1-1-2015

Diagonals of Tensor Products of Banach Lattices with Bases.

Byunghoon Lee
University of Mississippi

DIAGONALS OF TENSOR PRODUCTS OF BANACH LATTICES WITH BASES

A Dissertation
presented in partial fulfillment of requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
The University of Mississippi

by

BYUNGHOON LEE

May 2015

ABSTRACT

In this dissertation, we investigate diagonals of tensor products of Banach lattices with bases. We first consider questions on the positive tensor products of ℓ_p-spaces. We characterize the main diagonals of the positive projective tensor product $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ and the positive injective tensor product $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$. Then, by using these two main diagonals, we characterize the reflexivity, the property of being Kantorovich-Banach spaces, and the property of being order continuous of $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ and $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$, as well as the space of all regular n-linear forms on $\ell_{p_1}\times\cdots\times\ell_{p_n}$ and the space of all regular n-homogeneous polynomials on ℓ_p ($1\leqslant p<\infty$). We also obtain the following interesting result. Let $1<p_1,\cdots,p_n,q<\infty$. Then the Banach lattice $\mathcal{L}^r(\ell_{p_1},\cdots,\ell_{p_n};\ell_q)$, the space of all regular n-linear operators from $\ell_{p_1}\times\cdots\times\ell_{p_n}$ to ℓ_q, is order continuous if and only if $1/p_1+\cdots+1/p_n<1/q$. However, $\mathcal{K}^r(\ell_{p_1},\cdots,\ell_{p_n};\ell_q)$, the sublattice of $\mathcal{L}^r(\ell_{p_1},\cdots,\ell_{p_n};\ell_q)$ generated by all positive compact operators from $\ell_{p_1}\times\cdots\times\ell_{p_n}$ to ℓ_q, is always order continuous for any $1<p_1,\cdots,p_n,q<\infty$.

We next consider the diagonals of injective tensor products of Banach lattices with bases. Let E be a Banach lattice with a 1-unconditional basis $\{e_i : i\in\mathbb{N}\}$. Denote by $\Delta(\check{\otimes}_{n,\epsilon}E)$ (resp. $\Delta(\check{\otimes}_{n,s,\epsilon}E)$) the main diagonal space of the n-fold full (resp. symmetric) injective Banach space tensor product, and denote by $\Delta(\check{\otimes}_{n,|\epsilon|}E)$ (resp. $\Delta(\check{\otimes}_{n,s,|\epsilon|}E)$) the main diagonal space of the n-fold full (resp. symmetric) injective Banach lattice tensor product. We show that these four main diagonal spaces are pairwise isometrically isomorphic by using the 1-unconditionality of the tensor diagonal $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$. We also show that the tensor diagonal $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ is a 1-unconditional basic sequence in both $\check{\otimes}_{n,\epsilon}E$ and $\check{\otimes}_{n,s,\epsilon}E$.

DEDICATION

To Hyojung, for her adoration, friendship, and support.

ACKNOWLEDGEMENTS

I would first like to express gratitude to my adviser, Dr. Qingying Bu, for his encouragement, guidance, and patience over the past five years. I am grateful for his supportive and encouraging words and for challenging me to always keep improving and learning. His advice and counsel have been integral in developing the material presented here as well as my personal mathematical development. Thank you for helping me become a stronger researcher, writer, and teacher.

I appreciate Drs. Gerard Buskes, Iwo Labuda, Micah Milinovich and James Cizdziel for taking time out of their busy schedules to serve on my dissertation committee. Their suggestions were well received. I thank Mr. Marlow Dorrough for giving me the opportunity to teach a variety of courses as a graduate student. Several mathematicians have helped shape and strengthen my research and understanding. In particular, I thank Dr. Joseph Diestel for helpful conversations and feedback about my thesis.

I also have benefited from the professional advice and encouragement from a large group of mentors. In particular, I extend heartfelt thanks to my former advisers Dr. Chan Huh and Dr. Joonchul Han of the Pusan National University for striving to mold an immature and young student into a mathematician. Along the same line, I would like to thank Drs. Bing Wei, Miña-Diaz, James Reid, Laura Sheppardson and Sandra Spiroff of the University of Mississippi for introducing many wonderful topics of mathematics to me and providing an environment of thought and growth over the past several years.

It would have been impossible to successfully emerge from this process without the support of my friends and family. I warmly thank Matthew Joo and his family for their constant encouragement and support. I thank my classmates Caroline, Chris, and Stephan for being emphatic and supportive friends. I thank my parents and sibling Jaehyuk for their constant encouragement and love. In addition, I would like to thank my wife's family for their support, as well. Finally, I thank my wife, Hyojung, my source of constant support, encouragement, and inspiration. I want to thank my son, Joonshub, for making me laugh and smile every day.

TABLE OF CONTENTS

1 INTRODUCTION

The general theory of tensor products of Banach spaces dates back to Grothendieck's famous Résumé [23] and Memoir [24] which were published in the 1950s. These two papers have had a considerable influence on the development of Banach space theory. They contained a general theory of tensor norms on tensor products of Banach spaces, studied the duality theory of these tensor products, and characterized the linearization of multilinear operators through the tensor products. Grothendieck's Résumé and Memoir opened many directions in functional analysis.

In 1980, R. Ryan in his doctoral thesis [42] introduced symmetric tensor products for the study of polynomials on Banach spaces and then characterized the linearization of homogeneous polynomials through the symmetric tensor products. The symmetric tensor products and homogeneous polynomials play a key role in the theory of holomorphic functions. See, for example, Dineen's book [16] and Mujica's book [37]. In 2005, Grecu and Ryan [21], when they studied the homogeneous polynomials with unconditionally convergent monomial expansions, introduced regular homogeneous polynomials with respect to the Banach lattice structure of the domain. A regular homogeneous polynomial is defined to be the difference of two positive homogeneous polynomials. In turn, positive homogeneous polynomials are defined in terms of their associated symmetric positive multilinear operators.

The linearization of positive multilinear operators on Banach lattices dates back to 1970s when Fremlin [17, 18] introduced the positive tensor products of Banach lattices.

Later, Schep [47], Grobler and Labuschagne [22], and Buskes and Van Rooij [10] have made significant contributions in the theory of positive tensor products and positive multilinear operators on Banach lattices.

In 2012, Bu and Buskes [7] characterized the linearization of orthogonally additive homogeneous polynomials through the quotient of positive tensor products of Banach lattices. They extended Sundaresan's results in [50] in 1989 and Benyamini, Lassalle, and Llavona's results in [32] in 2006.

Arias and Farmer [2] characterized the main diagonal of the *projective tensor product* $\ell_{p_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{p_n}$ $(1 \leqslant p_1, \cdots, p_n < \infty)$ and used it to characterize the reflexivity of $\ell_{p_1}\hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{p_n}$ (also see [43]). In section 4.1 of this dissertation, we use the Rademacher averaging to characterize the main diagonal of the *positive projective tensor product* $\ell_{p_1}\hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$ and then use it to characterize the reflexivity of $\ell_{p_1}\hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$. From the positivity perspective, we show that $\ell_{p_1}\hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$ is a Kantorovich-Banach space and hence, is an order continuous Banach lattice for any $1 \leqslant p_1, \cdots, p_n < \infty$. However, we show that its dual $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$, the space of all regular n-linear forms on $\ell_{p_1} \times \cdots \times \ell_{p_n}$, is an order continuous Banach lattice if and only if it is reflexive.

Holub [25] characterized the main diagonal of the *injective tensor product* of $\ell_p \check{\otimes}_\epsilon \ell_q$ $(1 \leqslant p, q < \infty)$. In section 4.2 of this dissertation, we use the Rademacher averaging to characterize the main diagonal of the *positive injective tensor product* $\ell_{p_1}\check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n}$ $(1 \leqslant p_1, \cdots, p_n < \infty)$ and then use it to characterize the reflexivity of $\ell_{p_1}\check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n}$. From the positivity perspective, we show that $\ell_{p_1}\check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n}$ is an order continuous Banach lattice for any $1 \leqslant p_1, \cdots, p_n < \infty$. However, we show that it is a Kantorovich-Banach space if and only if it is reflexive.

In chapter 4 of this dissertation, we characterize the main diagonals of the positive projective tensor product $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ and the positive injective tensor product $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$. Then by using these two main diagonals, we characterize the reflexivity, the property of being Kantorovich-Banach spaces, and the property of being order continuous of $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ and $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$, as well as the space of all regular n-linear forms on $\ell_{p_1}\times\cdots\times\ell_{p_n}$ and the space of all regular n-homogeneous polynomials on ℓ_p $(1\leqslant p<\infty)$. In section 4.2 of this dissertation, we also obtain the following interesting result. Let $1<p_1,\cdots,p_n,q<\infty$. Then the Banach lattice $\mathcal{L}^r(\ell_{p_1},\cdots,\ell_{p_n};\ell_q)$, the space of all regular n-linear operators from $\ell_{p_1}\times\cdots\times\ell_{p_n}$ to ℓ_q, is order continuous if and only if $1/p_1+\cdots+1/p_n<1/q$. However, $\mathcal{K}^r(\ell_{p_1},\cdots,\ell_{p_n};\ell_q)$, the sublattice of $\mathcal{L}^r(\ell_{p_1},\cdots,\ell_{p_n};\ell_q)$ generated by all positive compact operators from $\ell_{p_1}\times\cdots\times\ell_{p_n}$ to ℓ_q, is always order continuous for any $1<p_1,\cdots,p_n,q<\infty$.

In section 5.2 of this dissertation, we show that all four main diagonal spaces $\Delta(\check{\otimes}_{n,\epsilon}E)$, $\Delta(\check{\otimes}_{n,s,\epsilon}E)$, $\Delta(\check{\otimes}_{n,|\epsilon|}E)$, and $\Delta(\check{\otimes}_{n,s,|\epsilon|}E)$ are pairwise isometrically isomorphic. We also show that the tensor diagonal $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ is a 1-unconditional basic sequence in both $\check{\otimes}_{n,\epsilon}E$ and $\check{\otimes}_{n,s,\epsilon}E$. Let X be a Banach space with a 1-unconditional basis $\{e_i : i\in\mathbb{N}\}$. Then $\{e_{i_1}\otimes\cdots\otimes e_{i_n} : (i_1,\ldots,i_n)\in\mathbb{N}^n\}$ is a basis of both the n-fold full injective tensor product $\check{\otimes}_{n,\epsilon}X$ and the n-fold full projective tensor product $\hat{\otimes}_{n,\pi}X$ (see, e.g., [19, 20]), and $\{e_{i_1}\otimes_s\cdots\otimes_s e_{i_n} : (i_1,\ldots,i_n)\in\mathbb{N}^n, i_1\geqslant\cdots\geqslant i_n\}$ is a basis of both the n-fold symmetric injective tensor product $\check{\otimes}_{n,s,\epsilon}X$ and the n-fold symmetric projective tensor product $\hat{\otimes}_{n,s,\pi}X$ (see, e.g., [20]). However, they are not necessary unconditional bases (see, e.g., [49, 40, 48, 13, 39, 11]). In particular, the tensor diagonal $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ is a 1-unconditional basic sequence in both $\hat{\otimes}_{n,\pi}X$ (see, e.g., [25, 46]) and $\hat{\otimes}_{n,s,\pi}X$ (see, e.g., [5]); and Holub [25] showed that the tensor diagonal $\{e_i\otimes e_i : i\in\mathbb{N}\}$ is a 1-unconditional basic sequence in $\check{\otimes}_{2,\epsilon}X$. By using Holub's method, it is easy to show that the tensor diagonal $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ is a 1-unconditional basic sequence in $\check{\otimes}_{n,\epsilon}X$. However, Holub's method does not work for $\check{\otimes}_{n,s,\epsilon}X$. In section 5.2, by using a result obtained in [5], we show that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence in $\check{\otimes}_{n,s,\epsilon}X$ and we also show that the diagonal projections on both $\check{\otimes}_{n,\epsilon}X$ and $\check{\otimes}_{n,s,\epsilon}X$ are contractive.

From the positivity perspective, let E be a Banach lattice with a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a 1-unconditional basis of the n-fold full positive projective tensor product $\hat{\otimes}_{n,|\pi|}E$, and $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geqslant \cdots \geqslant i_n\}$ is a 1-unconditional basis of the n-fold positive symmetric projective tensor product $\hat{\otimes}_{n,s,|\pi|}E$ (see, e.g., [8, 5]). In section 5.2 of this dissertation, we show that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a 1-unconditional basis of the n-fold full positive injective tensor product $\check{\otimes}_{n,|\epsilon|}E$, and $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geqslant \cdots \geqslant i_n\}$ is a 1-unconditional basis of the n-fold positive symmetric injective tensor product $\check{\otimes}_{n,s,|\epsilon|}E$.

Let $\Delta(\hat{\otimes}_{n,\pi}E)$ (resp. $\Delta(\hat{\otimes}_{n,s,\pi}E)$) denote the closed subspace generated by the tensor diagonals $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\hat{\otimes}_{n,\pi}E$ (resp. $\hat{\otimes}_{n,s,\pi}E$), and let $\Delta(\hat{\otimes}_{n,|\pi|}E)$ (resp. $\Delta(\hat{\otimes}_{n,s,|\pi|}E)$) denote the closed sublattice generated by the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\hat{\otimes}_{n,|\pi|}E$ (resp. $\hat{\otimes}_{n,s,\pi}E$). Bu and Buskes [5] showed that these four main diagonal spaces $\Delta(\hat{\otimes}_{n,\pi}E), \Delta(\hat{\otimes}_{n,s,\pi}E), \Delta(\hat{\otimes}_{n,|\pi|}E)$, and $\Delta(\hat{\otimes}_{n,s,|\pi|}E)$ are isometrically isomorphic.

Now let $\Delta(\check{\otimes}_{n,\epsilon}E)$ (resp. $\Delta(\check{\otimes}_{n,s,\epsilon}E)$) denote the closed subspace generated by the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\check{\otimes}_{n,\epsilon}E$ (resp. $\check{\otimes}_{n,s,\epsilon}E$), and let $\Delta(\check{\otimes}_{n,|\epsilon|}E)$ (resp. $\Delta(\check{\otimes}_{n,s,|\epsilon|}E)$) denote the closed sublattice generated by the tensor diagonals $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\check{\otimes}_{n,|\epsilon|}E$ (resp. $\check{\otimes}_{n,s,|\epsilon|}E$). In section 5.2, we use the 1-unconditionality of the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ to show that these four main diagonal spaces $\Delta(\check{\otimes}_{n,\epsilon}E), \Delta(\check{\otimes}_{n,s,\epsilon}E), \Delta(\check{\otimes}_{n,|\epsilon|}E)$, and $\Delta(\check{\otimes}_{n,s,|\epsilon|}E)$ are isometrically isomorphic.

LIST OF SYMBOLS

For two Banach spaces X, Y and Banach lattices $E_1, ..., E_n, E$ and F,

- $X\hat{\otimes}_\pi Y$: the completion of $X \otimes Y$ with respect to $\|\cdot\|_\pi$.
- $X\check{\otimes}_\epsilon Y$: the completion of $X \otimes Y$ with respect to $\|\cdot\|_\epsilon$.
- $\mathcal{L}(X;Y)$: the space of all bounded linear operators from X to Y.
- $\mathcal{L}(E_1 \times \cdots \times E_n; F)$: the space of n-linear mappings from $E_1 \times \cdots \times E_n$ to F.
- $K(X;Y)$: the space of all compact operators from X to Y.
- $\Delta(X\hat{\otimes}_\pi Y)$ (resp. $\Delta(X\check{\otimes}_\epsilon Y)$): the main diagonal of $X\hat{\otimes}_\pi Y$ (resp. $X\check{\otimes}_\epsilon Y$).
- $\hat{\otimes}_{n,\pi}X$ (resp. $\check{\otimes}_{n,\epsilon}X$): n-fold projective (resp. injective) tensor products of X.
- $\Delta(\hat{\otimes}_{n,\pi}X)$ (resp. $\Delta(\check{\otimes}_{n,\epsilon}X)$): the main diagonal of $\hat{\otimes}_{n,\pi}X$ (resp. $\check{\otimes}_{n,\epsilon}X$).
- $\otimes_{n,s}X$: the n-fold symmetric algebraic tensor product of X.
- $\bar{\otimes}_{n,s}E$: the n-fold vector lattice symmetric tensor product of E.
- $\hat{\otimes}_{n,s,\pi}X$ (resp. $\check{\otimes}_{n,s,\epsilon}X$): n-fold symmetric projective (resp. injective) tensor products of X.

- $\Delta(\hat{\otimes}_{n,s,\pi}X)$ (resp. $\Delta(\check{\otimes}_{n,s,\epsilon}X)$): the main diagonal of $\hat{\otimes}_{n,s,\pi}X$ (resp. $\check{\otimes}_{n,s,\epsilon}X$).

- $E\hat{\otimes}_{|\pi|}F$: the completion of $E \otimes F$ with respect to $\|\cdot\|_{|\pi|}$.

- $E\check{\otimes}_{|\epsilon|}F$: the completion of $E \otimes F$ with respect to $\|\cdot\|_{|\epsilon|}$.

- $\mathcal{L}^r(E;F)$: the space of all regular operators from E to F.

- $\mathcal{L}_b(E;F)$: the set of order bounded operators from E to F.

- $\mathcal{K}^r(E;F)$: the space of all compact regular operators from E to F.

- $\mathcal{P}(^nE;F)$: the space of all continuous n-homogeneous polynomials from E to F.

- $\mathcal{P}^r(^nE;F)$: the space of all regular n-homogeneous polynomials from E to F.

- $\hat{\otimes}_{n,|\pi|}E$ (resp. $\check{\otimes}_{n,|\epsilon|}E$): n-fold positive projective (resp. injective) tensor products of E.

- $\Delta(\hat{\otimes}_{n,|\pi|}E)$ (resp. $\Delta(\check{\otimes}_{n,|\epsilon|}E)$): the main diagonal of $\hat{\otimes}_{n,|\pi|}E$ (resp. $\check{\otimes}_{n,|\epsilon|}E$).

- $\hat{\otimes}_{n,s,|\pi|}E$ (resp. $\check{\otimes}_{n,s,|\epsilon|}E$): n-fold positive symmetric projective (resp. injective) tensor products of E.

- $\Delta(\hat{\otimes}_{n,s,|\pi|}E))$ (resp. $\Delta(\check{\otimes}_{n,s,|\epsilon|}E)$): the main diagonal of $\hat{\otimes}_{n,s,|\pi|}E$ (resp. $\check{\otimes}_{n,s,|\epsilon|}E$).

2 PRELIMINARIES

Each chapter is divided into sections numbered by two digits, the first one being the number of the chapter. Definitions, theorems, propositions, corollaries, lemmas and examples in each chapter are labeled by two digits, the first indicating the chapter. We will use the symbol $\square$ to mark the end of proof. We refer to the book of Meyer-Nieberg [36] and of Aliprantis and Burkinshaw [1] for basic properties of Riesz spaces and Banach lattices that we use.

Definition 2.1. A real vector space E is said to be an *ordered vector space* whenever it is equipped with an order relation $\geq$ (i.e. $\geq$ is reflexive, antisymmetric and transitive binary relation on E) that is compatible with the algebraic structure of E in the sense that it satisfies the following two axioms:

(i) If $x \geq y$, then $x + z \geq y + z$ holds for all $z \in E$.

(ii) If $x \geq y$, then $\alpha x \geq \alpha y$ holds for all $\alpha \geq 0$.

An alternate notation for $x \geq y$ is $y \leq x$.

Definition 2.2. An element x in an ordered vector space E is called *positive* whenever $x \geq 0$ holds. The set of all positive elements of E will be denoted by E^+, i.e. $E^+ = \{x \in E : x \geq 0\}$. The set E^+ of positive vectors is called the *positive cone* of E.

Definition 2.3. A mapping $T : E \to F$ between two vector spaces is called a *linear operator* if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ holds for all $x, y \in E$ and all $\alpha, \beta \in \mathbb{R}$.

Definition 2.4. An operator $T : E \to F$ between two ordered vector spaces is said to be *positive* (in symbols $T \geq 0$ or $0 \leq T$) whenever $T(x) \geq 0$ for all $x \geq 0$.

An operator $T : E \to F$ between two ordered vector spaces is, of course, positive if and only if $T(E^+) \subseteq F^+$ (and equivalently $T(x) \geq T(y)$ if $x \geq y$).

Definition 2.5. A *Riesz space* (or a *vector lattice*) is an ordered vector space E with the additional property that for each pair of elements $x, y \in E$ the supremum and infimum of the set $\{x, y\}$ both exist in E.

We shall write

$$x \vee y = \sup\{x, y\} \quad \textit{and} \quad x \wedge y = \inf\{x, y\}.$$

Typical examples of Riesz spaces are provided by the function spaces.

Definition 2.6. A *function space* is a vector space E of real-valued functions on a set Ω such that for each pair $f, g \in E$ the functions

$$(f \vee g)(\omega) = \max\{f(\omega), g(\omega)\}$$

and

$$(f \wedge g)(\omega) = \min\{f(\omega), g(\omega)\}$$

both belong to E.

Every function space E with the pointwise ordering (i.e., $f \geq g$ holds in E if and only if $f(\omega) \geq g(\omega)$ for all $\omega \in \Omega$) is a Riesz space. Here are some important examples of function spaces:

(i) $\mathbb{R}^\Omega$, all real-valued functions defined on a set Ω.

(ii) $C(\Omega)$, all continuous real-valued functions on a topological space Ω.

(iii) $C_b(\Omega)$, all bounded real-valued continuous functions on a topological space Ω.

(iv) $\ell_\infty(\Omega)$, all bounded real-valued functions on a set Ω.

(v) ℓ_p $(0 < p < \infty)$, all real sequences $(x_1, x_2, \cdots)$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

The class of L_p-space is another important class of Riesz spaces. If (X, Σ, μ) is a measure space and $(0 < p < \infty)$, then $L_p(\mu)$ is the vector space of all real-valued μ-measurable functions f on X such that $\int_X |f|^p < \infty$. Also $L_\infty(\mu)$ is the vector space of all real-valued μ-measurable functions f on X such that $\mathrm{esssup}|f| < \infty$. As usual, functions differing on a set of measure zero are treated as identical, i.e., $f = g$ in $L_p(\mu)$ means that $f(x) = g(x)$ for μ-almost all $x \in X$. It is true that under the ordering $f \leq g$ whenever $f(x) \leq g(x)$ holds for μ-almost all $x \in X$, each $L_p(\mu)$ is a Riesz space.

Definition 2.7. A net $\{x_\alpha\}$ in a Riesz space is said to be *decreasing* (in symbols, $x_\alpha \downarrow$) whenever $\alpha > \beta$ implies $x_\alpha \leq x_\beta$.

The notation $\{x_\alpha\} \downarrow x$ means that $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$ both hold.

Definition 2.8. A Riesz space E is called *Archimedean* whenever $n^{-1}x \downarrow 0$ holds in E for each $x \in E^+$, $n \in \mathbb{N}$.

For any vector x in a Riesz space we define

$$x^+ = x \vee 0, \qquad x^- = (-x) \vee 0, \qquad |x| = x \vee (-x).$$

The element x^+ is called the positive part, x^- the negative part and $|x|$ the absolute value of x. Vectors x^+, x^-, and $|x|$ are satisfied in the following important theorem:

Theorem 2.9. *If x is an element of a Riesz space, then we have*

(i) $x = x^+ - x^-$,

(ii) $|x| = x^+ + x^-$,

(iii) $x^+ \wedge x^- = 0$.

Moreover the decomposition in (i) is unique in the sense that if $x = y - z$ holds with $y \wedge z = 0$, then $y = x^+$ and $z = x^-$.

Definition 2.10. In a Riesz space E and $x, y \in E$. Two elements x and y are called *disjoint* (denoted by $x \perp y$) if $|x| \wedge |y| = 0$.

Definition 2.11. For an operator $T : E \to F$ between two Riesz spaces we say that its modulus $|T|$ exists whenever $|T| = T \vee (-T)$ exists (in the sense that $|T|$ is the supremum of the set $\{T, -T\}$ in the ordered vector space $\mathcal{L}(E; F)$).

In what follows, by operator we mean a linear map between Riesz spaces.

Definition 2.12. An operator $T : E \to F$ is said to be a *regular operator* if it can be written as the difference of two positive operators.

Definition 2.13. Let A be a subset of an ordered set M. A set A is called *order bounded* if it is bounded both from above and from below.

Definition 2.14. An operator $T : E \to F$ that maps order bounded subsets of E onto order bounded subsets of F is called *order bounded.* We denote the set of order bounded operators from E to F by $\mathcal{L}_b(E; F)$.

Definition 2.15. A Riesz space is called *Dedekind complete* if every nonempty subset that is bounded above has a supremum.

When F is Dedekind complete, the ordered vector space $\mathcal{L}^r(E;F)$ has the structure of a Riesz space. In fact, $\mathcal{L}^r(E;F)$ is a Dedekind complete Riesz space. This result was proved first by Riesz [44] for the case $F = \mathbb{R}$, and later Kantorovič [29, 30] extended it to the general setting.

Definition 2.16. A subset A of a Riesz space is said to be *upwards directed* (in symbols $A \uparrow$) whenever for each pair $x, y \in A$ there exists some $z \in A$ with $x \leq z$ and $y \leq z$. The symbol $A \uparrow x$ means that A is upwards directed and $x = \sup A$ holds.

Definition 2.17. A Riesz subspace G of a Riesz space E is said to be *order dense* in E whenever for each $0 < x \in E$, there exists $y \in G$ such that $0 < y \leq x$.

Definition 2.18. The vector space $E^{\sim}$ of all order bounded linear functionals on a Riesz space E is called the *order dual* of E, i.e. $E^{\sim} = \mathcal{L}_b(E;\mathbb{R})$.

Definition 2.19. Let E be a (real) Riesz space equipped with a norm. The norm in E is called a *Riesz norm* if $|x| \leq |y|$ in E implies $\|x\| \leq \|y\|$. A normed Riesz space which is complete with respect to the norm is called a *Banach lattice*.

Recall that a norm $\|\cdot\|$ on a Riesz space is said to be a Riesz norm (or lattice norm) whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Note that this implies that for any $x \in E$ the elements x and $|x|$ have the same norm. Any Riesz space, equipped with a Riesz norm, is called a *normed Riesz space*. If the normed Riesz space E is also norm complete, then E is referred to as a Banach lattice.

Example 2.20. Both ℓ_p $(1 \leq p \leq \infty)$ and c_0 are Dedekind complete Banach lattices where the order is defined pointwise. Assume that (Ω, Σ, μ) is a measure space. Then $L_p(\mu)$ $(1 \leq p < \infty)$ is Dedekind complete Banach lattice. If μ is σ-finite, then $L_\infty(\mu)$ is Dedekind complete.

Definition 2.21. A Banach lattice E is called *order continuous* if $\|x_\alpha\| \downarrow 0$ whenever $x_\alpha \downarrow 0$.

Definition 2.22. A subspace U of E is called a *sublattice* of E if $x \vee y \in U$ and $x \wedge y \in U$ for all $x, y \in U$.

Definition 2.23. A subset A of E is called *solid* if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$.

Definition 2.24. Every solid subspace I of E is called an *ideal* or *order ideal* in E.

Definition 2.25. An ideal B of E is called a *band* if $\sup(A) \in B$ for every subset $A \subset B$ which has a supremum in E.

Definition 2.26 A band B of E is called a *projection band* if there is a linear projection $P : E \to B$ such that $0 \leq Px \leq x$ for all $x \in E^+$. Such a projection is called a *band projection.*

Example 2.27.

(i) If c denotes the space of all convergent real sequences, then c is a sublattice of ℓ_∞, but fails to be an ideal.

(ii) The space c_0 is an ideal in ℓ_∞, but fails to be a band in ℓ_∞. If $v_n = e_1 + \cdots + e_n$ where e_k denotes the k-th natural basis vector of c_0, then $e = \sup\{v_n : n \in \mathbb{N}\}$ exists in ℓ_∞, but not in c_0.

It is clear that every ideal in E is a sublattice of E. Moreover, the intersection of any two sublattices, ideals, or bands, respectively, is a sublattice (resp. ideal and band). The sum of two ideals is also an ideal and the sum of two projection bands is a projection band. However, the sum of two sublattices need not to be a sublattice (see, p.12, [36]).

Definition 2.28. A sequence (x_n) in a Banach space X is said to be a *Schauder basis* (or simply a *basis*) whenever for each $x \in X$ there exists a unique sequence (α_n) of scalars satisfying $x = \sum_{n=1}^{\infty} \alpha_n x_n$ (where, as usual, the convergence of the series is assumed to be in the norm.)

Definition 2.29. A sequence (x_n) in a Banach space X is a (Schauder) *basic sequence* if it is a (Schauder) basis for the closed vector subspace it generates.
(denoted by $[\{x_n : n \in \mathbb{N}\}]$).

Example 2.30. If X is c_0 or ℓ_p such that $1 \leq p < \infty$, then the sequence (e_n) of standard unit vectors of X is a basis and that $(\alpha_n) = \sum_{n=1}^{\infty} \alpha_n e_n$ whenever $(\alpha_n) \in X$. However, the sequence (e_n) is not a basis for ℓ_∞. For example, there is no sequence (α_n) of scalars such that $(1, 1, 1, ...) = \sum_{n=1}^{\infty} \alpha_n e_n$.

Whenever the sequence (e_n) lies in the unit sphere of a Banach space of sequences of scalars and is a basis for the space, the sequence will be called the *standard unit vector basis* for the space.

Definition 2.31. A basis (x_n) for a Banach space X is *unconditional* if every convergent series of the form $\sum_{n=1}^{\infty} \alpha_n x_n$ is unconditionally convergent. A basis for a Banach space is *conditional* if it is not unconditional.

Example 2.32. Suppose that X is c_0 or ℓ_p such that $1 \leq p < \infty$. Then the standard unit vector basis for X is unconditional. The classical Schauder basis for $C[0, 1]$ and the Harr basis for $L_1[0, 1]$ are conditional bases.

Definition 2.33. For an unconditional basis (x_n) in X, there exists a constant C such that for every sequence of scalars $(a_n)_{n=1}^{\infty}$ and $(\epsilon_n)_{n=1}^{\infty}$ of modulus at most 1. We have

$$\left\| \sum_{n=1}^{\infty} \epsilon_n a_n x_n \right\| \leq C \left\| \sum_{n=1}^{\infty} a_n x_n \right\|$$

If C is given, we call the basis C-*unconditional*.

Definition 2.34. Two bases (x_n) and (y_n) for Banach spaces are *equivalent* if, for every sequence (α_n) of scalars, the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges if and only if $\sum_{n=1}^{\infty} \alpha_n y_n$ converges.

The equivalence of bases guarantees an isomorphism of the Banach spaces they span.

Definition 2.35. A basis (x_n) for a Banach space X is called *boundedly complete*, whenever a sequence (α_n) of scalars is such that

$$\sup_m \left\| \sum_{n=1}^{m} \alpha_n x_n \right\| < \infty,$$

the series $\sum_{n=1}^{\infty} \alpha_n x_n$ converges.

Example 2.36. Let (e_n) be the standard unit vector basis for ℓ_p, where $1 \leq p < \infty$. If (α_n) is a sequence of scalars such that $\sup_m \|\sum_{n=1}^{m} \alpha_n e_n\|_p$ is finite, then

$$\sum_{n=1}^{\infty} |\alpha_n|^p = \sup_m \sum_{n=1}^{m} |\alpha_n|^p = \sup_m \left\| \sum_{n=1}^{m} \alpha_n e_n \right\|_p^p < +\infty,$$

so $\sum_{n=1}^{\infty} \alpha_n e_n$ converges to the element (α_n) of ℓ_p. The basis (e_n) is therefore boundedly complete.

Definition 2.37 Let X be a subspace of a normed space Z. We say that X is *complemented* in Z if there exists a subspace Y such that $Z = X \oplus Y$.

Any closed subspace of a Hilbert space is complemented.

Definition 2.38 Let X be a Banach space. We say that $A \subseteq X$ is *relatively compact* if $\bar{A}$ is compact in X.

Definition 2.39 Let X and Y are Banach spaces. A linear operator T from X into Y is called *compact* if $T(B)$ is a relatively compact subset of Y whenever B is a bounded subset of X.

3 MULTILINEAR OPERATORS AND TENSOR PRODUCTS

3.1 Multilinear Operators and Tensor Products of Banach Spaces

Definition 3.1. For Banach spaces $X_1, \cdots, X_n$, and Y, an operator $T : X_1 \times \cdots \times X_n \to Y$ is called *n-linear operator* if it is linear in each variable $x_k, y_k \in X_k$, $k = 1, \cdots, n$.
That is, for fixed k with $1 \leq k \leq n$ and $\alpha, \beta \in \mathbb{R}$, we have

$$T(x_1, \cdots, x_{k-1}, \alpha x_k + \beta y_k, x_{k+1}, \cdots, x_n)$$

$$= \alpha T(x_1, \cdots, x_{k-1}, x_k, x_{k+1}, \cdots, x_n) + \beta T(x_1, \cdots, x_{k-1}, y_k, x_{k+1}, \cdots, x_n).$$

Definition 3.2. An n-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is called *bounded* if there exists a constant C such that $\|T(x_1, \cdots, x_n)\| \leq C\|x_1\| \cdots \|x_n\|$ for all $x_1 \in X_1, \cdots, x_n \in X_n$.

Definition 3.3. An n-linear operator T is called *continuous* if $T(x_k^1, \cdots, x_k^n) \to T(x_1, \cdots, x_n)$ whenever $x_k^1 \to x^1, \cdots, x_k^n \to x^n$ as $k \to \infty$.

It is well-known that an n-linear operator T is continuous if and only if T is bounded.

For Banach space X, let X^* denote its topological dual and B_X denote its closed unit ball. For Banach spaces $X_1, \cdots, X_n$, and Y, let $\mathcal{L}(X_1, \cdots, X_n; Y)$ denote the space of all continuous n-linear operators from $X_1 \times \cdots \times X_n$ to Y. If Y denotes the scalars then we denote this space by $\mathcal{L}(X_1, \cdots, X_n)$. In addition, if $X_j = X$ for $j = 1, \cdots, n$, then we denote this space by $\mathcal{L}(^nX)$.

For each $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$, the norm of a continuous n-linear operator can be defined as follows:

$$\|T\| = \sup\{\|T(x_1, \cdots, x_n)\| : x_1 \in B_{X_1}, \cdots, x_n \in B_{X_n}\}.$$

Then $\mathcal{L}(X_1, \cdots, X_n; Y)$ with this norm is a Banach space.

Definition 3.4. An n-linear operator $T : X \times \cdots \times X \to Y$ is called *symmetric* if

$$T(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = T(x_1, \cdots, x_n)$$

for every permutation σ on $\{1, \cdots, n\}$.

We refer to Dineen [16] for results on polynomials.

Definition 3.5. A mapping $P : X \to Y$ is called an *n-homogeneous polynomial* if there exists a symmetric n-linear operator $T_P : X \times \cdots \times X \to Y$ such that

$$P(x) = T_P(x, \cdots, x), \quad \forall x \in X.$$

Let $\mathcal{P}(^n X; Y)$ denote the space of all continuous n-homogeneous polynomials from X to Y. For each $P \in \mathcal{P}(^n X; Y)$, define

$$\|P\| = \sup\{\|P(x)\| : x \in B_X\}.$$

Then $\mathcal{P}(^n X; Y)$ with this norm is a Banach space and the following theorem holds.

Theorem 3.6. *(Polarization Formula) For every $x \in X$,*

$$P(x) = T_P(x, \cdots, x)$$

and for every $x_1, \cdots, x_n \in X$,

$$T_P(x_1, \cdots, x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \cdots \epsilon_n P\Big(\sum_{i=1}^{n} \epsilon_i x_i \Big)$$

with the norm inequalities

$$\|P\| \leq \|T_P\| \leq \frac{n^n}{n!} \|P\|.$$

Next we will introduce the projective tensor products. For Banach spaces $X_1, \ldots, X_n$, let $X_1 \otimes \cdots \otimes X_n$ denote the *n-fold algebraic tensor product* of $X_1, \cdots, X_n$.

Definition 3.7. The *projective tensor norm* on $X_1 \otimes \cdots \otimes X_n$ is defined by

$$\|u\|_\pi = \inf \Big\{ \sum_{k=1}^{m} \|x_{1,k}\| \cdots \|x_{n,k}\| : u = \sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k} \Big\} \tag{3.1}$$

for every $u \in X_1 \otimes \cdots \otimes X_n$. The completion of $X_1 \otimes \cdots \otimes X_n$ with respect to this norm is denoted by $X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n$ and called the *n-fold projective tensor product* of $X_1, \cdots, X_n$.

Define an n-linear operator $\otimes : X_1 \times \cdots \times X_n \to X_1 \otimes \cdots \otimes X_n$ by

$$\otimes(x_1, \cdots, x_n) = x_1 \otimes \cdots \otimes x_n \quad \forall\, (x_1, \cdots, x_n) \in X_1 \times \cdots \times X_n. \tag{3.2}$$

Then, for every $T \in \mathcal{L}(X_1, \cdots, X_n; Y)$, there exists a unique $T^\otimes$ in $\mathcal{L}(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n; Y)$ such that

$$T = T^\otimes \circ \otimes \quad \text{and} \quad \|T\| = \|T^\otimes\|. \tag{3.3}$$

That is, the following diagram commutes:

$$\begin{tikzcd}
X_1 \times \cdots \times X_n \arrow[r, "T"] \arrow[d, "\otimes"'] & Y \\
X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n \arrow[ur, "T^{\otimes}"'] &
\end{tikzcd}$$

Moreover, $\mathcal{L}(X_1, \cdots, X_n; Y)$ is isometrically isomorphic to $\mathcal{L}(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n; Y)$.

For $x_1 \otimes \cdots \otimes x_n \in \otimes_n X$, let $x_1 \otimes_s \cdots \otimes_s x_n$ denote its symmetrization. That is,

$$x_1 \otimes_s \cdots \otimes_s x_n = \frac{1}{n!} \sum_{\sigma \in \pi(n)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)}, \tag{3.4}$$

where $\pi(n)$ is the group of permutations of $\{1, \ldots, n\}$. Then (e.g., see [49])

$$x_1 \otimes_s \cdots \otimes_s x_n = \frac{1}{2^n n!} \sum_{\delta_i = \pm 1} \delta_1 \cdots \delta_n \Big(\sum_{i=1}^n \delta_i x_i \Big) \otimes \cdots \otimes \Big(\sum_{i=1}^n \delta_i x_i \Big). \tag{3.5}$$

We write $\otimes_{n,s} X$ for the *n-fold symmetric algebraic tensor product* of X, that is, the linear span of $\{x_1 \otimes_s \cdots \otimes_s x_n : x_1, \ldots, x_n \in X\}$ in $\otimes_n X$. Each $u \in \otimes_{n,s} X$ has a representation $u = \sum_{k=1}^m \lambda_k x_k \otimes_s \cdots \otimes_s x_k$ where $\lambda_1, \ldots, \lambda_m$ are scalars and $x_1, \ldots, x_m$ are vectors in X.

Definition 3.8. The *symmetric projective tensor norm* on $\otimes_{n,s} X$ is defined by

$$\|u\|_{s,\pi} = \inf \Big\{ \sum_{k=1}^m |\lambda_k| \cdot \|x_k\|^n : u = \sum_{k=1}^m \lambda_k x_k \otimes_s \cdots \otimes_s x_k \in \otimes_{n,s} X \Big\}, \quad u \in \otimes_{n,s} X.$$

Let $\hat{\otimes}_{n,s,\pi} X$ denote the completion of $(\otimes_{n,s} X, \| \cdot \|_{s,\pi})$, called the *n-fold symmetric projective tensor product* of X.

Define an n-homogeneous polynomial $\otimes : X \to \otimes_{n,s} X$ by

$$\otimes(x) = x \otimes \cdots \otimes x, \quad \forall x \in X. \tag{3.6}$$

Then for every $P \in \mathcal{P}(^nX;Y)$ there exists a unique $P^{\otimes}$ in $\mathcal{L}(\hat{\otimes}_{n,s,\pi}X;Y)$ such that

$$P = P^{\otimes} \circ \otimes \quad \text{and} \quad \|P\| = \|P^{\otimes}\|. \tag{3.7}$$

That is, the following diagram commutes:

$$\begin{tikzcd} X \arrow[r, "P"] \arrow[d, "\otimes"'] & Y \\ \hat{\otimes}_{n,s,\pi}X \arrow[ur, "P^{\otimes}"'] & \end{tikzcd}$$

Moreover, $\mathcal{P}(^nX;Y)$ is isometrically isomorphic to $\mathcal{L}(\hat{\otimes}_{n,s,\pi}X;Y)$.

Next we will introduce the injective tensor products. Let $X_1, \cdots, X_n$ be Banach spaces.

Definition 3.9. The *injective tensor norm* on $X_1 \otimes \cdots \otimes X_n$ is defined by

$$\|u\|_\epsilon = \sup\left\{ \left| \sum_{k=1}^m x_1^*(x_{1,k}) \cdots x_n^*(x_{n,k}) \right| : u = \sum_{k=1}^m x_{1,k} \otimes \cdots \otimes x_{n,k}, x_k^* \in B_{X_k^*} \right\} \tag{3.8}$$

for every $u \in X_1 \otimes \cdots \otimes X_n$. The completion of $X_1 \otimes \cdots \otimes X_n$ with respect to this norm is denoted by $X_1 \hat{\otimes}_\epsilon \cdots \hat{\otimes}_\epsilon X_n$ and called the *n-fold injective tensor product* of $X_1, \cdots, X_n$.

For each $u \in X_1 \otimes \cdots \otimes X_n$, say, $u = \sum_{k=1}^m x_{1,k} \otimes \cdots \otimes x_{n,k}$, define $T_u : X_1^* \times \cdots \times X_n^* \to \mathbb{R}$ by

$$T_u(x_1^*, \cdots, x_n^*) = \sum_{k=1}^m x_1^*(x_{1,k}) \cdots x_n^*(x_{n,k}), \quad \forall\, x_i^* \in X_i^*, (i = 1, \cdots, n). \tag{3.9}$$

Then T_u is a finite-rank n-linear operator (which does not depend on the representations of u) and hence, $T_u \in \mathcal{L}(X_1^*, \cdots, X_n^*; \mathbb{R})$ with $\|T_u\| = \|u\|_\epsilon$.

The following lemma is straightforward from the definition.

Lemma 3.10. *Let $T_i : X_i \to X_i$ be bounded linear operators for $i = 1, \ldots, n$. Then*

$$\left\| \sum_{k=1}^{m} T_1(x_{1,k}) \otimes \cdots \otimes T_n(x_{n,k}) \right\|_{\epsilon} \leqslant \|T_1\| \cdots \|T_n\| \cdot \left\| \sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k} \right\|_{\epsilon}$$

for every $x_{1,k} \in X_1, \ldots, x_{n,k} \in X_n$, $k = 1, \ldots, m$.

Definition 3.11. The *symmetric injective tensor norm* on $\otimes_{n,s}X$ is defined by

$$\|u\|_{s,\epsilon} = \sup \left\{ \left| \sum_{k=1}^{m} \lambda_k \cdot \Big(x^*(x_k)\Big)^n \right| : u = \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k, x^* \in B_{X^*} \right\}$$

for every $u \in \otimes_{n,s}X$. The completion of $\otimes_{n,s}X$ with respect to this norm is denoted by $\check{\otimes}_{n,s,\epsilon}X$ and called the *n-fold symmetric injective tensor product* of X.

For each $u \in \otimes_{n,s}X$, say, $u = \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k$, define $P_u : X^* \to \mathbb{R}$ by

$$P_u(x^*) = \sum_{k=1}^{m} \lambda_k \cdot \Big(x^*(x_k)\Big)^n, \quad \forall\, x^* \in X^*. \tag{3.10}$$

Then P_u is an n-homogeneous polynomial (which does not depend on the representations of u) and $P_u \in \mathcal{P}(^n X^*; \mathbb{R})$ with $\|P_u\| = \|u\|_{s,\epsilon}$.

The following lemma is straightforward from the definition.

Lemma 3.12. *Let $T : X \to X$ be a bounded linear operator. Then*

$$\left\| \sum_{k=1}^{m} \lambda_k T(x_k) \otimes \cdots \otimes T(x_k) \right\|_{s,\epsilon} \leqslant \|T\|^n \cdot \left\| \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k \right\|_{s,\epsilon}$$

for every $x_k \in X$, $k = 1, \ldots, m$.

By linearly extending, equation (3.4) defines a linear projection $s : \otimes_n X \to \otimes_{n,s}X$. The linear projection $s : \otimes_n X \to \otimes_{n,s}X$ can be extended to $\check{\otimes}_{n,\epsilon}X$ with $\check{\otimes}_{n,s,\epsilon}X$ as its range and for every $u \in \check{\otimes}_{n,\epsilon}X$ (see [31]), we have

$$\left\|s(u)\right\|_{s,\epsilon} \leqslant \left\|s(u)\right\|_{\epsilon} \leqslant \frac{n^n}{n!}\left\|s(u)\right\|_{s,\epsilon}. \tag{3.11}$$

For the basic knowledge about n-linear operators, n-homogeneous polynomials, n-fold (symmetric) projective tensor products, and n-fold(symmetric) injective tensor products, we refer to [16, 37, 42, 45, 31, 14].

3.2 Positive Multilinear Operators and Positive Tensor Products of Banach Lattices

Let $E_1, \cdots, E_n, E$ and F be an (Archimedean) vector lattice.

Definition 3.13. An n-linear operator $T : E_1 \times \cdots \times E_n \to F$ is called *positive* if $T(x_1, \cdots, x_n) \in F^+$ whenever $x_1 \in E_1^+, \cdots, x_n \in E_n^+$. And T is called *regular* if T is the difference of two positive n-linear operators.

Let $\mathcal{L}^r(E_1, \cdots, E_n; F)$ denote the space of all regular n-linear operators from $E_1 \times \cdots \times E_n$ to F. If, in addition, F is Dedekind complete then by [35, Lemma 2.12 and Proposition 2.14], $\mathcal{L}^r(E_1, \cdots, E_n; F)$ is a Dedekind complete vector lattice.
Let E, F be (Archimedean) vector lattices.

Definition 3.14. An n-homogeneous polynomial $P : E \to F$ is called *positive* if its corresponding symmetric n-linear operator T_P is positive. And P is called *regular* if it is the difference of two positive polynomials.

It is easy to see that P is regular if and only if T_P is regular. Let $\mathcal{P}^r({}^nE; F)$ denote the space of all regular n-homogeneous polynomials from E to F. If, in addition, F is Dedekind complete then by [35, Lemma 2.15 and Lemma 2.16], $\mathcal{P}^r({}^nE; F)$ is a Dedekind complete vector lattice.

If, in addition, $E_1, \cdots, E_n, E, F$ are Banach lattices such that F is Dedekind complete then $\mathcal{L}^r(E_1, \cdots, E_n; F)$ is a Banach lattice with the *regular operator norm* $\|T\|_r = \| \, |T| \, \|$ for every $T \in \mathcal{L}^r(E_1, \cdots, E_n; F)$. Then also $\mathcal{P}^r({}^nE; F)$ is a Banach lattice with the *regular polynomial norm* $\|P\|_r = \| \, |P| \, \|$ for every $P \in \mathcal{P}^r({}^nE; F)$(e.g., see [3]).

Lemma 3.15. *Let $E_1, \ldots, E_n, F$ be Banach lattices with F Dedekind complete. Then for any $T \in \mathcal{L}^r(E_1, \ldots, E_n; F)$,*

$$\|T\|_r = \inf\Big\{\|S\| : S \in \mathcal{L}^r(E_1, \ldots, E_n; F)^+,$$
$$|T(x_1, \ldots, x_n)| \leqslant S(|x_1|, \ldots, |x_n|),\ \forall\, x_1 \in E_1, \ldots,\ \forall\, x_n \in E_n\Big\}. \quad (3.12)$$

Moreover, $\|T\| \leqslant \|T\|_r$.

Proof. Let

$$\mathcal{A} := \Big\{S : S \in \mathcal{L}^r(E_1, \ldots, E_n; F)^+,$$
$$|T(x_1, \ldots, x_n)| \leqslant S(|x_1|, \ldots, |x_n|),\ \forall\, x_1 \in E_1, \ldots,\ \forall\, x_n \in E_n\Big\},$$

and let $a = \inf\{\|S\| : S \in \mathcal{A}\}$. For any $S \in \mathcal{A}$, it follows from (2.10) in [3] that $|T| \leqslant S$. Thus $\|\,|T|\,\| \leqslant \|S\|$ and hence, $\|T\|_r \leqslant a$. On the other hand, $|T(x_1, \ldots, x_n)| \leqslant |T|(|x_1|, \ldots, |x_n|)$ for every $x_1 \in E_1, \ldots, x_n \in E_n$. Thus $|T| \in \mathcal{A}$ and hence, $a \leqslant \|\,|T|\,\| = \|T\|_r$. Therefore, $\|T\|_r = a$. Also,

$$\begin{aligned}
\|T\| &= \sup\{\|T(x_1, \cdots, x_n)\| : x_1 \in B_{X_1}, \cdots, x_n \in B_{X_n}\} \\
&= \sup\{\||T(x_1, \cdots, x_n)|\| : x_1 \in B_{X_1}, \cdots, x_n \in B_{X_n}\} \\
&\leqslant \sup\{\|S(|x_1|, \cdots, |x_n|)\| : x_1 \in B_{X_1}, \cdots, x_n \in B_{X_n}\} \\
&\leqslant \sup\{\|S\| \cdot \||x_1|\| \cdots \||x_n|\| : x_1 \in B_{X_1}, \cdots, x_n \in B_{X_n}\} \\
&\leqslant \|S\|.
\end{aligned}$$

Since $a = \inf\{\|S\| : S \in \mathcal{A}\} = \|T\|_r$ and $\|T\| \leqslant \inf\{\|S\| : S \in \mathcal{A}\} = \|T\|_r$.
Thus we have $\|T\| \leqslant \|T\|_r$.

□

Similarly, we have

Lemma 3.16. *Let E and F be Banach lattices with F Dedekind complete. Then for any $P \in \mathcal{P}^r({}^nE; F)$,*

$$\|P\|_r = \inf \Big\{ \|R\| : \ R \in \mathcal{P}^r({}^nE; F)^+, |P(x)| \leqslant R(|x|), \ \forall \, x \in E \Big\}. \tag{3.13}$$

Moreover, $\|P\| \leqslant \|P\|_r$.

Next we will introduce n-fold positive projective tensor products.

Definition 3.17. Let $E_1, \cdots, E_n, E, F$ be vector lattices. An n-linear operator $T : E_1 \times \cdots \times E_n \to F$ is called a *lattice n-morphism* if $|T(x_1, \cdots, x_n)| = T(|x_1|, \cdots, |x_n|)$.

Let $(E_1 \bar{\otimes} \cdots \bar{\otimes} E_n, \otimes)$ denote the *n-fold (Archimedean) vector lattice tensor product* of $E_1, \cdots, E_n$. We collect the following well known facts about this tensor product.

(a) $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ is a vector lattice and $\otimes$ is a lattice n-morphism from $E_1 \times \cdots \times E_n$ to $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ defined by $\otimes(x_1, \cdots, x_n) = x_1 \otimes \cdots \otimes x_n$ for every $x_1 \in E_1, \cdots, x_n \in E_n$.

(b) For any (Archimedean) vector lattice F there is a one to one correspondence between lattice n-morphisms $T : E_1 \times \cdots \times E_n \to F$ and lattice homomorphisms $T^{\otimes} : E_1 \bar{\otimes} \cdots \bar{\otimes} E_n \to F$ given by $T = T^{\otimes} \circ \otimes$.

(c) For any uniformly complete (Archimedean) vector lattice F there is a one to one correspondence between positive n-linear maps $T : E_1 \times \cdots \times E_n \to F$ and increasing linear maps $T^{\otimes} : E_1 \bar{\otimes} \cdots \bar{\otimes} E_n \to F$ given by $T = T^{\otimes} \circ \otimes$.

(d) $E_1 \otimes \cdots \otimes E_n$ is dense in $E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ in the sense that for any $u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ there exist $x_1 \in E_1^+, \cdots, x_n \in E_n^+$ such that, for every $\delta > 0$, there is $v \in E_1 \otimes \cdots \otimes E_n$ with $|u - v| \leq \delta x_1 \otimes \cdots \otimes x_n$.

(e) If $u \in E_1 \bar{\otimes} \cdots \bar{\otimes} E_n$ then there exist $x_1 \in E_1^+, \cdots, x_n \in E_n^+$ such that $|u| \leq x_1 \otimes \cdots \otimes x_n$.

(f) $E_1\otimes\cdots\otimes E_n$ is order dense in $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ in the sense that if $u>0$ in $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ then there exist $x_1>0$ in E_1, $\cdots$, $x_n>0$ in E_n such that $u\geq x_1\otimes\cdots\otimes x_n>0$.

Definition 3.18. For Banach lattices $E_1,\cdots,E_n$, the *positive projective tensor norm* $\|\cdot\|_{|\pi|}$ on $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ is defined by

$$\|u\|_{|\pi|}=\inf\Big\{\sum_{k=1}^{m}\|x_{1,k}\|\cdots\|x_{n,k}\|:x_{i,k}\in E_i^+,|u|\leq\sum_{k=1}^{m}x_{1,k}\otimes\cdots\otimes x_{n,k}\Big\}$$

for every $u\in E_1\bar{\otimes}\cdots\bar{\otimes}E_n$.

Then $\|\cdot\|_{|\pi|}$ is a lattice norm on $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ and

(g) $E_1\otimes\cdots\otimes E_n$ is norm dense in $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$, and

(h) the cone generated by $\{x_1\otimes\cdots\otimes x_n : x_k\in E_k^+, 1\leq k\leq n\}$ is norm dense in $(E_1\bar{\otimes}\cdots\bar{\otimes}E_n)^+$.

Let $E_1\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}E_n$ denote the completion of $E_1\bar{\otimes}\cdots\bar{\otimes}E_n$ under the lattice norm $\|\cdot\|_{|\pi|}$. Then $E_1\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}E_n$ is a Banach lattice, called the n-fold Fremlin projective tensor product, or n-fold positive projective tensor product of $E_1,\cdots,E_n$.

Proposition 3.19. *[7] Let $E_1,\cdots,E_n$, F be Banach lattices such that F is Dedekind complete. Then $\mathcal{L}^r(E_1,\cdots,E_n;F)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r(E_1\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}E_n;F)$.*

Definition 3.20. For a Banach lattice E, let $\bar{\otimes}_{n,s}E$ denote the *n-fold vector lattice symmetric tensor product* of E. The *positive symmetric projective tensor norm* on $\bar{\otimes}_{n,s}E$ is defined by

$$\|u\|_{s,|\pi|}=\inf\Big\{\sum_{k=1}^{m}\|x_k\|^n : x_k\in E^+,|u|\leq\sum_{k=1}^{m}x_k\otimes\cdots\otimes x_k\Big\}\quad(\forall\ u\in\bar{\otimes}_{n,s}E).$$

Then $\|\cdot\|_{s,|\pi|}$ is a lattice norm on $\bar{\otimes}_{n,s}E$. Let $\hat{\otimes}_{n,s,|\pi|}E$ denote the completion of $\bar{\otimes}_{n,s}E$ under the lattice norm $\|\cdot\|_{s,|\pi|}$. Then $\hat{\otimes}_{n,s,|\pi|}E$ is a Banach lattice, called the *n-fold Fremlin symmetric tensor product* or the *n-fold positive symmetric projective tensor product* of E.

Proposition 3.21. *[7] Let E and F be Banach lattices such that F is Dedekind complete. Then $\mathcal{P}^r({}^nE;F)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r(\hat{\otimes}_{n,s,|\pi|}E;F)$.*

In particular, we have $(\hat{\otimes}_{n,s,|\pi|}E)^* = \mathcal{P}^r({}^nE;\mathbb{R})$.

Let $E_1,\cdots,E_n$ be Banach lattices. For any $u\in E_1\otimes\cdots\otimes E_n$, say, $u=\sum_{i=1}^m x_{i,1}\otimes\cdots\otimes x_{i,n}$, define $T_u : E_1^*\times\cdots\times E_n^*\to\mathbb{R}$ by

$$T_u(x_1^*,\cdots,x_n^*)=\sum_{i=1}^m x_1^*(x_{i,1})\cdots x_n^*(x_{i,n}),\quad \forall\, x_k^*\in E_k^*,\quad k=1,\cdots,n.$$

Then T_u is a finite-rank n-linear operator (which does not depend on the representations of u) and hence, $T_u\in\mathcal{L}^r(E_1^*,\cdots,E_n^*;\mathbb{R})$.

Definition 3.22. Let $E_1\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}E_n$ denote the closed sublattice generated by $E_1\otimes\cdots\otimes E_n$ in $\mathcal{L}^r(E_1^*,\cdots,E_n^*;\mathbb{R})$, called the *n-fold positive injective tensor product* of $E_1,\cdots,E_n$. The norm on $E_1\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}E_n$ is denoted by $\|\cdot\|_{|\epsilon|}$, that is, for every $u\in E_1\otimes\cdots\otimes E_n$, $\|u\|_{|\epsilon|}=\|T_u\|_r$.

By Lemma 3.15, $\|u\|_\epsilon\leqslant\|u\|_{|\epsilon|}$. In particular, if u is a positive element in $E_1\otimes\cdots\otimes E_n$, then

$$\|u\|_{|\epsilon|}=\sup\left\{\left|\sum_{k=1}^m x_1^*(x_{1,k})\cdots x_n^*(x_{n,k})\right| : u=\sum_{k=1}^m x_{1,k}\otimes\cdots\otimes x_{n,k}, x_i^*\in B^+_{E_i^*}, 1\leqslant i\leqslant n\right\}.$$

For any $u\in E_1\otimes\cdots\otimes E_n$, it is easy to see that the operator $T_u : E_1^*\times\cdots\times E_n^*\to\mathbb{R}$ defined in (3.9) is a regular n-linear operator and hence, $T_u\in\mathcal{L}^r(E_1^*,\dots,E_n^*;\mathbb{R})$.

If $E_1 = \cdots = E_n = E$, we write $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_n$ by $\check{\otimes}_{n,|\epsilon|} E$. For any $u \in \otimes_{n,s} E$, it is easy to see that the polynomial $P_u : E^* \to \mathbb{R}$ defined in (3.10) is a regular n-homogeneous polynomial and hence, $P_u \in \mathcal{P}^r({}^n E^*; \mathbb{R})$.

Definition 3.23. Let $\check{\otimes}_{n,s,|\epsilon|} E$ denote the closed sublattice generated by $\otimes_{n,s} E$ in $\mathcal{P}^r({}^n E^*; \mathbb{R})$, called the *$n$-fold positive symmetric injective tensor product* of E. The norm on $\check{\otimes}_{n,s,|\epsilon|} E$ is denoted by $\| \cdot \|_{s,|\epsilon|}$, that is, for every $u \in \otimes_{n,s} E$, $\|u\|_{s,|\epsilon|} = \|P_u\|_r$ and $\|u\|_{s,\epsilon} \leqslant \|u\|_{s,|\epsilon|}$.

In particular, if u is a positive element in $\otimes_{n,s} E$, then

$$\|u\|_{s,|\epsilon|} = \sup \left\{ \left| \sum_{k=1}^{m} \lambda_k \cdot \Big(x^*(x_k) \Big)^n \right| : u = \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k, x^* \in B_{E^*}^+ \right\}.$$

For the basic knowledge about positive n-linear operators, positive n-homogeneous polynomials, positive n-fold (symmetric) projective tensor products, and positive n-fold(symmetric) injective tensor products, we refer to [17, 18, 47, 21, 7].

4 POSITIVE TENSOR PRODUCTS OF ℓ_p-SPACES

* All of results in 4.1 and 4.2 have been published in my paper "On positive tensor products of ℓ_p-spaces". This is joint work with Qingying Bu and Donghai Ji.(See [28].)

4.1 Positive Projective Tensor Products of ℓ_p-spaces

For $1 \leqslant p < \infty$, let $\{e_i : i \in \mathbb{N}\}$ denote the standard unit vector basis of ℓ_p. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ with the *square order* (see [19, 42], or [20]) is a basis (not necessary an unconditional basis) of the projective tensor product $\ell_{p_1} \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \ell_{p_n}$. Similar to the proof of [8, Lemma 22], we present the following lemma about the basis of the Fremlin projective tensor product $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$.

Lemma 4.1. *Let $1 \leqslant p_1, \cdots, p_n < \infty$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ with any order is a 1-unconditional basis of $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$.*

Recall that a Banach lattice E is called *order continuous* if $\|x_\alpha\| \downarrow 0$ whenever $x_\alpha \downarrow 0$. Equivalently, E is order continuous if and only if every monotone order bounded sequence in E is norm convergent.

Definition 4.2. A Banach lattice E is said to be a *Kantorovich-Banach space* (KB-space in short) if every monotone norm bounded sequence in E is norm convergent.

Equivalently, E is a KB-space if and only if E does not contain any sublattice ismorphic to c_0.

Note that $\ell_{p_1}\hat{\otimes}_{|\pi|}\ell_{p_2}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n} = \ell_{p_1}\hat{\otimes}_{|\pi|}(\ell_{p_2}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})$. By induction, [4, Corollary 7.4] yields the following lemma.

Lemma 4.3. *Let $1 \leqslant p_1, \cdots, p_n < \infty$. Then $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ is a KB-space, and hence it is order continuous.*

The following Rademacher averaging is a generalization of Lemma 2.22 in [45, p.34].

Rademacher averaging: *Let $Z_1, \cdots, Z_n$ be vector spaces and $x_{i,k} \in Z_i$ for $i = 1, \cdots, n$ and $k = 1, \cdots, m$. Then*

$$\sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k} = \int_0^1 \left(\sum_{k=1}^{m} r_k(t) x_{1,k} \right) \otimes \cdots \otimes \left(\sum_{k=1}^{m} r_k(t) x_{n,k} \right) dt,$$

where $\{r_k(t)\}_1^\infty$ is the sequence of Rademacher functions on $[0, 1]$.

The main diagonal of the projective tensor product $\ell_{p_1}\hat{\otimes}_\pi\cdots\hat{\otimes}_\pi\ell_{p_n}$ was characterized by Arias and Farmer [2]. Next by using Rademacher averaging, we will chacterize the main diagonal of the Fremlin projective tensor product $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$. Let $\Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})$ denote the *main diagonal* of $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$, that is, the closed subspace spanned by the diagonal vectors $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$. Next we will use Rademacher averaging theorem to characterize the main diagonal $\Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})$.

Theorem 4.4. *Let $1 \leqslant p_1, \cdots, p_n < \infty$ and let $\frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i}$. Then $\Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})$ is isometrically lattice isomorphic to ℓ_p if $p > 1$ and ℓ_1 if $p \leqslant 1$.*
i.e.

$$\Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}) \equiv \begin{cases} \ell_p, & if\, p > 1 \\ \ell_1, & if\, p \leq 1. \end{cases}$$

Proof. Case 1: $p > 1$.

Take any positive element $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i$ in $\Delta(\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})$, where $a_i \geqslant 0$ for $i = 1, \cdots, m$. By Rademacher averaging,

$$u = \int_0^1 \left(\sum_{i=1}^{m} a_i^{\frac{p}{p_1}} r_i(t) e_i \right) \otimes \cdots \otimes \left(\sum_{i=1}^{m} a_i^{\frac{p}{p_n}} r_i(t) e_i \right) dt$$

and hence,

$$\begin{aligned} \|u\|_{|\pi|} &\leqslant \sup_{0 \leqslant t \leqslant 1} \left\| \sum_{i=1}^{m} a_i^{\frac{p}{p_1}} r_i(t) e_i \right\|_{\ell_{p_1}} \cdots \left\| \sum_{i=1}^{m} a_i^{\frac{p}{p_n}} r_i(t) e_i \right\|_{\ell_{p_n}} \\ &= \left(\sum_{i=1}^{m} a_i^p \right)^{\frac{1}{p_1}} \cdots \left(\sum_{i=1}^{m} a_i^p \right)^{\frac{1}{p_n}} = \left\| (a_i)_{i=1}^m \right\|_{\ell_p}. \end{aligned}$$

On the other hand, define a positive n-linear form T on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ by

$$T(x^{(1)}, \cdots, x^{(n)}) = \sum_{i=1}^{m} a_i^{p-1} x_i^{(1)} \cdots x_i^{(n)}, \quad \forall\, x^{(k)} = \left(x_i^{(k)}\right)_i \in \ell_{p_k}, k = 1, \cdots, n.$$

Let $\frac{1}{t} := 1 - \frac{1}{p}$. By Hölder inequality,

$$\left| T(x^{(1)}, ..., x^{(n)}) \right| \leqslant \left\| \left(a_i^{p-1}\right)_{i=1}^m \right\|_{\ell_t} \cdot \left\| \left(x_i^{(1)}\right)_i \right\|_{\ell_{p_1}} \cdots \left\| \left(x_i^{(n)}\right)_i \right\|_{\ell_{p_n}},$$

which implies that

$$\|T\|_r = \|T\| \leqslant \left\| \left(a_i^{p-1}\right)_{i=1}^m \right\|_{\ell_t} = \left(\sum_{i=1}^{m} a_i^p \right)^{\frac{1}{t}}.$$

Note that $T \in \mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R}) = (\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})^*$ and that

$$\langle u, T \rangle = \sum_{i=1}^{m} a_i T(e_i, \cdots, e_i) = \sum_{i=1}^{m} a_i^p.$$

Thus

$$\sum_{i=1}^{m} a_i^p = \langle u, T \rangle \leqslant \|u\|_{|\pi|} \cdot \|T\|_r \leqslant \|u\|_{|\pi|} \cdot \left(\sum_{i=1}^{m} a_i^p \right)^{\frac{1}{t}},$$

which implies that

$$\|u\|_{|\pi|} \geqslant \left(\sum_{i=1}^{m} a_i^p\right)^{\frac{1}{p}} = \left\|(a_i)_{i=1}^m\right\|_{\ell_p}.$$

Therefore, $\|u\|_{|\pi|} = \|(a_i)_{i=1}^m\|_{\ell_p}$ and hence, $\Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})$ is isometrically lattice isomorphic to ℓ_p.

Case 2: $p \leqslant 1$.

Take any positive element $u = \sum_{i=1}^m a_i e_i \otimes \cdots \otimes e_i$ in $\Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})$, where $a_i \geqslant 0$ for $i = 1, \cdots, m$. Define a positive n-linear form T on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ by

$$T(x^{(1)}, \cdots, x^{(n)}) = \sum_{i=1}^{m} x_i^{(1)} \cdots x_i^{(n)}, \quad \forall\, x^{(k)} = \left(x_i^{(k)}\right)_i \in \ell_{p_k}, k = 1, \cdots, n.$$

Take $q_k \geqslant p_k$ for $k = 1, \cdots, n$ such that $\frac{1}{q_1} + \cdots + \frac{1}{q_n} = 1$. By Hölder inequality,

$$\left|T(x^{(1)}, ..., x^{(n)})\right| \leqslant \left\|\left(x_i^{(1)}\right)_i\right\|_{\ell_{q_1}} \cdots \left\|\left(x_i^{(n)}\right)_i\right\|_{\ell_{q_n}} \leqslant \left\|\left(x_i^{(1)}\right)_i\right\|_{\ell_{p_1}} \cdots \left\|\left(x_i^{(n)}\right)_i\right\|_{\ell_{p_n}},$$

which implies that $\|T\|_r = \|T\| \leqslant 1$. Note that $\langle u, T\rangle = \sum_{i=1}^m a_i T(e_i, \cdots, e_i) = \sum_{i=1}^m a_i$. Thus

$$\sum_{i=1}^{m} a_i = \langle u, T\rangle \leqslant \|T\|_r \cdot \|u\|_{|\pi|} \leqslant \|u\|_{|\pi|}.$$

On the other hand, $\|u\|_{|\pi|} \leqslant \sum_{i=1}^m \|a_i e_i \otimes \cdots \otimes e_i\|_{|\pi|} = \sum_{i=1}^m a_i$. Therefore, $\|u\|_{|\pi|} = \|(a_i)_{i=1}^m\|_{\ell_1}$ and hence, $\Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})$ is isometrically lattice isomorphic to ℓ_1. $\square$

Remark 4.5. In Theorem 4.4, if some (or all) of p_i's are ∞, then we consider $\ell_{p_i} = c_0$ if $p_i = \infty$, and consider $\ell_p = c_0$ if $p = \infty$. In this case, Theorem 4.4 is still true. Thus we have the first special case, if all p_i's are ∞ then $\Delta(c_0\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}c_0)$ is isometrically lattice isomorphic to c_0. The second special case is that if at least one of p_i's is 1, then $\Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})$ is isometrically lattice isomorphic to ℓ_1.

Next we will use the main diagonal of $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ to characterize its reflexivity. First we need the following lemma, which might be known. We provide a proof here since we do not find one handy in the literature.

Lemma 4.6. *Let E be an order continuous Banach lattice. Then E is reflexive if and only if every positive linear functional on E attains its norm.*

Proof. Suppose that every positive linear functional on E attains its norm. By James' theorem, we only need to show that every continuous linear functional on E attains its norm. Take any $f \in E^*$. Then there exists $x \in B_E$ such that $\|f\| = \|\,|f|\,\| = |f|(x)$. Note that $|f|(x) = |f|(x^+) - |f|(x^-)$. Without loss of generality, we assume that $x \in E^+$. Let $N(f) = \{z \in E : |f|(|z|) = 0\}$ be the null ideal of f and let $C(f) = N(f)^\perp$. Since E is order continuous, both $N(f)$ and $C(f)$ are projection bands.
Let $P^+ : E \to C(f^+)$ and $P^- : E \to C(f^-)$ be the respective band projections. As, for example, $(I - P^+)x \in C(f^+)^\perp = N(f^+)$ we have

$$f^+(x) = f^+(P^+x) + f^+((I - P^+)x) = f^+(P^+x).$$

By [36, Theorem 1.4.11] we have, for example, $P^-x \in C(f^-) \subseteq N(f^+)$ so that $f^+(P^-x) = 0$. Similarly we have $f^-(x) = f^-(P^-x)$ and $f^-(P^+x) = 0$. Thus

$$\begin{aligned} f(P^+x - P^-x) &= (f^+ - f^-)(P^+x - P^-x) \\ &= f^+(P^+x) - f^+(P^-x) - f^-(P^+x) + f^-(P^-x) \\ &= f^+(x) + f^-(x) = |f|(x) = \|f\|. \end{aligned}$$

Also, as $P^+x \perp P^-x$ and $0 \leqslant P^+x, P^-x \leqslant x$ we have $|P^+x - P^-x| \leqslant x$ and therefore $\|P^+x - P^-x\| \leqslant \|x\| \leqslant 1$ and thus f attains its norm. □

Remark 4.7. In Lemma 4.6, the order continuity of E is essential. For instance, let K be a compact Hausdorff space and $C(K)$ be the Banach lattice of all continuous functions on K. Then Riesz Representation Theorem implies that every positive linear functional on $C(K)$ attains its norm. However, $C(K)$ is not reflexive unless K is a finite set.

Theorem 4.8. *Let $1 \leqslant p_1, \cdots, p_n < \infty$. Then $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ is reflexive if and only if $1/p_1 + \cdots + 1/p_n < 1$.*

Proof. If $1/p_1 + \cdots + 1/p_n \geqslant 1$, then Theorem 4.4 implies that $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ is not reflexive. If $1/p_1 + \cdots + 1/p_n < 1$, then each $p_i > 1$ and hence, each ℓ_{p_i} is reflexive. Take any positive $T \in \mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}) = (\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})^*$. Since

$$\|T\|_r = \|T\| = \sup\left\{\left|T(x^{(1)}, \cdots, x^{(n)})\right| : x^{(i)} \in B_{\ell_{p_i}}, 1 \leqslant i \leqslant n\right\},$$

there exist sequences $\{x_k^{(i)}\}_{k=1}^{\infty}$ in $B_{\ell_{p_i}}$ $(1 \leqslant i \leqslant n)$ such that

$$\lim_k \left|T(x_k^{(1)}, \cdots, x_k^{(n)})\right| = \|T\|_r.$$

Moreover, there exist subsequences $\{x_{i_k}^{(i)}\}_{k=1}^{\infty}$ of $\{x_k^{(i)}\}_{k=1}^{\infty}$ and $x_0^{(i)}$ in ℓ_{p_i} $(1 \leqslant i \leqslant n)$ such that $\lim_k x_{i_k}^{(i)} = x_0^{(i)}$ weakly in ℓ_{p_i}. By [43, Theorem 4.1], T is weakly sequentially continuous. Thus

$$\|T\|_r = \lim_k \left|T(x_{1_k}^{(1)}, \cdots, x_{n_k}^{(n)})\right| = \left|T(x_0^{(1)}, \cdots, x_0^{(n)})\right|$$

and hence, T attains its norm. It follows from Lemma 4.3 and Lemma 4.6 that $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ is reflexive. □

Theorem 4.4 and Theorem 4.8 yield the following corollary.

Corollary 4.9 *Let $1 \leqslant p_1, \cdots, p_n < \infty$. Then $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ does not contain any complemented sublattice isomorphic to ℓ_1 if and only if $1/p_1 + \cdots + 1/p_n < 1$.*

Next we will show that in the space of all regular n-linear forms on $\ell_{p_1} \times \cdots \times \ell_{p_n}$, the reflexivity, the property of being a KB-space, and the property of being order continuous are all equivalent.

Theorem 4.10 *Let $1 \leqslant p_1, \cdots, p_n < \infty$. Then the following statements are equivalent:*

(i) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ *is reflexive.*

(ii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ *is a KB-space.*

(iii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ *is order continuous.*

(iv) $1/p_1 + \cdots + 1/p_n < 1$.

Proof. Note that $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R}) = (\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})^*$. If $1/p_1 + \cdots + 1/p_n < 1$, then Theorem 4.8 implies that $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ is reflexive, and (iv) $\Rightarrow$ (i) follows. It is trivial that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Now suppose (iii) holds. By [36, p.93, Theorem 2.4.14], $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ does not contain any sublattice isomorphic to ℓ_∞ and hence, $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$ does not contain any complemented sublattice isomorphic to ℓ_1. Corollary 4.9 implies that $1/p_1 + \cdots + 1/p_n < 1$ and (iv) follows. □

Note that $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q) = \mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}, \ell_{q'}; \mathbb{R})$ for any $1 < q < \infty$. Theorem 4.10 yields the following corollary.

Corollary 4.11. *Let $1 \leqslant p_1, \cdots, p_n < \infty$ and let $1 < q < \infty$. Then the following statements are equivalent:*

(i) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ *is reflexive.*

(ii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ *is a KB-space.*

(iii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ *is order continuous.*

(iv) $1/p_1 + \cdots + 1/p_n < 1/q$.

To consider the case $q = 1$ in Corollary 4.11, we need the following lemma.

Lemma 4.12. *Let λ be a reflexive Banach sequence lattice and F a Banach lattice. Then $\mathcal{L}^r(\lambda; F^*)$ is a KB-space if and only if F^* is a KB-space and every positive linear operator from λ to F^* is compact.*

Proof. Note that $\mathcal{L}^r(\lambda; F^*) = (\lambda\hat{\otimes}_{|\pi|}F)^*$. The theorem follows from [9, Theorem 6.9] and [36, p.93, Theorem 2.4.14]. □

The case $q = 1$ in Corollary 4.11 is as follows.

Corollary 4.13. *Let $1 \leqslant p_1, \cdots, p_n < \infty$. Then the following statements are equivalent:*

(i) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_1)$ *is a KB-space.*

(ii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_1)$ *is order continuous.*

(iii) $1/p_1 + \cdots + 1/p_n < 1$.

Proof. Note that

$$\mathcal{L}^r(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}; \ell_1) = \mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_1) = (\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}\hat{\otimes}_{|\pi|}c_0)^*.$$

Suppose (iii). Then Theorem 4.8 implies that $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ is reflexive. Thus every positive linear operator from $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$ to ℓ_1 is compact and hence, (i) follows from Lemmas 4.1 and 4.12. It is trivial that (i) $\Rightarrow$ (ii). Now suppose (ii). Then $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R}) = (\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n})^*$ is order continuous and hence, Theorem 4.10 implies (iii). □

Let $\Delta(\hat{\otimes}_{n,s,|\pi|}\ell_p)$ denote the *main diagonal* of $\hat{\otimes}_{n,s,|\pi|}\ell_p$, that is, the closed subspace spanned by the diagonal vectors $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\hat{\otimes}_{n,s,|\pi|}\ell_p$. Parallel to the Fremlin projective tensor product $\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}$, we have the same results for the Fremlin projective symmetric tensor product $\hat{\otimes}_{n,s,|\pi|}\ell_p$ as follows.

Theorem 4.14. *Let $1 \leqslant p < \infty$.*

(i) *$\Delta(\hat{\otimes}_{n,s,|\pi|}\ell_p)$ is isometrically lattice isomorphic to $\ell_{\frac{p}{n}}$ if $p > n$ and ℓ_1 if $p \leqslant n$.*

(ii) *$\hat{\otimes}_{n,s,|\pi|}\ell_p$ is reflexive if and only if $p > n$.*

(iii) *$\hat{\otimes}_{n,s,|\pi|}\ell_p$ does not contain any complemented sublattice isomorphic to ℓ_1 if and only if $p > n$.*

(iv) *$\hat{\otimes}_{n,s,|\pi|}\ell_p$ is a KB-space and hence, it is order continuous.*

Proof. (i) Note that $\Delta(\hat{\otimes}_{n,s,|\pi|}\ell_p) = \Delta(\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p)$. Let $\frac{1}{q} = \sum_{i=1}^{n}\frac{1}{p} = \frac{n}{p}$, then Theorem 4.4 implies that

$$\Delta(\hat{\otimes}_{n,s,|\pi|}\ell_p) \equiv \begin{cases} \ell_q, & \text{if } q > 1 \\ \ell_1, & \text{if } q \leq 1. \end{cases}$$

where $q = \frac{p}{n}$.

(ii) $\hat{\otimes}_{n,s,|\pi|}\ell_p = (\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p)$ is reflexive if and only if $\frac{n}{p} < 1$ from Theorem 4.8. If $p > n$ then $\hat{\otimes}_{n,s,|\pi|}\ell_p$ is isometrically lattice isomorphic to $\ell_{\frac{p}{n}}$ and it is reflexive.

(iii) $\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p$ does not contain any complemented sublattice isomorphic to ℓ_1 if and only if $1/p + \cdots + 1/p = \frac{n}{p} < 1$ from Corollary 4.9.

(iv) Since $\ell_p(1 \leq p < \infty)$ is a KB-space and $\ell_p\hat{\otimes}_{|\pi|}\ell_p$ is a KB-space (see [4]). Note that $\ell_p\hat{\otimes}_{|\pi|}\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p = \ell_p\hat{\otimes}_{|\pi|}(\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p)$. Therefore $\hat{\otimes}_{n,s,|\pi|}\ell_p$ is a KB-space and hence, it is order continuous. □

Parallel to the space $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$, we have the same results for the space $\mathcal{P}^r({}^n\ell_p; \mathbb{R})$ of all regular n-homogeneous polynomials on ℓ_p as follows.

Theorem 4.15. *Let $1 \leqslant p < \infty$. Then the following statements are equivalent.*

(i) *$\mathcal{P}^r({}^n\ell_p; \mathbb{R})$ is reflexive.*

(ii) *$\mathcal{P}^r({}^n\ell_p; \mathbb{R})$ is a KB-space.*

(iii) *$\mathcal{P}^r({}^n\ell_p; \mathbb{R})$ is order continuous.*

(iv) *$p > n$.*

Proof. Note that $(\hat{\otimes}_{n,s,|\pi|}\ell_p)^* = \mathcal{P}^r({}^n\ell_p;\mathbb{R})$. If $p > n$, then Theorem 4.14 implies that $\hat{\otimes}_{n,s,|\pi|}\ell_p$ is reflexive. Thus $(\hat{\otimes}_{n,s,|\pi|}\ell_p)^* = \mathcal{P}^r({}^n\ell_p;\mathbb{R})$ is reflexive, and (iv) $\Rightarrow$ (i) follows. It is trivial that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Now suppose (iii) holds. By [36, p.93, Theorem 2.4.14], $\mathcal{P}^r({}^n\ell_p;\mathbb{R})$ does not contain any sublattice isomorphic to ℓ_∞ and hence, $\hat{\otimes}_{n,s,|\pi|}\ell_p$ does not contain any complemented sublattice isomorphic to ℓ_1. Corollary 4.9 implies that $1/p+\cdots+1/p = \frac{n}{p} < 1$ and (iv) follows. $\square$

Lemma 4.16. *Let $1 \leqslant p_1, \cdots, p_n < \infty$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ with any order is a 1-unconditional basis of $\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n}$.*

Proof. It is easy to see that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ is a disjoint sequence and hence, is a 1-unconditional basic sequence in $\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n}$. Take any $u = x_1 \otimes \cdots \otimes x_n \in \ell_{p_1} \otimes \cdots \otimes \ell_{p_n}$. Write

$$x_k = \sum_{i=1}^{\infty} a_{i,k} e_i, \quad k = 1, \cdots, n.$$

Then

$$u = \sum_{i_1, \cdots, i_n = 1}^{\infty} a_{i,1} \cdots a_{i,n} e_{i_1} \otimes \cdots \otimes e_{i_n},$$

which shows that the linear span of $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ is dense in $\ell_{p_1} \otimes \cdots \otimes \ell_{p_n}$ and hence, dense in $\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n}$. Therefore, $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ is a 1-unconditional basis of $\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n}$. □

By induction, [4, Theorem 7.3] and [9, Proposition 2.2, Theorems 3.1, 5.2] yield the following lemma.

Lemma 4.17. *Let $1 < p_1, \cdots, p_n < \infty$. Then $(\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n})^* = \ell_{p_1'} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n'}$.*

Hölder Inequality. *Let $1 \leqslant p_1, p_2 \leqslant \infty$ such that $1/p_1 + 1/p_2 = 1$. Then for every $(b_i^{(k)})_i \in \ell_{p_k}$, $k = 1, 2$, we have*

$$\sum_{i=1}^{\infty} \left| b_i^{(1)} \cdot b_i^{(2)} \right| \leqslant \left\| \left(b_i^{(1)}\right)_i \right\|_{\ell_{p_1}} \cdot \left\| \left(b_i^{(2)}\right)_i \right\|_{\ell_{p_2}}. \tag{4.1}$$

We can prove this by using Young's inequality for $n = 2$. Next we need the following inequality, for which we were unable to find a reference in the literature.

Generalized Hölder Inequality: *Let* $1 \leqslant p_1, \cdots, p_n \leqslant \infty$ *such that* $1/p_1 + \cdots + 1/p_n = 1$. *Then for every* $(b_i^{(k)})_i \in \ell_{p_k}$, $k = 1, \cdots, n$, *we have*

$$\sum_{i=1}^{\infty} \left| b_i^{(1)} \cdots b_i^{(n)} \right| \leqslant \left\| \left(b_i^{(1)}\right)_i \right\|_{\ell_{p_1}} \cdots \left\| \left(b_i^{(n)}\right)_i \right\|_{\ell_{p_n}} . \tag{4.2}$$

Proof. We use induction on n.

(i) When $n = 2$, (4.2) is reduced to (4.1), so it is true.

(ii) Suppose (4.2) holds for some $n \geq 2$. We need to show that it also holds for $n + 1$.

Case 1: $1 < p_1, \cdots p_{n+1} < \infty$

Let $\sum_{k=1}^{n+1} \frac{1}{p_k} = 1$ and let $(b_i^{(k)})_i \in \ell_{p_k}$, $k = 1, \cdots, n+1$.

In particular, we have $p_1 > 0$, $\frac{p_1}{p_1 - 1} > 0$ and $\frac{1}{p_1} + \frac{p_1 - 1}{p_1} = 1$. Then by (4.1),

$$\begin{aligned} \sum_{i=1}^{\infty} \left| b_i^{(1)} \cdots b_i^{(n+1)} \right| &= \sum_{i=1}^{\infty} \left| b_i^{(1)} \right| \cdot \left| b_i^{(2)} \cdots b_i^{(n+1)} \right| \\ &\leq \left\| \left(b_i^{(1)}\right)_i \right\|_{\ell_{p_1}} \left\| \left(b_i^{(2)} \cdots b_i^{(n+1)}\right)_i \right\|_{\ell_{\frac{p_1}{p_1-1}}} \\ &= \left\| \left(b_i^{(1)}\right)_i \right\|_{\ell_{p_1}} \left(\sum_{i=1}^{\infty} \left| b_i^{(2)} \cdots b_i^{(n+1)} \right|^{\frac{p_1}{p_1-1}} \right)^{\frac{p_1-1}{p_1}} \end{aligned} \tag{$*$}$$

Since $\frac{p_1}{p_1-1} > 0$ and $p_k > 0$ for $k = 1, \cdots, n+1$. We have $p_k \cdot \left(\frac{p_1-1}{p_1}\right) > 0$ and

$$\sum_{k=2}^{n+1} \frac{p_1}{p_k(p_1 - 1)} = \left(\frac{p_1}{p_1 - 1}\right) \cdot \sum_{k=2}^{n+1} \frac{1}{p_k} = \left(\frac{p_1}{p_1 - 1}\right)\left(1 - \frac{1}{p_1}\right) = 1$$

By induction hypothesis and $(*)$, we have

$$
\begin{aligned}
\sum_{i=1}^{\infty}\left|b_i^{(1)}\cdots b_i^{(n+1)}\right| &\leq \left\|\left(b_i^{(1)}\right)_i\right\|_{\ell_{p_1}}\cdot\Big[\prod_{k=2}^{n+1}\Big(\sum_{i=1}^{\infty}\big(\left|b_i^{(k)}\right|^{\frac{p_1}{p_1-1}}\big)^{\frac{p_k(p_1-1)}{p_1}}\Big)^{\frac{p_1}{p_k(p_1-1)}}\Big]^{\frac{p_1-1}{p_1}}\\
&= \left\|\left(b_i^{(1)}\right)_i\right\|_{\ell_{p_1}}\cdot\prod_{k=2}^{n+1}\Big(\sum_{i=1}^{\infty}\left|b_i^{(k)}\right|^{p_k}\Big)^{\frac{1}{p_k}}\\
&= \left\|\left(b_i^{(1)}\right)_i\right\|_{\ell_{p_1}}\cdot\prod_{k=2}^{n+1}\left\|\left(b_i^{(k)}\right)_i\right\|_{\ell_{p_k}}\\
&= \left\|(b_i^{(1)})_i\right\|_{\ell_{p_1}}\cdots\left\|(b_i^{(n)})_i\right\|_{\ell_{p_{n+1}}}
\end{aligned}
$$

Case 2: If $p_1 = 1$ then $p_k = \infty$ for all $k = 2, \cdots, n+1$

$$
\begin{aligned}
\sum_{i=1}^{N}\left|b_i^{(1)}\cdots b_i^{(n+1)}\right| &\leq \sum_{i=1}^{N}\left|b_i^{(1)}\right|\cdot max_{1\leq i\leq N}\left|b_i^{(2)}\right|\cdots max_{1\leq i\leq N}\left|b_i^{(n+1)}\right|\\
&\leq \left\|(b_i^{(1)})_i\right\|_{\ell_{p_1}}\left\|(b_i^{(2)})_i\right\|_{\ell_{p_\infty}}\cdots\left\|(b_i^{(n+1)})_i\right\|_{\ell_{p_\infty}}
\end{aligned}
$$

Therefore, the series converges and Hölder Inequality follows by taking the limit as $N \to \infty$. $\square$

The main diagonal of the injective tensor product of $\ell_{p_1}\check{\otimes}_\epsilon\cdots\check{\otimes}_\epsilon\ell_{p_n}$ was characterized by Holub [25]. Next by using Hölder inequality, we will chacterize the main diagonal of the positive injective tensor product $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$. Let $\Delta(\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n})$ denote the *main diagonal* of $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$, that is, the closed subspace spanned by the diagonal vectors $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ in $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$.

Theorem 4.18. *Let $1 \leqslant p_1, \cdots, p_n < \infty$ and let $\frac{1}{s} = \sum_{i=1}^{n}\frac{1}{p_i'}$. Then $\Delta(\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n})$ is isometrically lattice isomorphic to $\ell_{s'}$ if $s > 1$ and c_0 if $s \leq 1$.*

i.e.

$$
\Delta(\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}) \equiv \begin{cases} \ell_{s'}, & if\, s > 1\\ c_0 & if\, s \leq 1.\end{cases}
$$

Proof. Case 1: $s > 1$.

Take any positive element $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \Delta(\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n})$, where $a_i \geqslant 0$ for $i = 1, \cdots, m$. By Hölder inequality,

$$\begin{aligned}
\|u\|_{|\epsilon|} &= \sup\left\{\left|\sum_{i=1}^{m} a_i x_1^*(e_i)\cdots x_n^*(e_i)\right| : \ x_k^* \in B^+_{\ell_{p'_k}}, 1 \leqslant k \leqslant n\right\} \\
&= \sup\left\{\left|\sum_{i=1}^{m} a_i b_i^{(1)} \cdots b_i^{(n)}\right| : \ (b_i^{(k)})_i := x_k^* \in B^+_{\ell_{p'_k}}, 1 \leqslant k \leqslant n\right\} \\
&\leqslant \sup\left\{\left\|(a_i)_{i=1}^{m}\right\|_{\ell_{s'}} \cdot \left\|\left(b_i^{(1)}\right)_i\right\|_{\ell_{p'_1}} \cdots \left\|\left(b_i^{(n)}\right)_i\right\|_{\ell_{p'_n}} : (b_i^{(k)})_i \in B^+_{\ell_{p'_k}}, 1 \leqslant k \leqslant n\right\} \\
&\leqslant \left\|(a_i)_{i=1}^{m}\right\|_{\ell_{s'}}.
\end{aligned}$$

On the other hand, take any $(b_i)_i \in B^+_{\ell_s}$. If $p_k = 1$ for some k, let $x_k^* = (1, 1, \cdots)$. Then $\|x_k^*\|_{\ell_{p'_k}} = \|x_k^*\|_{\ell_\infty} = 1$. If $p_k > 1$ for some k, let $x_k^* = (b_i^{\frac{s}{p'_k}})_i$. Then $\|x_k^*\|_{\ell_{p'_k}} = \left(\|(b_i)_i\|_{\ell_s}\right)^{\frac{s}{p'_k}} \leqslant 1$. Without loss of generality, assume that $p_1 = \cdots = p_k = 1$ and $p_{k+1} > 1, \cdots, p_n > 1$. Since $\sum_{i=k+1}^{n} \frac{s}{p'_i} = \sum_{i=1}^{n} \frac{s}{p'_i} = 1$, it follows that

$$\left|\sum_{i=1}^{m} a_i b_i\right| = \left|\sum_{i=1}^{m} a_i \cdot 1 \cdots 1 \cdot b_i^{\frac{s}{p'_{k+1}}} \cdots b_i^{\frac{s}{p'_n}}\right| = \left|\sum_{i=1}^{m} a_i x_1^*(e_i) \cdots x_n^*(e_i)\right| \leqslant \|u\|_{|\epsilon|}.$$

Thus

$$\left\|(a_i)_{i=1}^{m}\right\|_{\ell_{s'}} = \sup\left\{\left|\sum_{i=1}^{m} a_i b_i\right| : (b_i)_i \in B^+_{\ell_s}\right\} \leqslant \|u\|_{|\epsilon|}.$$

Therefore, $\|u\|_{|\epsilon|} = \|(a_i)_{i=1}^{m}\|_{\ell_{s'}}$ and hence, $\Delta(\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n})$ is isometrically lattice isomorphic to $\ell_{s'}$.

Case 2: $s \leqslant 1$.

Take any positive element $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \Delta(\ell_{p_1} \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} \ell_{p_n})$, where $a_i \geqslant 0$ for $i = 1, \cdots, m$. Take $q_k \geqslant p'_k$ for $k = 1, \cdots, n$ such that $\frac{1}{q_1} + \cdots + \frac{1}{q_n} = 1$. By

Hölder inequality,

$$
\begin{aligned}
\|u\|_{|\epsilon|} &= \sup\left\{\left|\sum_{i=1}^{m} a_i x_1^*(e_i)\cdots x_n^*(e_i)\right| : x_k^* \in B^+_{\ell_{p'_k}}, 1\leqslant k\leqslant n\right\}\\
&= \sup\left\{\left|\sum_{i=1}^{m} a_i b_i^{(1)}\cdots b_i^{(n)}\right| : (b_i^{(k)})_i := x_k^* \in B^+_{\ell_{p'_k}}, 1\leqslant k\leqslant n\right\}\\
&\leqslant \left\|(a_i)_{i=1}^m\right\|_{c_0}\cdot \sup\left\{\left\|\left(b_i^{(1)}\right)_i\right\|_{\ell_{q_1}}\cdots\left\|\left(b_i^{(n)}\right)_i\right\|_{\ell_{q_n}} : (b_i^{(k)})_i \in B^+_{\ell_{p'_k}}, 1\leqslant k\leqslant n\right\}\\
&\leqslant \left\|(a_i)_{i=1}^m\right\|_{c_0}\cdot \sup\left\{\left\|\left(b_i^{(1)}\right)_i\right\|_{\ell_{p'_1}}\cdots\left\|\left(b_i^{(n)}\right)_i\right\|_{\ell_{p'_n}} : (b_i^{(k)})_i \in B^+_{\ell_{p'_k}}, 1\leqslant k\leqslant n\right\}\\
&\leqslant \left\|(a_i)_{i=1}^m\right\|_{c_0}.
\end{aligned}
$$

On the other hand, for each fixed j, $1\leqslant j\leqslant m$. Let $x_k^* = e_j \in B^+_{\ell_{p'_k}}$ for $k=1,\cdots,n$. Then

$$
\|u\|_{|\epsilon|} \geqslant \left|\sum_{i=1}^{m} a_i x_1^*(e_i)\cdots x_n^*(e_i)\right| = a_j,
$$

which implies that $\|u\|_{|\epsilon|} \geqslant \|(a_i)_{i=1}^m\|_{c_0}$. Therefore, $\|u\|_{|\epsilon|} = \|(a_i)_{i=1}^m\|_{c_0}$ and hence, $\Delta(\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n})$ is isometrically lattice isomorphic to c_0. □

Remark 4.19. In Theorem 4.18, if some (or all) of p_i's are ∞, then $s\leqslant 1$. In this case, we consider $\ell_{p_i} = c_0$ if $p_i=\infty$, and consider $\ell_p = c_0$ if $p=\infty$. The Theorem 4.18 is still true. Thus we have the first special case, if at lease one of p_i's is ∞, then $\Delta(\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n})$ is isometrically lattice isomorphic to c_0. The second special case is that if all p_i's are 1, then $\Delta(\ell_1\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_1)$ is isometrically lattice isomorphic to ℓ_1.

Next we will use the main diagonal of $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$ to characterize its reflexivity as well as being a KB-space.

Theorem 4.20. *Let $1<p_1,\cdots,p_n<\infty$. Then the following statements are equivalent:*

(i) $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$ *is reflexive.*

(ii) $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$ *is a KB-space.*

(iii) $1/p'_1+\cdots+1/p'_n<1$.

Proof. If $1/p_1' + \cdots + 1/p_n' < 1$, then $\ell_{p_1'}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n'}$ is reflexive. Lemma 4.17 implies that $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$ is reflexive and (iii) $\Rightarrow$ (i) follows. It is trivial that (i) $\Rightarrow$ (ii). Now suppose (ii) holds. Then by [36, p.93, Theorem 2.4.14], $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$ does not contain any sublattice isomorphic to c_0. Theorem 4.18 implies that $1/p_1' + \cdots + 1/p_n' < 1$ and (iii) follows. $\square$

Note that $\ell_{p_1}\check{\otimes}_{|\epsilon|}\ell_{p_2}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n} = \ell_{p_1}\check{\otimes}_{|\epsilon|}(\ell_{p_2}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n})$. By induction, [9, Theorems 6.7, 6.8] yield the following corollary.

Corollary 4.21. *Let $1 < p_1, \cdots, p_n < \infty$. Then $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$ contains any sublattice neither isomorphic to ℓ_1 nor isomorphic to ℓ_∞.*

Note that Lemma 4.16 implies that $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$ is σ-Dedekind complete. Corollary 4.21 and [36, p.87, Corollary 2.4.3] yield the following corollary.

Corollary 4.22. *Let $1 < p_1, \cdots, p_n < \infty$. Then $\ell_{p_1}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n}$ is order continuous.*

For Banach lattices $E_1, \cdots, E_n, F$ with F Dedekind complete, let $\mathcal{K}^r(E_1, \cdots, E_n; F)$ denote the sublattice of $\mathcal{L}^r(E_1, \cdots, E_n; F)$ generated by all positive compact n-linear operators from $E_1\times\cdots\times E_n$ to F. It is easy to see that $\mathcal{K}^r(E_1, E_2, \cdots, E_n; F) = \mathcal{K}^r(E_1; \mathcal{K}^r(E_2, \cdots, E_n; F))$. By [9, Proposition 2.1 and Theorem 5.2], for any $1 < p < \infty$, $\mathcal{K}^r(\ell_p; F)$ is isometrically lattice isomorphic to $\ell_{p'}\check{\otimes}_{|\epsilon|}F$. By induction, it follows that for any $1 < p_1, \cdots, p_n < \infty$, $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; F)$ is isometrically lattice isomorphic to $\ell_{p_1'}\check{\otimes}_{|\epsilon|}\cdots\check{\otimes}_{|\epsilon|}\ell_{p_n'}\check{\otimes}_{|\epsilon|}F$. Thus Theorem 4.20 and Corollaries 4.21, 4.22 yield the following theorem.

Theorem 4.23. *Let $1 < p_1, \cdots, p_n, q < \infty$.*

(i) $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ *contains any sublattice neither isomorphic to ℓ_1 nor isomorphic to ℓ_∞.*

(ii) $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ *is order continuous.*

(iii) *The following statements are equivalent:*

(a) $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ *is reflexive.*

(b) $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ *is a KB-space.*

(c) $1/p_1 + \cdots + 1/p_n < 1/q$.

Pitt [41] proved that every continuous linear operator from ℓ_p to ℓ_q is compact if and only if $p > q$. Chen and Wickstead [12] proved that if every positive linear operator from ℓ_p to ℓ_q is compact then we still have $p > q$. For the n-linear operator case, Alencar and Floret [43] proved that every continuous n-linear operator from $\ell_{p_1} \times \cdots \times \ell_{p_n}$ to ℓ_q is compact if and only if $1/p_1 + \cdots + 1/p_n < 1/q$ (also see [15]). Next we will give the same characterization for the positive compact n-linear operators on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ as follows.

Theorem 4.24. *Let* $1 \leqslant p_1, \cdots, p_n, q < \infty$. *Then every positive n-linear operator from* $\ell_{p_1} \times \cdots \times \ell_{p_n}$ *to* ℓ_q *is compact if and only if* $1/p_1 + \cdots + 1/p_n < 1/q$.

Proof. It follows from [12, Theorem 4.9] that the theorem is true for $n = 1$. Now suppose that the theorem is true for $n - 1$. To show the theorem is true for n, we only need to show that $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q) = \mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ implies that $1/p_1 + \cdots + 1/p_n < 1/q$.

Now suppose that $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q) = \mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$. Then $\mathcal{K}^r(\ell_{p_1}; \ell_q) = \mathcal{L}^r(\ell_{p_1}; \ell_q)$ and hence, $p_1 > q \geqslant 1$. Also then $\mathcal{K}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q) = \mathcal{L}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)$. By induction hypothesis, $1/p_2 + \cdots + 1/p_n < 1/q$, which, by Corollaries 4.11 and 4.13, implies that $\mathcal{L}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)$ is a KB-space. Note that

$$\begin{aligned}
\mathcal{K}^r(\ell_{p_1}; \mathcal{L}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)) &= \mathcal{K}^r(\ell_{p_1}; \mathcal{K}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)) \\
&= \mathcal{K}^r(\ell_{p_1}, \ell_{p_2} \cdots, \ell_{p_n}; \ell_q) \\
&= \mathcal{L}^r(\ell_{p_1}, \ell_{p_2} \cdots, \ell_{p_n}; \ell_q) \\
&= \mathcal{L}^r(\ell_{p_1}; \mathcal{L}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)).
\end{aligned}$$

It follows from Lemma 4.12 that $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is a KB-space. Thus Corollaries 4.11 and 4.13 imply that $1/p_1 + \cdots + 1/p_n < 1/q$. $\square$

From Corollary 4.11 and Theorems 4.23, 4.24, we have the following interesting results. For any $1 < p_1, \cdots, p_n, q < \infty$, $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is always order continuous, but $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is order continuous if and only if $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q) = \mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$. For instance, if $1/p_1 + \cdots + 1/p_n \geqslant 1/q$ then $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is order continuous but $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is not order continuous.

5 DIAGONALS OF TENSOR PRODUCTS OF BANACH LATTICES WITH BASES

5.1 Diagonals of Projective Tensor Products of Banach Lattices

In this section we assume that X is a Banach space with a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. Gelbaum and Lamadrid [19] showed that $\{e_i \otimes e_j : (i,j) \in \mathbb{N}^2\}$ with the square order is a basis of $\hat{\otimes}_{2,\pi} X$ (it is not necessarily an unconditional basis). For instance, Kwapien and Pelczynski [49] showed that $\{e_i \otimes e_j : (i,j) \in \mathbb{N}^2\}$ is not an unconditional basis of $\hat{\otimes}_{2,\pi} \ell_2$. In general, Grecu and Ryan [20] established that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ with the order defined in [20] is a basis of $\hat{\otimes}_{n,\pi} X$. They also showed that $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n, i_1 \geqslant \cdots \geqslant i_n\}$ with the order defined in [20] is a basis of $\hat{\otimes}_{n,s,\pi} X$.

Definition 5.1. Let $\Delta(\hat{\otimes}_{n,\pi} X)$ (resp. $\Delta(\hat{\otimes}_{n,s,\pi} X)$) denote the *main diagonal space* of $\hat{\otimes}_{n,\pi} X$ (resp. $\hat{\otimes}_{n,s,\pi} X$), that is, the closed subspace spanned in $\hat{\otimes}_{n,\pi} X$ (resp. in $\hat{\otimes}_{n,s,\pi} X$) by the *tensor diagonal* $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$.

Holub [25] proved that the tensor diagonal $\{e_i \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\hat{\otimes}_{2,\pi} X)$ and Sanchez [46] generalized the Holub's result. He showed that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\hat{\otimes}_{n,\pi} X)$.

Bu and Buskes [5] proved that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\hat{\otimes}_{n,s,\pi} X)$ and Bu, Buskes, Popov, Tcaciuc, and Troitsky [8] showed that the tensor basis $\{e_i \otimes e_j : (i,j) \in \mathbb{N}^2\}$ with any order is a 1-unconditional basis of $\hat{\otimes}_{2,|\pi|} E$.

By a *Banach lattice with a Schauder basis* we mean a Banach lattice in which the unit vectors form a basis and the order is defined coordinatewise. It follows that such a Schauder basis is 1-unconditional. Conversely, every Banach space with a 1-unconditional basis is a Banach lattice with the order defined coordinatewise. In what follows, E is a Banach lattice with a basis $\{e_i : i \in \mathbb{N}\}$. As a special case of [8, Lemma 22], the set $\{e_i \otimes e_j : (i,j) \in \mathbb{N}^2\}$ with any order is a (1-unconditional) basis of $\hat{\otimes}_{2,|\pi|}E$.

In a similar way, Bu and Buskes [5] also proved that the following lemma.

Lemma. *The tensor basis* $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ *with any order is a (*1-*unconditional) basis of* $\hat{\otimes}_{n,|\pi|}E$*, and the tensor basis* $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geqslant \cdots \geqslant i_n\}$ *with any order is a (*1-*unconditional) basis of* $\hat{\otimes}_{n,s,|\pi|}E$*.*

Definition 5.2. Let $\Delta(\hat{\otimes}_{n,|\pi|}E)$ (resp. $\Delta(\hat{\otimes}_{n,s,|\pi|}E)$) denote the *main diagonal space* of $\hat{\otimes}_{n,|\pi|}E$ (resp. $\hat{\otimes}_{n,s,|\pi|}E$), that is, the closed subspace spanned in $\hat{\otimes}_{n,|\pi|}E$ (resp. in $\hat{\otimes}_{n,s,|\pi|}E$) by the *tensor diagonal* $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$. It follows from the above lemma that $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of both $\Delta(\hat{\otimes}_{n,|\pi|}E)$ and $\Delta(\hat{\otimes}_{n,s,|\pi|}E)$.

In [5], Bu and Buskes proved that all four main diagonal spaces $\Delta(\hat{\otimes}_{n,\pi}E)$, $\Delta(\hat{\otimes}_{n,s,\pi}E)$, $\Delta(\hat{\otimes}_{n,|\pi|}E)$, and $\Delta(\hat{\otimes}_{n,s,|\pi|}E)$ are pairwise isometrically isomorphic.

It is natural question to ask whether this result holds for all four main diagonal spaces such as $\Delta(\check{\otimes}_{n,\epsilon}E)$, $\Delta(\check{\otimes}_{n,s,\epsilon}E)$, $\Delta(\check{\otimes}_{n,|\epsilon|}E)$, and $\Delta(\check{\otimes}_{n,s,|\epsilon|}E)$.

5.2 Diagonals of Injective Tensor Products of Banach Lattices

* All of results in this section have been proved in my paper "Diagonals of Injective Tensor Products of Banach Lattices with Bases". This is joint work with Qingying Bu and Donghai Ji.(See [27].)

In this section we assume that X is a Banach space with a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. Gelbaum and Lamadrid [19] showed that $\{e_i \otimes e_j : (i,j) \in \mathbb{N}^2\}$ with the square order is a basis of $\check{\otimes}_{2,\epsilon} X$. In general, Grecu and Ryan [20] showed that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ with the order defined in [20] is a basis of $\check{\otimes}_{n,\epsilon} X$. They also showed that $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geqslant \cdots \geqslant i_n\}$ with the order defined in [20] is a basis of $\check{\otimes}_{n,s,\epsilon} X$.

Let $\Delta(\check{\otimes}_{n,\epsilon} X)$ (resp. $\Delta(\check{\otimes}_{n,s,\epsilon} X)$) denote the *main diagonal space* of $\check{\otimes}_{n,\epsilon} X$ (resp. $\check{\otimes}_{n,s,\epsilon} X$), that is, the closed subspace spanned in $\check{\otimes}_{n,\epsilon} X$ (resp. in $\check{\otimes}_{n,s,\epsilon} X$) by the *tensor diagonal* $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$. It is known that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ and $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geqslant \cdots \geqslant i_n\}$, respectively, is not necessarily an unconditional basis of $\check{\otimes}_{n,\epsilon} X$ and $\check{\otimes}_{n,s,\epsilon} X$ (see, e.g., [49, 40, 48, 13, 39, 11]). Next we will use the following Rademacher averaging formula (see, e.g., [45, Lemma 2.22]) to show that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is an unconditional basis of both $\Delta(\check{\otimes}_{n,\epsilon} X)$ and $\Delta(\check{\otimes}_{n,s,\epsilon} X)$, and their diagonal projections are contractive.

Lemma 5.3. *The tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\check{\otimes}_{n,\epsilon} X)$ and the projection $Q : \check{\otimes}_{n,\epsilon} X \to \Delta(\check{\otimes}_{n,\epsilon} X)$ defined by*

$$Q(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \begin{cases} e_{i_1} \otimes \cdots \otimes e_{i_n}, & \textit{if } i_1 = \cdots = i_n, \\ 0, & \textit{otherwise.} \end{cases}$$

is bounded with $\|Q\| \leqslant 1$.

Proof. First we adopt Holub's proof of Theorem 3.12 in [25] to prove that $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ is a 1-unconditional basic sequence of $\check{\otimes}_{n,\epsilon}X$. Let $I: X\to X$ be the identity operator and for $\theta_i=\pm1$ ($i\in\mathbb{N}$), define $T: X\to X$ by $T(\sum_{i=1}^\infty a_ie_i)=\sum_{i=1}^\infty\theta_ia_ie_i$ for every $x=\sum_{i=1}^\infty a_ie_i\in X$. Then $\|T\|\leqslant 1$. Now for any $m\in\mathbb{N}$, by Lemma 3.10,

$$\begin{aligned}\left\|\sum_{i=1}^m\theta_ia_ie_i\otimes\cdots\otimes e_i\right\|_\epsilon &= \left\|\sum_{i=1}^m a_iT(e_i)\otimes I(e_i)\otimes\cdots\otimes I(e_i)\right\|_\epsilon\\ &\leqslant \left\|\sum_{i=1}^m a_ie_i\otimes\cdots\otimes e_i\right\|_\epsilon.\end{aligned}$$

Thus $\{e_i\otimes\cdots\otimes e_i : i\in\mathbb{N}\}$ is a 1-unconditional basic sequence of $\check{\otimes}_{n,\epsilon}X$.

Next we show that Q is well-defined and bounded with $\|Q\|\leqslant 1$. Take any $u=\sum_{i_1,\ldots,i_n}b_{i_1,\cdots,i_n}e_{i_1}\otimes\cdots\otimes e_{i_n}\in\check{\otimes}_{n,\epsilon}X$. For every $p,q\in\mathbb{N}$ with $p<q$, let

$$u_{p,q}=\sum_{i_1,\ldots,i_n=p}^q b_{i_1,\cdots,i_n}e_{i_1}\otimes\cdots\otimes e_{i_n}.$$

Then there exist $x_{j,k}=\sum_{i=1}^\infty a_{i,j,k}e_i\in X$, $k=1,\ldots,m$ and $j=1,\ldots,n$ such that

$$u_{p,q}=\sum_{k=1}^m x_{1,k}\otimes\cdots\otimes x_{n,k}.$$

Thus

$$b_{i,\cdots,i}=\sum_{k=1}^m a_{i,1,k}\cdots a_{i,n,k},\quad p\leqslant i\leqslant q.$$

By Rademacher averaging,

$$\begin{aligned}
\left\|\sum_{i=p}^{q} b_{i,\cdots,i} e_i \otimes \cdots \otimes e_i\right\|_\epsilon &= \left\|\sum_{k=1}^{m}\sum_{i=p}^{q} a_{i,1,k}\cdots a_{i,n,k} e_i \otimes \cdots \otimes e_i\right\|_\epsilon \\
&= \left\|\sum_{k=1}^{m}\sum_{i=p}^{q} (a_{i,1,k}e_i) \otimes \cdots \otimes (a_{i,n,k}e_i)\right\|_\epsilon \\
&= \left\|\sum_{k=1}^{m}\int_0^1 \Big(\sum_{i=p}^{q} a_{i,1,k} r_i(t) e_i\Big) \otimes \cdots \otimes \Big(\sum_{i=p}^{q} a_{i,n,k} r_i(t) e_i\Big)\, dt\right\|_\epsilon \\
&\leqslant \int_0^1 \left\|\sum_{k=1}^{m} \Big(\sum_{i=p}^{q} a_{i,1,k} r_i(t) e_i\Big) \otimes \cdots \otimes \Big(\sum_{i=p}^{q} a_{i,n,k} r_i(t) e_i\Big)\right\|_\epsilon \, dt \\
&= \int_0^1 \left\|\sum_{k=1}^{m} \Big(T_t(x_{1,k})\Big) \otimes \cdots \otimes \Big(T_t(x_{n,k})\Big)\right\|_\epsilon \, dt \\
&\leqslant \int_0^1 \|T_t\|^n \cdot \left\|\sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k}\right\|_\epsilon \, dt \\
&\leqslant \left\|\sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k}\right\|_\epsilon = \left\|u_{p,q}\right\|_\epsilon,
\end{aligned}$$

where $T_t : X \to X$ is defined by $T_t(x) = \sum_{i=1}^{\infty} a_i r_i(t) e_i$ for every $x = \sum_{i=1}^{\infty} a_i e_i \in X$ and every $t \in [0,1]$. Therefore, for every $p, q \in \mathbb{N}$ with $p < q$,

$$\left\|\sum_{i=p}^{q} b_{i,\cdots,i} e_i \otimes \cdots \otimes e_i\right\|_\epsilon \leqslant \left\|\sum_{i_1,\cdots,i_n=p}^{q} b_{i_1,\cdots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}\right\|_\epsilon,$$

which implies that Q is well-defined and bounded with $\|Q\| \leqslant 1$. □

Definition 5.4. An n-homogeneous polynomial $P : E \to Y$ is called *orthogonally additive* if $P(x+y) = P(x) + P(y)$ whenever $x, y \in E$ with $x \perp y$.

Lemma 5.5. *The tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a* 1*-unconditional basis of* $\Delta(\check{\otimes}_{n,s,\epsilon} X)$ *and the projection* $Q_s : \check{\otimes}_{n,s,\epsilon} X \to \Delta(\check{\otimes}_{n,s,\epsilon} X)$ *defined by*

$$Q_s(e_{i_1} \otimes_s \cdots \otimes_s e_{i_n}) = \begin{cases} e_{i_1} \otimes_s \cdots \otimes_s e_{i_n}, & \text{if } i_1 = \cdots = i_n, \\ 0, & \text{otherwise.} \end{cases}$$

is bounded with $\|Q_s\| \leqslant 1$.

Proof. For any $m \in \mathbb{N}$, let $u = \sum_{i=1}^m a_i e_i \otimes \cdots \otimes e_i$, and let $P_u : X^* \to \mathbb{R}$ be the n-homogeneous polynomial defined in (2.5), that is,

$$P_u(x^*) = \sum_{i=1}^{m} a_i \cdot \Big(x^*(e_i)\Big)^n, \quad \forall\, x^* \in X^*. \tag{5.1}$$

For every $a \in \mathbb{R}$ and every $p > 0$ we define $a^p = \operatorname{sign}(a) \cdot |a|^p$. Note that X has a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. It can be a Banach lattice with the order defined coordinatewise. Also note that X^* is a sequence space via $x^* \leftrightarrow (x^*(e_1), x^*(e_2), \dots)$ for every $x^* \in E^*$. Thus for every $x^*, y^* \in X^*$, from functional calculation, $\big(x^{*p} + y^{*p}\big)^{\frac{1}{p}}$ is defined (coordinatewise) to be an element of X^* (see, e.g., [34, Section 1.d]), that is,

$$\big(x^{*p} + y^{*p}\big)^{\frac{1}{p}}(e_i) = \big(x^*(e_i)^p + y^*(e_i)^p\big)^{\frac{1}{p}}, \quad i = 1, 2, \dots.$$

If, moreover, $x^* \perp y^*$, then $\big(x^{*p} + y^{*p}\big)^{\frac{1}{p}} = x^* + y^*$ by [8, Lemma 3]. Thus

$$\big(x^*(e_i) + y^*(e_i)\big)^p = x^*(e_i)^p + y^*(e_i)^p, \quad i = 1, 2, \dots. \tag{5.2}$$

It follows from (5.1) and (5.2) that $P_u(x^* + y^*) = P_u(x^*) + P_u(y^*)$ for every $x^*, y^* \in X^*$ with $x^* \perp y^*$. Thus P_u is orthogonally additive (see [3, 51, 38, 26, 33]). Let $T_u : X^* \times \cdots \times X^* \to \mathbb{R}$

be the n-linear operator defined in (3.9), that is,

$$T_u(x_1^*, \ldots, x_n^*) = \sum_{i=1}^{m} a_i \cdot x_1^*(e_i) \cdots x_n^*(e_i), \quad \forall\, x_1^*, \ldots, x_n^* \in X^*.$$

Then T_u is the symmetric n-linear operator associated to P_u. It follows from [5, Theorem 5.4] that $\|T_u\| = \|P_u\|$ and hence, $\|u\|_{s,\epsilon} = \|P_u\| = \|T_u\| = \|u\|_\epsilon$.

Now let $\theta_i = \pm 1$ ($i \in \mathbb{N}$). Then by (3.11) and Lemma 5.3,

$$\begin{aligned} \left\| \sum_{i=1}^{m} \theta_i a_i e_i \otimes \cdots \otimes e_i \right\|_{s,\epsilon} &\leqslant \left\| \sum_{i=1}^{m} \theta_i a_i e_i \otimes \cdots \otimes e_i \right\|_{\epsilon} \\ &\leqslant \left\| \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \right\|_{\epsilon} = \|u\|_\epsilon \\ &= \|u\|_{s,\epsilon} = \left\| \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \right\|_{s,\epsilon}. \end{aligned}$$

Thus $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence of $\check{\otimes}_{n,s,\epsilon} X$.

Similarly to the proof of Lemma 5.3, we can use the Rademacher averaging formula to show that Q_s is well-defined and bounded with $\|Q_s\| \leqslant 1$. $\square$

By a *Banach lattice with a Schauder basis* we mean a Banach lattice in which the unit vectors form a basis and the order is defined coordinatewise. It follows that such a Schauder basis is 1-unconditional. Conversely, every Banach space with a 1-unconditional basis is a Banach lattice with the order defined coordinatewise. In what follows $E_1, \ldots, E_n$ are Banach lattices with (1-unconditional) basis $\{e_i^{(1)} : i \in \mathbb{N}\}, \ldots, \{e_i^{(n)} : i \in \mathbb{N}\}$, respectively.

Theorem 5.6. *The tensor basis $\{e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ with any order is a (1-unconditional) basis of $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_n$.*

Proof. First, we will show that $(e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}) \perp (e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)})$ provided $(i_1, \ldots, i_n) \neq (k_1, \ldots, k_n)$. Note that $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_n$ can be considered as a closed sublattice of $\mathcal{L}^r(E_1^*, \ldots, E_n^*; \mathbb{R})$.

It suffices to show that

$$\langle (e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}) \wedge (e^{(1)}_{k_1} \otimes \cdots \otimes e^{(n)}_{k_n}), (x^*_1, \ldots, x^*_n) \rangle = 0$$

for every $x^*_1 \in E^{*+}_1, \ldots, x^*_n \in E^{*+}_n$. Let

$$\alpha_1 = x^*_1(e^{(1)}_{k_1}), \ldots, \alpha_n = x^*_n(e^{(n)}_{k_n}) \quad \text{and} \quad u^*_{1,1} = \alpha_1 f^{(1)}_{k_1}, \ldots, u^*_{n,1} = \alpha_n f^{(n)}_{k_n},$$

where $\{f^{(1)}_i : i \in \mathbb{N}\}, \ldots, \{f^{(n)}_i : i \in \mathbb{N}\}$ are, respectively, the appropriate bi-orthogonal functionals on $E^*_1, \ldots, E^*_n$. Then for every $x = \sum_{i=1}^{\infty} b_i e^{(1)}_i \in E^+_1$,

$$u^*_{1,1}(x) = \alpha_1 f^{(1)}_{k_1}(x) = x^*_1(e^{(1)}_{k_1}) b_{k_1} \leqslant \sum_{i=1}^{\infty} b_i x^*_1(e^{(1)}_i) = x^*_1(x).$$

It follows that $u^*_{1,1} \leqslant x^*_1$ and similarly, $u^*_{2,1} \leqslant x^*_2, \ldots, u^*_{n,1} \leqslant x^*_n$.
Let $u^*_{1,2} = x^*_1 - u^*_{1,1}, \ldots, u^*_{n,2} = x^*_n - u^*_{n,1}$. Then $(u^*_{1,1}, u^*_{1,2}), \ldots, (u^*_{n,1}, u^*_{n,2})$ are partitions of $x^*_1, \ldots, x^*_n$ respectively.

By [6, Proposition 2.2],

$$\begin{aligned} & \langle (e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}) \wedge (e^{(1)}_{k_1} \otimes \cdots \otimes e^{(n)}_{k_n}), (x^*_1, \ldots, x^*_n) \rangle \\ \leqslant & \sum_{j_1, \ldots, j_n = 1}^{2} \langle e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}, (u^*_{1,j_1}, \ldots, u^*_{n,j_n}) \rangle \wedge \langle e^{(1)}_{k_1} \otimes \cdots \otimes e^{(n)}_{k_n}, (u^*_{1,j_1}, \ldots, u^*_{n,j_n}) \rangle. \end{aligned}$$

If all j_m's are 1, then the general term

$$\begin{aligned} & \langle e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}, (u^*_{1,j_1}, \ldots, u^*_{n,j_n}) \rangle \wedge \langle e^{(1)}_{k_1} \otimes \cdots \otimes e^{(n)}_{k_n}, (u^*_{1,j_1}, \ldots, u^*_{n,j_n}) \rangle \\ \leqslant & \langle e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}, (u^*_{1,1}, \ldots, u^*_{n,1}) \rangle = u^*_{1,1}(e^{(1)}_{i_1}) \cdots u^*_{n,1}(e^{(n)}_{i_n}) \\ = & \alpha_1 f^{(1)}_{k_1}(e^{(1)}_{i_1}) \cdots \alpha_n f^{(n)}_{k_n}(e^{(n)}_{i_n}) = 0 \end{aligned}$$

since $(i_1, \ldots, i_n) \neq (k_1, \ldots, k_n)$. If at least one of j_m's is 2, say $j_1 = 2$, then the general term

$$\begin{aligned}
& \langle e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}, (u_{1,j_1}^*, \ldots, u_{n,j_n}^*) \rangle \wedge \langle e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)}, (u_{1,j_1}^*, \ldots, u_{n,j_n}^*) \rangle \\
\leqslant \; & \langle e_{k_1}^{(1)} \otimes e_{k_2}^{(2)} \otimes \cdots \otimes e_{k_n}^{(n)}, (u_{1,2}^*, u_{2,j_2}^*, \ldots, u_{n,j_n}^*) \rangle \\
= \; & u_{1,2}^*(e_{k_1}^{(1)}) \cdot u_{2,j_2}^*(e_{k_2}^{(2)}) \cdots u_{n,j_n}^*(e_{k_n}^{(n)}) \\
= \; & \left(\alpha_1 - \alpha_1 f_{k_1}^{(1)}(e_{k_1}^{(1)}) \right) \cdot u_{2,j_2}^*(e_{k_2}^{(2)}) \cdots u_{n,j_n}^*(e_{k_n}^{(n)}) = 0.
\end{aligned}$$

Therefore,

$$\langle (e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}) \wedge (e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)}), (x_1^*, \ldots, x_n^*) \rangle = 0$$

and hence, $(e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}) \perp (e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)})$.

Being a disjoint sequence in a Banach lattice, $\{e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a 1-unconditional basic sequence of $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_n$. It is left to show that its span is dense in $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_n$.

Take any $x_i \in E_i$ with $\|x_i\| \leqslant 1$ for $i = 1, \ldots, n$. Given any $\sigma \in (0, 1)$, we can find basis projections P_i on E_i, respectively, such that

$$y_i = P_i(x_i) \quad \text{and} \quad \|y_i - x_i\| \leqslant \sigma, \quad i = 1, \ldots, n.$$

Then

$$\begin{aligned}
\|x_1 \otimes x_2 - y_1 \otimes y_2\|_{|\epsilon|} &= \|x_1 \otimes (x_2 - y_2) + (x_1 - y_1) \otimes y_2\|_{|\epsilon|} \\
&\leqslant \|x_1\| \cdot \|x_2 - y_2\| + \|x_1 - y_1\| \cdot \|y_2\| \leqslant 2\sigma,
\end{aligned}$$

and similarly,

$$\|x_1 \otimes \cdots \otimes x_n - y_1 \otimes \cdots \otimes y_n\|_{|\epsilon|} \leqslant n\sigma.$$

Since $y_1 \otimes \cdots \otimes y_n$ is in the span $\{e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$, it follows that $x_1 \otimes \cdots \otimes x_n$ is in the closure of this span. Thus this span is dense in $E_1 \otimes \cdots \otimes E_n$ and hence, dense in $E_1 \check{\otimes}_{|\epsilon|} \cdots \check{\otimes}_{|\epsilon|} E_n$. $\square$

In what follows E is a Banach lattice with a (1-unconditional) basis $\{e_i : i \in \mathbb{N}\}$. The following consequence comes straightforward from the previous theorem.

Corollary 5.7. *The tensor basis* $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ *with any order is a (1-unconditional) basis of* $\check{\otimes}_{n,|\epsilon|} E$*, and the tensor basis* $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geqslant \cdots \geqslant i_n\}$ *with any order is a (1-unconditional) basis of* $\check{\otimes}_{n,s,|\epsilon|} E$*.*

Definition 5.8. Let $\Delta(\check{\otimes}_{n,|\epsilon|} E)$ (resp. $\Delta(\check{\otimes}_{n,s,|\epsilon|} E)$) denote the *main diagonal space* of $\check{\otimes}_{n,|\epsilon|} E$ (resp. $\check{\otimes}_{n,s,|\epsilon|} E$), that is, the closed subspace spanned in $\check{\otimes}_{n,|\epsilon|} E$ (resp. in $\check{\otimes}_{n,s,|\epsilon|} E$) by the *tensor diagonal* $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$.

It follows from the above corollary that $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a (1-unconditional) basis of both $\Delta(\check{\otimes}_{n,|\epsilon|} E)$ and $\Delta(\check{\otimes}_{n,s,|\epsilon|} E)$.

Recall that we already have other two main diagonal spaces $\Delta(\check{\otimes}_{n,\epsilon} E)$ and $\Delta(\check{\otimes}_{n,s,\epsilon} E)$ introduced at the beginning of this section. Next we will show that all four main diagonal spaces are pairwise isometrically isomorphic. First we need the Hölder inequality (3.2) and then a lemma.

Lemma 5.9. *For* $x_1^*, \ldots, x_n^* \in E^{*+}$*, define* $x^* : E \to \mathbb{R}$ *by*

$$x^*(x) = \sum_{i=1}^{\infty} a_i \big(x_1^*(e_i)\big)^{\frac{1}{n}} \cdots \big(x_n^*(e_i)\big)^{\frac{1}{n}}$$

for every $x = \sum_{i=1}^{\infty} a_i e_i \in E$*. Then* $x^* \in E^{*+}$ *with* $\|x^*\| \leqslant \big\|x_1^*\big\|^{\frac{1}{n}} \cdots \big\|x_n^*\big\|^{\frac{1}{n}}$*.*

Proof. For any $m \in \mathbb{N}$, by Hölder inequality,

$$\begin{aligned} & \sum_{i=1}^{m} \left| a_i \big(x_1^*(e_i)\big)^{\frac{1}{n}} \cdots \big(x_n^*(e_i)\big)^{\frac{1}{n}} \right| \\ = \;& \sum_{i=1}^{m} \left| a_i x_1^*(e_i) \right|^{\frac{1}{n}} \cdots \left| a_i x_n^*(e_i) \right|^{\frac{1}{n}} \\ \leqslant \;& \left(\sum_{i=1}^{m} |a_i x_1^*(e_i)| \right)^{\frac{1}{n}} \cdots \left(\sum_{i=1}^{m} |a_i x_n^*(e_i)| \right)^{\frac{1}{n}} \\ = \;& \left(x_1^*\big(\sum_{i=1}^{m} \pm a_i e_i \big) \right)^{\frac{1}{n}} \cdots \left(x_n^*\big(\sum_{i=1}^{m} \pm a_i e_i \big) \right)^{\frac{1}{n}} \\ \leqslant \;& \left\| x_1^* \right\|^{\frac{1}{n}} \cdots \left\| x_n^* \right\|^{\frac{1}{n}} \cdot \left\| \sum_{i=1}^{m} a_i e_i \right\|. \end{aligned}$$

Thus x^* is well-defined and $x^* \in E^{*+}$ with $\|x^*\| \leqslant \left\|x_1^*\right\|^{\frac{1}{n}} \cdots \left\|x_n^*\right\|^{\frac{1}{n}}$. □

Theorem 5.10. *All four main diagonal spaces $\Delta(\check{\otimes}_{n,\epsilon} E)$, $\Delta(\check{\otimes}_{n,s,\epsilon} E)$, $\Delta(\check{\otimes}_{n,|\epsilon|} E)$, and $\Delta(\check{\otimes}_{n,s,|\epsilon|} E)$ are pairwise isometrically isomorphic.*

Proof. First we show that $\Delta(\check{\otimes}_{n,|\epsilon|} E)$ is isometrically isomorphic to $\Delta(\check{\otimes}_{n,s,|\epsilon|} E)$. Since $\{e_i \otimes \cdots \otimes e_i : \; i \in \mathbb{N}\}$ is a basis of both $\Delta(\check{\otimes}_{n,|\epsilon|} E)$ and $\Delta(\check{\otimes}_{n,s,|\epsilon|} E)$, it suffices to show that $\|u\|_{|\epsilon|} = \|u\|_{s,|\epsilon|}$ for every $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \otimes_n E$. Without loss of generality, we assume that $u \geqslant 0$, that is, $a_i \geqslant 0$ for $i = 1, \ldots, m$. For any $\sigma > 0$ there exist $x_1^*, \ldots, x_n^* \in B_{E^{*+}}$ such that

$$\|u\|_{|\epsilon|} \leqslant \sum_{i=1}^{m} a_i x_1^*(e_i) \cdots x_n^*(e_i) + \sigma.$$

Let x^* be defined in Lemma 5.9. Then $x^* \in B_{E^{*+}}$ and for each $i \in \mathbb{N}$,

$$x^*(e_i) = \big(x_1^*(e_i)\big)^{\frac{1}{n}} \cdots \big(x_n^*(e_i)\big)^{\frac{1}{n}}.$$

Thus

$$\|u\|_{s,|\epsilon|} \geqslant \sum_{i=1}^{m} a_i\big(x^*(e_i)\big)^n = \sum_{i=1}^{m} a_i x_1^*(e_i)\cdots x_n^*(e_i) \geqslant \|u\|_{|\epsilon|} - \sigma.$$

It follows that $\|u\|_{s,|\epsilon|} \geqslant \|u\|_{|\epsilon|}$. From their definitions it is trivial that $\|u\|_{s,|\epsilon|} \leqslant \|u\|_{|\epsilon|}$ and hence, $\|u\|_{s,|\epsilon|} = \|u\|_{|\epsilon|}$.

Secondly, we show that $\Delta(\check{\otimes}_{n,\epsilon}E)$ is isometrically isomorphic to $\Delta(\check{\otimes}_{n,|\epsilon|}E)$. Since $\{e_i \otimes \cdots \otimes e_i :\ i \in \mathbb{N}\}$ is a basis of both $\Delta(\check{\otimes}_{n,\epsilon}E)$ and $\Delta(\check{\otimes}_{n,|\epsilon|}E)$, it suffices to show that $\|u\|_{\epsilon} = \|u\|_{|\epsilon|}$ for every $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \otimes_n E$. By Lemma 5.3,

$$\begin{aligned}
\big\|u\big\|_{|\epsilon|} &= \big\||u|\big\|_{|\epsilon|} = \left\|\sum_{i=1}^{m} |a_i| e_i \otimes \cdots \otimes e_i\right\|_{|\epsilon|} \\
&= \left\|\sum_{i=1}^{m} |a_i| e_i \otimes \cdots \otimes e_i\right\|_{\epsilon} = \left\|\sum_{i=1}^{m} \pm a_i e_i \otimes \cdots \otimes e_i\right\|_{\epsilon} \\
&\leqslant \left\|\sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i\right\|_{\epsilon} = \big\|u\big\|_{\epsilon}.
\end{aligned}$$

On the other hand, it follows from their definitions that $\|u\|_{\epsilon} \leqslant \|u\|_{|\epsilon|}$ and hence, $\|u\|_{\epsilon} = \|u\|_{|\epsilon|}$.

Finally we show that $\Delta(\check{\otimes}_{n,\epsilon}E)$ is isometrically isomorphic to $\Delta(\check{\otimes}_{n,s,\epsilon}E)$. Since $\{e_i \otimes \cdots \otimes e_i :\ i \in \mathbb{N}\}$ is a basis of both $\Delta(\check{\otimes}_{n,\epsilon}E)$ and $\Delta(\check{\otimes}_{n,s,\epsilon}E)$, it suffices to show that $\|u\|_{\epsilon} = \|u\|_{s,\epsilon}$ for every $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \otimes_n E$, which was proved in the proof of Lemma 5.5. $\square$

BIBLIOGRAPHY

[1] C.D. Aliprantis and O. Burkinshaw, *Positive operators*, vol. 119, Academic Press, 1985.

[2] A. Arias and J. D. Farmer, *On the structure of tensor products of ℓ_p-spaces*, Pacific J. Math. **175** (1996), 13–37.

[3] K. Boulabiar and G. Buskes, *Vector lattice powers: f-algebras and functional calculus*, Comm. Algebra **34** (2006), 1435–1442.

[4] Q. Bu and G. Buskes, *Schauder decompositions and the Fremlin pojective tensor product of Banach lattices*, J. Math. Anal. appl. **355** (2009), 335–351.

[5] Q. Bu and G. Buskes, *Diagonals of projective tensor products and orthogonally additive polynomials*, Studia Math. **201** (2014), 1–15.

[6] Q. Bu, G. Buskes, and Y. Li, *AM- and AL-spaces of polynomials on Banach lattices*, Proc. Edinburgh Math. Soc.

[7] Q. Bu and G. Buskes, *Polynomials on Banach lattices and positive tensor products*, J. Math. Anal. Appl. **388** (2012), 845–862.

[8] Q. Bu, A. Tcaciuc G. Buskes, A.I. Popov, and V.G. Troitsky, *The 2-concavification of a banach lattice equals the diagonal of the Fremlin tensor square*, Positivity **17** (2013), 283–298.

[9] Q. Bu and N.C. Wong, *Some geometric properties inherited by the positive tensor products of atomic Banach lattices*, Indag. Math. **23** (2012), 199–213.

[10] G. Buskes and A. Van Rooij, *Bounded variation and tensor products of Banach lattices*, Positivity **7** (2003), 47–59.

[11] D. Carando and D. Galicer, *Unconditionality in tensor products and ideas of polynomials, multilinear forms and operators*, Quart. J. Math. **62** (2011), 845–869.

[12] Z.L. Chen and A.W. Wickstead, *Some applications of Rademacher sequences in Banach lattice*, Positivity **2** (1998), 171–191.

[13] A. Defant, J.C. Díaz, D. Garcia, and M. Maestre, *Unconditional basis and Gordon-Lewis constants for spaces of polynomials*, J. Funct. Anal. **182** (2001), 119–145.

[14] A. Defant and K. Floret, *Tensor Norms and Operator Ideals*, North-Holland, Amsterdam (1993).

[15] V. Dimant and I. Zalduendo, *Bases in spaces of multilinear forms over Banach spaces*, J. Math. Anal. Appl. **200** (1996), 548–566.

[16] S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer, 1999.

[17] D.H. Fremlin, *Tensor products of Archimedean vector lattices*, Amer. J. Math. **94** (1972), 778–798.

[18] D.H. Fremlin, *Tensor products of Banach lattices*, Math. Ann. **211** (1974), 87–106.

[19] B.R. Gelbaum and J.Gil de Lamadrid, *Bases of tensor products of Banach spaces*, Pacific J. Math. **11** (1961), 1281–1286.

[20] B.C. Grecu and R.A. Ryan, *Schauder bases for symmetric tensor products*, Publ. RIMS, Kyoto Univ. **41** (2005), 459–469.

[21] B.C. Grecu and R.A. Ryan, *Polynomials on Banach spaces with unconditional bases*, Proc. Amer. Math. Soc. **133** (2005), 1083–1091.

[22] J. J. Grobler and C. C. A. Labuschagne, *The tensor product of Archimedean ordered vector spaces*, Proc. Cambridge Philos. Soc. **104** (1988), 331–345.

[23] A. Grothendieck, *Résumé de la théorie métrique des produits tensoriels topologiques*, Bol. Soc. Mat. São Paulo **8** (1953/54), 1–79.

[24] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc. **16** (1955).

[25] J. R. Holub, *Tensor product bases and tensor diagonals*, Trans. Amer. Math. Soc. **151** (1970), 563–579.

[26] A. Ibort, P. Linares, and J. G. Llavona, *A representation theorem for orthogonally additive polynomials on Riesz spaces*, Rev. Mat. Complut. (2010).

[27] Donghai Ji, Byunghoon Lee, and Qingying Bu, *Diagnoals of injective tensor products of Banach lattices with bases*, Preprint available on the arXiv at `http://arxiv.org/abs/1502.05012` (2015).

[28] Donghai Ji, Byunghoon Lee, and Qingying Bu, *On positive tensor products of ℓ_p- spaces*, Indagationes Mathematicae **25** (2014), 563–574.

[29] L. V. Kantorovič, *On the moment problem for a finite interval*, Dokl.Akad.Nauk SSSR **14** (1937), 531–537.

[30] L. V. Kantorovič, *Linear operators in semi-ordered spaces*, Mat. Sb.(N. S.) **49** (1940), 209–284.

[31] K.Floret, *Natural norms on symmetric tensor products of normed spaces*, Note Mat. **17** (1997), 153–188.

[32] S. Lassalle, Y. Benyamini, and J. G. Llavona, *Homogeneous orthogonally additive polynomials on banach lattices*, Bull. London Math. Soc. **38** (2006), 459–469.

[33] P. Linares, *Orthogonally Addtive Polynomials and Applications*, Ph.D. Dissertation, University of Madrid, 2009.

[34] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces II-Function Spaces*, Springer, 1979.

[35] J. Loane, *Polynomials on vector lattices*, Ph.D. Dissertation, National University of Ireland, Galway, 2007.

[36] P. Meyer-Nieberg, *Banach Lattices*, Springer-Verlag, 1991.

[37] J. Mujica, *Complex Analysis inBanach Spaces*, Math. Studies, North-Holland, Amsterdam **120** (1986).

[38] D. Pérez-García and I. Villanueva, *Orthogonally additive polynomials on spaces of continuous functions*, J. Math. Anal. Appl. **306** (2005), 97–105.

[39] D. Pérez-García and I. Villanueva, *Unconditional bases in tensor products of Hilbert spaces*, Math. Scand. **96** (2005), 280–288.

[40] G. Pisier, *Some result on Banach spaces without unconditional structure*, Compositio Math. **37** (1978), 3–9.

[41] H. R. Pitt, *A note on bilinear forms*, J. Lond. Math. Soc. **11** (1936), 174–180.

[42] R.A. Ryan, *Applications of Topological Tensor Products to Infinite Dimensional Holomorphy*, Ph.D. thesis, Trinity College, Dublin (1980).

[43] R.Alencar and K.Floret, *Weak–strong continuity of multilinear mappings and the Pełczński–Pitt theorem*, J. Math. Anal. Appl. **206** (1997), 532–546.

[44] F. Riesz, *Sur la décomposition des opérations linéaires*, Atti Congr. Internaz. Mat., Bologna **1928, 3** (1930), 143–148.

[45] R.A. Ryan, *Introduction to Tensor Products of Banach Spaces*, Springer, 2002.

[46] F.C. Sanchez, *Complemented subspaces of spaces of multilinear forms and tensor products*, J. Math. Anal. Appl. **254** (2001), 645–653.

[47] A. R. Schep, *Factorization of positive multilinear maps*, Illinois J. Math. **28** (1984), 579–591.

[48] C. Schütt, *Unconditionality in tensor products*, Israel J. Math. **31** (1978), 209–216.

[49] S.Kwapień and A.Pelczyński, *The main triangle projection in matrix spaces and its applications*, Studia Math. **34** (1970), 43–68.

[50] K. Sundaresan, *Geometry of spaces of homogeneous polynomials on Banach lattices*, DIMACS Ser. Discrete Math. Theoret. Comput. Sci. and Amer. Math. Soc. **4** (1991), 571–586.

[51] M. A. Toumi, *Orthogonally Additive Polynomials on Dedekind σ-complete vector lattices*, Math. Proc. Royal. Irish Acad. **110A** (2010), 83–94.

VITA

The author was born on September 26, 1975 in Pusan, South Korea. He attended at Peniel high school in Pusan and graduated in 1994. He then enrolled at the Kosin University in Pusan, South Korea where he graduated *summa cum laude* with a B.A. in Mathematics in 1998. In the spring of 2000, he earned an M.S. in Mathematics from the Pusan National University. There he wrote a thesis in abstract algebra under the supervision of Professor Chan Huh. After graduation, he spent 5 semesters as a lecturer at the Kosin University and performed his military service in the Korea army as an artillery officer between 2001–2004.

After finishing his military service, he decided to study abroad and enrolled at the University of Alabama in 2007. In the summer of 2010, he earned an M.A. in Mathematics from the University of Alabama in Huntsville. In the fall of 2010, he began his graduate studies at the University of Mississippi in functional analysis under the supervision of Professor Qingying Bu. He is the recipient of the university's 2012, 2013, and 2014 Summer Research Grant and the Dissertation Fellowship for the Spring 2015.