For every $z = 2\cos\theta$, $0 \leq \theta \leq \arccos(R^2\mu^2/2)$, we have

$$p_n(z) = \sqrt{n+1}\left(\frac{2}{R}\right)^{n+1} \cos^n(\theta/2)\left(\frac{\sin((n+2)\theta/2)}{2\sin\theta} + \epsilon_n(z)\right), \tag{28}$$

where $\epsilon_n(z)$ decays geometrically fast on compact subsets of $(R^2\mu^2, 2]$, while $\epsilon_n(z) = O(1/n)$ uniformly on $[R^2\mu^2, 2]$ as $n \to \infty$;

(c) With $\lambda(z) := (z-\mu)/(\mu z - 1)$, we have

$$\begin{aligned} p_n(z) =& \frac{\mu^n}{\sqrt{n}} \cdot \frac{(1-\mu\lambda(v_{z,1}))^2(1-\mu\lambda(v_{z,2}))^2\mu^4}{(1+\mu^2)^2} \cdot \frac{\Psi_{n-1}(\lambda(v_{z,1})) - \Psi_{n-1}(\lambda(v_{z,2}))}{\lambda(v_{z,1}) - \lambda(v_{z,2})} \\ &+ O(\mu^n/n^{3/2}) \end{aligned}$$

locally uniformly as $n \to \infty$ in the interior of Σ_0.

Thus we see that $p_n(z)$ decreases like $\mu^n/\sqrt{n}$ in the interior of Σ_0, but the limit $\lim_{n\to\infty} \sqrt{n}\mu^{-n}p_n(z)$ does not exist. In fact, our next theorem characterizes the limit points of the sequence $\{\sqrt{n}\mu^{-n}p_n(z)\}_{n=0}^{\infty}$.

Theorem 3.3. *A function f is the normal limit in the interior of Σ_0 of some subsequence of $\{\sqrt{n}\mu^{-n}p_n\}$ if and only if f is of the form*

$$f(z) = \frac{\mu(1-\mu\lambda(v_{z,1}))^2(1-\mu\lambda(v_{z,2}))^2}{(1+\mu^2)} \cdot \frac{\chi_q(\lambda(v_{z,1})) - \chi_q(\lambda(v_{z,2}))}{\lambda(v_{z,1}) - \lambda(v_{z,2})} \tag{29}$$

for some $0 \le q \le 1$.

Moreover, for a value of $q \in [0,1]$ and a subsequence $\{n_k\}$ of the natural numbers, $\sqrt{n_k}\mu^{-n_k}p_{n_k}$ converges to the function f in (29) as $k \to \infty$ if and only if

$$\lim_{k\to\infty} \langle -\log_{\mu^4}(n_k - 1)\rangle = q.$$

The proof of Theorem 3.2 is extensive. Some of its aspects will be stated and proven as independent facts in the next two subsections. Theorem 3.3 is rather a Corollary of Theorem 3.2, and its proof is simpler.

3.2 Meromorphic continuation of h_φ

As above,

$$\mu = \frac{R - \sqrt{R^2 - 4}}{2} \quad \text{and} \quad \lambda(z) = \frac{z - \mu}{\mu z - 1}.$$

Notice that $\lambda(z)$ is its own inverse.

Proposition 3.4. *Let φ be a conformal map of G_1 onto $\mathbb{D}_1$.*

(a) *The function $\varphi(\psi(w))$, originally defined on $\rho < |w| < 1/\rho$, admits a meromorphic continuation, denoted by $h_\varphi(w)$, to all of $\mathbb{C} \setminus \{\mu, 1/\mu\}$. Moreover, μ and $1/\mu$ are both non-isolated singularities of h_φ of "essential type", in the sense that in every punctured neighborhood of either one of these two points, the function h_φ attains every value of the extended complex plane.*

(b) *$h_\varphi \circ \lambda$ is meromorphic in $\mathbb{C} \setminus \{0\}$, and for all $k \in \mathbb{Z} \setminus \{0\}$,*

$$(h_\varphi \circ \lambda)(t) = \begin{cases} \dfrac{1}{\overline{(h_\varphi \circ \lambda)(\bar{t}/\mu^{2k})}}, & \mu^{2k+2} \le |t| \le \mu^{2k},\ k \text{ odd}, \\ \\ (h_\varphi \circ \lambda)(t) = (h_\varphi \circ \lambda)(t/\mu^{2k}), & \mu^{2k+2} \le |t| \le \mu^{2k},\ k \text{ even}. \end{cases} \tag{30}$$

(c) *For every $z \in G_1$, the solutions that the equation $(h_\varphi \circ \lambda)(t) = \varphi(z)$ has in $0 < |t| < 1$ are the elements of the two sequences $\{\mu^{4k} t_{z,1}\}_{k=0}^{\infty}$ and $\{\mu^{4k} t_{z,2}\}_{k=0}^{\infty}$, with both*

$$t_{z,1} := \lambda(v_{z,1}) \quad \text{and} \quad t_{z,2} := \lambda(v_{z,2})$$

lying in $\mu^2 < |t| < 1$.

Moreover

$$\begin{aligned}(h_\varphi \circ \lambda)'(\mu^{4k} t_{z,1}) &= -\frac{(1-\mu^4)\varphi'(z)(t_{z,1}-t_{z,2})}{(1-\mu t_{z,1})^2(1-\mu t_{z,2})^2\mu^{4k+1}t_{z,1}}, \quad k \geq 0,\\ (h_\varphi \circ \lambda)'(\mu^{4k} t_{z,2}) &= -\frac{(1-\mu^4)\varphi'(z)(t_{z,2}-t_{z,1})}{(1-\mu t_{z,1})^2(1-\mu t_{z,2})^2\mu^{4k+1}t_{z,2}}, \quad k \geq 0.\end{aligned} \tag{31}$$

Proof. Observe first that $v_{z,1}$ and $\overline{v_{z,2}}$ are reflections of each other about the circle $|w - 1/R| = 1/R$, given that $Rv_{z,1} - 1$ and $R\overline{v_{z,2}} - 1$ are reflections of each other about the unit circle. Since the reflection of the unit circle about $|w - 1/R| = 1/R$ is the circle $|w - R/(R^2-1)| = 1/(R^2-1)$, we then have that ψ maps

$$\mathfrak{D} := \{w : |w - R/(R^2-1)| > 1/(R^2-1),\ |w| < 1\}$$

onto G_1, and $\partial\mathfrak{D}$ onto L_1. Because φ is analytic in a neighborhood of $\overline{G}_1$, the composition $h_\varphi(w) = \varphi(\psi(w))$ makes sense and is analytic in $\overline{\mathfrak{D}}$.

On the other hand, using that $\mu = (R-\mu)^{-1}$, it is easy to see that λ maps the annulus $\mu^2 \leq |t| \leq 1$ conformally onto $\overline{\mathfrak{D}}$. In effect, λ is an automorphism of the unit circle, it maps $|t| = \mu$ onto $|w - 1/R| = 1/R$, and preserves reflections about circles, so that it maps $|t| = \mu^2$ onto $|w - R/(R^2-1)| = 1/(R^2-1)$. Thus, $h_\varphi \circ \lambda$ is analytic on $\mu^2 \leq |t| \leq 1$, mapping the boundary of this annulus onto the unit circle, and since

$$(\psi \circ \lambda)(\mu^2 t) = \overline{(\psi \circ \lambda)(t)} = (\psi \circ \lambda)(\bar{t}), \quad |t| = 1,$$

we then have

$$(h_\varphi \circ \lambda)(\mu^2 t) = \frac{1}{\overline{(h_\varphi \circ \lambda)(\bar{t})}}, \quad |t| = 1.$$

Hence, we can extend $h_\varphi \circ \lambda$ meromorphically to all of $\mathbb{C} \setminus \{0\}$ as specified by (30). Now we see from (30) that $h_\varphi \circ \lambda$ maps $\mu^2 \le |t| \le 1$ onto $\overline{\mathbb{D}}_1$ and $\mu^4 \le |t| \le \mu^2$ onto $\overline{\mathbb{C}} \setminus \mathbb{D}_1$, and that

$$(h_\varphi \circ \lambda)(\mu^4 t) = (h_\varphi \circ \lambda)(t). \tag{32}$$

Hence, $h_\varphi \circ \lambda$ maps every annulus $\mu^{4(k+1)} \le |t| \le \mu^{4k}$, $k \in \mathbb{Z}$, onto $\overline{\mathbb{C}}$, and thus composing back with λ (which is its own inverse) we obtain Part (a) of the proposition.

It also follows directly from (30) that if $z \in G_1$, the solutions that the equation $(h_\varphi \circ \lambda)(t) = \varphi(z)$ has in $0 < |t| < 1$ are the elements of the two sequences $\{\mu^{4k} t_{z,1}\}_{k=0}^{\infty}$ and $\{\mu^{4k} t_{z,2}\}_{k=0}^{\infty}$, with $t_{z,1} = \lambda(v_{z,1})$ and $t_{z,2} = \lambda(v_{z,2})$ being the only solutions that said equation has in $\mu^2 < |t| < 1$.

Finally, in virtue of (32) we have

$$(h_\varphi \circ \lambda)'(\mu^{4k} t_{z,j}) = \mu^{-4k} (h_\varphi \circ \lambda)'(t_{z,j}) = \mu^{-4k} \varphi'(z) (\psi \circ \lambda)'(t_{z,j}), \quad j = 1, 2,$$

which combined with the fact that $t_{z,2} = \mu^2 / t_{z,1}$ and that

$$(\psi \circ \lambda)'(t) = \left[\mu^{-1} \frac{t - \mu^3}{\mu t - 1} + \mu \frac{\mu t - 1}{t - \mu^3} \right]' = -\frac{(1 - \mu^4)}{\mu t} \frac{t - (\mu^2/t)}{(1 - \mu t)^2 (1 - \mu(\mu^2/t))^2}$$

yields (31). $\square$

3.3 Proof of Theorem 3.1

The fact that $\mu = (R - \sqrt{R^2 - 4})/2$ is the smallest number with the property that $\varphi(\psi(w))$ admits a meromorphic continuation $h_\varphi(w)$ to the annulus $\mu < |w| < 1$ now emerges clearly from Proposition 3.4(a).

We infer from Part (c) of Proposition 3.4 that for every $z \in G_1$, the solutions that $h_\varphi(w) = \varphi(z)$

has in $\mathbb{D}_1 \setminus \{\mu\}$ are the elements of the sequences $\{\lambda(\mu^{4k}t_{z,1})\}_{k=0}^{\infty}$ and $\{\lambda(\mu^{4k}t_{z,2})\}_{k=0}^{\infty}$. Since $\lambda(t)$ maps the disk $|t - 1/R| \leq 1/R$ onto $|w| \leq \mu$ and $|v_{z,1}| \geq |v_{z,2}|$, we find that $h_{\varphi}(w) = \varphi(z)$ has solutions in $\mu < |w| < 1$ if and only if $t_{z,1} \in \mathbb{D}_1 \setminus \{t : |t-1/R| \leq 1/R\}$, or equivalently, $|v_{z,1}| > 1$. To finish the proof, it suffices to show that if $\mu < |v_{z,1} = \lambda(t_{z,1})| < 1$, then $|\lambda(\mu^{4k}t_{z,1})| < |v_{z,1}|$ for every $k \geq 1$, with the same assertion being true for $v_{z,2}$. But this follows clearly from the fact that $\lambda(t)$ maps $(-\infty, 0]$ and $[0, \infty)$ onto $[\mu, \infty)$ and $(-\infty, \mu]$, respectively, while any other ray departing from the origin gets mapped onto a circle passing through the points μ and $1/\mu$, and the two points of this circle that are closest to and farthest from the origin lie, respectively, inside $\mathbb{T}_\mu$ and outside $\mathbb{T}_1$.

3.4 Auxiliary lemmas

The following lemmas have been set apart because they are rather technical and may obscure the central idea of the proof of Theorem 3.2. For a first reading, we recommend the reader to trust the validity of Lemma 3.7 below and move on to the next section.

Lemma 3.5. *For every compact set $E \subset \{t : |1 - Rt| < 1\}$ there exist positive constants m and M such that for every integer $n \geq 1$,*

$$\left| e^{-(\mu^{-1}-\mu)ts} - \mu^{-n}\lambda^n(ts/n) \right| \leq \frac{Ms^2 e^{-ms}}{n}, \quad t \in E, \quad 0 \leq s \leq n\,.$$

Proof. It is easy to see that for every $z \in \mathbb{C}$, the function $\kappa(s) := |1 - sz|$ is convex in $\mathbb{R}$. Indeed,

$$\kappa''(s) = \frac{[\Im(z)]^2}{|1 - sz|^3} \geq 0.$$

Hence, for $z \in D_1 := \{z : |1-z| < 1\}$ and $n \geq 1$

$$\left|1 - \frac{sz}{n}\right| \leq \left|1 - (1-|1-z|)\frac{s}{n}\right| \leq e^{-(1-|1-z|)s/n}, \quad 0 \leq s \leq n. \tag{33}$$

Next, suppose t is such that $|\mu^{-1}\lambda(t)| < 1$ and consider the Möbius transformation

$$\sigma_t(s) := \frac{(1-\mu^2)t}{\mu(1-\mu t s)}, \quad s \in \mathbb{R}.$$

Since

$$\mu^{-1}\lambda(t) = 1 - \frac{(1-\mu^2)t}{\mu(1-\mu t)}, \tag{34}$$

we readily see that $\sigma_t(1) \in D_1$. Also, since $|\mu^{-1}\lambda(t)| < 1$ iff $|1 - Rt| < 1$, and $\mu^{-1}(1 - \mu^2) = (1-\mu^2)(1+\mu^2)^{-1}R < R$, it follows that $\sigma_t(0) = \mu^{-1}(1-\mu^2)t \in D_1$. Then, given that $\sigma_t(\infty) = 0 \in \partial D_1$ and that σ maps the real line conformally onto a circle, we conclude that σ_t must maps the segment $[0,1]$ onto a circular arc that lies inside D_1. Therefore, by (34) and (33), we have

$$\begin{aligned} |\mu^{-1}\lambda(ts/n)| &= \left|1 - \frac{s}{n}\,\sigma_t(s/n)\right| \\ &\leq e^{-(1-|1-\sigma_t(s/n)|)s/n}, \quad 0 \leq s \leq n, \quad |1-Rt| < 1. \end{aligned} \tag{35}$$

Now set $\alpha := \mu^{-1} - \mu$, so that for $0 \leq s \leq n$ and $|1 - Rt| < 1$ we have

$$\begin{aligned} \left|e^{-\alpha ts/n} - \mu^{-1}\lambda(ts/n)\right| &= \left|e^{-\alpha ts/n} - \left(1 - \frac{\alpha ts/n}{1-\mu ts/n}\right)\right| \\ &= \frac{s^2}{n^2}\left|\frac{\mu\alpha t^2}{1-\mu ts/n} + \sum_{j=2}^{\infty}\frac{(-\alpha t)^j}{j!}\left(\frac{s}{n}\right)^{j-2}\right| \\ &\leq \frac{s^2}{n^2}\left(\frac{\mu\alpha|t|^2}{1-\mu|t|} + e^{\alpha|t|} - \alpha|t| - 1\right). \end{aligned} \tag{36}$$

Thus, for a compact set $E \subset \{t : |1 - Rt| < 1\}$, and with the constants

$$M := \max_{t \in E} \left(\frac{\mu\alpha|t|^2}{1 - \mu|t|} + e^{\alpha|t|} - \alpha|t| - 1 \right),$$

$$m_1 := \alpha \min_{t \in E} \Re(t) > 0, \quad m_2 := \min_{(t,s) \in E \times [0,n]} (1 - |1 - \sigma_t(s/n)|) > 0,$$

and

$$m := 2^{-1} \min\{m_1, m_2\},$$

we obtain from (35) and (36) that

$$\begin{aligned} \left|e^{-\alpha ts} - \mu^{-n}\lambda^n(ts/n)\right| &\leq \frac{Ms^2}{n^2} \sum_{\ell=1}^{n} e^{-\alpha\Re(t)s(\ell-1)/n} \left|\mu^{-1}\lambda(ts/n)\right|^{n-\ell} \\ &\leq \frac{Ms^2}{n^2} \sum_{\ell=1}^{n} e^{-\alpha\Re(t)s(\ell-1)/n} e^{-(1-|1-\sigma_t(s/n)|)s(n-\ell)/n} \\ &\leq \frac{Ms^2 e^{-ms}}{n}, \quad t \in E, \quad 0 \leq s \leq n\,. \end{aligned}$$

□

Lemma 3.6. *With $G_n(t) := \mu^{-n}\lambda^n(t)\lambda'(t)$, $n \geq 1$ and $\Psi_n(t)$ defined by (27), we have*

$$\int_0^n \mu^{-4\langle \log_{\mu^4}(s/n) \rangle} s G'_{n+1}(st/n) ds = \frac{(n+1)(1-\mu^2)^2 \Psi_n(t)}{\mu t^2} + O(1)$$

locally uniformly on $\{t : |1 - Rt| < 1\}$ as $n \to \infty$.

Proof. Let $E \subset \{t : |1 - Rt| < 1\}$ be compact. First we notice that

$$G'_{n+1}(st/n) = (n+1)\mu^{-n-1}\lambda^n(ts/n)\mathfrak{L}_n(st/n) \tag{37}$$

with

$$\mathfrak{L}_n(st/n) = \frac{(1-\mu^2)^2}{(1-\mu ts/n)^4} + \frac{2\mu(1-\mu^2)(ts/n-\mu)}{(n+1)(1-\mu ts/n)^4} \\ = (1-\mu^2)^2\left[1+O(s/n)+O(1/n)\right] \tag{38}$$

uniformly for $(t,s) \in E \times [0,n]$ as $n \to \infty$. Hence, with $\alpha = \mu^{-1} - \mu$ and given that $1 \le \mu^{-4\langle \log_{\mu^4}(s/n)\rangle} \le \mu^{-4}$, we get

$$\int_0^n \mu^{-4\langle \log_{\mu^4}(s/n)\rangle} s e^{-\alpha ts} \mathfrak{L}_n(st/n) ds = (1-\mu^2)^2 \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n)\rangle} s e^{-\alpha ts} ds \\ + O(1/n) \tag{39}$$

uniformly for $(t,s) \in E \times [0,n]$ as $n \to \infty$.

Combining (37), (38), (39) and Lemma 3.5 we readily see that there exist positive constants M' and m such that

$$\left| \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n)\rangle} s G'_{n+1}(st/n) ds - \frac{(n+1)(1-\mu^2)^2}{\mu} \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n)\rangle} s e^{-\alpha ts} ds \right| \\ \le \frac{n+1}{\mu} \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n)\rangle} s \left|\mu^{-n}\lambda^n(ts/n) - e^{-\alpha ts}\right| |\mathfrak{L}_n(st/n)| ds + O(1) \\ \le M' \int_0^\infty s^3 e^{-ms} ds + O(1)$$

uniformly in $t \in E$ as $n \to \infty$. This concludes the proof of Lemma 3.6. □

Lemma 3.7. *For $G_n(t)$ defined as in Lemma 3.6, we have:*

(a)

$$\sum_{k=0}^{\infty} \mu^{4k} G_{n+1}(\mu^{4k} t) = \frac{-\mu^3(1-\mu^2)\left[\Psi_n(t) + O(1/n)\right]}{(n+1)(1+\mu^2)t}$$

locally uniformly on $\{t : |1 - Rt| < 1\}$ as $n \to \infty$.

(b) *For every compact* $K \subset \{t : |1 - Rt| \le 1,\ t \ne 0\}$*, there exists an open set* A *with*

$$K \subset A \subset \overline{A} \subset \{t : |t| < 1,\ \Re(t) > 0\},$$

such that

$$\begin{aligned}\sum_{k=0}^{\infty} \mu^{4k} G_{n+1}(\mu^{4k} t) &= \mu^{-n-1}\lambda^{n+1}(t)\lambda'(t) - \frac{\mu^3(1-\mu^2)\left[\Psi_n(t) + O(1/n)\right]}{(n+1)(1+\mu^2)t} \\ &= \mu^{-n-1}\lambda^{n+1}(t)\lambda'(t) + O(1/n)\end{aligned}$$

uniformly for $t \in A$ *as* $n \to \infty$.

Proof. Using summation by parts we find

$$\begin{aligned}\sum_{k=0}^{K} \mu^{4k} G_{n+1}(\mu^{4k} t) &= \frac{1-\mu^{4(K+1)}}{1-\mu^4} G_{n+1}(\mu^{4K} t) \\ &\quad - \sum_{k=0}^{K-1} \frac{1-\mu^{4(k+1)}}{1-\mu^4}\left[G_{n+1}(\mu^{4(k+1)}t) - G_{n+1}(\mu^{4k}t)\right] \\ &= \frac{G_{n+1}(t)}{1-\mu^4} - \frac{\mu^{4(K+1)} G_{n+1}(\mu^{4K}t)}{1-\mu^4} \\ &\quad + \sum_{k=0}^{K-1} \frac{\mu^{4(k+1)}}{1-\mu^4}\left[G_{n+1}(\mu^{4(k+1)}t) - G_{n+1}(\mu^{4k}t)\right]\end{aligned}$$

Letting $K \to \infty$ and using Lemma 3.6 we obtain

$$\begin{aligned}\sum_{k=0}^{\infty} \mu^{4k} G_{n+1}(\mu^{4k} t) &= \frac{G_{n+1}(t)}{1-\mu^4} + \sum_{k=0}^{\infty} \frac{\mu^{4(k+1)}}{1-\mu^4} \int_{\mu^{4k}}^{\mu^{4(k+1)}} \frac{\partial G_{n+1}(st)}{\partial s} ds \\ &= \frac{G_{n+1}(t)}{1-\mu^4} - \frac{\mu^4 t}{(1-\mu^4)n^2} \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n)\rangle} s G'_{n+1}(st/n) ds \\ &= \frac{-\mu^3(1-\mu^2)}{(n+1)(1+\mu^2)}\left[t \int_0^n \mu^{-4\langle \log_{\mu^4}(s/n)\rangle} s e^{-\alpha t s} ds + O(1/n)\right]\end{aligned}$$

locally uniformly on $\{t : |1 - Rt| < 1\}$ as $n \to \infty$. This proof Part (a).

As for Part (b), it is clear that every compact set $K \subset \{t : |1 - Rt| \leq 1,\ t \neq 0\}$ has a sufficiently small neighborhood A with the required properties and such that for all $t \in \overline{A}$, $\mu^4 t \in \{t : |1 - Rt| < 1\}$. Then, Part (b) follows from Part (a) by noticing that we can write

$$\sum_{k=0}^{\infty} \mu^{4k} G_{n+1}(\mu^{4k} t) = G_{n+1}(t) + \mu^4 \sum_{k=0}^{\infty} \mu^{4k} G_{n+1}(\mu^{4k}(\mu^4 t))\,,$$

that $\Psi_n(\mu^4 t) = \Psi_n(t)$ and that Ψ_n is bounded on A. $\square$

3.5 Proof of Theorem 3.2

We begin by proving Part (b), for which it suffices to show that for every $z \in [R^2\mu^2, 2]$, there exists an $\epsilon > 0$ such that for every $\zeta = 2\cos\theta \in D_\epsilon(z) \cap [R^2\mu^2, 2]$,

$$p_n(\zeta) = \sqrt{n+1}\left(\frac{2}{R}\right)^{n+1} \cos^n(\theta/2)\left(\frac{\sin((n+2)\theta/2)}{2\sin\theta} + \epsilon_n(\zeta)\right), \tag{40}$$

with $\epsilon_n(\zeta)$ decaying geometrically fast if $z \in (R^2\mu^2, 2]$, while $\epsilon_n(\zeta) = O(1/n)$ as $n \to \infty$ when $z = R^2\mu^2$. We shall only prove this for $z = R^2\mu^2$ and $z = 2$, since the proof for $z \in (R^2\mu^2, 2)$ follows a similar (actually simpler) argument.

First let $z = 2$, so that according to Theorem 3.1, the equation (8) has in $\mu < |w| < 1$ one solution $v_{z,1} = 2/R$ of largest modulus and $h'_\varphi(v_{z,1}) = 0$, $h''_\varphi(v_{z,1}) \neq 0$. For this z, we choose corresponding numbers μ' and δ satisfying the hypothesis of Lemma 2.3, and in addition, satisfying that

$$(|v_{z,1}| + \delta)\rho^2 < |v_{z,1}| - \delta\,.$$

We can then choose τ and r such that $|v_{z,1}|+\delta<\tau<1$, $\rho<r<1$, and

$$\tau r^2 < |v_{z,1}| - \delta. \tag{41}$$

Now, Lemma 2.3 guarantees that we can find an $\epsilon>0$ such that for every $\zeta\in D^*_\epsilon(z)\cap(R^2\mu^2,2]$, the equation $h_\varphi(\zeta)=\varphi(\zeta)$ has in $\mu'\le|w|<1$ no solutions other than $v_{\zeta,1}$ and $v_{\zeta,2}$, which are distinct and contained in $D_\delta(v_{z,1})$. Then, for every $\zeta\in D_\epsilon(z)\cap(R^2\mu^2,2]$, we obtain from Corollary 2.2 and the residue theorem that

$$\begin{aligned} p_n(\zeta) = {} & \frac{\sqrt{n+1}\,\varphi'(\zeta)}{2\pi i}\oint_{\mathbb{T}_{\mu'}}\frac{w^n dw}{h_\varphi(w)-\varphi(\zeta)} + \frac{\sqrt{n+1}\,\varphi'(\zeta)}{2\pi i}\oint_{\mathbb{T}_\tau}\frac{w^n K_n(w)dw}{h_\varphi(w)-\varphi(\zeta)} \\ & + \sqrt{n+1}\,\varphi'(\zeta)\times\begin{cases}\dfrac{v^n_{\zeta,1}}{h'_\varphi(v_{\zeta,1})}+\dfrac{v^n_{\zeta,2}}{h'_\varphi(v_{\zeta,2})}, & \zeta\ne z,\\ \dfrac{nv^{n-1}_{z,1}}{h''_\varphi(v_{z,1})}+O(|v_{z,1}|^{n-1}), & \zeta=z.\end{cases} \end{aligned} \tag{42}$$

Now,

$$h'_\varphi(v_{\zeta,j}) = \varphi'(\psi(v_{\zeta,j}))\psi'(v_{\zeta,j}) = \varphi'(\zeta)\psi'(v_{\zeta,j}),\quad j=1,2,$$

$$h''_\varphi(v_{z,1}) = \varphi'(z)\psi''(v_{z,1}),$$

and $\psi(w)=Rw-1+(Rw-1)^{-1}$, so that making the substitution

$$v_{\zeta,1} = 1/R + e^{i\theta}/R = \frac{2e^{i\theta/2}\cos(\theta/2)}{R},\quad 0\le\theta\le\arccos(R^2\mu^2/2) \tag{43}$$

and taking into account that $v_{\zeta,2} = \overline{v_{\zeta,1}}$ for $\zeta \in [-2,2]$, we find from (42) that

$$p_n(2\cos\theta) = O(\sqrt{n}(\mu')^n) + O(\sqrt{n}\tau^n r^{2n}) \\ + \sqrt{n+1} \times \begin{cases} \dfrac{(2/R)^{n+1}\sin((n+2)\theta/2)\cos^n(\theta/2)}{2\sin\theta}, & 2-\epsilon < 2\cos\theta < 2, \\ n(2/R)^{n+1}/4 + O((2/R)^n), & \theta = 1, \end{cases}$$

as $n \to \infty$, and the assertion follows by noticing that by (41),

$$\max\{\mu', \tau r^2\} < |v_{z,1}| - \delta < |v_{\zeta,1}| = 2\cos(\theta/2)/R, \quad 2-\epsilon < 2\cos\theta \le 2.$$

The completion of Part (b) as well as the proof of the other parts of the theorem require a common argument that we develop now. First we fix numbers r and η such that $\rho < r < 1$ and

$$\eta r^2 < \mu < \eta < 1.$$

Define the compact sets

$$E := \{w : |w - 1/R| \ge 1/R,\ |w| \le \mu\}, \quad K := \lambda(E \cup E^*),$$

where E^* denotes the reflection of E about the circle $|w - 1/R| = 1/R$, and fix an open set $A \supset K$ satisfying the hypothesis of Lemma 3.7(b).

Given that for every $z \in \Sigma_0$, $v_{z,1} \in E$, we have that $\{v_{z,1}, v_{z,2}\} \subset E \cup E^*$ for all $z \in \Sigma_0$ and we can choose an open set $U \supset \Sigma_0$ such that (recall that $t_{z,j} := \lambda(v_{z,j})$, $j = 1,2$)

$$|v_{z,1}| < \frac{\mu + \eta}{2}, \quad \{t_{z,1}, t_{z,2}\} \subset A, \quad z \in U.$$

Then, by Theorem 3.1, if $z \in U$, the equation $h_\varphi(w) - \varphi(z)$ has no solutions in $(\mu+\eta)/2 \le |w| < 1$, and we obtain from Corollary 2.2 that

$$\begin{aligned} p_n(z) &= \frac{\sqrt{n+1}\varphi'(z)}{2\pi i} \oint_{\mathbb{T}_1} \frac{w^n dw}{\varphi(\psi(w)) - \varphi(z)} + \frac{\sqrt{n+1}\varphi'(z)}{2\pi i} \oint_{\mathbb{T}_\eta} \frac{w^n K_n(w) dw}{\varphi(\psi(w)) - \varphi(z)} \\ &= \frac{\sqrt{n+1}\varphi'(z)}{2\pi i} \oint_{\mathbb{T}_1} \frac{w^n dw}{\varphi(\psi(w)) - \varphi(z)} + O(\sqrt{n}(\eta r^2)^n) \end{aligned} \tag{44}$$

uniformly for $z \in U$ as $n \to \infty$. We shall now analyze the behavior as $n \to \infty$ of the integral

$$I_n(z) := \frac{1}{2\pi i} \int_{|w|=1} \frac{w^n dw}{h_\varphi(w) - \varphi(z)}.$$

From Proposition 3.4, making the change of variables $w = \lambda(t)$ and applying the residue theorem we get that for all integer $N \ge 1$ and $z \in G_1 \setminus \{-2, 2\}$,

$$\begin{aligned} I_n(z) :=& \frac{1}{2\pi i} \int_{|t|=1} \frac{[\lambda(t)]^n \lambda'(t) dt}{(h_\varphi \circ \lambda)(t) - \varphi(z)} \\ =& - \frac{\mu(1-\mu t_{z,1})^2 (1-\mu t_{z,2})^2}{(1-\mu^4)\varphi'(z)} \\ & \times \frac{\sum_{k=0}^{N-1} \mu^{4k} t_{z,1} \lambda^n(\mu^{4k} t_{z,1}) \lambda'(\mu^{4k} t_{z,1}) - \sum_{k=0}^{N-1} \mu^{4k} t_{z,2} \lambda^n(\mu^{4k} t_{z,2}) \lambda'(\mu^{4k} t_{z,2})}{(t_{z,1} - t_{z,2})} \\ & + \frac{1}{2\pi i} \int_{|t|=\mu^{2N}} \frac{[\lambda(t)]^n \lambda'(t) dt}{(h_\varphi \circ \lambda)(t) - \varphi(z)}. \end{aligned}$$

But $|\lambda(t)| \le 1$ for $|t| \le 1$ and $|(h_\varphi \circ \lambda)(t)| = 1$ for $|t| = \mu^{2N}$, so that

$$\left| \frac{1}{2\pi i} \int_{|t|=\mu^{2N}} \frac{[\lambda(t)]^n \lambda'(t) dt}{(h_\varphi \circ \lambda)(t) - \varphi(z)} \right| \le \frac{(1-\mu^2)\mu^{2N}}{(1-|\varphi(z)|)(1-\mu^{2N+1})^2} \underset{N\to\infty}{\to} 0,$$

and thus

$$I_n(z) = \frac{\mu(1-\mu t_{z,1})^2(1-\mu t_{z,2})^2\mu^n}{(1-\mu^4)\varphi'(z)} \cdot \frac{t_{z,1}\sum_{k=0}^{\infty}\mu^{4k}G_n(\mu^{4k}t_{z,1}) - t_{z,2}\sum_{k=0}^{\infty}\mu^{4k}G_n(\mu^{4k}t_{z,2})}{t_{z,1}-t_{z,2}}, \tag{45}$$

with $G_n(t) = \mu^{-n}\lambda^n(t)\lambda'(t)$. Notice that this equality must also be true in a limiting sense for $z = -2$.

Since $\{t_{z,1}, t_{z,2}\} \subset A$ for all $z \in U$, we find from Lemma 3.7(b) , (45) and (31) that

$$\begin{aligned} I_n(z) &= \frac{\lambda'(t_{z,1})\lambda^n(t_{z,1})}{(h_\varphi \circ \lambda)'(t_{z,1})} + \frac{\lambda'(t_{z,2})\lambda^n(t_{z,2})}{(h_\varphi \circ \lambda)'(t_{z,2})} + O(\mu^n/n) \\ &= \frac{v_{z,1}^n}{\varphi'(z)\psi'(v_{z,1})} + \frac{v_{z,2}^n}{\varphi'(z)\psi'(v_{z,2})} + O(\mu^n/n) \end{aligned} \tag{46}$$

uniformly for $z \in U$ as $n \to \infty$.

Now, if $z = R^2\mu^2$, we have that for every $\zeta \in U \cap [R^2\mu^2, 2]$, $|v_{\zeta,1} = \overline{v_{\zeta,2}}| \geq \mu$, so that making once again the substitution (43), we obtain from (46) and (44) that

$$p_n(\zeta) = \sqrt{n+1}\left(\frac{2}{R}\right)^{n+1}\cos^n(\theta/2)\left(\frac{\sin((n+2)\theta/2)}{2\sin\theta} + O(1/n)\right),$$

uniformly for $\zeta = 2\cos\theta \in U \cap [R^2\mu^2, 2]$ as $n \to \infty$. This completes the proof of Part (b) of the theorem.

To prove Part (a), we observe that if $z \in \partial\Sigma_0 \setminus R^2\mu^2$, then $\mu = |v_{z,1}| > |v_{z,2}|$, and so we can find $\epsilon > 0$ with $D_\epsilon(z) \subset U$ and such that $\max_{\zeta \in D_\epsilon(z)} |v_{\zeta,2}| < \mu$. Given that $|v_{\zeta,1}| \geq \mu$ for $\zeta \in D_\epsilon(z) \cap (\Sigma_1 \cup \partial\Sigma_0)$, we obtain from (46) and (44) that

$$p_n(\zeta) = \sqrt{n+1}(v_{\zeta,1})^n\left(\frac{1}{\psi'(v_{\zeta,1})} + O(1/n)\right)$$

uniformly for $\zeta \in D_\epsilon(z) \cap (\Sigma_1 \cup \partial\Sigma_0)$ as $n \to \infty$, which together with Theorem 1.2 yields Part (a) of the theorem.

Finally, Part (c) follows directly from (45), (44) and Lemma 3.7(a) since when we let z vary over some fixed compact subset of the interior of Σ_0, the points $t_{z,1}$ and $t_{z,2}$ stay on some compact subset of $\{t : |t - 1/R| < 1/R\}$.

3.6 Proof of Theorem 3.3

Theorem 3.3 will follow directly from Theorem 3.2 if we prove that the sequence $\{\langle \log_{\mu^4}(1/n)\rangle\}_{n=1}^{\infty}$ is dense in $[0,1]$, and that if $q \in [0,1]$ and $\{n_j\}_{j=1}^{\infty}$ is a subsequence of the natural numbers such that $\langle \log_{\mu^4}(1/n_j)\rangle \to q$ as $j \to \infty$, then

$$\lim_{j\to\infty} t^2 \int_0^\infty \mu^{-4\langle \log_{\mu^4}(s/n_j)\rangle} s e^{-(\mu^{-1}-\mu)ts} ds = (1+\mu^2)\mu t \sum_{k=-\infty}^{\infty} \mu^{4(k-q)} e^{-(\mu^{-1}-\mu)t\mu^{4(k+1-q)}} . \quad (47)$$

It is a well-known theorem in diophantine approximation that if β is an irrational number, then the sequence $\{\langle j\beta\rangle\}_{j=1}^{\infty}$ is dense in $[0,1]$. If $0 < \mu^4 < 1$, then there exists a positive integer $\mathfrak{n}$ such that $\log_{\mu^4}(\mathfrak{n})$ is irrational because $\log_{\mu^4}(\mathfrak{n})$ and $\log_{\mu^4}(\mathfrak{n}+1)$ cannot be both rational. Hence the sequence $\{\langle \log_{\mu^4}(\mathfrak{n}^j)\rangle\}_{j=1}^{\infty}$ is dense in $[0,1]$, and so is the sequence $\{\langle \log_{\mu^4}(1/n)\rangle\}_{n=1}^{\infty}$.

Now if $\langle \log_{\mu^4}(1/n_j)\rangle \to q \in [0,1]$, then for every positive number $s \neq \mu^{4(k+1-q)}$, $k \in \mathbb{Z}$,

$$\begin{aligned}
\lim_{j\to\infty} \mu^{-4\langle \log_{\mu^4}(s/n_j)\rangle} &= \lim_{j\to\infty} \mu^{-4\langle\langle \log_{\mu^4}(s)\rangle + \langle \log_{\mu^4}(1/n_j)\rangle\rangle} \\
&= \mu^{-4\langle\langle \log_{\mu^4}(s)\rangle + q\rangle} \\
&= \begin{cases} s^{-1}\mu^{4(k+1-q)}, & \mu^{4(k+1)} < s < \mu^{4(k+1-q)}, \\ s^{-1}\mu^{4(k-q)}, & \mu^{4(k+1-q)} < s \leq \mu^{4k}, \end{cases}
\end{aligned}$$

and we obtain (47) by an easy application of Lebesgue's dominated convergence theorem.

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