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Well-covered Graphs, Unique Colorability, and Covering Range

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WELL-COVERED GRAPHS, UNIQUE COLORABILITY, AND COVERING RANGE

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ABSTRACT

In 1970 Michael Plummer introduced the notion of well-coveredness of graphs [13]. A graph is called *well-covered* if all of its maximal independent sets have the same cardinality. A generalization of trees was defined as *k-trees* by Beineke and Pippert in 1968 [1]. This dissertation gives a characterization of well-covered k-trees.

The concept of *unique colorability* was introduced by Cartwright and Harary in 1967 [5]. A graph is said to be uniquely χ-colorable if, modulo permutations of colors, it has exactly one proper χ-coloring. The k-trees with at least $k+1$ vertices are minimal uniquely $(k+1)$-colorable, i.e., they have the minimal number of edges necessary for uniquely $(k+1)$-colorable graphs. In this dissertation we introduce the *k-frames*, a new class of minimal uniquely $(k+1)$-colorable graphs that generalizes the k-trees.

We present a parameter that measures how far a graph is from being well-covered. The *covering range* of a graph is the difference between the cardinality of a largest maximal independent set of a graph and the cardinality of a smallest maximal independent set of the graph. We give the covering range for some cubic graphs and a class of k-regular graphs.

DEDICATION

I dedicate this dissertation to my brilliant and inspiring daughter, Maia, and to my adviser, William Staton, as he embarks upon a new life adventure.

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1 Introduction

A graph is said to be well-covered if all of its maximal independent sets are also maximum. The concept of well-coveredness was introduced by Michael Plummer in 1970 [13] and has since been studied extensively yielding numerous results. Chapter 2 surveys some results for well-covered graphs, including a characterization of well-covered bipartite graphs given by Ravindra in 1977 [14].

The class of graphs known as the k-trees was introduced by Bieneke and Pippert in 1968 [1]. This class of graphs which generalizes the trees has likewise been the focal point of much research. Extrapolated from the characterization of well-covered bipartite graphs is a characterization of well-covered trees. A major emphasis of this dissertation is the generalization of this characterization to a characterization of well-covered k-trees. Chapter 3 is fully devoted to this characterization.

Unique colorability will be discussed in Chapter 4. A graph is uniquely χ-colorable if, modulo permutations of colors, there is only one proper χ-coloring of its vertex set. The notion of unique colorability was introduced by Cartwright and Harary in 1967 [5]. The k-trees are uniquely $(k+1)$-colorable, and this observation led to the definition of a new class of graphs called the k-frames.

The idea of measuring the degree to which a graph fails to be well-covered led to the defining of what seems to be a new parameter, the covering range of a graph. The covering range of a graph is the difference between the size of a largest maximal independent set of a graph and the size of a smallest maximal independent set of the graph. In Chapter 5, bounds are given for the covering range of several classes of graphs.

1.1 Definitions and Notations

The following definitions will be used throughout this work. The reader is referred to West [17] for definitions not mentioned here.

Definition 1.1. A *graph* G is an ordered pair $G = (V; E)$, in which V is a non-empty finite set and E is a collection of unordered pairs from V. Each element of V is called a *vertex*, and each element of E is called an *edge*. The graphs considered here are simple and undirected. For an edge $e = uv$, u and v are *adjacent* vertices, denoted $u \sim v$.

Definition 1.2. A *subgraph* H of a graph G has vertex set $V(H) \subseteq V(G)$ and edge set $E(H) \subseteq E(G) \cap (V(H) \times V(H))$; an *induced subgraph* H of a graph G has vertex set $V(H) \subseteq V(G)$ and edge set $\{uv : u, v \in V(H),\ uv \in E(G)\}$.

Definition 1.3. A *bipartite graph* G is a graph whose vertex set can be partitioned into two subsets X and Y such that each edge of G has one vertex in X and the other in Y. In a *complete bipartite graph* G every vertex of X is joined by an edge to every vertex of Y. In this case G is denoted by $K_{m,n}$ if $|X| = m$ and $|Y| = n$.

Definition 1.4. A vertex set I in a graph G is called *independent* if no two vertices of I are joined by an edge. The *independence number* of a graph G, denoted $\alpha(G)$, is the cardinality of a maximum independent set of G. An independent set I in a graph G is a *maximal independent set* if I is not a proper subset of any independent set of G.

Definition 1.5. A graph G is *well-covered* if all maximal independent sets have the same cardinality.

Definition 1.6. A vertex set C in a graph G is called a *clique* if all of the vertices of C are pairwise adjacent. The cardinality of a largest clique in a graph G is called the *clique number* of G and is denoted $\omega(G)$.

Definition 1.7. Let G be a graph and $v \in V(G)$. The *neighborhood* of v, $N(v)$, is defined to be $\{u \in V(G) : uv \in E(G)\}$ and $N_H(v)$ denotes the neighborhood of vertex v in subgraph H; the *closed neighborhood* of v, $N[v]$ is defined to be $N(v) \cup \{v\}$. The *degree* of a vertex v is the size of the neighborhood of v, $d(v) = |N(v)|$. If $d(v) = r$ for every $v \in V(G)$, then G is said to be *r-regular*. An edge e of G is a *pendant edge* if e is incident with a vertex of degree one.

Definition 1.8. A set Γ of edges in a graph G is said to be a *matching* if no two edges of Γ are incident with a common vertex. A matching Γ in G is a *perfect matching* if every vertex of G is incident with an edge of Γ.

1.2 k-Trees

Definition 1.9. Let k be a positive integer. For $n \geq k$, *k-trees* on n vertices are defined recursively as follows:

i. The complete graph K_k is the smallest k-tree.

ii. A k-tree with $n + 1$ vertices is formed by adjoining a new vertex v to every vertex of a k-clique of a k-tree with n vertices.

The class of k-trees was introduced by Beineke and Pippert [1]. When $k = 1$, this class coincides with the class of trees.

A graph is called *chordal* if every cycle of length exceeding three has a chord. Hence, k-trees are a subclass of chordal graphs.

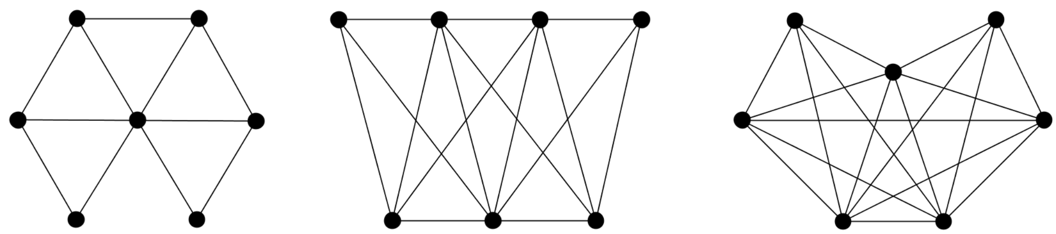


Figure 1: k-Trees

1.3 Colorability

Definition 1.10. If r is a positive integer, then a proper r*-coloring* of a graph G is a partition of the vertex set of G into r nonempty independent sets. The smallest r for which there is an r-coloring of G is called the *chromatic number of* G and is denoted "$\chi(G)$". A graph G with chromatic number χ is *uniquely colorable* if there is, modulo permutations of the colors, exactly one χ-coloring of the vertices of G. The *chromatic polynomial* of a graph G, commonly denoted $\chi_G(\lambda)$, counts the number of colorings of G as a function of λ colors [2].

Two easily observable facts follow.

Fact 1.1. *If T be a k-tree with n vertices, $n \geq k+1$, then T is uniquely $(k+1)$-colorable.*

Proof. If $n = k+1$, then T is isomorphhic to the graph K_{k+1}, so the color classes of G are singletons. We proceed by induction. If k-trees with n vertices are uniquely $(k+1)$-colorable and T is a k-tree with $n+1$ vertices, then let v be a vertex that is adjacent to exactly k vertices. Due to the recursive definition of k-trees, such a vertex exists. Then $T \setminus \{v\}$ is uniquely $(k+1)$-colorable, and v is adjacent to a k-clique, hence to vertices of k distinct colors, leaving a unique choice of color for v. □

Fact 1.2. *If G is a connected bipartite graph with $n \geq 2$ vertices, then G is uniquely 2-colorable.*

Proof. If $n = 2$, then G is isomorphic to the graph K_2, and the result is obvious. If $n > 2$, let v be a non-cut vertex of G. Then $G \setminus v$ is connected and bipartite, so $G \setminus v$ is uniquely 2-colorable by induction. Now v has at least one neighbor, so only one color is available for v. □

Note that we have twice shown that the trees are uniquely 2-colorable.

2 Well-covered Graphs

Since its introduction, the class of well-covered graphs has been the focus of much research. Determining that a graph is not well-covered means showing two maximal independent sets of different sizes. Determining that a graph is well-covered means all maximal independent sets must be compared. A well-covered graph can be constructed from any graph G by partitioning the vertex set of G into disjoint cliques $C_1, C_2, \ldots, C_r$ and adjoining to G r disjoint cliques $H_1, H_2, \ldots, H_r$ by joining every vertex of H_i to every vertex of C_i for each i. Each maximal independent set contains exactly one vertex from each $H_i \cup C_i$ [11].

In this chapter we discuss some existing results on well-covered graphs as related to girth and degree and well-covered bipartite graphs. In our discussion of well-covered bipartite graphs, a characterization of reduced well-covered bipartite graphs is given.

2.1 Well-coveredness and Girth

The *girth* of a graph is the length of a shortest cycle in the graph. The study of well-coveredness and its relationship to girth was introduced by Finbow and Hartnell [7] in 1983. In their attempt to determine a winning strategy for a two-person game, they obtained a characterization of well-covered graphs with girth eight or greater. Later Finbow, Hartnell, and Nowakowski [8] characterized well-covered graphs with girth at least five and then well-covered graphs with no 4- or 5-cycles. Their results are stated below.

Theorem 2.1. [7] *Let G be a graph with girth at least 8. Then G is well-covered if and only if its pendant edges form a perfect matching.*

Theorem 2.2. [8] *Let G be a connected graph of girth at least 5. Then G is well-covered if and only if G is constructed in the following manner: take a collection of disjoint* 5*-cycles and edges, and join them up so that at least one vertex in each original edge still has degree one, and each of the original* 5*-cycles has no two adjacent vertices of degree three or more or G is one of the six exceptions shown in* Figure 2*.*

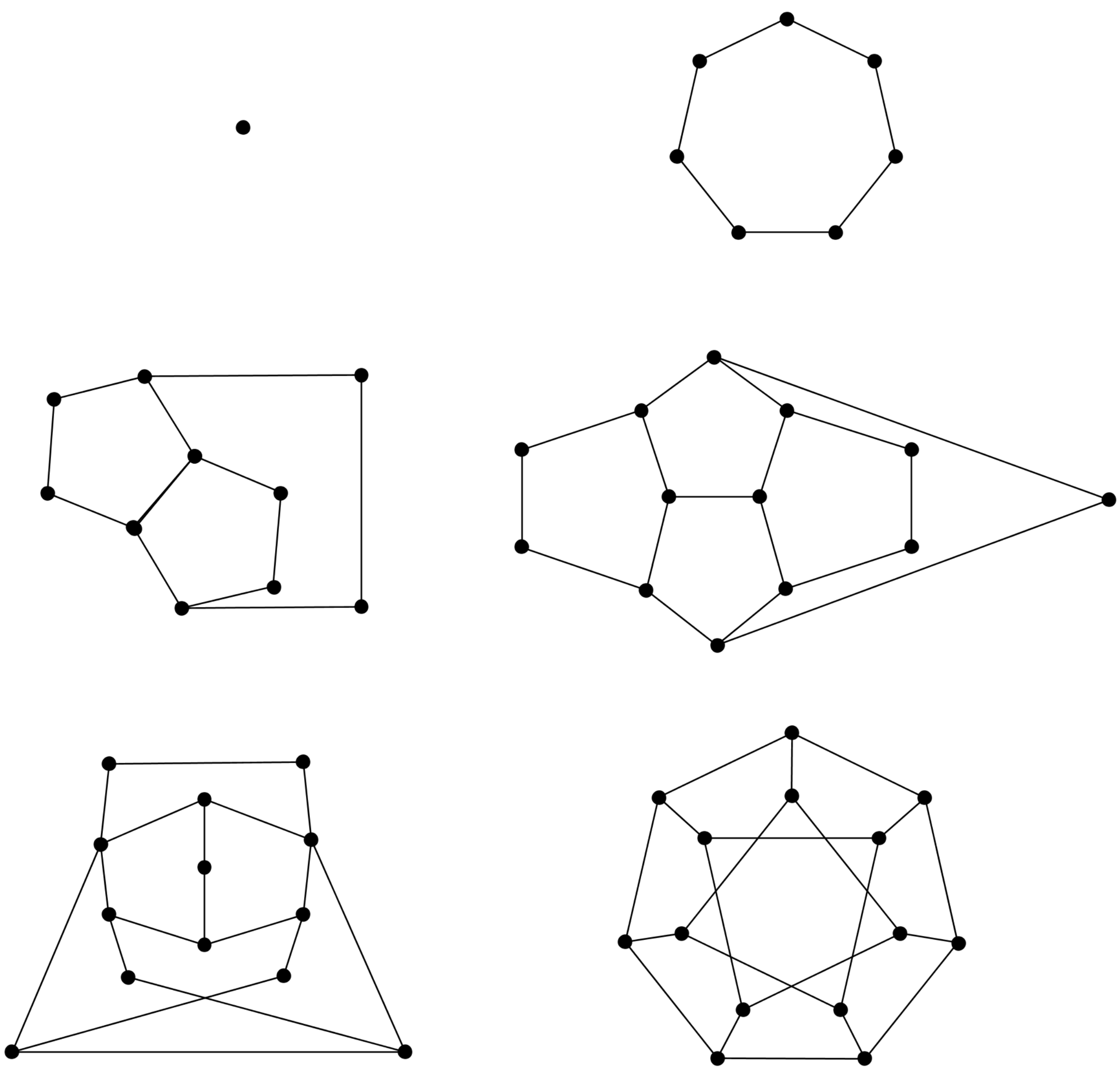


Figure 2: Well-covered graphs with girth at least 5

This corollary to Theorem 2.2 follows when the six aforementioned exceptions are not included in the characterization.

Corollary 2.1. *Let $G \neq K_1$, C_7 have girth at least 6. Then G is well-covered if and only if its pendant edges form a perfect matching.*

Before the next result of Finbow, Hartnell, and Nowakowski it is necessary to define the following family of graphs.

Definition 2.1. Let $\mathcal{F}$ be the family of graphs G such that there exists vertex set $\{v_1, v_2, \ldots, v_k\} \subseteq V(G)$ where for each $i = 1, \ldots, k$, $N[v_i]$ is a complete graph with $|N[v_i]| \leq 3$ and $N[v_1] \cup N[v_2] \cup \ldots \cup N[v_k]$ is a partition of the vertex set of G.

Theorem 2.3. [9] *A graph G containing no 4- or 5-cycles is well-covered if and only if $G \in \mathcal{F}$ or G is one of the two exceptions in* Figure 3.

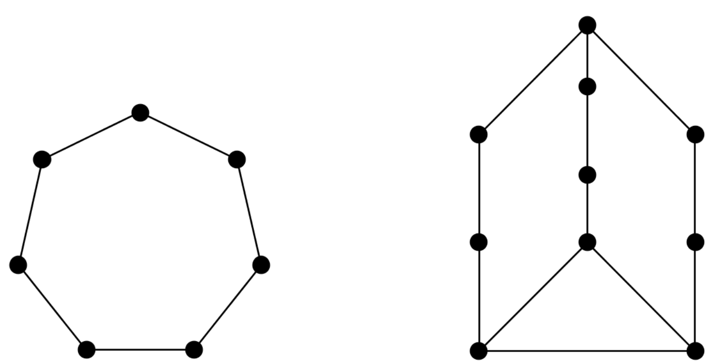


Figure 3: Well-covered graphs with no C_4 or C_5

2.2 Well-coveredness and Degree

The following result is widely used in many proofs involving well-coveredness.

Theorem 2.4. [13] *If* G *is a well-covered graph, then for any* $v \in V(G)$, $G \setminus N[v]$ *is also well-covered.*

The next results deal with graphs that are cubic, i.e., they are 3-regular. The topic of well-covered cubic graphs is relevant here as there are some results in a later chapter of this dissertation about measuring how far a particular class of cubic graphs is from being well-covered and some results on an infinite family of well-covered k-regular graphs.

Plummer and Campbell [4] studied well-covered cubic graphs and in 1988 established the next characterization.

Theorem 2.5. [4] *There are precisely four cubic 3-connected well-covered planar graphs. They are shown in* Figure 4.

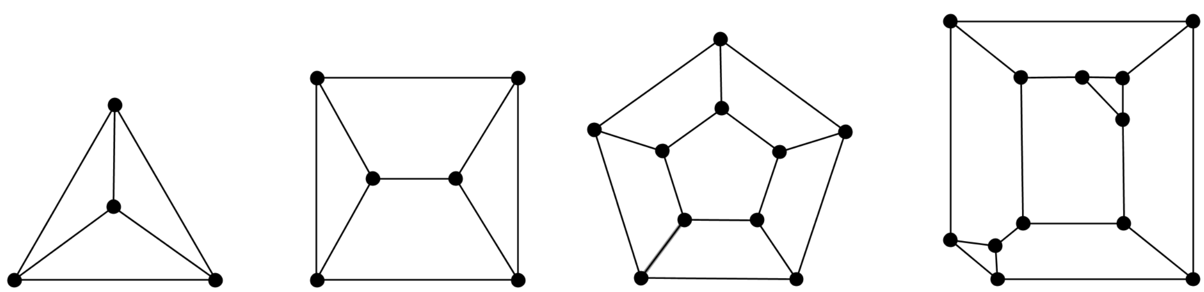


Figure 4: The well-covered cubic planar graphs

In 1993 Campbell, Ellingham, and Royle [3] identified all cubic well-covered graphs. They first constructed an infinite family of well-covered cubic graphs.

Theorem 2.6. [3] *Let* W *denote the class of cubic graphs constructed as follows. Given a collection of copies of* A, B, *and* C *shown in* Figure 5, *join every terminal pair by two edges to a terminal pair in another, possibly the same, so that the result is cubic. Then every graph in* W *is well-covered.*

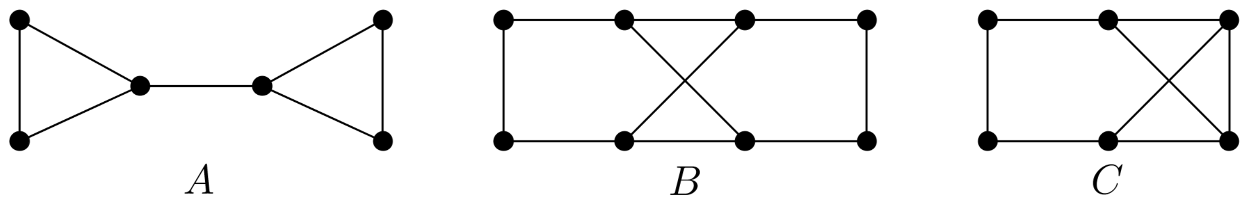


Figure 5: Components of an infinite family of well-covered cubic graphs

Theorem 2.7. [3] *Let G be a connected cubic graph. Then G is well-covered if and only if $G \in W$ or G is one of the graphs in* Figure 6*.*

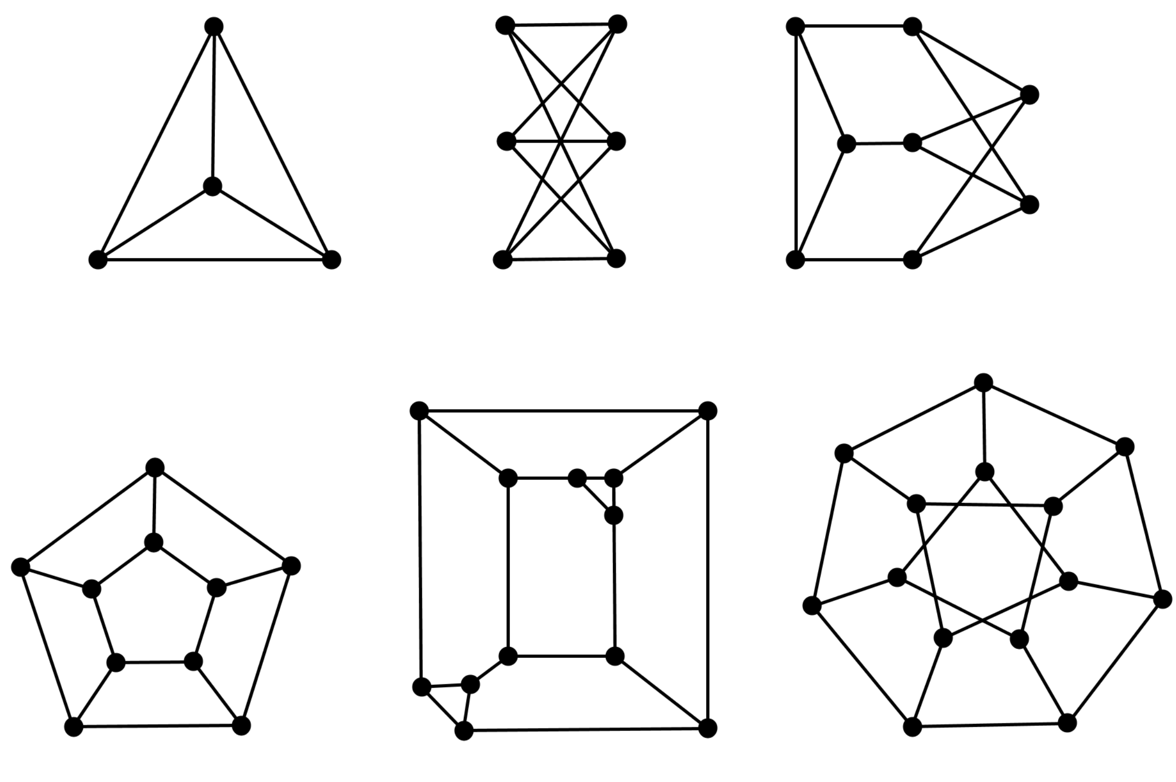


Figure 6: Well-covered cubic graphs

2.3 Well-covered Bipartite Graphs

Ravindra proved the following result about well-covered bipartite graphs. Here G_e denotes the induced subgraph of G on $N(u) \cup N(v)$ where $e = uv$.

Theorem 2.8. [14] *A bipartite graph G without isolated vertices is well-covered if and only if G has a perfect matching M and for every edge $e \in M$, G_e is a complete bipartite graph.*

The following corollary to the above theorem provided the motivation for a main result of this dissertation. The next chapter is devoted to the discussion of this result.

Corollary 2.2. [14] *A tree T is well-covered if and only if T has a perfect matching Γ consisting entirely of pendant edges.*

We continue our discussion of well-covered bipartite graphs with the following definitions.

Definition 2.2. Let G be a graph, and v, w vertices of G. We say that v and w are *clones* if $N(v) = N(w)$.

Definition 2.3. Let G and H be graphs. We say that G is *reduced* if G has no pair of clones. The *reduction* of H is the graph obtained by collapsing clone sets to single vertices.

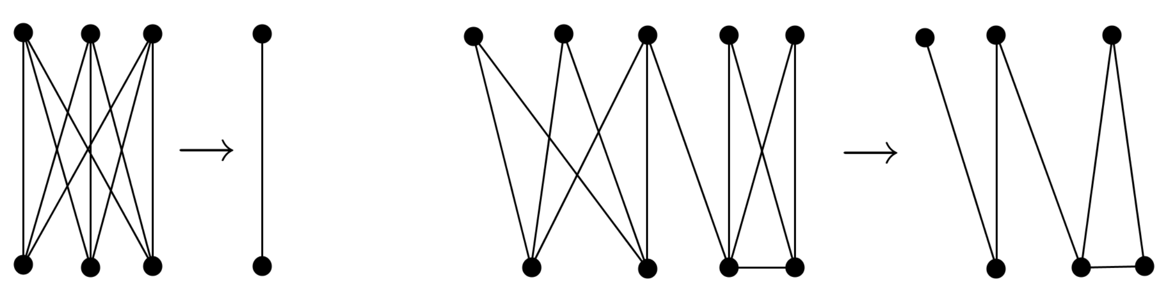


Figure 7: Graphs and reductions

Reduced well-covered bipartite graphs seem to capture the essence of Ravindra's Theorem.

Theorem 2.9. *If G is a reduced, well-covered, bipartite graph, then G has a vertex of degree one.*

Proof. Suppose G is a well-covered, bipartite, and reduced graph. Let v be a vertex of maximum degree, and let w be adjacent to v with edge uv in the perfect matching M. Then the subgraph induced by $N[v] \cup N[w]$ is complete bipartite by Theorem 2.8. If vertex $u \neq v$ is a neighbor of w, then u is adjacent to every neighbor of v; and since $d(v) = \Delta(G)$, u has no other neighbor. So u is a clone of v, which contradicts G being reduced. It follows that vertex w has no neighbor other than v. Hence $d(w) = 1$. □

Theorem 2.10. *The reduction of a well-covered bipartite graph is well-covered.*

Proof. Let G' be the reduction of a well-covered bipartite graph G. Without loss of generality, G' is connected. We shall proceed by induction on the number of vertices. If G' has two vertices then $G' \cong K_2$, which is well-covered. If G' has four vertices then $G' \cong P_4$, which is well-covered. Assume the reduction of a well-covered bipartite graph is well-covered if the reduction has fewer than $2n$ vertices, and let G' have $2n$ vertices. Choose a vertex v of degree one in G', and delete it and its neighbor u. Vertex v in G' represents a set of clone vertices $\{v_1, v_2, \dots v_r\} = V$ in G. Now $G \setminus N[V]$ is a well-covered bipartite graph, and $G' \setminus N[v]$ is its reduction. Furthermore $G' \setminus N[v]$ has fewer than $2n$ vertices, and thus it is well-covered by the induction hypothesis. The vertices in $G' \setminus N[v]$ that are adjacent to vertex u in G', $N_{G' \setminus N[v]}(u)$, are pairwise nonadjacent, so a perfect matching M in $G' \setminus N[v]$ must have a distinct edge for each of these $d(u) - 1$ vertices. Graph G' is built by putting back the edges from $N_{G' \setminus N[v]}(u)$ to u and edge uv. Now in G' there exists a perfect matching $M' = M \cup uv$, and for edge uv the subgraph induced by $N(u) \cup N(v)$ is the complete bipartite graph $K_{1,|N(u)|}$. The other edges of M' satisfy this criterion as they were edges of M. Hence, G' is well-covered. ☐

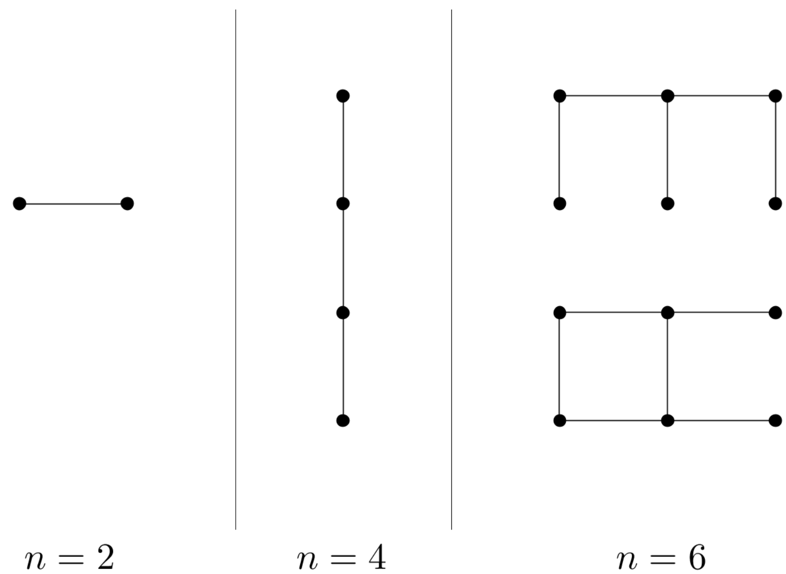


Figure 8: Reduced well-covered bipartite graphs on n vertices

Theorem 2.11. *If G is a reduced well-covered bipartite graph, then G has a unique perfect matching.*

Proof. Suppose that a reduced well-covered bipartite graph G has two perfect matchings, M_1 and M_2. Then there exists an edge $x_1y_1 \in M_1$ such that $x_1y_1 \notin M_2$ and $x_1y_2 \in M_2$ for some $y_2 \in V(G)$. In M_1 vertex y_2 is adjacent to another vertex, say x_2. Since G is a well-covered bipartite graph, $N(x_1) \cup N(y_2)$ is a complete bipartite graph. So every neighbor of x_1 is also a neighbor of x_2, and every neighbor of y_2 is also a neighbor of y_1. Hence, vertices x_1 and x_2 are clones, and vertices y_1 and y_2 are clones. This contradicts G being reduced. ☐

Very well-covered graphs

Definition 2.4. If the size of every maximal independent set of a graph G on n vertices is $\frac{1}{2}n$, then G is said to be *very well-covered*.

The class of very-well covered graphs encompasses the well-covered bipartite graphs. In 1981 Odile Favaron [6] characterized the very well-covered graphs. She showed that the reduction of a very well-covered graph has a unique perfect matching. The method of proof offers an alternative to methods above in the results for reduced well-covered bipartite graphs. The results of Favaron follow this definition attributable to Plummer.

Definition 2.5. [13] Suppose a graph has perfect matching $M = \{u_1v_1, u_1v_2, \ldots, u_nv_n\}$. Then M has property P if

i. no vertex $w \in V(G)$ satisfies $w \sim u_i$ and $w \sim v_i$ for $u_iv_i \in M$ and

ii. no set of two independent vertices $\{x, y\} \subseteq V(G)$ satisfies $x \sim u_i$ and $y \sim v_i$ for $u_iv_i \in M$.

Theorem 2.12. [6] *For a graph G the following statements are equivalent:*

a. G is very well-covered.

b. There exists a perfect matching in G which satisfies the property P.

c. There exists at least one perfect matching in G, and every perfect matching satisfies property P.

Theorem 2.13. [6] *The following statements are equivalent:*

a. G is a very well-covered irreducible graph.

b. G has a perfect matching which satisfies property P, and G does not contain a C_4 with two edges in M.

c. G has a unique perfect matching which satisfies property P.

3 Well-covered k-Trees

In this chapter we investigate k-trees and present a major result of this dissertation, a characterization of well-covered k-trees. The following definition is essential to this characterization.

Definition 3.1. A vertex v in a k-tree T is said to be a *simplicial vertex* if the vertices adjacent to v form a k-clique. If C is a $(k+1)$-clique of T, then C is called a *simplicial $(k+1)$-clique* if C contains a simplicial vertex.

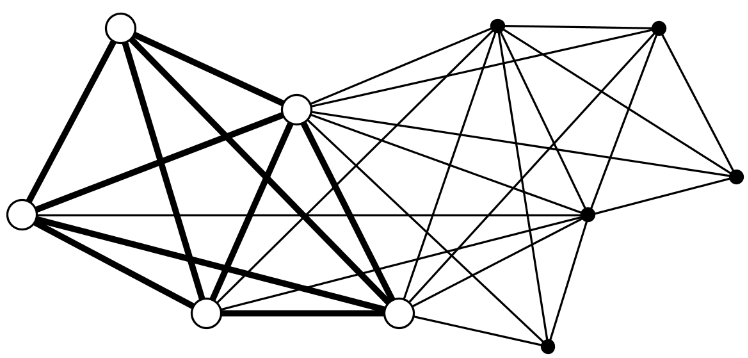


Figure 9: A simplicial 5-clique in a 4-tree

Due to the recursive definition of k-trees, in a k-tree with at least $k+1$ vertices, there is at least one simplicial vertex.

We now state some fundamental facts about k-trees. The ideas presented here are necessary to proving a main result of this research.

Fact 3.1. *If T is a k-tree with n vertices, $n \geq k+1$, then* *(i.)* *the clique number* $\omega(T) = k + 1$ *[15];*

(ii.) *the chromatic number* $\chi(T) = k + 1$*;*

(iii.) *the minimum degree* $\delta(T) = k$*; and*

(iv.) T *has exactly* $kn - \binom{k+1}{2}$ *edges.*

Fact 3.2. *If* T *is a* k*-tree and* v *a simplicial vertex in* T*, then* $T \setminus v$ *is a* k*-tree.*

In a tree, a simplicial 2-clique is a pendant edge, and thus the characterization of well-covered trees given by Ravindra could be restated as follows.

Theorem 3.1. *A tree* T *is well-covered if and only if its vertex set is the disjoint union of simplicial* 2*-cliques.*

When the theorem is formulated this way it is natural to ask whether this case for $k = 1$ can be generalized for any k. To do so the following lemmas are necessary.

Lemma 3.1. *If* T *is a well-covered* k*-tree with* n *vertices,* $n \geq k+1$*, then simplicial cliques are disjoint.*

Proof. Let $v \in C_1 \cap C_2$, for simplicial $(k+1)$-cliques C_1 and C_2 in a well-covered k-tree T; and let v_1 and v_2 be simplicial vertices in C_1 and C_2, respectively. Since for $i = 1, 2$, vertex v_i is adjacent to every vertex in C_i and only those vertices, a maximal independent set I can be built using v_1 and v_2 in addition to some other vertices not in $C_1 \cup C_2$. Since vertex v is adjacent to every vertex in C_1 and C_2, a maximal independent set J can be built using vertex v and perhaps even fewer vertices not in $C_1 \cup C_2$. Hence, $|I| > |J| + 1$. This contradicts the well-coveredness of T. □

Lemma 3.1 clearly states a necessary condition for the proposed generalization.

Lemma 3.2. *If T is a well-covered k-tree with n vertices, $n \geq k+1$, then the independence number $\alpha(T) = \dfrac{n}{k+1}$.*

Proof. Let T be a well-covered k- tree. Since T is $(k+1)$-colorable, $\alpha(T) \geq \dfrac{n}{k+1}$. Let $S_1, S_2, ..., S_{k+1}$ be the $k+1$ color classes of T, and let v be a vertex in one of them. Each vertex in a k-tree is a member of a $(k+1)$-clique. So v has a neighbor in each of the other k color classes, and if added to another color class would violate the independence of that color class. Hence each color class is a maximal independent set, and as such $\alpha(T) = \dfrac{n}{k+1}$. □

Lemma 3.2 is important because as simplicial vertices and their neighbors are successively deleted from a k-tree, a k-tree might not necessarily be obtained at each stage, but the independence numbers are maintained.

Lemma 3.3. *If T is a well-covered k-tree with n vertices, $n \geq k+1$, then T is the union of disjoint $(k+1)$-cliques.*

Proof. Let v be a simplicial vertex in a well-covered k-tree T. Consider $H = T \setminus N[v]$. The subgraph H may not be a k-tree, but it is a chordal graph, and as such H has a simplicial vertex. Let w be a simplicial vertex in H. If $d_H(w) \geq k+1$, then $N[w]$ forms a clique of size $k+2$ or greater, but $\omega(T) = k+1$. If $d_H(w) < k$, then consider $J = H \setminus N[w]$. As a subgraph of a $(k+1)$-colorable graph, J is $(k+1)$-colorable, and J has at least $n - 2k - 1$ vertices. So $\alpha(J) \geq \dfrac{n-2k-1}{k+1}$. The independence number of T is two greater than the independence number of J due to the contribution of vertices v and w. Thus $\alpha(T) \geq \dfrac{n+1}{k+1}$. However, we have just shown that $\alpha(T) = \dfrac{n}{k+1}$. So $d_H(w) = k$, and $N[w]$ is a $(k+1)$-clique disjoint from $N[v]$. This process of deleting a simplicial vertex terminates when only a final simplicial vertex and its necessary k neighbors remain. □

We are now ready to state a main result of the research in this dissertation. It is a characterization of well-covered k-trees that generalizes the result of Ravindra for well-covered trees.

Theorem 3.2. *Let T be a k-tree with $n \geq k+1$ vertices. Then T is well-covered if and only if the vertex set of T is the disjoint union of simplicial $(k+1)$-cliques.*

Proof. Let T be a well-covered k-tree. By Lemma 3, T is the disjoint union of $(k+1)$-cliques. It remains to show that these cliques are indeed simplicial. Let C be a clique of T as described in Lemma 3, and suppose to contradiction that C is not simplicial. There exists a vertex $v \notin C$ adjacent to k of the vertices of C and a vertex $w \in V(C)$ that is not adjacent to v. As C is not simplicial, w is adjacent to a vertex $x \notin V(C)$. There exists a vertex $y \in V(C)$ that is not adjacent to x, for if x were adjacent to every vertex of C then $C \cup \{x\}$ would be a clique of size $k+2$ which is greater than $\omega(T)$. Now suppose that v is not adjacent to x. The set $\{v, x\}$ is adjacent to every vertex in C. So a maximal independent set I can be built avoiding any vertex of C. Thus $|I| \leq \dfrac{n}{k+1} - 1$; this contradicts the well-coveredness of T as $\alpha(T) = \dfrac{n}{k+1}$. So vertex v must be adjacent to vertex x. Now there exists the four-cycle $v - y - w - x - v$ without a chord; a contradiction to the chordality of T. Hence vertex w is a simplicial vertex, and T is the disjoint union of simplicial $(k+1)$-cliques.

□

Well-covered k-trees have a unique decomposition into $\dfrac{n}{k+1}$ simplicial cliques and a unique decomposition into $k+1$ independent sets, i.e., they are uniquely $(k+1)$-colorable. The unique colorability of k-trees will be discussed in Chapter 4.

4 Unique Colorability

In this chapter we discuss unique colorability and introduce a new family of uniquely colorable graphs. The existing literature on unique colorability is vast. The terms "coloring" and "chromatic" refer to the informal notion of coloring vertices so that adjacent vertices have different colors. We begin with a few basic properties of uniquely colorable graphs.

Proposition 4.1. *If a graph G is uniquely r-colorable, then in any r-coloring of G every vertex v of G is adjacent with at least one point of every color different from the color assigned to v.*

Proposition 4.2. *If a graph G is uniquely r-colorable, then $\chi(G) = r$.*

Unique colorability can also be shown in terms of the chromatic polynomial of a graph. A graph G is uniquely r-colorable if and only if the chromatic polynomial $\chi_G(r) = r!$. It is well known that trees T with n vertices have chromatic polynomial

$$\chi_T(\lambda) = \lambda(\lambda - 1)^{n-1}.$$

Hence trees with n vertices have two 2-colorings. Allowing for permutations, this means there is only one 2-coloring. For k-trees the situation is described by the following theorem. Here $\lambda^{\underline{k}}$ is read "λ to the falling power of k" and means k consecutive factors beginning with λ and ending with $\lambda - (k-1)$.

Theorem 4.1. *Let T be a k-tree with n vertices. Then $\chi_T(\lambda) = \lambda^{\underline{k}}(\lambda - k)^{n-k}$*

Proof. If T has $k+1$ vertices, then $T = K_{k+1}$ and

$$\chi_T(\lambda) = \lambda(\lambda-1)\ldots(\lambda-k) = \lambda^{\underline{k+1}} = \lambda^{\underline{k}}(\lambda-k)^{k+1-k}.$$

Suppose k-trees T_n with n vertices have chromatic polynomial

$$\chi_{T_n}(\lambda) = \lambda^{\underline{k}}(\lambda-k)^{n-k}.$$

Let T be a k-tree with $n+1$ vertices. Then T arose from joining a vertex v to a k-clique in the k-tree $T \setminus v$ which has n vertices and chromatic polynomial

$$\chi_{T\setminus v}(\lambda) = \lambda^{\underline{k}}(\lambda-k)^{n-k}.$$

Any coloring of T with λ colors induces one of the $\lambda^{\underline{k}}(\lambda-k)^{n-k}$ colorings of $T \setminus v$. The neighbors of v, a clique, have k of these colors, leaving $(\lambda - k)$ choices for v. So

$$\chi_T(\lambda) = \lambda^{\underline{k}}(\lambda-k)^{n-k}(\lambda-k) = \lambda^{\underline{k}}(\lambda-k)^{n+1-k}.$$

□

The next result is a consequence of a graph being uniquely colorable.

Theorem 4.2. [16] *If graph G with n vertices is uniquely $(k+1)$-colorable, then G has at least $kn - \binom{k+1}{2}$ edges.*

As noted in Chapter 3, k-trees with n vertices have exactly $kn - \binom{k+1}{2}$ edges. Hence k-trees are minimal uniquely $(k+1)$-colorable graphs, minimal in the sense of having fewest edges. It is natural to ask whether these are the only minimal uniquely $(k+1)$-colorable graphs. This question is answered in the following section.

4.1 k-Frames

Definition 4.1. Let k be a positive integer. The smallest *k-frame* is K_{k+1}, which is uniquely $(k+1)$-colorable. If G is a k-frame with n vertices, then a k-frame with $n+1$ vertices is formed by coloring G with $k+1$ colors, choosing vertices $v_1, v_2, \cdots, v_k$ of k distinct colors, and joining a new vertex v to each v_i, $1 \leq i \leq k$.

This definition of a seemingly new class of graphs is suggested by the recursive definition of k-trees and the fact that k-trees are uniquely $(k+1)$-colorable. Every k-tree is a k-frame; the converse is not true. In fact Figure 9 shows a 2-frame that is not a 2-tree.

Fact 4.1. *If G is a k-frame with n vertices, $n \geq k+1$, then*

(i.) the clique number $\omega(G) = k+1$;

(ii.) the chromatic number $\chi(G) = k+1$;

(iii.) the minimum degree $\delta(G) = k$; and

(iv.) G has exactly $kn - \binom{k+1}{2}$ edges.

Fact 4.2. *Let G be a k-frame with n vertices, $n \geq k+1$. Then G is uniquely $(k+1)$-colorable.*

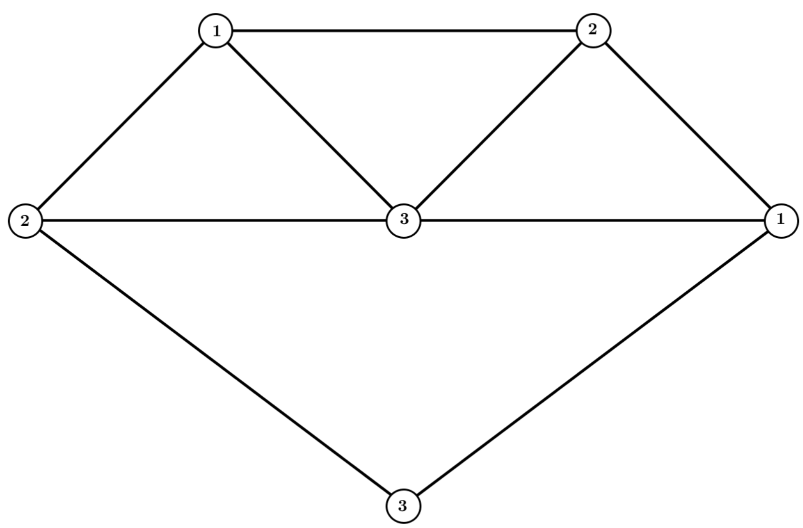


Figure 10: A 2-frame

Proof. If G is a k-frame on $k+1$ vertices, then G is a K_{k+1}, so G is uniquely $(k+1)$-colorable. If k-frames on n vertices are uniquely $(k+1)$-colorable and G is a k-frame on $n+1$ vertices, then G arose by joining a vertex v to vertices of k distinct colors in a k-frame with n vertices, which is uniquely $(k+1)$-colorable. Hence only one color remains for v. □

The k-frames with n vertices have the same number of edges as k-trees with n vertices, and thus, provide a larger class of minimal uniquely $(k+1)$-colorable graphs.

Uniquely colorable planar graphs have been studied by Fowler [10]. Informally a graph is said to be planar if it can be drawn in a plane with no two edges "crossing" each other. Fowler showed that the uniquely 4-colorable planar graphs are precisely the planar 3-trees [10]. It is natural to ask whether a similar property holds for the class of uniquely 3-colorable planar graphs. Every 2-tree is planar and uniquely 3-colorable, but no theorem comparable to Fowler's is possible for $\chi = 3$ because the class of 2-frames provides examples for every $n \geq 6$ of planar uniquely 3-colorable graphs on n vertices which are not 2-trees. The graph in Figure 10 is the smallest such graph.

4.2 Well-covered k-Frames

It is natural to ask whether there are well-covered k-frames other than the well-covered k-trees characterized in Chapter 3, and if so, whether these can be characterized. In fact, the smallest candidate for a well-covered proper k-frame would have $2k+2$ vertices. There are proper k-frames for every $n \geq k+4$ vertices, but n must be a multiple of $k+1$ in order for a k-frame to be well-covered. Construction is as follows.

Let $\{v_1, v_2, \ldots, v_{k+1}\}$ be a $(k+1)$-clique. Now $\forall\ i = 1, \ldots, k+1$ adjoin vertex w_i likewise:

a. $\forall\ i \neq k$, w_i is adjacent to v_j for all $j > i$ and w_i is adjacent to w_j for all $j < i$;

b. w_k is adjacent to v_i for all $i \neq k$.

It is evident that the resulting graph W_{k+1} is a k-frame since each vertex was adjoined to k vertices of k distinct colors. To see that W_{k+1} is not a k-tree, note that vertex w_{k+1}, when adjoined, is not joined to a k-clique since $w_1 \nsim w_k$. To see that W_{k+1} is well-covered, note that it is the union of a $(k+1)$-clique, a k-clique, and $\{w_1\}$. Hence any independent set of more than two vertices must contain vertex w_1. Similarly, any independent set of more than two vertices must contain vertex w_k. But w_1 and w_k dominate the $(k+1)$-clique consisting of the v_i. So W_{k+1} is a well-covered k-frame which is not a k-tree. Figure 11 shows a W_3 and a W_4.

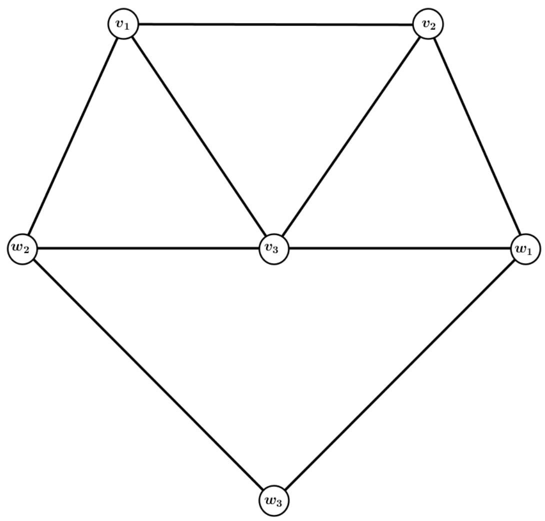
We now show that if $n \geq k+1$ is a multiple of $k+1$, then there is a proper k-frame on n vertices which is well-covered.

Definition 4.2. If W_{k+1} is a subgraph of a k-frame G in such a way that vertices w_{k+1} and v_{k+1} in the construction above have, respectively, degrees k and $2k$ in G, then W_{k+1} will be called a *simple* W_{k+1}.

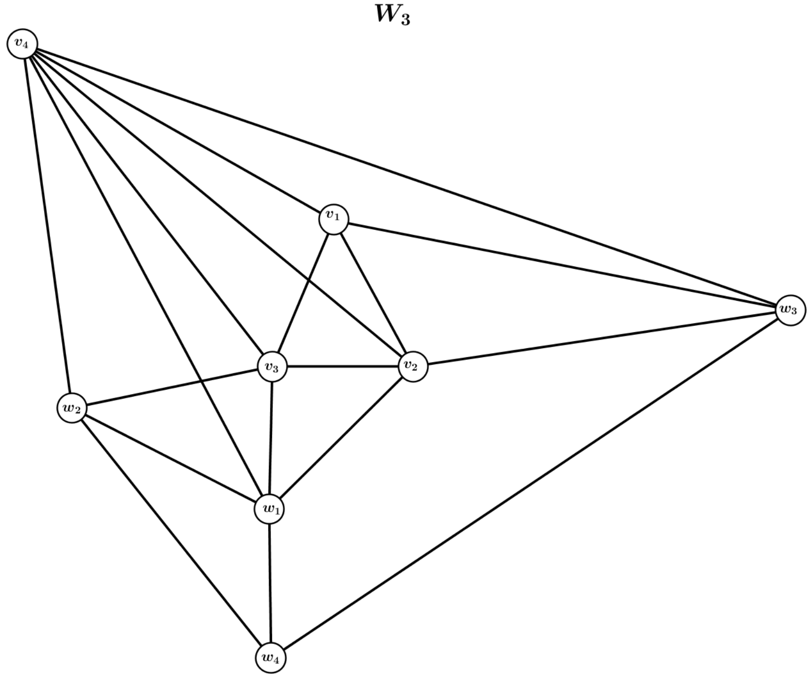
Theorem 4.3. *Let G be a well-covered k-frame with n vertices, and suppose there exists simplicial clique $C = \{v_1, v_2, \ldots, v_{k+1}\}$ with v_{k+1} the simplicial vertex. Adjoin vertices w_1, w_2, ..., w_{k+1} to build C into W_{k+1} as in the construction above. The resulting graph G' is a well covered k-frame with $n+k+1$ vertices.*

Proof. If a maximal independent set of G' contains a vertex of C other than v_k it must not contain w_k, and hence must contain exactly one of the w_i's, since the w_i's other than w_k are a clique. If a maximal independent set contains no vertex of C, it clearly contains at most two w_i's. The vertices v_{k+1}, w_1, w_2, ..., w_{k+1} are not dominated externally. Only w_k and w_{k+1} dominate w_k, so one of these must be chosen. If w_{k+1} is chosen, then v_{k+1} is not dominated.

If w_k is chosen, then w_{k-1} is not dominated. It follows that every maximal independent set contains exactly two vertices from W_{k+1}, and G' is well-covered. $\square$



$\boldsymbol{W_3}$



$\boldsymbol{W_4}$

Figure 11: Well-covered k-frames

It is reasonable to ask whether the graphs obtained from the construction are in fact the only well-covered k-frames. More precisely:

Question: Is it true that a k-frame G is well-covered if and only if its vertex set is a disjoint union of simplicial $(k+1)$-cliques and graphs that are isomorphic to simple W_{k+1}'s?

5 Covering Range

The idea of measuring the degree to which a graph fails to be well-covered led to the defining of what seems to be a new parameter.

Definition 5.1. Let G be a graph. Define the *covering range* of G, $CR(G)$), to be max $\{|I| - |J| : I, J$ are maximal independent sets in $G\}$ and the *covering spectrum* of G, $CS(G)$, to be the set $\{|I| : I$ is a maximal independent set in $G\}$.

By the definition above G is well-covered if and only if $CR(G) = 0$ if and only if $CS(G)$ is a singleton.

The next results show bounds on $CR(G)$ in terms of degree and precise values for $CR(G)$ for certain classes of graphs.

Theorem 5.1. *Let G be a connected graph.*

i. If G has n vertices, $n \geq 2$, then $CR(G) \leq n-2$. The $CR(G) = n-2$ if and only if $G = K_{1,n-1}$.

ii. If G is k-regular and connected, then $CR(G) \leq \left(\frac{k-1}{2(k+1)}\right) n$.

iii. For $n \equiv 0 \bmod 8$, there exist cubic graphs G with n vertices and $CR(G) = \frac{n}{4}$.

Proof. i. If $CR(G) = n$, then G has a maximal independent set of size 0. If $CR(G) = n-1$, then there exist a vertex which is adjacent to every other vertex and a set of n isolated vertices. Hence $CR(G) \leq n-2$. The following statements are equivalent:

$CR(G) = n - 2$; for maximal independents I and J set in G, $|I| = 1$ and $|J| = n - 1$; and $G = K_{1,n-1}$.

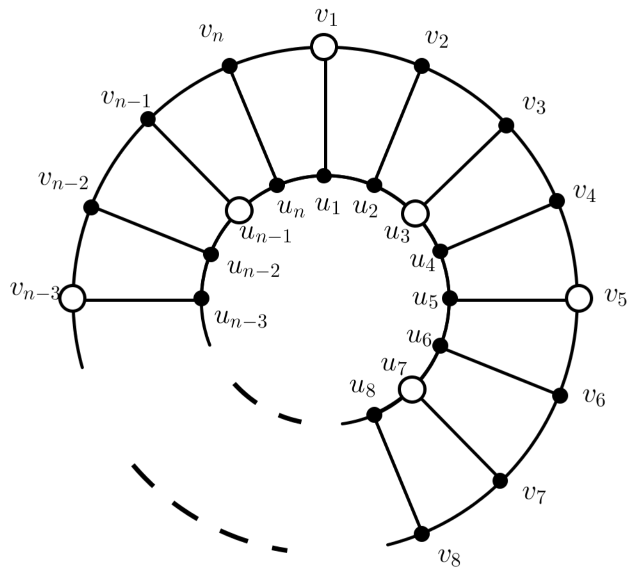
ii. For a k-regular graph G, a vertex v can cover at most $k+1$ vertices. Thus a smallest possible maximal independent set could have cardinality $\frac{n}{k+1}$. A largest maximal independent set must have cardinality at most $\frac{n}{2}$. If a maximum independent set I were larger than $\frac{n}{2}$, then the vertices in the complement of I would have degrees larger than degrees of the vertices of I. Hence

$$CR(G) \leq \frac{n}{2} - \frac{n}{k+1} = \left(\frac{k-1}{2(k+1)}\right) n.$$

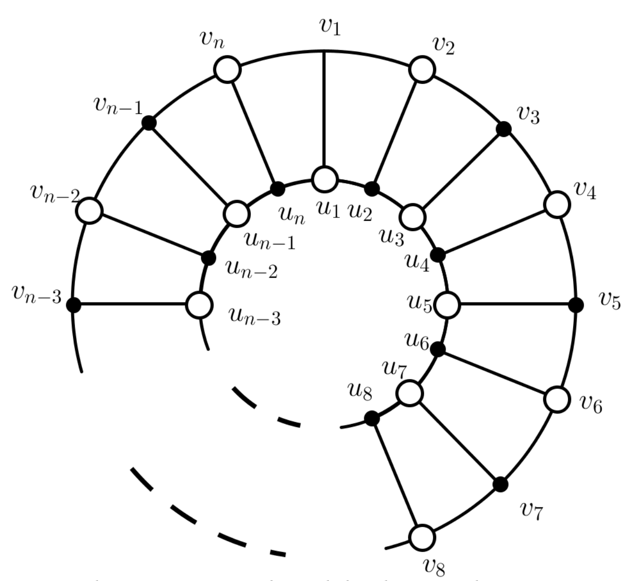
iii. Denote by $CY(n)$ the class of cubic graphs called the cylinders which have vertex set $\{v_1, v_2, \ldots, v_n\} \cup \{u_1, u_2, \ldots, u_n\}$ and edge set $\{v_i \sim v_{i+1}\} \cup \{v_i \sim u_i\} \cup \{u_i \sim u_{i+1}\}$ where subscript addition is modulo n. Let G be $CY(n)$ for $n \equiv 0$ modulo 4. In this case $|V(G)| \equiv 0$ modulo 8. A smallest maximal independent set of G can be built with vertices $\{v_i\colon\ i \equiv 1 \bmod 4\}$ and vertices $\{u_i\colon\ i \equiv 3 \bmod 4\}$. The set $\{v_i\colon\ i \equiv 1 \bmod 4\}$ is a set of pairwise nonadjacent vertices that cover themselves, all of the even indexed vertices of the cycle induced by the v_i's, and all of the u_i's for $i \equiv 1$ modulo 4. The set $\{u_i\colon\ i \equiv 3 \bmod 4\}$ cover the remaining vertices with no duplication for a smallest maximal independent set of size $\frac{|V(G)|}{4}$. A largest maximal independent set with size $\frac{|V(G)|}{2}$ is built with the even indexed v_i's and the odd indexed u_i's. So $CR(G) = \frac{|V(G)|}{2} - \frac{|V(G)|}{4} = \frac{|V(G)|}{4}$.

□

A largest and a smallest maximal independent set of $CY(n)$, $|V(CY(n))| \equiv 0$ modulo 8, are shown in Figure 12.



A smallest maximal independent set



A largest maximal independent set

Figure 12: Cylinder with $|V(G)| \equiv 0 \bmod 8$

Existing literature identifies all well-covered cubic graphs. We now show results for the covering range of cylinders, noting when they are well-covered.

Theorem 5.2. *Let graph $G = CY(n)$. Then*

i. $CR(G) = \dfrac{n}{2}$ *for* $n \equiv 0 \mod 4$*;*

ii. $CR(G) = \dfrac{n-5}{2}$ *for* $n \equiv 1 \mod 4$*;*

iii. $CR(G) = \dfrac{n-2}{2}$ *for* $n \equiv 2 \mod 4$*; and*

iv. $CR(G) = \dfrac{n-3}{2}$ *for* $n \equiv 3 \mod 4$*.*

Proof. Index the vertices modulo 4. A smallest and a largest maximal independent set will be found in a manner similar to the previous result. Note that $CS(G)$ is comprised of even numbers only.

i. $n \equiv 0$ mod 4: The result has previously been shown.

ii. $n \equiv 1$ mod 4: A largest maximal independent set I is built with vertex set $\{v_i : i \equiv 0, 2\} \cup \{u_i : i \equiv 1, 3 \text{ and } i \neq n\}$; so

$$|I| = \frac{n-1}{2} + \frac{n-1}{2} = n - 1.$$

A smallest maximal independent set J is built with vertices $\{v_i : i \equiv 1, i \neq n\} \cup \{u_i : i \equiv 3\} \cup \{u_n, v_{n-1}\}$. So

$$|J| = \frac{n-1}{4} + \frac{n-1}{4} + 2 = \frac{n+3}{2}.$$

The maximal independent sets I and J are best possible because of the bounds on smallest and largest maximal independent sets. Thus

$$CR(G) = |I| - |J| = (n-1) - \frac{n+3}{2} = \frac{n-5}{2} \quad \text{for} \quad n \equiv 1 \mod 4.$$

iii. $n \equiv 2 \bmod 4$: A largest maximal independent set I is built with vertex set $\{v_i : i \equiv 0, 2\} \cup \{u_i : i \equiv 1, 3\}$; so

$$|I| = n.$$

A smallest maximal independent set J is built with vertices $\{v_i : i \equiv 1, i \neq n-1\} \cup \{u_i : i \equiv 3\} \cup \{u_{n-1}, v_{n-2}\}$. So

$$|J| = \frac{n-2}{4} + \frac{n-2}{4} + 2 = \frac{n+2}{2}.$$

This is best possible because a smaller maximal independent set would have cardinality less than $\frac{|V(G)|}{4}$. Hence

$$CR(G) = |I| - |J| = n - \frac{n+2}{2} = \frac{n-2}{2} \quad \text{for} \quad n \equiv 2 \mod 4.$$

iv. $n \equiv 3 \bmod 4$: A largest maximal independent set I is built with vertex set $\{v_i : i \equiv 0, 2\} \cup \{u_i : i \equiv 1, 3 \text{ and } i \neq n\}$; so

$$|I| = \frac{n-1}{2} + \frac{n-1}{2} = n - 1.$$

A smallest maximal independent set J is built with vertices $\{v_i : i \equiv 1\} \cup \{u_i : i \equiv 3\}$. So

$$|J| = \frac{n-3}{4} + 1 + \frac{n-3}{4} + 1 = \frac{n+1}{2}.$$

The maximal independent sets I and J are best possible because of the bounds on smallest and largest maximal independent sets. Thus

$$CR(G) = |I| - |J| = (n-1) - \frac{n+1}{2} = \frac{n-3}{2} \quad \text{for} \quad n \equiv 3 \mod 4.$$

□

From the result above it is easy to identify the well-covered cylinders. The graphs $CY(3)$ and $CY(5)$ are well-covered and were identified in the well-covered cubic graphs characterizations in Chapter 2.

The cylinders are a subclass of a larger family of cubic graphs defined below.

5.1 Generalized Petersen Graphs

Definition 5.2. Let $n \geq 3$ and $1 \leq k \leq \frac{n}{2}$ with k relatively prime to n. Then the *Generalized Petersen Graph* $P(n,k)$ has vertex set $\{v_1, v_2, \cdots, v_n\} \cup \{u_1, w_2, \cdots, u_n\}$ and edge set $\{v_i \sim v_{i+1}\} \cup \{v_i \sim u_i\} \cup \{u_i \sim u_{i+k}\}$, $\forall i$, where subscript addition is modulo n.

Figure 13 shows the Classical Petersen Graph, $P(5,2)$, and $P(6,1)$. Note that $P(n,1) = CY(n)$.

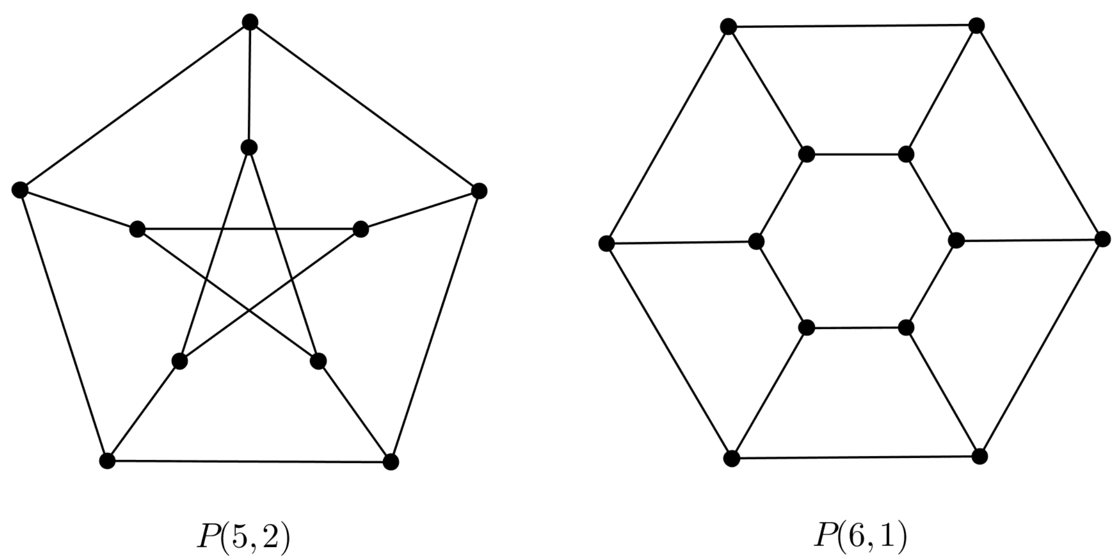


Figure 13: Generalized Petersen Graphs

It is routine to show $CR(P(5,2)) = 1$. I believe determining $CR(P(n,k))$ to be a substantial question. For n even the question is fully answered by the following theorem.

Theorem 5.3. *Let $G = P(n,k)$ and k be odd. Then*

i. $CR(G) = \dfrac{n}{2}$ *for $n \equiv 0$ mod 4 and*

ii. $CR(G) = \dfrac{n-2}{2}$ *for $n \equiv 2$ mod 4.*

Proof. Index the vertices modulo 4.

i. $n \equiv 0$ mod 4: A largest maximal independent set I can be constructed with vertices $\{v_i : i \equiv 1,3 \mod 4\} \cup \{u_i : i \equiv 2,4 \mod 4\}$. Since each v_i is only adjacent to v_{i+1}, v_{i-1}, and u_i and each u_i is only adjacent to v_i, u_{i+k}, and u_{i-k}, the vertices chosen to construct I are pairwise nonadjacent. Furthermore, the vertices of I are collectively adjacent to every vertex not in I. Lastly,

$$|I| = \frac{n}{2} + \frac{n}{2} = n.$$

Now for a smallest maximal independent set J choose vertices $\{v_i : i \equiv 1 \mod 4\} \cup \{u_i : i \equiv 3 \mod 4\}$. This set is pairwise nonadjacent, and all even index vertices are covered by J and all odd indexed vertice are either chosen as elements of J or covered by J. So

$$|J| = \frac{n}{4} + \frac{n}{4} = \frac{n}{2}.$$

Thus

$$CR(G) = |I| - |J| = n - \frac{n}{2} = \frac{n}{2} \quad \text{for} \quad n \equiv 0 \mod 4.$$

ii. $n \equiv 2$ mod 4: To construct a largest maximal independent set I use vertex set $\{v_i : i \equiv 1,3 \mod 4\} \cup \{u_i : i \equiv 2,4 \mod 4\}$. Since all even indexed v_i's are chosen, the cycle induced by the v_i's is completely covered. Likewise, the cycle induced by the u_i's is covered by the odd indexed u_i's. Since even indexed v_i's are only adjacent to odd indexed v_i's and odd indexed u_i's are only adjacent to even indexed u_i's the vertices of I are pairwise nonadjacent. Now

$$|I| = \frac{n}{2} + \frac{n}{2} = n.$$

A smallest maximal independent set J is built with vertices $\{v_i : i \equiv 1 \mod 4\} \cup \{u_i : i \equiv 3 \mod 4\}$. This set is pairwise nonadjacent, and all even index vertices are covered by J and all odd indexed vertice are either chosen as elements of J or covered by J. So

$$|J| = \frac{n-2}{4} + \frac{n-2}{4} + 2 = \frac{n+2}{2}.$$

Thus

$$CR(G) = |I| - |J| = n - \frac{n+2}{2} = \frac{n-2}{2} \quad \text{for} \quad n \equiv 2 \mod 4.$$

□

I have investigated $CR(P(n,2))$ for small values of n and have found the following to be true.

- $CR(P(7,2)) = 0$.
- $CR(P(5,2)) = CR(P(9,2)) = CR(P(11,2)) = 1$.
- $CR(P(13,2)) = CR(P(15,2)) = 2$.

Note that the graph $P(7,2)$ is well-covered and was identified in Chapter 2 as a well-covered cubic graph.

5.2 k-Graphs

Studying covering range and well-covered graphs led to interesting examples of k-regular graphs whose vertex sets are disjoint unions of k-cliques.

Definition 5.3. Let $k \geq 2$. A connected graph G is said to be a *k-graph* if G is k-regular and the vertices of G can be partitioned into disjoint k-cliques.

The graphs in Figure 14 are a 2-graph, a 3-graph, and a 4-graph, respectively. Each of the graphs pictured is well-covered. Theorem 5.4 characterizes completely the well-covered k-graphs.

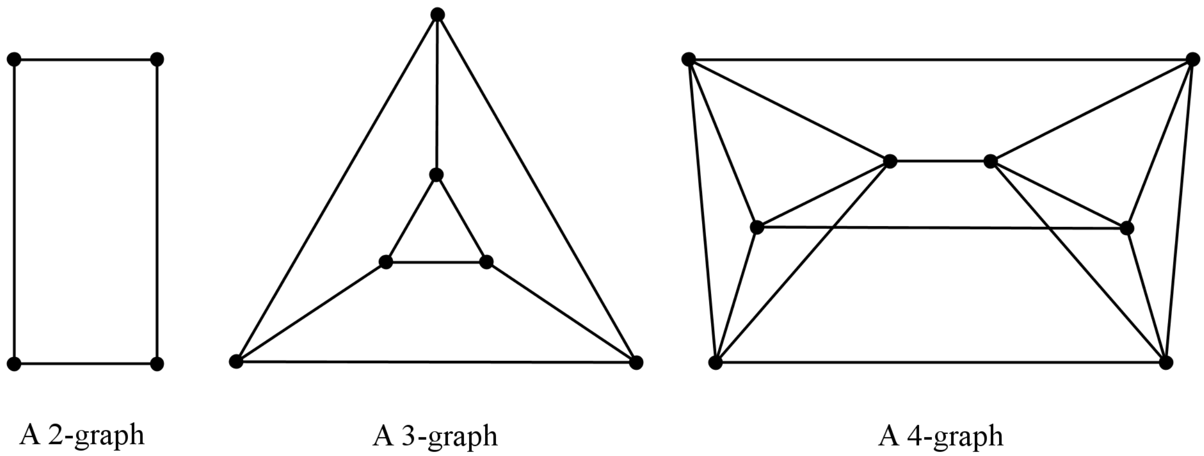


Figure 14: k-Graphs

Theorem 5.4. *Let G be a k-graph with $k \geq 3$. Then G is well-covered if and only if for each k-clique C of G, the vertices of C have neighbors in at most $k-1$ of the k-cliques other than C.*

Proof. Suppose G is a well-covered k-graph. If the condition of the theorem is not satisfied, then each vertex v_i of a k-clique C has one neighbor u_i in a k-clique $C_i \neq C$ for $i = 1, \ldots, k$. So a maximal independent set external to C can be built. This provides a "small" independent set; a "large" independent set is provided by Brooks' Theorem.

Suppose the condition is satisfied and that I is a maximal independent set with fewer than $\frac{n}{k}$ vertices. Then one of the $\frac{n}{k}$ disjoint k-cliques, say C, contains no vertex of I. But the vertices of C have neighbors in at most $k-1$ of those k-cliques, each of which can contain at most one vertex of I. Each element of I can have at most one neighbor in C, so I can be adjacent to at most $k-1$ vertices of C, contradicting the maximality of I. $\square$

The well-covered k-graphs provide a large class of well-covered graphs with a high degree of symmetry. In fact, all k-graphs with fewer than $k^2 + k$ vertices are well-covered. Theorem 5.5 shows the construction of an infinite family of well-covered k-graphs.

Theorem 5.5. *Let $k \geq 3$ and $m \geq 2$. Then*

(i.) there exists a connected well-covered k-graph on $2mk$ vertices and

(ii.) if k is even, there exists a connected well-covered k-graph on mk vertices.

Proof. (*i.*) Let H have vertices $\{v_1, v_2, \ldots, v_k\} \cup \{w_1, w_2, \ldots, w_k\}$, where the v_i and w_i induce, respectively, k-cliques, and where $v_i \sim w_i$ for $i \neq 1$. So H has $2k - 2$ vertices of degree k and 2 vertices of degree $k - 1$. Let H_i be a copy of H for $1 \leq i \leq m$, and let x_i and y_i be the vertices of degree $k - 1$ in H_i. Now, take the disjoint union of the H_i and add edges $x_1 \sim x_2$, $y_2 \sim y_3$, $x_3 \sim x_4$, $\ldots$ where the last edge added may be either $x_m \sim y_1$ or $y_m \sim y_1$. The resulting graph G is a k-graph on $2mk$ vertices and each k-clique has neighbors in exactly 2 other k-cliques. Hence by Theorem 5.4, G is well-covered if $k \geq 3$.
(*ii.*) If $k = 2r$ is even, let H_1, H_2, $\ldots$, H_m be k-cliques with $H_i = V_i \cup W_i$ where V_i and W_i are r-cliques. Now, take the disjoint union of H_i and adjoin

(a) a matching between V_i and V_{i+1} if i is odd and $i < m$,

(b) a matching between W_i and W_{i+1} if i is even and $i < m$, and

(c) a matching between W_1 and the unmatched k-clique in H_m.

The resulting graph G is a k-graph on mk vertices, and each k-clique has neighbors in only 2 other k-cliques, so again G is well-covered by Theorem 5.4. □

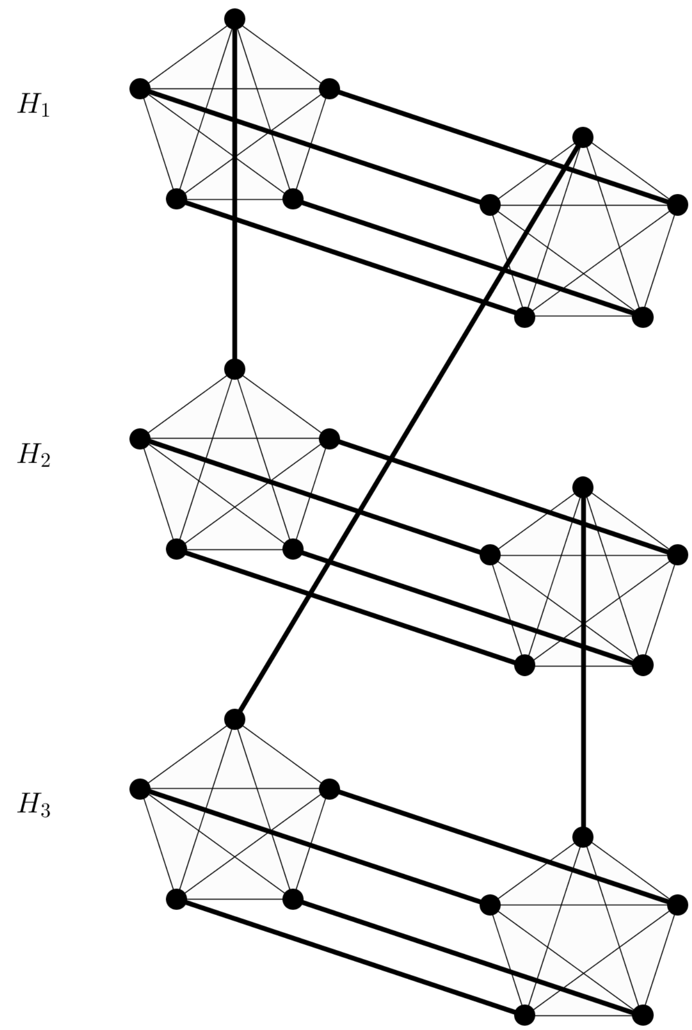


Figure 15: Construction (i) of a well-covered 5-graph, m=3

Figures 15 and 16 depict the constructions of well-covered k-graphs for case (i) and case (ii) of Theorem 5.5.

Not every k-graph is well-covered. For example, each even cycle C_{2n} is a 2-graph, but only C_4 is well-covered. Figure 17 shows the smallest 3-graph which fails to be well-covered. The covering range for k-graphs tends to be narrow, especially for large k.

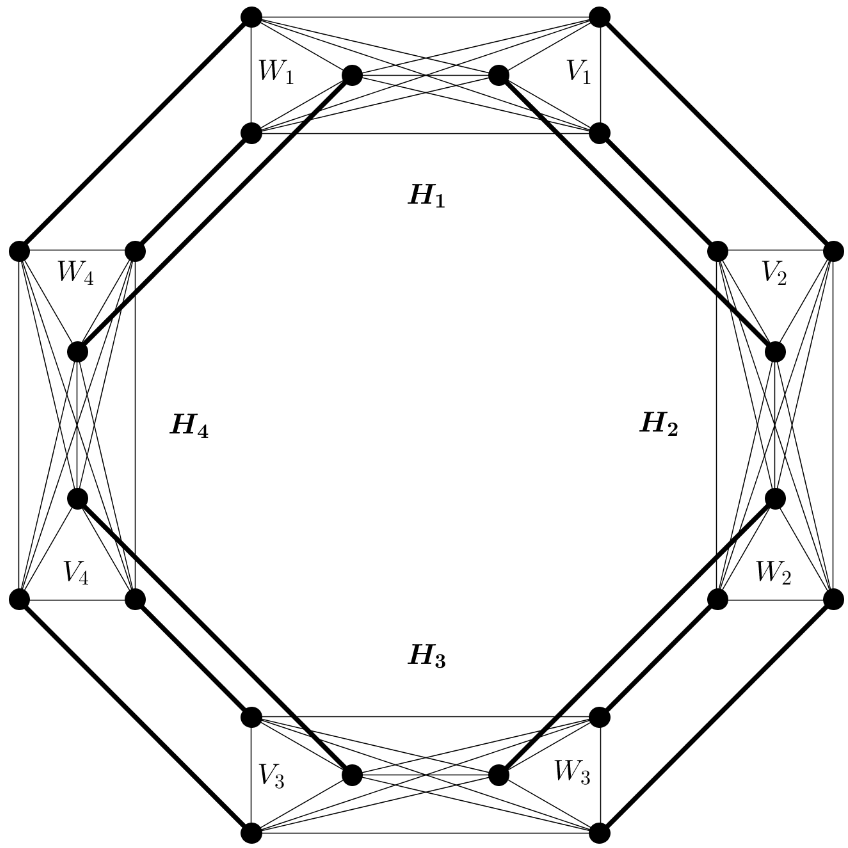


Figure 16: Construction (*ii*) of a well-covered 6-graph, m=4

Theorem 5.6. *Let G be a k-graph with n vertices. Then* $CR(G) \leq \dfrac{n}{k^2+k}$.

Proof. Since a maximal independent set may contain at most one vertex in each k-clique of the disjoint collection of k-cliques, every maximal independent set has at most $\dfrac{n}{k}$ vertices. For a fixed maximal independent set I, let us say that a clique containing a vertex of I is good, otherwise bad. Let C be a bad k-clique. Then each vertex of C is adjacent to some vertex of I, so C is "adjacent" to k other k-cliques, each of which is good. Hence each bad

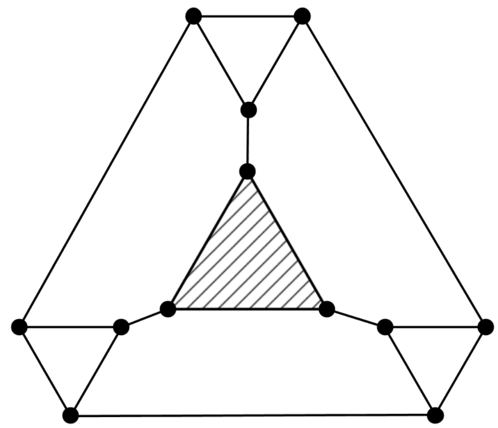


Figure 17: A "bad" 3-clique in a 3-graph

k-clique determines k good k-cliques. There is no duplication, since a vertex can have only one neighbor outside its k-clique. If there are g good and b bad vertices, we have $kb \leq g$. Now

$$|I| = \frac{n}{k} - b = g.$$

Thus

$$g \geq \frac{n}{k} - \frac{g}{k}$$

$$g(1 + \frac{1}{k}) \geq \frac{n}{k}$$

$$g \geq \frac{n}{k} \bullet \frac{k}{k+1} = \frac{n}{k+1}.$$

It follows that

$$\frac{n}{k+1} \leq |I|.$$

So from above,

$$\frac{n}{k+1} \leq |I| \leq \frac{n}{k}.$$

Subtracting yields

$$CR(G) \leq \frac{n}{k^2 + k}.$$

□

An infinite family of k-graphs with the maximum covering range can be constructed in the following manner.

(i.) Let H be $k+1$ k-cliques, $C_1, \ldots, C_{k+1}$, such that cliques C_i for $2 \leq i \leq k$ have neighbors in every other k-clique. Now there is one vertex v in C_1 with $d(v) = k - 1$, and there is one vertex w in C_{k+1} with $d(w) = k - 1$. The degree of all other vertices is k.

(ii.) Let H_i be a copy of H for $1 \leq i \leq m$, and let v_i and w_i be the vertices of degree $k - 1$ in H_i.

(iii.) To this disjoint union of the H_i's add edges $v_i \sim v_{i+1}$ if i is odd and edges $w_i \sim w_{i+1}$ if i is even with the last edge being either $w_1 \sim w_m$ or $w_1 \sim v_m$. Now each vertex has degree k.

The resulting graph G is a k-graph with $m(k^2 + k)$ vertices. This construction is illustrated in Figure 18.

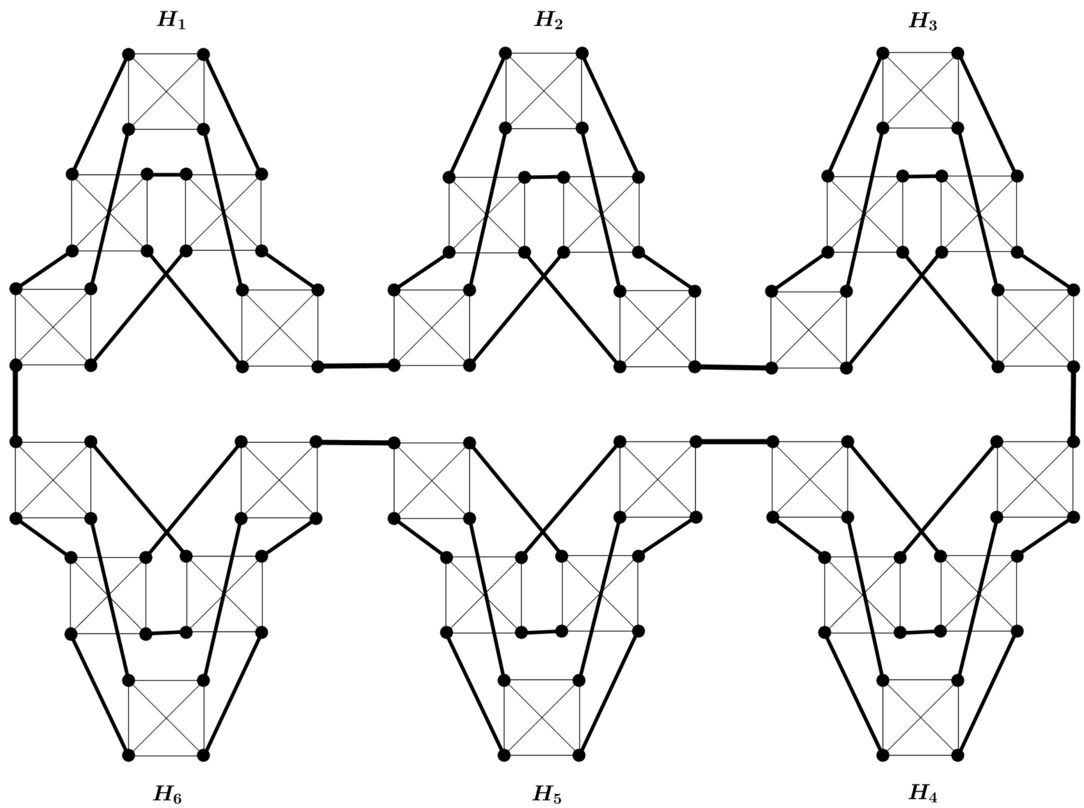


Figure 18: Construction of 4-graph, m=6

6 Remaining Questions

In Chapter 3 a characterization of well-covered k-trees was given, and in Chapter 4 the k-frames were introduced as a larger class of graphs that encompasses the k-trees. The characterization of well-covered k-frames other than k-trees remains an open question. Since not all k-frames are chordal, a similar proof would not work. A construction of well-covered k-frames has been provided, but the question of whether that construction yields all of the well-covered k-frames is not settled.

Conjecture. *Let G be a k-frame. Then G is well-covered if and only if its vertex set is the disjoint union of simplicial $(k+1)$-cliques and simple W_{k+1}'s.*

Existing literature identifies all of the well-covered cubic graphs. Since no characterization of k-regular well-covered graphs for $k \geq 4$ is known, it is natural to wonder whether these graphs or any subclass of these graphs can be identified. The k-graphs offer a large class of k-regular well-covered graphs. Are there other large classes of well-covered k-regular graphs? In identifying all well-covered cubic graphs, Campbell, Ellingham, and Royle [3] introduced a construction of an infinite family of well-covered cubic graphs using three components. One of the components, shown in Figure 19, produces an infinite family of well-covered 3-graphs when adjoined in the manner as described in their result. Is it possible to prove a similar result for $k = 4$ using the construction of well-covered k-graphs and the characterization for well-covered k-graphs?

A well-covered graph is 1-well-covered if and only if the deletion of any vertex from the

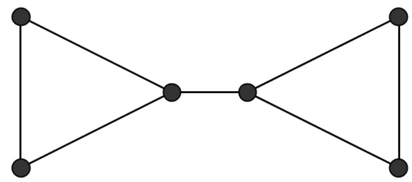


Figure 19: Component A

vertex set of the graph is also well-covered. In 1993 Pinter [12] identified the only 3-connected 4-regular planar 1-well-covered graph; it is depicted in Figure 20. Similar questions might be proposed for degrees 3 and 5.

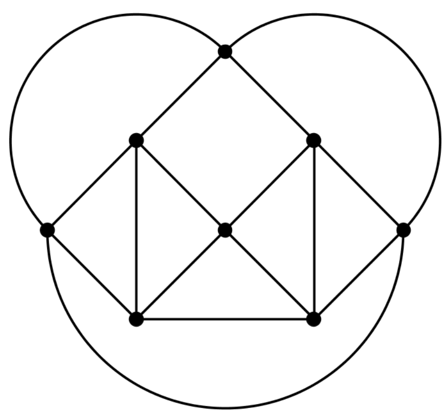


Figure 20: The 3-connected 4-regular planar 1-well-covered graph

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VITA

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