Diagonals of Tensor Products of Banach Lattices with Bases.

Byunghoon Lee

University of Mississippi

Follow this and additional works at: https://egrove.olemiss.edu/etd

Part of the Mathematics Commons

Recommended Citation
https://egrove.olemiss.edu/etd/1442

This Dissertation is brought to you for free and open access by the Graduate School at eGrove. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of eGrove. For more information, please contact egrove@olemiss.edu.
DIAGONALS OF TENSOR PRODUCTS OF
BANACH LATTICES WITH BASES

A Dissertation
presented in partial fulfillment of requirements
for the degree of Doctor of Philosophy
in the Department of Mathematics
The University of Mississippi

by
BYUNGHOON LEE

May 2015
ABSTRACT

In this dissertation, we investigate diagonals of tensor products of Banach lattices with bases. We first consider questions on the positive tensor products of $\ell_p$-spaces. We characterize the main diagonals of the positive projective tensor product $\ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n}$ and the positive injective tensor product $\ell_{p_1} \check{\otimes} |\epsilon| \cdots \check{\otimes} |\epsilon| \ell_{p_n}$. Then, by using these two main diagonals, we characterize the reflexivity, the property of being Kantorovich-Banach spaces, and the property of being order continuous of $\ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n}$ and $\ell_{p_1} \check{\otimes} |\epsilon| \cdots \check{\otimes} |\epsilon| \ell_{p_n}$, as well as the space of all regular $n$-linear forms on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ and the space of all regular $n$-homogeneous polynomials on $\ell_p$ ($1 \leq p < \infty$). We also obtain the following interesting result. Let $1 < p_1, \cdots, p_n, q < \infty$. Then the Banach lattice $L^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$, the space of all regular $n$-linear operators from $\ell_{p_1} \times \cdots \times \ell_{p_n}$ to $\ell_q$, is order continuous if and only if $1/p_1 + \cdots + 1/p_n < 1/q$. However, $K^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$, the sublattice of $L^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ generated by all positive compact operators from $\ell_{p_1} \times \cdots \times \ell_{p_n}$ to $\ell_q$, is always order continuous for any $1 < p_1, \cdots, p_n, q < \infty$.

We next consider the diagonals of injective tensor products of Banach lattices with bases. Let $E$ be a Banach lattice with a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. Denote by $\Delta(\hat{\otimes}_{n,\epsilon} E)$ (resp. $\Delta(\check{\otimes}_{n,\epsilon} E)$) the main diagonal space of the $n$-fold full (resp. symmetric) injective Banach space tensor product, and denote by $\Delta(\hat{\otimes}_{n,|\epsilon|} E)$ (resp. $\Delta(\check{\otimes}_{n,|\epsilon|} E)$) the main diagonal space of the $n$-fold full (resp. symmetric) injective Banach lattice tensor product. We show that these four main diagonal spaces are pairwise isometrically isomorphic by using the 1-unconditionality of the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$. We also show that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence in both $\hat{\otimes}_{n,\epsilon} E$ and $\check{\otimes}_{n,\epsilon} E$. 


DEDICATION

To Hyojung, for her adoration, friendship, and support.
ACKNOWLEDGEMENTS

I would first like to express gratitude to my adviser, Dr. Qingying Bu, for his encouragement, guidance, and patience over the past five years. I am grateful for his supportive and encouraging words and for challenging me to always keep improving and learning. His advice and counsel have been integral in developing the material presented here as well as my personal mathematical development. Thank you for helping me become a stronger researcher, writer, and teacher.

I appreciate Drs. Gerard Buskes, Iwo Labuda, Micah Milinovich and James Cizdziel for taking time out of their busy schedules to serve on my dissertation committee. Their suggestions were well received. I thank Mr. Marlow Dorrough for giving me the opportunity to teach a variety of courses as a graduate student. Several mathematicians have helped shape and strengthen my research and understanding. In particular, I thank Dr. Joseph Diestel for helpful conversations and feedback about my thesis.

I also have benefited from the professional advice and encouragement from a large group of mentors. In particular, I extend heartfelt thanks to my former advisers Dr. Chan Huh and Dr. Joonchul Han of the Pusan National University for striving to mold an immature and young student into a mathematician. Along the same line, I would like to thank Drs. Bing Wei, Miña-Diaz, James Reid, Laura Sheppardson and Sandra Spiroff of the University of Mississippi for introducing many wonderful topics of mathematics to me and providing an environment of thought and growth over the past several years.

It would have been impossible to successfully emerge from this process without the support of my friends and family. I warmly thank Matthew Joo and his family for their constant encouragement and support. I thank my classmates Caroline, Chris, and Stephan
for being emphatic and supportive friends. I thank my parents and sibling Jaehyuk for their constant encouragement and love. In addition, I would like to thank my wife’s family for their support, as well. Finally, I thank my wife, Hyojung, my source of constant support, encouragement, and inspiration. I want to thank my son, Joonshub, for making me laugh and smile every day.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Abstract</strong></td>
<td>ii</td>
</tr>
<tr>
<td><strong>Dedication</strong></td>
<td>iii</td>
</tr>
<tr>
<td><strong>Acknowledgements</strong></td>
<td>iv</td>
</tr>
<tr>
<td><strong>Introduction</strong></td>
<td>1</td>
</tr>
<tr>
<td><strong>Preliminaries</strong></td>
<td>7</td>
</tr>
<tr>
<td><strong>Multilinear Operators and Tensor Products</strong></td>
<td>15</td>
</tr>
<tr>
<td>3.1 Multilinear Operators and Tensor Products of Banach Spaces</td>
<td>15</td>
</tr>
<tr>
<td>3.2 Positive Multilinear Operators and Positive Tensor Products of Banach Lattices</td>
<td>22</td>
</tr>
<tr>
<td><strong>Positive Tensor Products of (\ell_p)-Spaces</strong></td>
<td>28</td>
</tr>
<tr>
<td>4.1 Positive Projective Tensor Products of (\ell_p)-Spaces</td>
<td>28</td>
</tr>
<tr>
<td>4.2 Positive Injective Tensor Products of (\ell_p)-Spaces</td>
<td>38</td>
</tr>
<tr>
<td><strong>Diagonals of Tensor Products of Banach Lattices with Bases</strong></td>
<td>46</td>
</tr>
<tr>
<td>5.1 Diagonals of Projective Tensor Products of Banach Lattices</td>
<td>46</td>
</tr>
<tr>
<td>5.2 Diagonals of Injective Tensor Products of Banach Lattices</td>
<td>48</td>
</tr>
<tr>
<td><strong>Bibliography</strong></td>
<td>58</td>
</tr>
<tr>
<td><strong>Vita</strong></td>
<td>62</td>
</tr>
</tbody>
</table>


1 INTRODUCTION

The general theory of tensor products of Banach spaces dates back to Grothendieck’s famous Résumé [23] and Memoir [24] which were published in the 1950s. These two papers have had a considerable influence on the development of Banach space theory. They contained a general theory of tensor norms on tensor products of Banach spaces, studied the duality theory of these tensor products, and characterized the linearization of multilinear operators through the tensor products. Grothendieck’s Résumé and Memoir opened many directions in functional analysis.

In 1980, R. Ryan in his doctoral thesis [42] introduced symmetric tensor products for the study of polynomials on Banach spaces and then characterized the linearization of homogeneous polynomials through the symmetric tensor products. The symmetric tensor products and homogeneous polynomials play a key role in the theory of holomorphic functions. See, for example, Dineen’s book [16] and Mujica’s book [37]. In 2005, Grecu and Ryan [21], when they studied the homogeneous polynomials with unconditionally convergent monomial expansions, introduced regular homogeneous polynomials with respect to the Banach lattice structure of the domain. A regular homogeneous polynomial is defined to be the difference of two positive homogeneous polynomials. In turn, positive homogeneous polynomials are defined in terms of their associated symmetric positive multilinear operators.

The linearization of positive multilinear operators on Banach lattices dates back to 1970s when Fremlin [17, 18] introduced the positive tensor products of Banach lattices.
Later, Schep [47], Grobler and Labuschagne [22], and Buskes and Van Rooij [10] have made significant contributions in the theory of positive tensor products and positive multilinear operators on Banach lattices.


Arias and Farmer [2] characterized the main diagonal of the projective tensor product \( \ell_{p_1} \hat{\otimes} \pi \cdots \hat{\otimes} \pi \ell_{p_n} \) (\( 1 \leq p_1, \cdots, p_n < \infty \)) and used it to characterize the reflexivity of \( \ell_{p_1} \hat{\otimes} \pi \cdots \hat{\otimes} \pi \ell_{p_n} \) (also see [43]). In section 4.1 of this dissertation, we use the Rademacher averaging to characterize the main diagonal of the positive projective tensor product \( \ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n} \) and then use it to characterize the reflexivity of \( \ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n} \). From the positivity perspective, we show that \( \ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n} \) is a Kantorovich-Banach space and hence, is an order continuous Banach lattice for any \( 1 \leq p_1, \cdots, p_n < \infty \). However, we show that its dual \( \mathcal{L}^\prime(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R}) \), the space of all regular \( n \)-linear forms on \( \ell_{p_1} \times \cdots \times \ell_{p_n} \), is an order continuous Banach lattice if and only if it is reflexive.

Holub [25] characterized the main diagonal of the injective tensor product of \( \ell_p \hat{\otimes} \ell_q \) (\( 1 \leq p, q < \infty \)). In section 4.2 of this dissertation, we use the Rademacher averaging to characterize the main diagonal of the positive injective tensor product \( \ell_{p_1} \hat{\otimes} |\epsilon| \cdots \hat{\otimes} |\epsilon| \ell_{p_n} \) (\( 1 \leq p_1, \cdots, p_n < \infty \)) and then use it to characterize the reflexivity of \( \ell_{p_1} \hat{\otimes} |\epsilon| \cdots \hat{\otimes} |\epsilon| \ell_{p_n} \). From the positivity perspective, we show that \( \ell_{p_1} \hat{\otimes} |\epsilon| \cdots \hat{\otimes} |\epsilon| \ell_{p_n} \) is an order continuous Banach lattice for any \( 1 \leq p_1, \cdots, p_n < \infty \). However, we show that it is a Kantorovich-Banach space if and only if it is reflexive.
In chapter 4 of this dissertation, we characterize the main diagonals of the positive projective tensor product $\ell_{p_1} \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} \ell_{p_n}$ and the positive injective tensor product $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$. Then by using these two main diagonals, we characterize the reflexivity, the property of being Kantorovich-Banach spaces, and the property of being order continuous of $\ell_{p_1} \widehat{\otimes}_{|\pi|} \cdots \widehat{\otimes}_{|\pi|} \ell_{p_n}$ and $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$, as well as the space of all regular $n$-linear forms on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ and the space of all regular $n$-homogeneous polynomials on $\ell_p$ ($1 \leq p < \infty$). In section 4.2 of this dissertation, we also obtain the following interesting result. Let $1 < p_1, \cdots, p_n, q < \infty$. Then the Banach lattice $L^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$, the space of all regular $n$-linear operators from $\ell_{p_1} \times \cdots \times \ell_{p_n}$ to $\ell_q$, is order continuous if and only if $1/p_1 + \cdots + 1/p_n < 1/q$. However, $K^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$, the sublattice of $L^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ generated by all positive compact operators from $\ell_{p_1} \times \cdots \times \ell_{p_n}$ to $\ell_q$, is always order continuous for any $1 < p_1, \cdots, p_n, q < \infty$.

In section 5.2 of this dissertation, we show that all four main diagonal spaces $\Delta(\hat{\otimes}_{n, \epsilon} E)$, $\Delta(\hat{\otimes}_{n, s, \epsilon} E)$, $\Delta(\hat{\otimes}_{n, |\epsilon|} E)$, and $\Delta(\hat{\otimes}_{n, s, |\epsilon|} E)$ are pairwise isometrically isomorphic. We also show that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence in both $\hat{\otimes}_{n, \epsilon} E$ and $\hat{\otimes}_{n, s, \epsilon} E$. Let $X$ be a Banach space with a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a basis of both the $n$-fold full injective tensor product $\hat{\otimes}_{n, \epsilon} X$ and the $n$-fold full projective tensor product $\hat{\otimes}_{n, \pi} X$ (see, e.g., [19, 20]), and $\{e_{i_1} \otimes s \cdots \otimes s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geq \cdots \geq i_n\}$ is a basis of both the $n$-fold symmetric injective tensor product $\hat{\otimes}_{n, s, \epsilon} X$ and the $n$-fold symmetric projective tensor product $\hat{\otimes}_{n, s, \pi} X$ (see, e.g., [20]). However, they are not necessary unconditional bases (see, e.g., [49, 40, 48, 13, 39, 11]). In particular, the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence in both $\hat{\otimes}_{n, \pi} X$ (see, e.g., [25, 46]) and $\hat{\otimes}_{n, s, \pi} X$ (see, e.g., [5]); and Holub [25] showed that the tensor diagonal $\{e_i \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence in $\hat{\otimes}_{2, \epsilon} X$. By using Holub’s method, it is easy to show that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence in $\hat{\otimes}_{n, \epsilon} X$. However, Holub’s method
does not work for $\hat{\otimes}_{n,s,\epsilon}X$. In section 5.2, by using a result obtained in [5], we show that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence in $\hat{\otimes}_{n,s,\epsilon}X$ and we also show that the diagonal projections on both $\hat{\otimes}_{n,\epsilon}X$ and $\hat{\otimes}_{n,s,\epsilon}X$ are contractive.

From the positivity perspective, let $E$ be a Banach lattice with a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a 1-unconditional basis of the $n$-fold full positive projective tensor product $\hat{\otimes}_{n,|\pi|}E$, and $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geq \cdots \geq i_n\}$ is a 1-unconditional basis of the $n$-fold positive symmetric projective tensor product $\hat{\otimes}_{n,s,|\pi|}E$ (see, e.g., [8, 5]). In section 5.2 of this dissertation, we show that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a 1-unconditional basis of the $n$-fold full positive injective tensor product $\hat{\otimes}_{n,\epsilon}E$, and $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geq \cdots \geq i_n\}$ is a 1-unconditional basis of the $n$-fold positive symmetric injective tensor product $\hat{\otimes}_{n,s,\epsilon}E$.

Let $\Delta(\hat{\otimes}_{n,\pi}E)$ (resp. $\Delta(\hat{\otimes}_{n,s,\pi}E)$) denote the closed subspace generated by the tensor diagonals $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\hat{\otimes}_{n,\pi}E$ (resp. $\hat{\otimes}_{n,s,\pi}E$), and let $\Delta(\hat{\otimes}_{n,|\pi|}E)$ (resp. $\Delta(\hat{\otimes}_{n,s,|\pi|}E)$) denote the closed sublattice generated by the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\hat{\otimes}_{n,|\pi|}E$ (resp. $\hat{\otimes}_{n,s,|\pi|}E$). Bu and Buskes [5] showed that these four main diagonal spaces $\Delta(\hat{\otimes}_{n,\pi}E), \Delta(\hat{\otimes}_{n,s,\pi}E), \Delta(\hat{\otimes}_{n,|\pi|}E)$, and $\Delta(\hat{\otimes}_{n,s,|\pi|}E)$ are isometrically isomorphic.

Now let $\Delta(\hat{\otimes}_{n,\epsilon}E)$ (resp. $\Delta(\hat{\otimes}_{n,s,\epsilon}E)$) denote the closed subspace generated by the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\hat{\otimes}_{n,\epsilon}E$ (resp. $\hat{\otimes}_{n,s,\epsilon}E$), and let $\Delta(\hat{\otimes}_{n,|\epsilon|}E)$ (resp. $\Delta(\hat{\otimes}_{n,s,|\epsilon|}E)$) denote the closed sublattice generated by the tensor diagonals $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\hat{\otimes}_{n,|\epsilon|}E$ (resp. $\hat{\otimes}_{n,s,|\epsilon|}E$). In section 5.2, we use the 1-unconditionality of the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ to show that these four main diagonal spaces $\Delta(\hat{\otimes}_{n,\epsilon}E), \Delta(\hat{\otimes}_{n,s,\epsilon}E), \Delta(\hat{\otimes}_{n,|\epsilon|}E)$, and $\Delta(\hat{\otimes}_{n,s,|\epsilon|}E)$ are isometrically isomorphic.
LIST OF SYMBOLS

For two Banach spaces $X$, $Y$ and Banach lattices $E_1, ..., E_n, E$ and $F$,

- $X\hat{\otimes}_\pi Y$: the completion of $X \otimes Y$ with respect to $\| \cdot \|_\pi$.

- $X\hat{\otimes}_\epsilon Y$: the completion of $X \otimes Y$ with respect to $\| \cdot \|_\epsilon$.

- $\mathcal{L}(X; Y)$: the space of all bounded linear operators from $X$ to $Y$.

- $\mathcal{L}(E_1 \times \cdots \times E_n; F)$: the space of $n$-linear mappings from $E_1 \times \cdots \times E_n$ to $F$.

- $K(X; Y)$: the space of all compact operators from $X$ to $Y$.

- $\Delta(X\hat{\otimes}_\pi Y)$ (resp. $\Delta(X\hat{\otimes}_\epsilon Y)$): the main diagonal of $X\hat{\otimes}_\pi Y$ (resp. $X\hat{\otimes}_\epsilon Y$).

- $\hat{\otimes}_{n,\pi} X$ (resp. $\hat{\otimes}_{n,\epsilon} X$): $n$-fold projective (resp. injective) tensor products of $X$.

- $\Delta(\hat{\otimes}_{n,\pi} X)$ (resp. $\Delta(\hat{\otimes}_{n,\epsilon} X)$): the main diagonal of $\hat{\otimes}_{n,\pi} X$ (resp. $\hat{\otimes}_{n,\epsilon} X$).

- $\otimes_{n,s} X$: the $n$-fold symmetric algebraic tensor product of $X$.

- $\tilde{\otimes}_{n,s} E$: the $n$-fold vector lattice symmetric tensor product of $E$.

- $\hat{\otimes}_{n,s,\pi} X$ (resp. $\hat{\otimes}_{n,s,\epsilon} X$): $n$-fold symmetric projective (resp. injective) tensor products of $X$. 
• $\Delta(\hat{\otimes}_{n,\pi}X)$ (resp. $\Delta(\tilde{\otimes}_{n,\pi}X)$): the main diagonal of $\hat{\otimes}_{n,\pi}X$ (resp. $\tilde{\otimes}_{n,\pi}X$).

• $E\hat{\otimes}_{|\pi|}F$: the completion of $E \otimes F$ with respect to $\| \cdot \|_{|\pi|}$.

• $E\tilde{\otimes}_{|\epsilon|}F$: the completion of $E \otimes F$ with respect to $\| \cdot \|_{|\epsilon|}$.

• $\mathcal{L}^r(E; F)$: the space of all regular operators from $E$ to $F$.

• $\mathcal{L}_b(E; F)$: the set of order bounded operators from $E$ to $F$.

• $\mathcal{K}^r(E; F)$: the space of all compact regular operators from $E$ to $F$.

• $\mathcal{P}(^nE; F)$: the space of all continuous $n$-homogeneous polynomials from $E$ to $F$.

• $\mathcal{P}^r(^nE; F)$: the space of all regular $n$-homogeneous polynomials from $E$ to $F$.

• $\hat{\otimes}_{n,|\pi|}E$ (resp. $\tilde{\otimes}_{n,|\epsilon|}E$): $n$-fold positive projective (resp. injective) tensor products of $E$.

• $\Delta(\hat{\otimes}_{n,|\pi|}E)$ (resp. $\Delta(\tilde{\otimes}_{n,|\epsilon|}E)$): the main diagonal of $\hat{\otimes}_{n,|\pi|}E$ (resp. $\tilde{\otimes}_{n,|\epsilon|}E$).

• $\hat{\otimes}_{n,s,|\pi|}E$ (resp. $\tilde{\otimes}_{n,s,|\epsilon|}E$): $n$-fold positive symmetric projective (resp. injective) tensor products of $E$.

• $\Delta(\hat{\otimes}_{n,s,|\pi|}E)$ (resp. $\Delta(\tilde{\otimes}_{n,s,|\epsilon|}E)$): the main diagonal of $\hat{\otimes}_{n,s,|\pi|}E$ (resp. $\tilde{\otimes}_{n,s,|\epsilon|}E$).
2 PRELIMINARIES

Each chapter is divided into sections numbered by two digits, the first one being the number of the chapter. Definitions, theorems, propositions, corollaries, lemmas and examples in each chapter are labeled by two digits, the first indicating the chapter. We will use the symbol □ to mark the end of proof. We refer to the book of Meyer-Nieberg [36] and of Aliprantis and Burkinshaw [1] for basic properties of Riesz spaces and Banach lattices that we use.

Definition 2.1. A real vector space $E$ is said to be an ordered vector space whenever it is equipped with an order relation $\geq$ (i.e. $\geq$ is reflexive, antisymmetric and transitive binary relation on $E$) that is compatible with the algebraic structure of $E$ in the sense that it satisfies the following two axioms:

(i) If $x \geq y$, then $x + z \geq y + z$ holds for all $z \in E$.

(ii) If $x \geq y$, then $\alpha x \geq \alpha y$ holds for all $\alpha \geq 0$.

An alternate notation for $x \geq y$ is $y \leq x$.

Definition 2.2. An element $x$ in an ordered vector space $E$ is called positive whenever $x \geq 0$ holds. The set of all positive elements of $E$ will be denoted by $E^+$, i.e. $E^+ = \{x \in E : x \geq 0\}$. The set $E^+$ of positive vectors is called the positive cone of $E$.

Definition 2.3. A mapping $T : E \to F$ between two vector spaces is called a linear operator if and only if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ holds for all $x, y \in E$ and all $\alpha, \beta \in \mathbb{R}$.
Definition 2.4. An operator \( T : E \to F \) between two ordered vector spaces is said to be positive (in symbols \( T \geq 0 \) or \( 0 \leq T \)) whenever \( T(x) \geq 0 \) for all \( x \geq 0 \).

An operator \( T : E \to F \) between two ordered vector spaces is, of course, positive if and only if \( T(E^+) \subseteq F^+ \) (and equivalently \( T(x) \geq T(y) \) if \( x \geq y \)).

Definition 2.5. A Riesz space (or a vector lattice) is an ordered vector space \( E \) with the additional property that for each pair of elements \( x, y \in E \) the supremum and infimum of the set \( \{x, y\} \) both exist in \( E \).

We shall write

\[
x \vee y = \sup\{x, y\} \quad \text{and} \quad x \land y = \inf\{x, y\}.
\]

Typical examples of Riesz spaces are provided by the function spaces.

Definition 2.6. A function space is a vector space \( E \) of real-valued functions on a set \( \Omega \) such that for each pair \( f, g \in E \) the functions

\[
(f \vee g)(\omega) = \max\{f(\omega), g(\omega)\}
\]

and

\[
(f \land g)(\omega) = \min\{f(\omega), g(\omega)\}
\]

both belong to \( E \).

Every function space \( E \) with the pointwise ordering (i.e., \( f \geq g \) holds in \( E \) if and only if \( f(\omega) \geq g(\omega) \) for all \( \omega \in \Omega \)) is a Riesz space. Here are some important examples of function spaces:

(i) \( \mathbb{R}^\Omega \), all real-valued functions defined on a set \( \Omega \).
(ii) $C(\Omega)$, all continuous real-valued functions on a topological space $\Omega$.

(iii) $C_b(\Omega)$, all bounded real-valued continuous functions on a topological space $\Omega$.

(iv) $\ell_\infty(\Omega)$, all bounded real-valued functions on a set $\Omega$.

(v) $\ell_p (0 < p < \infty)$, all real sequences $(x_1, x_2, \cdots)$ with $\sum_{n=1}^{\infty} |x_n|^p < \infty$.

The class of $L_p$-space is another important class of Riesz spaces. If $(X, \Sigma, \mu)$ is a measure space and $(0 < p < \infty)$, then $L_p(\mu)$ is the vector space of all real-valued $\mu$-measurable functions $f$ on $X$ such that $\int_X |f|^p < \infty$. Also $L_\infty(\mu)$ is the vector space of all real-valued $\mu$-measurable functions $f$ on $X$ such that $\text{esssup}|f| < \infty$. As usual, functions differing on a set of measure zero are treated as identical, i.e., $f = g$ in $L_p(\mu)$ means that $f(x) = g(x)$ for $\mu$-almost all $x \in X$. It is true that under the ordering $f \leq g$ whenever $f(x) \leq g(x)$ holds for $\mu$-almost all $x \in X$, each $L_p(\mu)$ is a Riesz space.

**Definition 2.7.** A net $\{x_\alpha\}$ in a Riesz space is said to be *decreasing* (in symbols, $x_\alpha \downarrow$) whenever $\alpha > \beta$ implies $x_\alpha \leq x_\beta$.

The notation $\{x_\alpha\} \downarrow x$ means that $x_\alpha \downarrow$ and $\inf\{x_\alpha\} = x$ both hold.

**Definition 2.8.** A Riesz space $E$ is called *Archimedean* whenever $n^{-1}x \downarrow 0$ holds in $E$ for each $x \in E^+$, $n \in \mathbb{N}$.

For any vector $x$ in a Riesz space we define

$$x^+ = x \vee 0, \quad x^- = (-x) \vee 0, \quad |x| = x \vee (-x).$$

The element $x^+$ is called the positive part, $x^-$ the negative part and $|x|$ the absolute value of $x$. Vectors $x^+, x^-$, and $|x|$ are satisfied in the following important theorem:

**Theorem 2.9.** If $x$ is an element of a Riesz space, then we have

(i) $x = x^+ - x^-$. 

9
(ii) $|x| = x^+ + x^-$,

(iii) $x^+ \land x^- = 0$.

Moreover the decomposition in (i) is unique in the sense that if $x = y - z$ holds with $y \land z = 0$, then $y = x^+$ and $z = x^-$.

**Definition 2.10.** In a Riesz space $E$ and $x, y \in E$. Two elements $x$ and $y$ are called **disjoint** (denoted by $x \perp y$) if $|x| \land |y| = 0$.

**Definition 2.11.** For an operator $T : E \rightarrow F$ between two Riesz spaces we say that its modulus $|T|$ exists whenever $|T| = T \lor (-T)$ exists (in the sense that $|T|$ is the supremum of the set $\{T, -T\}$ in the ordered vector space $\mathcal{L}(E; F)$).

In what follows, by operator we mean a linear map between Riesz spaces.

**Definition 2.12.** An operator $T : E \rightarrow F$ is said to be a **regular operator** if it can be written as the difference of two positive operators.

**Definition 2.13.** Let $A$ be a subset of an ordered set $M$. A set $A$ is called **order bounded** if it is bounded both from above and from below.

**Definition 2.14.** An operator $T : E \rightarrow F$ that maps order bounded subsets of $E$ onto order bounded subsets of $F$ is called **order bounded**. We denote the set of order bounded operators from $E$ to $F$ by $\mathcal{L}_b(E; F)$.

**Definition 2.15.** A Riesz space is called **Dedekind complete** if every nonempty subset that is bounded above has a supremum.
When $F$ is Dedekind complete, the ordered vector space $\mathcal{L}^r(E;F)$ has the structure of a Riesz space. In fact, $\mathcal{L}^r(E;F)$ is a Dedekind complete Riesz space. This result was proved first by Riesz [44] for the case $F = \mathbb{R}$, and later Kantorovič [29, 30] extended it to the general setting.

**Definition 2.16.** A subset $A$ of a Riesz space is said to be *upwards directed* (in symbols $A \uparrow$) whenever for each pair $x, y \in A$ there exists some $z \in A$ with $x \leq z$ and $y \leq z$. The symbol $A \uparrow x$ means that $A$ is upwards directed and $x = \text{sup} A$ holds.

**Definition 2.17.** A Riesz subspace $G$ of a Riesz space $E$ is said to be *order dense* in $E$ whenever for each $0 < x \in E$, there exists $y \in G$ such that $0 < y \leq x$.

**Definition 2.18.** The vector space $E^\sim$ of all order bounded linear functionals on a Riesz space $E$ is called the *order dual* of $E$, i.e. $E^\sim = \mathcal{L}_b(E;\mathbb{R})$.

**Definition 2.19.** Let $E$ be a (real) Riesz space equipped with a norm. The norm in $E$ is called a *Riesz norm* if $|x| \leq |y|$ in $E$ implies $\|x\| \leq \|y\|$. A normed Riesz space which is complete with respect to the norm is called a *Banach lattice*.

Recall that a norm $\|\cdot\|$ on a Riesz space is said to be a Riesz norm (or lattice norm) whenever $|x| \leq |y|$ implies $\|x\| \leq \|y\|$. Note that this implies that for any $x \in E$ the elements $x$ and $|x|$ have the same norm. Any Riesz space, equipped with a Riesz norm, is called a *normed Riesz space*. If the normed Riesz space $E$ is also norm complete, then $E$ is referred to as a Banach lattice.

**Example 2.20.** Both $\ell_p$ ($1 \leq p \leq \infty$) and $c_0$ are Dedekind complete Banach lattices where the order is defined pointwise. Assume that $(\Omega, \Sigma, \mu)$ is a measure space. Then $L_p(\mu)$ ($1 \leq p < \infty$) is Dedekind complete Banach lattice. If $\mu$ is $\sigma$-finite, then $L_\infty(\mu)$ is Dedekind complete.

**Definition 2.21.** A Banach lattice $E$ is called *order continuous* if $\|x_\alpha\| \downarrow 0$ whenever $x_\alpha \downarrow 0$. 

11
Definition 2.22. A subspace $U$ of $E$ is called a sublattice of $E$ if $x \lor y \in U$ and $x \land y \in U$ for all $x, y \in U$.

Definition 2.23. A subset $A$ of $E$ is called solid if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$.

Definition 2.24. Every solid subspace $I$ of $E$ is called an ideal or order ideal in $E$.

Definition 2.25. An ideal $B$ of $E$ is called a band if $\sup(A) \in B$ for every subset $A \subset B$ which has a supremum in $E$.

Definition 2.26. A band $B$ of $E$ is called a projection band if there is a linear projection $P : E \to B$ such that $0 \leq Px \leq x$ for all $x \in E^+$. Such a projection is called a band projection.

Example 2.27.

(i) If $c$ denotes the space of all convergent real sequences, then $c$ is a sublattice of $\ell_\infty$, but fails to be an ideal.

(ii) The space $c_0$ is an ideal in $\ell_\infty$, but fails to be a band in $\ell_\infty$. If $v_n = e_1 + \cdots + e_n$ where $e_k$ denotes the $k$-th natural basis vector of $c_0$, then $e = \sup\{v_n : n \in \mathbb{N}\}$ exists in $\ell_\infty$, but not in $c_0$.

It is clear that every ideal in $E$ is a sublattice of $E$. Moreover, the intersection of any two sublattices, ideals, or bands, respectively, is a sublattice (resp. ideal and band). The sum of two ideals is also an ideal and the sum of two projection bands is a projection band. However, the sum of two sublattices need not to be a sublattice (see, p.12, [36]).

Definition 2.28. A sequence $(x_n)$ in a Banach space $X$ is said to be a Schauder basis (or simply a basis) whenever for each $x \in X$ there exists a unique sequence $(\alpha_n)$ of scalars satisfying $x = \sum_{n=1}^{\infty} \alpha_n x_n$ (where, as usual, the convergence of the series is assumed to be in the norm.)
Definition 2.29. A sequence \((x_n)\) in a Banach space \(X\) is a (Schauder) basic sequence if it is a (Schauder) basis for the closed vector subspace it generates.

(denoted by \([\{x_n : n \in \mathbb{N}\}]\).

Example 2.30. If \(X\) is \(c_0\) or \(\ell_p\) such that \(1 \leq p < \infty\), then the sequence \((e_n)\) of standard unit vectors of \(X\) is a basis and that \((\alpha_n) = \sum_{n=1}^{\infty} \alpha_n e_n\) whenever \((\alpha_n) \in X\). However, the sequence \((e_n)\) is not a basis for \(\ell_\infty\). For example, there is no sequence \((\alpha_n)\) of scalars such that \((1,1,1,...) = \sum_{n=1}^{\infty} \alpha_n e_n\).

Whenever the sequence \((e_n)\) lies in the unit sphere of a Banach space of sequences of scalars and is a basis for the space, the sequence will be called the standard unit vector basis for the space.

Definition 2.31. A basis \((x_n)\) for a Banach space \(X\) is unconditional if every convergent series of the form \(\sum_{n=1}^{\infty} \alpha_n x_n\) is unconditionally convergent. A basis for a Banach space is conditional if it is not unconditional.

Example 2.32. Suppose that \(X\) is \(c_0\) or \(\ell_p\) such that \(1 \leq p < \infty\). Then the standard unit vector basis for \(X\) is unconditional. The classical Schauder basis for \(C[0,1]\) and the Harr basis for \(L_1[0,1]\) are conditional bases.

Definition 2.33. For an unconditional basis \((x_n)\) in \(X\), there exists a constant \(C\) such that for every sequence of scalars \((a_n)_{n=1}^{\infty}\) and \((\epsilon_n)_{n=1}^{\infty}\) of modulus at most 1. We have

\[
\left\| \sum_{n=1}^{\infty} \epsilon_n a_n x_n \right\| \leq C \left\| \sum_{n=1}^{\infty} a_n x_n \right\|
\]

If \(C\) is given, we call the basis \(C\)-unconditional.

Definition 2.34. Two bases \((x_n)\) and \((y_n)\) for Banach spaces are equivalent if, for every sequence \((\alpha_n)\) of scalars, the series \(\sum_{n=1}^{\infty} \alpha_n x_n\) converges if and only if \(\sum_{n=1}^{\infty} \alpha_n y_n\) converges.
The equivalence of bases guarantees an isomorphism of the Banach spaces they span.

**Definition 2.35.** A basis \((x_n)\) for a Banach space \(X\) is called *boundedly complete*, whenever a sequence \((\alpha_n)\) of scalars is such that

\[
\sup_m \left\| \sum_{n=1}^{m} \alpha_n x_n \right\| < \infty,
\]

the series \(\sum_{n=1}^{\infty} \alpha_n x_n\) converges.

**Example 2.36.** Let \((e_n)\) be the standard unit vector basis for \(\ell_p\), where \(1 \leq p < \infty\). If \((\alpha_n)\) is a sequence of scalars such that \(\sup_m \|\sum_{n=1}^{m} \alpha_n e_n\|_p\) is finite, then

\[
\sum_{n=1}^{\infty} |\alpha_n|^p = \sup_m \sum_{n=1}^{m} |\alpha_n|^p = \sup_m \left\| \sum_{n=1}^{m} \alpha_n e_n \right\|_p < +\infty,
\]

so \(\sum_{n=1}^{\infty} \alpha_n e_n\) converges to the element \((\alpha_n)\) of \(\ell_p\). The basis \((e_n)\) is therefore boundedly complete.

**Definition 2.37** Let \(X\) be a subspace of a normed space \(Z\). We say that \(X\) is *complemented* in \(Z\) if there exists a subspace \(Y\) such that \(Z = X \oplus Y\).

Any closed subspace of a Hilbert space is complemented.

**Definition 2.38** Let \(X\) be a Banach space. We say that \(A \subseteq X\) is *relatively compact* if \(\bar{A}\) is compact in \(X\).

**Definition 2.39** Let \(X\) and \(Y\) are Banach spaces. A linear operator \(T\) from \(X\) into \(Y\) is called *compact* if \(T(B)\) is a relatively compact subset of \(Y\) whenever \(B\) is a bounded subset of \(X\).
3 MULTILINEAR OPERATORS AND TENSOR PRODUCTS

3.1 Multilinear Operators and Tensor Products of Banach Spaces

Definition 3.1. For Banach spaces $X_1, \ldots, X_n$, and $Y$, an operator $T : X_1 \times \cdots \times X_n \to Y$ is called an $n$-linear operator if it is linear in each variable $x_k, y_k \in X_k$, $k = 1, \ldots, n$. That is, for fixed $k$ with $1 \leq k \leq n$ and $\alpha, \beta \in \mathbb{R}$, we have

$$T(x_1, \ldots, x_{k-1}, \alpha x_k + \beta y_k, x_{k+1}, \ldots, x_n)$$

$$= \alpha T(x_1, \ldots, x_{k-1}, x_k, x_{k+1}, \ldots, x_n) + \beta T(x_1, \ldots, x_{k-1}, y_k, x_{k+1}, \ldots, x_n).$$

Definition 3.2. An $n$-linear operator $T : X_1 \times \cdots \times X_n \to Y$ is called bounded if there exists a constant $C$ such that $\|T(x_1, \ldots, x_n)\| \leq C \|x_1\| \cdots \|x_n\|$ for all $x_1 \in X_1, \ldots, x_n \in X_n$.

Definition 3.3. An $n$-linear operator $T$ is called continuous if $T(x_1^1, \ldots, x_n^1) \to T(x_1, \ldots, x_n)$ whenever $x_k^1 \to x^1, \ldots, x_k^n \to x^n$ as $k \to \infty$.

It is well-known that an $n$-linear operator $T$ is continuous if and only if $T$ is bounded.

For Banach space $X$, let $X^*$ denote its topological dual and $B_X$ denote its closed unit ball. For Banach spaces $X_1, \ldots, X_n$, and $Y$, let $\mathcal{L}(X_1, \ldots, X_n; Y)$ denote the space of all continuous $n$-linear operators from $X_1 \times \cdots \times X_n$ to $Y$. If $Y$ denotes the scalars then we denote this space by $\mathcal{L}(X_1, \ldots, X_n)$. In addition, if $X_j = X$ for $j = 1, \cdots, n$, then we denote this space by $\mathcal{L}^n(X)$. 

15
For each $T \in \mathcal{L}(X_1, \ldots, X_n; Y)$, the norm of a continuous $n$-linear operator can be defined as follows:

$$
\|T\| = \sup\{\|T(x_1, \cdots, x_n)\| : x_1 \in B_{X_1}, \cdots, x_n \in B_{X_n}\}.
$$

Then $\mathcal{L}(X_1, \cdots, X_n; Y)$ with this norm is a Banach space.

**Definition 3.4.** An $n$-linear operator $T : X \times \cdots \times X \to Y$ is called *symmetric* if

$$
T(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) = T(x_1, \cdots, x_n)
$$

for every permutation $\sigma$ on $\{1, \cdots, n\}$.

We refer to Dineen [16] for results on polynomials.

**Definition 3.5.** A mapping $P : X \to Y$ is called an $n$-homogeneous polynomial if there exists a symmetric $n$-linear operator $T_P : X \times \cdots \times X \to Y$ such that

$$
P(x) = T_P(x, \cdots, x), \quad \forall x \in X.
$$

Let $\mathcal{P}(^nX; Y)$ denote the space of all continuous $n$-homogeneous polynomials from $X$ to $Y$. For each $P \in \mathcal{P}(^nX; Y)$, define

$$
\|P\| = \sup\{\|P(x)\| : x \in B_X\}.
$$

Then $\mathcal{P}(^nX; Y)$ with this norm is a Banach space and the following theorem holds.
Theorem 3.6. (Polarization Formula) For every \( x \in X \),

\[
P(x) = T_P(x, \cdots, x)
\]

and for every \( x_1, \cdots, x_n \in X \),

\[
T_P(x_1, \cdots, x_n) = \frac{1}{2^n n!} \sum_{\epsilon_i = \pm 1} \epsilon_1 \cdots \epsilon_n P\left( \sum_{i=1}^n \epsilon_i x_i \right)
\]

with the norm inequalities

\[
\|P\| \leq \|T_P\| \leq \frac{n^n}{n!} \|P\|.
\]

Next we will introduce the projective tensor products. For Banach spaces \( X_1, \ldots, X_n \), let \( X_1 \otimes \cdots \otimes X_n \) denote the \( n \)-fold algebraic tensor product of \( X_1, \cdots, X_n \).

Definition 3.7. The projective tensor norm on \( X_1 \otimes \cdots \otimes X_n \) is defined by

\[
\|u\|_{\pi} = \inf \left\{ \sum_{k=1}^m \|x_{1,k}\| \cdots \|x_{n,k}\| : u = \sum_{k=1}^m x_{1,k} \otimes \cdots \otimes x_{n,k} \right\} \quad (3.1)
\]

for every \( u \in X_1 \otimes \cdots \otimes X_n \). The completion of \( X_1 \otimes \cdots \otimes X_n \) with respect to this norm is denoted by \( \hat{X}_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi \hat{X}_n \) and called the \( n \)-fold projective tensor product of \( X_1, \cdots, X_n \).

Define an \( n \)-linear operator \( \otimes : X_1 \times \cdots \times X_n \to X_1 \otimes \cdots \otimes X_n \) by

\[
\otimes(x_1, \cdots, x_n) = x_1 \otimes \cdots \otimes x_n \quad \forall (x_1, \cdots, x_n) \in X_1 \times \cdots \times X_n. \quad (3.2)
\]

Then, for every \( T \in \mathcal{L}(X_1, \cdots, X_n; Y) \), there exists a unique \( T^\otimes \) in \( \mathcal{L}(X_1 \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi X_n; Y) \) such that

\[
T = T^\otimes \circ \otimes \quad \text{and} \quad \|T\| = \|T^\otimes\|. \quad (3.3)
\]
That is, the following diagram commutes:

\[
\begin{array}{c}
X_1 \times \cdots \times X_n \xrightarrow{T} Y \\
\bigotimes \quad \Downarrow \quad \Downarrow \\
X_1 \hat{\otimes}_{\pi} \cdots \hat{\otimes}_{\pi} X_n
\end{array}
\]

Moreover, \( L(X_1, \cdots, X_n; Y) \) is isometrically isomorphic to \( L(X_1 \hat{\otimes}_{\pi} \cdots \hat{\otimes}_{\pi} X_n; Y) \).

For \( x_1 \otimes \cdots \otimes x_n \in \otimes_n X \), let \( x_1 \otimes_s \cdots \otimes_s x_n \) denote its symmetrization. That is,

\[
x_1 \otimes_s \cdots \otimes_s x_n = \frac{1}{n!} \sum_{\sigma \in \pi(n)} x_{\sigma(1)} \otimes \cdots \otimes x_{\sigma(n)},
\]

(3.4)

where \( \pi(n) \) is the group of permutations of \( \{1, \ldots, n\} \). Then (e.g., see [49])

\[
x_1 \otimes_s \cdots \otimes_s x_n = \frac{1}{2^n n!} \sum_{\delta_i = \pm 1} \delta_1 \cdots \delta_n \left( \sum_{i=1}^{n} \delta_i x_i \right) \otimes \cdots \otimes \left( \sum_{i=1}^{n} \delta_i x_i \right).
\]

(3.5)

We write \( \otimes_{n,s} X \) for the \emph{n-fold symmetric algebraic tensor product} of \( X \), that is, the linear span of \( \{x_1 \otimes_s \cdots \otimes_s x_n : x_1, \ldots, x_n \in X\} \) in \( \otimes_n X \). Each \( u \in \otimes_{n,s} X \) has a representation \( u = \sum_{k=1}^{m} \lambda_k x_k \otimes_s \cdots \otimes_s x_k \) where \( \lambda_1, \ldots, \lambda_m \) are scalars and \( x_1, \ldots, x_m \) are vectors in \( X \).

**Definition 3.8.** The **symmetric projective tensor norm** on \( \otimes_{n,s} X \) is defined by

\[
\|u\|_{s,\pi} = \inf \left\{ \sum_{k=1}^{m} |\lambda_k| \cdot \|x_k\|^n : u = \sum_{k=1}^{m} \lambda_k x_k \otimes_s \cdots \otimes_s x_k \in \otimes_{n,s} X \right\}, \quad u \in \otimes_{n,s} X.
\]

Let \( \otimes_{n,s,\pi} X \) denote the completion of \( (\otimes_{n,s} X, \| \cdot \|_{s,\pi}) \), called the **n-fold symmetric projective tensor product** of \( X \).

Define an \( n \)-homogeneous polynomial \( \otimes : X \to \otimes_{n,s} X \) by

\[
\otimes(x) = x \otimes \cdots \otimes x, \quad \forall x \in X.
\]

(3.6)
Then for every \( P \in \mathcal{P}(^n X; Y) \) there exists a unique \( P^\otimes \) in \( \mathcal{L}(\hat{\otimes}_{n,s,\pi} X; Y) \) such that

\[
P = P^\otimes \circ \otimes \quad \text{and} \quad \| P \| = \| P^\otimes \|.
\] (3.7)

That is, the following diagram commutes:

\[
\begin{array}{ccc}
X & \xrightarrow{P} & Y \\
\otimes & \downarrow & \otimes \\
\hat{\otimes}_{n,s,\pi} X & & \\
\end{array}
\]

Moreover, \( \mathcal{P}(^n X; Y) \) is isometrically isomorphic to \( \mathcal{L}(\hat{\otimes}_{n,s,\pi} X; Y) \).

Next we will introduce the injective tensor products. Let \( X_1, \ldots, X_n \) be Banach spaces.

**Definition 3.9.** The *injective tensor norm* on \( X_1 \otimes \cdots \otimes X_n \) is defined by

\[
\| u \|_\epsilon = \sup \left\{ \left| \sum_{k=1}^m x_1^*(x_{1,k}) \cdots x_n^*(x_{n,k}) \right| : u = \sum_{k=1}^m x_{1,k} \otimes \cdots \otimes x_{n,k}, x^*_k \in B_{X_k^*} \right\}
\] (3.8)

for every \( u \in X_1 \otimes \cdots \otimes X_n \). The completion of \( X_1 \otimes \cdots \otimes X_n \) with respect to this norm is denoted by \( X_1 \hat{\otimes} \cdots \hat{\otimes} \epsilon X_n \) and called the *n-fold injective tensor product* of \( X_1, \ldots, X_n \).

For each \( u \in X_1 \hat{\otimes} \cdots \hat{\otimes} X_n \), say, \( u = \sum_{k=1}^m x_{1,k} \otimes \cdots \otimes x_{n,k} \), define \( T_u : X_1^* \times \cdots \times X_n^* \to \mathbb{R} \) by

\[
T_u(x_1^*, \ldots, x_n^*) = \sum_{k=1}^m x_1^*(x_{1,k}) \cdots x_n^*(x_{n,k}), \quad \forall x^*_i \in X_i^*, (i = 1, \ldots, n). \tag{3.9}
\]

Then \( T_u \) is a finite-rank \( n \)-linear operator (which does not depend on the representations of \( u \)) and hence, \( T_u \in \mathcal{L}(X_1^*, \cdots, X_n^*, \mathbb{R}) \) with \( \| T_u \| = \| u \|_\epsilon \).

The following lemma is straightforward from the definition.
Lemma 3.10. Let \( T_i : X_i \to X_i \) be bounded linear operators for \( i = 1, \ldots, n \). Then

\[
\left\| \sum_{k=1}^{m} T_1(x_{1,k}) \otimes \cdots \otimes T_n(x_{n,k}) \right\|_\epsilon \leq \| T_1 \| \cdots \| T_n \| \cdot \left\| \sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k} \right\|_\epsilon
\]

for every \( x_{1,k} \in X_1, \ldots, x_{n,k} \in X_n, k = 1, \ldots, m \).

Definition 3.11. The symmetric injective tensor norm on \( \otimes_{n,s} X \) is defined by

\[
\| u \|_{s,\epsilon} = \sup \left\{ \left\| \sum_{k=1}^{m} \lambda_k \cdot (x^*(x_k))^n \right\| : u = \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k, x^* \in B_{X^*} \right\}
\]

for every \( u \in \otimes_{n,s} X \). The completion of \( \otimes_{n,s} X \) with respect to this norm is denoted by \( \tilde{\otimes}_{n,s,\epsilon} X \) and called the \( n \)-fold symmetric injective tensor product of \( X \).

For each \( u \in \otimes_{n,s} X \), say, \( u = \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k \), define \( P_u : X^* \to \mathbb{R} \) by

\[
P_u(x^*) = \sum_{k=1}^{m} \lambda_k \cdot (x^*(x_k))^n, \quad \forall x^* \in X^*.
\]

Then \( P_u \) is an \( n \)-homogeneous polynomial (which does not depend on the representations of \( u \)) and \( P_u \in \mathcal{P}(n X^*; \mathbb{R}) \) with \( \| P_u \| = \| u \|_{s,\epsilon} \).

The following lemma is straightforward from the definition.

Lemma 3.12. Let \( T : X \to X \) be a bounded linear operator. Then

\[
\left\| \sum_{k=1}^{m} \lambda_k T(x_k) \otimes \cdots \otimes T(x_k) \right\|_{s,\epsilon} \leq \| T \|^n \cdot \left\| \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k \right\|_{s,\epsilon}
\]

for every \( x_k \in X, k = 1, \ldots, m \).

By linearly extending, equation (3.4) defines a linear projection \( s : \otimes_n X \to \otimes_{n,s} X \). The linear projection \( s : \otimes_n X \to \otimes_{n,s} X \) can be extended to \( \tilde{\otimes}_{n,\epsilon} X \) with \( \tilde{\otimes}_{n,s,\epsilon} X \) as its range.
and for every $u \in \tilde{\otimes}_{n,\epsilon} X$ (see [31]), we have

$$
\| s(u) \|_{s,\epsilon} \leq \| s(u) \|_{\epsilon} \leq \frac{n^n}{n!} \| s(u) \|_{s,\epsilon}.
$$

(3.11)

For the basic knowledge about $n$-linear operators, $n$-homogeneous polynomials, $n$-fold (symmetric) projective tensor products, and $n$-fold(symmetric) injective tensor products, we refer to [16, 37, 42, 45, 31, 14].
Let $E_1, \cdots, E_n, E$ and $F$ be an (Archimedean) vector lattice.

**Definition 3.13.** An $n$-linear operator $T : E_1 \times \cdots \times E_n \to F$ is called *positive* if $T(x_1, \cdots, x_n) \in F^+$ whenever $x_1 \in E_1^+, \cdots, x_n \in E_n^+$. And $T$ is called *regular* if $T$ is the difference of two positive $n$-linear operators.

Let $\mathcal{L}^r(E_1, \cdots, E_n; F)$ denote the space of all regular $n$-linear operators from $E_1 \times \cdots \times E_n$ to $F$. If, in addition, $F$ is Dedekind complete then by [35, Lemma 2.12 and Proposition 2.14], $\mathcal{L}^r(E_1, \cdots, E_n; F)$ is a Dedekind complete vector lattice.

Let $E, F$ be (Archimedean) vector lattices.

**Definition 3.14.** An $n$-homogeneous polynomial $P : E \to F$ is called *positive* if its corresponding symmetric $n$-linear operator $T_P$ is positive. And $P$ is called *regular* if it is the difference of two positive polynomials.

It is easy to see that $P$ is regular if and only if $T_P$ is regular. Let $\mathcal{P}^r(nE; F)$ denote the space of all regular $n$-homogeneous polynomials from $E$ to $F$. If, in addition, $F$ is Dedekind complete then by [35, Lemma 2.15 and Lemma 2.16], $\mathcal{P}^r(nE; F)$ is a Dedekind complete vector lattice.

If, in addition, $E_1, \cdots, E_n, E, F$ are Banach lattices such that $F$ is Dedekind complete then $\mathcal{L}^r(E_1, \cdots, E_n; F)$ is a Banach lattice with the *regular operator norm* $\|T\|_r = \||T||$ for every $T \in \mathcal{L}^r(E_1, \cdots, E_n; F)$. Then also $\mathcal{P}^r(nE; F)$ is a Banach lattice with the *regular polynomial norm* $\|P\|_r = \||P||$ for every $P \in \mathcal{P}^r(nE; F)$(e.g., see [3]).
Lemma 3.15. Let $E_1, \ldots, E_n, F$ be Banach lattices with $F$ Dedekind complete. Then for any $T \in \mathcal{L}^r(E_1, \ldots, E_n; F)$,

$$
\|T\|_r = \inf \left\{ \|S\| : S \in \mathcal{L}^r(E_1, \ldots, E_n; F)^+, \right. \\
T(x_1, \ldots, x_n) \leq S(|x_1|, \ldots, |x_n|), \ \forall x_1 \in E_1, \ldots, x_n \in E_n \left. \right\}. \quad (3.12)
$$

Moreover, $\|T\| \leq \|T\|_r$.

Proof. Let

$$
A := \left\{ S : S \in \mathcal{L}^r(E_1, \ldots, E_n; F)^+, \right. \\
T(x_1, \ldots, x_n) \leq S(|x_1|, \ldots, |x_n|), \ \forall x_1 \in E_1, \ldots, x_n \in E_n \left. \right\},
$$

and let $a = \inf\{\|S\| : S \in A\}$. For any $S \in A$, it follows from (2.10) in [3] that $|T| \leq S$. Thus $\|T\| \leq \|S\|$ and hence, $\|T\|_r \leq a$. On the other hand, $|T(x_1, \ldots, x_n)| \leq |T|(|x_1|, \ldots, |x_n|)$ for every $x_1 \in E_1, \ldots, x_n \in E_n$. Thus $|T| \in A$ and hence, $a \leq \|T\| = \|T\|_r$. Therefore, $\|T\|_r = a$. Also,

$$
\|T\| = \sup \{ \|T(x_1, \ldots, x_n)\| : x_1 \in B_{X_1}, \ldots, x_n \in B_{X_n} \}
$$

$$
= \sup \{ \|T(x_1, \ldots, x_n)\| : x_1 \in B_{X_1}, \ldots, x_n \in B_{X_n} \}
$$

$$
\leq \sup \{ \|S(|x_1|, \ldots, |x_n|)\| : x_1 \in B_{X_1}, \ldots, x_n \in B_{X_n} \}
$$

$$
\leq \sup \{ \|S\| \cdot \|x_1\| \cdots \|x_n\| : x_1 \in B_{X_1}, \ldots, x_n \in B_{X_n} \}
$$

$$
\leq \|S\|.
$$

Since $a = \inf\{\|S\| : S \in A\} = \|T\|_r$ and $\|T\| \leq \inf\{\|S\| : S \in A\} = \|T\|_r$.

Thus we have $\|T\| \leq \|T\|_r$. \qed

23
Similarly, we have

**Lemma 3.16.** Let $E$ and $F$ be Banach lattices with $F$ Dedekind complete. Then for any $P \in \mathcal{P}^r(nE; F)$,

$$
\|P\|_r = \inf \left\{ \|R\| : R \in \mathcal{P}^r(nE; F)^+, |P(x)| \leq R(|x|), \forall x \in E \right\}.
$$

(3.13)

Moreover, $\|P\| \leq \|P\|_r$.

Next we will introduce n-fold positive projective tensor products.

**Definition 3.17.** Let $E, \cdots, E_n, E, F$ be vector lattices. An $n$-linear operator $T : E_1 \times \cdots \times E_n \to F$ is called a lattice n-morphism if $|T(x_1, \cdots, x_n)| = T(|x_1|, \cdots, |x_n|)$.

Let $(E_1 \otimes \cdots \otimes E_n, \otimes)$ denote the $n$-fold (Archimedean) vector lattice tensor product of $E_1, \cdots, E_n$. We collect the following well known facts about this tensor product.

(a) $E_1 \otimes \cdots \otimes E_n$ is a vector lattice and $\otimes$ is a lattice n-morphism from $E_1 \times \cdots \times E_n$ to $E_1 \otimes \cdots \otimes E_n$ defined by $\otimes(x_1, \cdots, x_n) = x_1 \otimes \cdots \otimes x_n$ for every $x_1 \in E_1, \cdots, x_n \in E_n$.

(b) For any (Archimedean) vector lattice $F$ there is a one to one correspondence between lattice n-morphisms $T : E_1 \times \cdots \times E_n \to F$ and lattice homomorphisms $T^\otimes : E_1 \otimes \cdots \otimes E_n \to F$ given by $T = T^\otimes \circ \otimes$.

(c) For any uniformly complete (Archimedean) vector lattice $F$ there is a one to one correspondence between positive $n$-linear maps $T : E_1 \times \cdots \times E_n \to F$ and increasing linear maps $T^\otimes : E_1 \otimes \cdots \otimes E_n \to F$ given by $T = T^\otimes \circ \otimes$.

(d) $E_1 \otimes \cdots \otimes E_n$ is dense in $E_1 \otimes \cdots \otimes E_n$ in the sense that for any $u \in E_1 \otimes \cdots \otimes E_n$ there exist $x_1 \in E_1^+, \cdots, x_n \in E_n^+$ such that, for every $\delta > 0$, there is $v \in E_1 \otimes \cdots \otimes E_n$ with $|u - v| \leq \delta x_1 \otimes \cdots \otimes x_n$.

(e) If $u \in E_1 \otimes \cdots \otimes E_n$ then there exist $x_1 \in E_1^+, \cdots, x_n \in E_n^+$ such that $|u| \leq x_1 \otimes \cdots \otimes x_n$. 

24
(f) $E_1 \otimes \cdots \otimes E_n$ is order dense in $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$ in the sense that if $u > 0$ in $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$ then there exist $x_1 > 0$ in $E_1$, \ldots, $x_n > 0$ in $E_n$ such that $u \geq x_1 \otimes \cdots \otimes x_n > 0$.

**Definition 3.18.** For Banach lattices $E_1, \ldots, E_n$, the *positive projective tensor norm* $\| \cdot \|_{\pi}$ on $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$ is defined by

$$\| u \|_{\pi} = \inf \left\{ \sum_{k=1}^{m} \| x_{1,k} \| \cdots \| x_{n,k} \| : x_{i,k} \in E_i^+, |u| \leq \sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k} \right\}$$

for every $u \in E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$.

Then $\| \cdot \|_{\pi}$ is a lattice norm on $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$ and

(g) $E_1 \otimes \cdots \otimes E_n$ is norm dense in $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$, and

(h) the cone generated by $\{ x_1 \otimes \cdots \otimes x_n : x_k \in E_k^+, 1 \leq k \leq n \}$ is norm dense in $(E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n)^+$.

Let $E_1 \hat{\tilde{\otimes}}_{\| \cdot \|_{\pi}} \cdots \hat{\tilde{\otimes}}_{\| \cdot \|_{\pi}} E_n$ denote the completion of $E_1 \tilde{\otimes} \cdots \tilde{\otimes} E_n$ under the lattice norm $\| \cdot \|_{\pi}$. Then $E_1 \hat{\tilde{\otimes}}_{\| \cdot \|_{\pi}} \cdots \hat{\tilde{\otimes}}_{\| \cdot \|_{\pi}} E_n$ is a Banach lattice, called the $n$-fold Fremlin projective tensor product, or $n$-fold positive projective tensor product of $E_1, \ldots, E_n$.

**Proposition 3.19.** [7] Let $E_1, \ldots, E_n, F$ be Banach lattices such that $F$ is Dedekind complete. Then $\mathcal{L}^r(E_1, \cdots, E_n; F)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r(E_1 \hat{\tilde{\otimes}}_{\| \cdot \|_{\pi}} \cdots \hat{\tilde{\otimes}}_{\| \cdot \|_{\pi}} E_n; F)$.

**Definition 3.20.** For a Banach lattice $E$, let $\tilde{\otimes}_{n,s} E$ denote the $n$-fold vector lattice symmetric tensor product of $E$. The *positive symmetric projective tensor norm* on $\tilde{\otimes}_{n,s} E$ is defined by

$$\| u \|_{s,\| \cdot \|_{\pi}} = \inf \left\{ \sum_{k=1}^{m} \| x_k \|^n : x_k \in E^+, |u| \leq \sum_{k=1}^{m} x_k \otimes \cdots \otimes x_k \right\} \quad (\forall u \in \tilde{\otimes}_{n,s} E).$$
Then $\| \cdot \|_{s,|\pi|}$ is a lattice norm on $\bar{\otimes}_{n,s} E$. Let $\hat{\otimes}_{n,s,|\pi|} E$ denote the completion of $\bar{\otimes}_{n,s} E$ under the lattice norm $\| \cdot \|_{s,|\pi|}$. Then $\hat{\otimes}_{n,s,|\pi|} E$ is a Banach lattice, called the $n$-fold Fremlin symmetric tensor product or the $n$-fold positive symmetric projective tensor product of $E$.

**Proposition 3.21.** [7] Let $E$ and $F$ be Banach lattices such that $F$ is Dedekind complete. Then $\mathcal{P}^r(nE;F)$ is isometrically isomorphic and lattice homomorphic to $\mathcal{L}^r(\hat{\otimes}_{n,s,|\pi|} E;F)$.

In particular, we have $(\hat{\otimes}_{n,s,|\pi|} E)^* = \mathcal{P}^r(nE;\mathbb{R})$.

Let $E_1, \ldots, E_n$ be Banach lattices. For any $u \in E_1 \otimes \cdots \otimes E_n$, say, $u = \sum_{i=1}^m x_{i,1} \otimes \cdots \otimes x_{i,n}$, define $T_u : E_1^* \times \cdots \times E_n^* \to \mathbb{R}$ by

$$T_u(x_1^*, \ldots, x_n^*) = \sum_{i=1}^m x_1^*(x_{i,1}) \cdots x_n^*(x_{i,n}), \quad \forall x_k^* \in E_k^*, \ k = 1, \ldots, n.$$ 

Then $T_u$ is a finite-rank $n$-linear operator (which does not depend on the representations of $u$) and hence, $T_u \in \mathcal{L}^r(E_1^*, \ldots, E_n^*; \mathbb{R})$.

**Definition 3.22.** Let $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_n$ denote the closed sublattice generated by $E_1 \otimes \cdots \otimes E_n$ in $\mathcal{L}^r(E_1^*, \ldots, E_n^*; \mathbb{R})$, called the $n$-fold positive injective tensor product of $E_1, \ldots, E_n$. The norm on $E_1 \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} E_n$ is denoted by $\| \cdot \|_{|\pi|}$, that is, for every $u \in E_1 \otimes \cdots \otimes E_n$, $\|u\|_{|\pi|} = \|T_u\|_r$.

By Lemma 3.15, $\|u\|_e \leq \|u\|_{|\pi|}$. In particular, if $u$ is a positive element in $E_1 \otimes \cdots \otimes E_n$, then

$$\|u\|_{|\pi|} = \sup \left\{ \left| \sum_{k=1}^m x_1^*(x_{1,k}) \cdots x_n^*(x_{n,k}) \right| : u = \sum_{k=1}^m x_{1,k} \otimes \cdots \otimes x_{n,k}, x_i^* \in B_{E_i}^+, 1 \leq i \leq n \right\}.$$ 

For any $u \in E_1 \otimes \cdots \otimes E_n$, it is easy to see that the operator $T_u : E_1^* \times \cdots \times E_n^* \to \mathbb{R}$ defined in (3.9) is a regular $n$-linear operator and hence, $T_u \in \mathcal{L}^r(E_1^*, \ldots, E_n^*; \mathbb{R})$. 

26
If $E_1 = \cdots = E_n = E$, we write $E_1 \otimes_{|\epsilon|} \cdots \otimes_{|\epsilon|} E_n$ by $\otimes_{n,|\epsilon|} E$. For any $u \in \otimes_{n,s} E$, it is easy to see that the polynomial $P_u : E^* \to \mathbb{R}$ defined in (3.10) is a regular $n$-homogeneous polynomial and hence, $P_u \in \mathcal{P}^r(n E^*; \mathbb{R})$.

**Definition 3.23.** Let $\otimes_{n,s,|\epsilon|} E$ denote the closed sublattice generated by $\otimes_{s} E$ in $\mathcal{P}^r(n E^*; \mathbb{R})$, called the $n$-fold positive symmetric injective tensor product of $E$. The norm on $\otimes_{n,s,|\epsilon|} E$ is denoted by $\| \cdot \|_{s,|\epsilon|}$, that is, for every $u \in \otimes_{n,s} E$, $\|u\|_{s,|\epsilon|} = \|P_u\|_r$ and $\|u\|_{s,\epsilon} \leq \|u\|_{s,|\epsilon|}$.

In particular, if $u$ is a positive element in $\otimes_{n,s} E$, then

$$\|u\|_{s,|\epsilon|} = \sup \left\{ \left\| \sum_{k=1}^{m} \lambda_k \cdot \left( x^* (x_k) \right)^n \right\| : u = \sum_{k=1}^{m} \lambda_k x_k \otimes \cdots \otimes x_k, x^* \in B_{E^*}^+ \right\}.$$

For the basic knowledge about positive $n$-linear operators, positive $n$-homogeneous polynomials, positive $n$-fold (symmetric) projective tensor products, and positive $n$-fold(symmetric) injective tensor products, we refer to [17, 18, 47, 21, 7].
4 POSITIVE TENSOR PRODUCTS OF $\ell_p$-SPACES

* All of results in 4.1 and 4.2 have been published in my paper “On positive tensor products of $\ell_p$-spaces”. This is joint work with Qingying Bu and Donghai Ji. (See [28].)

4.1 Positive Projective Tensor Products of $\ell_p$-spaces

For $1 \leq p < \infty$, let $\{e_i : i \in \mathbb{N}\}$ denote the standard unit vector basis of $\ell_p$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ with the square order (see [19], [42], or [20]) is a basis (not necessary an unconditional basis) of the projective tensor product $\ell_{p_1} \hat{\otimes}_{\pi} \cdots \hat{\otimes}_{\pi} \ell_{p_n}$. Similar to the proof of [8, Lemma 22], we present the following lemma about the basis of the Fremlin projective tensor product $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$.

**Lemma 4.1.** Let $1 \leq p_1, \cdots, p_n < \infty$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ with any order is a 1-unconditional basis of $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$.

Recall that a Banach lattice $E$ is called order continuous if $\|x_\alpha\| \downarrow 0$ whenever $x_\alpha \downarrow 0$. Equivalently, $E$ is order continuous if and only if every monotone order bounded sequence in $E$ is norm convergent.

**Definition 4.2.** A Banach lattice $E$ is said to be a **Kantorovich-Banach space** (KB-space in short) if every monotone norm bounded sequence in $E$ is norm convergent.

Equivalently, $E$ is a KB-space if and only if $E$ does not contain any sublattice isomorphic to $c_0$. 

28
Note that $\ell_{p_1} \hat{\otimes}_{|\pi|} \ell_{p_2} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n} = \ell_{p_1} \hat{\otimes}_{|\pi|} (\ell_{p_2} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})$. By induction, [4] Corollary 7.4] yields the following lemma.

**Lemma 4.3.** Let $1 \leq p_1, \cdots, p_n < \infty$. Then $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$ is a KB-space, and hence it is order continuous.

The following Rademacher averaging is a generalization of Lemma 2.22 in [45, p.34].

**Rademacher averaging:** Let $Z_1, \cdots, Z_n$ be vector spaces and $x_{i,k} \in Z_i$ for $i = 1, \cdots, n$ and $k = 1, \cdots, m$. Then

$$
\sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k} = \int_{0}^{1} \left( \sum_{k=1}^{m} r_k(t) x_{1,k} \right) \otimes \cdots \otimes \left( \sum_{k=1}^{m} r_k(t) x_{n,k} \right) dt,
$$

where $\{r_k(t)\}_1^\infty$ is the sequence of Rademacher functions on $[0, 1]$.

The main diagonal of the projective tensor product $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$ was characterized by Arias and Farmer [2]. Next by using Rademacher averaging, we will characterize the main diagonal of the Fremlin projective tensor product $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$. Let $\Delta(\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})$ denote the main diagonal of $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$, that is, the closed subspace spanned by the diagonal vectors $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ in $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$. Next we will use Rademacher averaging theorem to characterize the main diagonal $\Delta(\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})$.

**Theorem 4.4.** Let $1 \leq p_1, \cdots, p_n < \infty$ and let $\frac{1}{p} = \sum_{i=1}^{n} \frac{1}{p_i}$. Then $\Delta(\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})$ is isometrically lattice isomorphic to $\ell_p$ if $p > 1$ and $\ell_1$ if $p \leq 1$.

i.e.

$$
\Delta(\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}) \equiv \begin{cases} 
\ell_p, & \text{if } p > 1 \\
\ell_1, & \text{if } p \leq 1.
\end{cases}
$$

**Proof.** Case 1: $p > 1$. 

29
Take any positive element $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i$ in $\Delta(\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})$, where $a_i \geq 0$ for $i = 1, \cdots, m$. By Rademacher averaging,

$$u = \int_0^1 \left( \sum_{i=1}^{m} a_i^{\frac{1}{p_1}} r_i(t) e_i \right) \otimes \cdots \otimes \left( \sum_{i=1}^{m} a_i^{\frac{1}{p_n}} r_i(t) e_i \right) dt$$

and hence,

$$\|u\|_{|\pi|} \leq \sup_{0 \leq t \leq 1} \left\| \sum_{i=1}^{m} a_i^{\frac{1}{p_1}} r_i(t) e_i \right\|_{\ell_{p_1}} \cdots \left\| \sum_{i=1}^{m} a_i^{\frac{1}{p_n}} r_i(t) e_i \right\|_{\ell_{p_n}} = \left( \sum_{i=1}^{m} a_i^{p_1} \right)^{\frac{1}{p_1}} \cdots \left( \sum_{i=1}^{m} a_i^{p_n} \right)^{\frac{1}{p_n}} = \| a_i \|_{\ell_p}.$$ 

On the other hand, define a positive $n$-linear form $T$ on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ by

$$T(x^{(1)}, \ldots, x^{(n)}) = \sum_{i=1}^{m} a_i^{p-1} x_i^{(1)} \cdots x_i^{(n)}, \ \forall x^{(k)} = (x_i^{(k)})_i \in \ell_{p_k}, k = 1, \cdots, n.$$ 

Let $\frac{1}{q} := 1 - \frac{1}{p}$. By Hölder inequality,

$$|T(x^{(1)}, \ldots, x^{(n)})| \leq \left\| \left( a_i^{p-1} \right)_{i=1}^{m} \right\|_{\ell_{q}} \cdots \left\| \left( x_i^{(k)} \right)_i \right\|_{\ell_{p_k}} \cdots \left\| \left( x_i^{(n)} \right)_i \right\|_{\ell_{p_n}},$$

which implies that

$$\|T\|_r = \|T\| \leq \left\| \left( a_i^{p-1} \right)_{i=1}^{m} \right\|_{\ell_{q}} = \left( \sum_{i=1}^{m} a_i^{p} \right)^{\frac{1}{q}}.$$ 

Note that $T \in L^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R}) = (\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})^*$ and that

$$\langle u, T \rangle = \sum_{i=1}^{m} a_i T(e_i, \ldots, e_i) = \sum_{i=1}^{m} a_i^p.$$ 

Thus

$$\sum_{i=1}^{m} a_i^p = \langle u, T \rangle \leq \|u\|_{|\pi|} \cdot \|T\|_r \leq \|u\|_{|\pi|} \cdot \left( \sum_{i=1}^{m} a_i^{p} \right)^{\frac{1}{q}},$$

30
which implies that
\[ \|u\|_{\pi} \geq \left( \sum_{i=1}^{m} a_i^p \right)^{\frac{1}{p}} = \|(a_i)_{i=1}^{m}\|_{\ell_p}. \]

Therefore, \( \|u\|_{\pi} = \|(a_i)_{i=1}^{m}\|_{\ell_p} \) and hence, \( \Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}) \) is isometrically lattice isomorphic to \( \ell_p \).

Case 2: \( p \leq 1 \).

Take any positive element \( u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \) in \( \Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}) \), where \( a_i \geq 0 \) for \( i = 1, \cdots, m \). Define a positive \( n \)-linear form \( T \) on \( \ell_{p_1} \times \cdots \times \ell_{p_n} \) by
\[
T(x^{(1)}, \cdots, x^{(n)}) = \sum_{i=1}^{m} x_i^{(1)} \cdots x_i^{(n)}, \quad \forall x^{(k)} = (x_i^{(k)})_i \in \ell_{p_k}, k = 1, \cdots, n.
\]

Take \( q_k \geq p_k \) for \( k = 1, \cdots, n \) such that \( \frac{1}{q_1} + \cdots + \frac{1}{q_n} = 1 \). By Hölder inequality,
\[
|T(x^{(1)}, \cdots, x^{(n)})| \leq \left\| (x_i^{(1)})_i \right\|_{\ell_{q_1}} \cdots \left\| (x_i^{(n)})_i \right\|_{\ell_{q_n}} \leq \left\| (x_i^{(1)})_i \right\|_{\ell_{p_1}} \cdots \left\| (x_i^{(n)})_i \right\|_{\ell_{p_n}},
\]
which implies that \( \|T\|_r = \|T\| \leq 1 \). Note that \( \langle u, T \rangle = \sum_{i=1}^{m} a_i T(e_i, \cdots, e_i) = \sum_{i=1}^{m} a_i \).

Thus
\[
\sum_{i=1}^{m} a_i = \langle u, T \rangle \leq \|T\|_r \cdot \|u\|_{\pi} \leq \|u\|_{\pi}.
\]

On the other hand, \( \|u\|_{\pi} \leq \sum_{i=1}^{m} \|a_i e_i \otimes \cdots \otimes e_i\|_{\pi} = \sum_{i=1}^{m} a_i \). Therefore, \( \|u\|_{\pi} = \|(a_i)_{i=1}^{m}\|_{\ell_1} \) and hence, \( \Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}) \) is isometrically lattice isomorphic to \( \ell_1 \).

\[ \Box \]

**Remark 4.5.** In Theorem 4.4, if some (or all) of \( p_i \)'s are \( \infty \), then we consider \( \ell_{p_i} = c_0 \) if \( p_i = \infty \), and consider \( \ell_p = c_0 \) if \( p = \infty \). In this case, Theorem 4.4 is still true. Thus we have the first special case, if all \( p_i \)'s are \( \infty \) then \( \Delta(c_0\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}c_0) \) is isometrically lattice isomorphic to \( c_0 \). The second special case is that if at least one of \( p_i \)'s is \( 1 \), then \( \Delta(\ell_{p_1}\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_{p_n}) \) is isometrically lattice isomorphic to \( \ell_1 \).
Next we will use the main diagonal of $\ell_{p_{i}} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_{n}}$ to characterize its reflexivity.

First we need the following lemma, which might be known. We provide a proof here since we do not find one handy in the literature.

**Lemma 4.6.** Let $E$ be an order continuous Banach lattice. Then $E$ is reflexive if and only if every positive linear functional on $E$ attains its norm.

**Proof.** Suppose that every positive linear functional on $E$ attains its norm. By James’ theorem, we only need to show that every continuous linear functional on $E$ attains its norm. Take any $f \in E^*$. Then there exists $x \in B_{E}$ such that $\|f\| = \|f\| = |f|(x)$. Note that $|f|(x) = |f|(x^+) - |f|(x^-)$. Without loss of generality, we assume that $x \in E^+$. Let $N(f) = \{z \in E : |f|(|z|) = 0\}$ be the null ideal of $f$ and let $C(f) = N(f)^\perp$. Since $E$ is order continuous, both $N(f)$ and $C(f)$ are projection bands.

Let $P^+ : E \to C(f^+)$ and $P^- : E \to C(f^-)$ be the respective band projections. As, for example, $(I - P^+)x \in C(f^+) = N(f^+)$ we have

$$f^+(x) = f^+(P^+x) + f^+((I - P^+)x) = f^+(P^+x).$$

By [36] Theorem 1.4.11] we have, for example, $P^-x \in C(f^-) \subseteq N(f^+)$ so that $f^+(P^-x) = 0$. Similarly we have $f^-(x) = f^-(P^-x)$ and $f^-(P^+x) = 0$. Thus

$$f(P^+x - P^-x) = (f^+ - f^-)(P^+x - P^-x) = f^+(P^+x) - f^+(P^-x) - f^-(P^+x) + f^-(P^-x) = f^+(x) + f^-(x) = |f|(x) = \|f\|.$$  

Also, as $P^+x \perp P^-x$ and $0 \leq P^+x, P^-x \leq x$ we have $|P^+x - P^-x| \leq x$ and therefore $\|P^+x - P^-x\| \leq \|x\| \leq 1$ and thus $f$ attains its norm. \qed
Remark 4.7. In Lemma 4.6, the order continuity of $E$ is essential. For instance, let $K$ be a compact Hausdorff space and $C(K)$ be the Banach lattice of all continuous functions on $K$. Then Riesz Representation Theorem implies that every positive linear functional on $C(K)$ attains its norm. However, $C(K)$ is not reflexive unless $K$ is a finite set.

Theorem 4.8. Let $1 \leq p_1, \cdots, p_n < \infty$. Then $\ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n}$ is reflexive if and only if $1/p_1 + \cdots + 1/p_n < 1$.

Proof. If $1/p_1 + \cdots + 1/p_n \geq 1$, then Theorem 4.4 implies that $\ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n}$ is not reflexive. If $1/p_1 + \cdots + 1/p_n < 1$, then each $p_i > 1$ and hence, each $\ell_{p_i}$ is reflexive. Take any positive $T \in L^r(\ell_{p_1}, \cdots, \ell_{p_n}) = (\ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n})^*$. Since

$$ \|T\|_r = \|T\| = \sup \{|T(x^{(1)}, \cdots, x^{(n)})| : x^{(i)} \in B_{\ell_{p_i}}, 1 \leq i \leq n\}, $$

there exist sequences $\{x^{(i)}_k\}_{k=1}^\infty$ in $B_{\ell_{p_i}}$ ($1 \leq i \leq n$) such that

$$ \lim_{k} \left| T(x^{(1)}_k, \cdots, x^{(n)}_k) \right| = \|T\|_r. $$

Moreover, there exist subsequences $\{x^{(i)}_{i_k}\}_{k=1}^\infty$ of $\{x^{(i)}_k\}_{k=1}^\infty$ and $x^{(i)}_0$ in $\ell_{p_i}$ ($1 \leq i \leq n$) such that $\lim_k x^{(i)}_{i_k} = x^{(i)}_0$ weakly in $\ell_{p_i}$. By [43, Theorem 4.1], $T$ is weakly sequentially continuous. Thus

$$ \|T\|_r = \lim_{k} \left| T(x^{(1)}_{i_k}, \cdots, x^{(n)}_{i_k}) \right| = \left| T(x^{(1)}_0, \cdots, x^{(n)}_0) \right| $$

and hence, $T$ attains its norm. It follows from Lemma 4.3 and Lemma 4.6 that $\ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n}$ is reflexive. \qed

Theorem 4.4 and Theorem 4.8 yield the following corollary.

Corollary 4.9 Let $1 \leq p_1, \cdots, p_n < \infty$. Then $\ell_{p_1} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n}$ does not contain any complemented sublattice isomorphic to $\ell_1$ if and only if $1/p_1 + \cdots + 1/p_n < 1$. 

33
Next we will show that in the space of all regular $n$-linear forms on $\ell_{p_1} \times \cdots \times \ell_{p_n}$, the reflexivity, the property of being a KB-space, and the property of being order continuous are all equivalent.

**Theorem 4.10** Let $1 \leq p_1, \ldots, p_n < \infty$. Then the following statements are equivalent:

(i) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ is reflexive.

(ii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ is a KB-space.

(iii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ is order continuous.

(iv) $1/p_1 + \cdots + 1/p_n < 1$.

**Proof.** Note that $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R}) = (\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n})^*$. If $1/p_1 + \cdots + 1/p_n < 1$, then Theorem 4.8 implies that $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ is reflexive, and (iv) $\Rightarrow$ (i) follows. It is trivial that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Now suppose (iii) holds. By [36, p.93, Theorem 2.4.14], $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$ does not contain any sublattice isomorphic to $\ell_\infty$ and hence, $\ell_{p_1} \hat{\otimes}_{|\pi|} \cdots \hat{\otimes}_{|\pi|} \ell_{p_n}$ does not contain any complemented sublattice isomorphic to $\ell_1$. Corollary 4.9 implies that $1/p_1 + \cdots + 1/p_n < 1$ and (iv) follows.

Note that $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q) = \mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}, \ell_q; \mathbb{R})$ for any $1 < q < \infty$. Theorem 4.10 yields the following corollary.

**Corollary 4.11.** Let $1 \leq p_1, \cdots, p_n < \infty$ and let $1 < q < \infty$. Then the following statements are equivalent:

(i) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is reflexive.

(ii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is a KB-space.

(iii) $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is order continuous.

(iv) $1/p_1 + \cdots + 1/p_n < 1/q$.

To consider the case $q = 1$ in Corollary 4.11, we need the following lemma.
Lemma 4.12. Let \( \lambda \) be a reflexive Banach sequence lattice and \( F \) a Banach lattice. Then \( \mathcal{L}^*(\lambda; F^*) \) is a KB-space if and only if \( F^* \) is a KB-space and every positive linear operator from \( \lambda \) to \( F^* \) is compact.

Proof. Note that \( \mathcal{L}^*(\lambda; F^*) = (\lambda \otimes |\pi| F)^* \). The theorem follows from [9, Theorem 6.9] and [36, p.93, Theorem 2.4.14].

The case \( q = 1 \) in Corollary 4.11 is as follows.

Corollary 4.13. Let \( 1 \leq p_1, \cdots, p_n < \infty \). Then the following statements are equivalent:

(i) \( \mathcal{L}^*(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_1) \) is a KB-space.

(ii) \( \mathcal{L}^*(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_1) \) is order continuous.

(iii) \( 1/p_1 + \cdots + 1/p_n < 1 \).

Proof. Note that

\[
\mathcal{L}^*(\ell_{p_1} \otimes_{|\pi|} \cdots \otimes_{|\pi|} \ell_{p_n}; \ell_1) = \mathcal{L}^*(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_1) = (\ell_{p_1} \otimes_{|\pi|} \cdots \otimes_{|\pi|} \ell_{p_n} \otimes_{|\pi|} c_0)^*.
\]

Suppose (iii). Then Theorem 4.8 implies that \( \ell_{p_1} \otimes_{|\pi|} \cdots \otimes_{|\pi|} \ell_{p_n} \) is reflexive. Thus every positive linear operator from \( \ell_{p_1} \otimes_{|\pi|} \cdots \otimes_{|\pi|} \ell_{p_n} \) to \( \ell_1 \) is compact and hence, (i) follows from Lemmas 4.1 and 4.12. It is trivial that (i) \( \Rightarrow \) (ii). Now suppose (ii). Then \( \mathcal{L}^*(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R}) = (\ell_{p_1} \otimes_{|\pi|} \cdots \otimes_{|\pi|} \ell_{p_n})^* \) is order continuous and hence, Theorem 4.10 implies (iii). 

Let \( \Delta(\otimes_{n,s,|\pi|} \ell_p) \) denote the main diagonal of \( \otimes_{n,s,|\pi|} \ell_p \), that is, the closed subspace spanned by the diagonal vectors \( \{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\} \) in \( \otimes_{n,s,|\pi|} \ell_p \). Parallel to the Fremlin projective tensor product \( \ell_{p_1} \otimes_{|\pi|} \cdots \otimes_{|\pi|} \ell_{p_n} \), we have the same results for the Fremlin projective symmetric tensor product \( \otimes_{n,s,|\pi|} \ell_p \) as follows.
Theorem 4.14. Let $1 \leq p < \infty$.

(i) $\Delta(\hat{\otimes}_{n,s,|\pi|\ell_p})$ is isometrically lattice isomorphic to $\ell_{\frac{n}{p}}$ if $p > n$ and $\ell_1$ if $p \leq n$.

(ii) $\hat{\otimes}_{n,s,|\pi|\ell_p}$ is reflexive if and only if $p > n$.

(iii) $\hat{\otimes}_{n,s,|\pi|\ell_p}$ does not contain any complemented sublattice isomorphic to $\ell_1$ if and only if $p > n$.

(iv) $\hat{\otimes}_{n,s,|\pi|\ell_p}$ is a KB-space and hence, it is order continuous.

Proof. (i) Note that $\Delta(\hat{\otimes}_{n,s,|\pi|\ell_p}) = \Delta(\ell_{\frac{n}{p}}\hat{\otimes}_{|\pi|\ell_p})$. Let $\frac{1}{q} = \sum_{i=1}^{n} \frac{1}{p} = \frac{n}{p}$, then Theorem 4.4 implies that

$$\Delta(\hat{\otimes}_{n,s,|\pi|\ell_p}) \equiv \begin{cases} \ell_q, & \text{if } q > 1 \\ \ell_1, & \text{if } q \leq 1 \end{cases}$$

where $q = \frac{p}{n}$.

(ii) $\hat{\otimes}_{n,s,|\pi|\ell_p} = (\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p)$ is reflexive if and only if $\frac{n}{p} < 1$ from Theorem 4.8. If $p > n$ then $\hat{\otimes}_{n,s,|\pi|\ell_p}$ is isometrically lattice isomorphic to $\ell_{\frac{n}{p}}$ and it is reflexive.

(iii) $\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p$ does not contain any complemented sublattice isomorphic to $\ell_1$ if and only if $1/p + \cdots + 1/p = \frac{n}{p} < 1$ from Corollary 4.9.

(iv) Since $\ell_p(1 \leq p < \infty)$ is a KB-space and $\ell_p\hat{\otimes}_{|\pi|}\ell_p$ is a KB-space (see [4]). Note that $\ell_p\hat{\otimes}_{|\pi|}\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p = \ell_p\hat{\otimes}_{|\pi|}(\ell_p\hat{\otimes}_{|\pi|}\cdots\hat{\otimes}_{|\pi|}\ell_p)$. Therefore $\hat{\otimes}_{n,s,|\pi|\ell_p}$ is a KB-space and hence, it is order continuous.

Parallel to the space $L^r(\ell_{p_1}, \cdots, \ell_{p_n}; \mathbb{R})$, we have the same results for the space $\mathcal{P}^r(n\ell_p; \mathbb{R})$ of all regular $n$-homogeneous polynomials on $\ell_p$ as follows.

Theorem 4.15. Let $1 \leq p < \infty$. Then the following statements are equivalent.

(i) $\mathcal{P}^r(n\ell_p; \mathbb{R})$ is reflexive.

(ii) $\mathcal{P}^r(n\ell_p; \mathbb{R})$ is a KB-space.

(iii) $\mathcal{P}^r(n\ell_p; \mathbb{R})$ is order continuous.

(iv) $p > n$. 

36
Proof. Note that $(\hat{\otimes}_{n,s|\pi} \ell_p)^* = \mathcal{P}^r(n\ell_p; \mathbb{R})$. If $p > n$, then Theorem 4.14 implies that $\hat{\otimes}_{n,s|\pi} \ell_p$ is reflexive. Thus $(\hat{\otimes}_{n,s|\pi} \ell_p)^* = \mathcal{P}^r(n\ell_p; \mathbb{R})$ is reflexive, and (iv) $\Rightarrow$ (i) follows. It is trivial that (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). Now suppose (iii) holds. By [36, p.93, Theorem 2.4.14], $\mathcal{P}^r(n\ell_p; \mathbb{R})$ does not contain any sublattice isomorphic to $\ell_\infty$ and hence, $\hat{\otimes}_{n,s|\pi} \ell_p$ does not contain any complemented sublattice isomorphic to $\ell_1$. Corollary 4.9 implies that $1/p + \cdots + 1/p = \frac{n}{p} < 1$ and (iv) follows. 

\[\square\]
4.2 Positive Injective Tensor Products of $\ell_p$-spaces

**Lemma 4.16.** Let $1 \leq p_1, \ldots, p_n < \infty$. Then $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ with any order is a 1-unconditional basis of $\ell_{p_1} \hat{\otimes} |e| \cdots \hat{\otimes} |e| \ell_{p_n}$.

**Proof.** It is easy to see that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a disjoint sequence and hence, is a 1-unconditional basic sequence in $\ell_{p_1} \hat{\otimes} |e| \cdots \hat{\otimes} |e| \ell_{p_n}$. Take any $u = x_1 \otimes \cdots \otimes x_n \in \ell_{p_1} \otimes \cdots \otimes \ell_{p_n}$. Write

$$x_k = \sum_{i=1}^{\infty} a_{i,k} e_i, \quad k = 1, \ldots, n.$$

Then

$$u = \sum_{i_1, \ldots, i_n=1}^{\infty} a_{i_1} \cdots a_{i_n} e_{i_1} \otimes \cdots \otimes e_{i_n},$$

which shows that the linear span of $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is dense in $\ell_{p_1} \otimes \cdots \otimes \ell_{p_n}$ and hence, dense in $\ell_{p_1} \hat{\otimes} |e| \cdots \hat{\otimes} |e| \ell_{p_n}$. Therefore, $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ is a 1-unconditional basis of $\ell_{p_1} \hat{\otimes} |e| \cdots \hat{\otimes} |e| \ell_{p_n}$.

By induction, [4, Theorem 7.3] and [9, Proposition 2.2, Theorems 3.1, 5.2] yield the following lemma.

**Lemma 4.17.** Let $1 < p_1, \ldots, p_n < \infty$. Then $(\ell_{p_1} \hat{\otimes} |e| \cdots \hat{\otimes} |e| \ell_{p_n})^* = \ell_{p_1'} \hat{\otimes} |\pi| \cdots \hat{\otimes} |\pi| \ell_{p_n'}$.

**Hölder Inequality.** Let $1 \leq p_1, p_2 \leq \infty$ such that $1/p_1 + 1/p_2 = 1$. Then for every $(b_i^{(k)})_i \in \ell_{p_k}, k = 1, 2$, we have

$$\sum_{i=1}^{\infty} \left| b_i^{(1)} \cdot b_i^{(2)} \right| \leq \left\| (b_i^{(1)})_i \right\|_{\ell_{p_1}} \cdot \left\| (b_i^{(2)})_i \right\|_{\ell_{p_2}}. \quad (4.1)$$

We can prove this by using Young’s inequality for $n = 2$. Next we need the following inequality, for which we were unable to find a reference in the literature.
Generalized Hölder Inequality: Let \(1 \leq p_1, \ldots, p_n \leq \infty\) such that \(1/p_1 + \cdots + 1/p_n = 1\). Then for every \((b_i^{(k)})_i \in \ell_{p_k}, k = 1, \ldots, n\), we have

\[
\sum_{i=1}^{\infty} |b_i^{(1)} \cdots b_i^{(n)}| \leq \left\| (b_i^{(1)})_i \right\|_{\ell_{p_1}} \cdots \left\| (b_i^{(n)})_i \right\|_{\ell_{p_n}}. \tag{4.2}
\]

Proof. We use induction on \(n\).

(i) When \(n = 2\), (4.2) is reduced to (4.1), so it is true.

(ii) Suppose (4.2) holds for some \(n \geq 2\). We need to show that it also holds for \(n + 1\).

Case 1: \(1 < p_1, \ldots, p_{n+1} < \infty\)

Let \(\sum_{k=1}^{n+1} \frac{1}{p_k} = 1\) and let \((b_i^{(k)})_i \in \ell_{p_k}, k = 1, \ldots, n + 1\).

In particular, we have \(p_1 > 0\), \(\frac{p_1}{p_1 - 1} > 0\) and \(\frac{1}{p_1} + \frac{p_1 - 1}{p_1} = 1\). Then by (4.1),

\[
\sum_{i=1}^{\infty} |b_i^{(1)} \cdots b_i^{(n+1)}| = \sum_{i=1}^{\infty} \left| b_i^{(1)} \right| \cdot \left| b_i^{(2)} \cdots b_i^{(n+1)} \right| \\
\leq \left\| (b_i^{(1)})_i \right\|_{\ell_{p_1}} \left\| (b_i^{(2)} \cdots b_i^{(n+1)})_i \right\|_{\ell_{\frac{p_1}{p_1 - 1}}} \\
= \left\| (b_i^{(1)})_i \right\|_{\ell_{p_1}} \left( \sum_{i=1}^{\infty} \left| b_i^{(2)} \cdots b_i^{(n+1)} \right| \right)^{\frac{1}{p_1}} \cdot \left(\frac{p_1 - 1}{p_1}\right) \left\| (b_i^{(2)} \cdots b_i^{(n+1)})_i \right\|_{\ell_{\frac{p_1}{p_1 - 1}}} = \frac{p_1}{p_1 - 1} \left(\frac{p_1 - 1}{p_1}\right) \left(\frac{p_1 - 1}{p_1}\right) \left(\frac{p_1 - 1}{p_1}\right) = 1
\]

Since \(\frac{p_1}{p_1 - 1} > 0\) and \(p_k > 0\) for \(k = 1, \ldots, n + 1\). We have \(p_k \cdot \left(\frac{p_1 - 1}{p_1}\right) > 0\) and

\[
\sum_{k=2}^{n+1} \frac{p_1}{p_k(p_1 - 1)} = \left(\frac{p_1}{p_1 - 1}\right) \cdot \sum_{k=2}^{n+1} \frac{1}{p_k} = \left(\frac{p_1}{p_1 - 1}\right) \left(1 - \frac{1}{p_1}\right) = 1
\]

By induction hypothesis and (*), we have
\[
\sum_{i=1}^{\infty} \left| b_{i}^{(1)} \cdots b_{i}^{(n+1)} \right| \leq \left\| (b_{i}^{(1)})_{i} \right\|_{\ell_{p_{1}}} \cdot \left[ \prod_{k=2}^{n+1} \left( \sum_{i=1}^{\infty} \left| b_{i}^{(k)} \right| \right)^{p_{k}} \right]^{\frac{1}{p_{k}}}
\]

\[
= \left\| (b_{i}^{(1)})_{i} \right\|_{\ell_{p_{1}}} \cdot \prod_{k=2}^{n+1} \left( \sum_{i=1}^{\infty} \left| b_{i}^{(k)} \right| \right)^{p_{k}}
\]

\[
= \left\| (b_{i}^{(1)})_{i} \right\|_{\ell_{p_{1}}} \cdot \left\| (b_{i}^{(2)})_{i} \right\|_{\ell_{p_{2}}} \cdots \left\| (b_{i}^{(n+1)})_{i} \right\|_{\ell_{p_{n+1}}}
\]

Case 2: If \( p_{1} = 1 \) then \( p_{k} = \infty \) for all \( k = 2, \cdots, n+1 \)

\[
\sum_{i=1}^{N} \left| b_{i}^{(1)} \cdots b_{i}^{(n+1)} \right| \leq \sum_{i=1}^{N} \left| b_{i}^{(1)} \right| \cdot \max_{1 \leq i \leq N} \left| b_{i}^{(2)} \right| \cdots \max_{1 \leq i \leq N} \left| b_{i}^{(n+1)} \right|
\]

\[
\leq \left\| (b_{i}^{(1)})_{i} \right\|_{\ell_{p_{1}}} \left\| (b_{i}^{(2)})_{i} \right\|_{\ell_{p_{2}}} \cdots \left\| (b_{i}^{(n+1)})_{i} \right\|_{\ell_{p_{n+1}}}
\]

Therefore, the series converges and Hölder Inequality follows by taking the limit as \( N \to \infty \). \( \square \)

The main diagonal of the injective tensor product of \( \ell_{p_{1}} \otimes \cdots \otimes \ell_{p_{n}} \) was characterized by Holub [25]. Next by using Hölder inequality, we will characterize the main diagonal of the positive injective tensor product \( \ell_{p_{1}} \otimes |v| \cdots \otimes |v| \ell_{p_{n}} \). Let \( \Delta(\ell_{p_{1}} \otimes |v| \cdots \otimes |v| \ell_{p_{n}}) \) denote the main diagonal of \( \ell_{p_{1}} \otimes |v| \cdots \otimes |v| \ell_{p_{n}} \), that is, the closed subspace spanned by the diagonal vectors \( \{ e_{i} \otimes \cdots \otimes e_{i} : i \in \mathbb{N} \} \) in \( \ell_{p_{1}} \otimes |v| \cdots \otimes |v| \ell_{p_{n}} \).

**Theorem 4.18.** Let \( 1 \leq p_{1}, \cdots, p_{n} < \infty \) and let \( \frac{1}{s} = \sum_{i=1}^{n} \frac{1}{p_{i}} \). Then \( \Delta(\ell_{p_{1}} \otimes |v| \cdots \otimes |v| \ell_{p_{n}}) \) is isometrically lattice isomorphic to \( \ell_{s} \) if \( s > 1 \) and \( c_{0} \) if \( s \leq 1 \).

i.e.

\[
\Delta(\ell_{p_{1}} \otimes |v| \cdots \otimes |v| \ell_{p_{n}}) \equiv \begin{cases} 
\ell_{s'}, & \text{if } s > 1 \\
c_{0}, & \text{if } s \leq 1.
\end{cases}
\]
Proof. Case 1: $s > 1$.

Take any positive element $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \Delta(\ell_{p_1} \otimes_{|\cdot|} \cdots \otimes_{|\cdot|} \ell_{p_n})$, where $a_i \geq 0$ for $i = 1, \ldots, m$. By Hölder inequality,

$$
\|u\|_{\ell_s} = \sup \left\{ \left\| \sum_{i=1}^{m} a_i x_i^*(e_i) \right\|_{\ell_{\| s \|}} : x_i^* \in B_{\ell_{p_i}}^+, 1 \leq k \leq n \right\} = \sup \left\{ \left\| \sum_{i=1}^{m} a_i b_i^{(1)} \cdots b_i^{(n)} \right\| : (b_i^{(k)})_i := x_i^* \in B_{\ell_{p_i}}^+, 1 \leq k \leq n \right\} \leq \sup \left\{ \| (a_i)^m_{i=1} \|_{\ell_{\ell'}} \cdot \left\| (b_i^{(1)})_i \right\|_{\ell_{p_1}} \cdots \left\| (b_i^{(n)})_i \right\|_{\ell_{p_n}} : (b_i^{(k)})_i \in B_{\ell_{p_i}}^+, 1 \leq k \leq n \right\} \leq \| (a_i)^m_{i=1} \|_{\ell_{\ell'}}.
$$

On the other hand, take any $(b_i)_i \in B_{\ell_{\ell'}}^+$. If $p_k = 1$ for some $k$, let $x_k^* = (1, 1, \cdots)$. Then $\| x_k^* \|_{\ell_{p_k}} = \| x_k^* \|_{\ell_{\infty}} = 1$. If $p_k > 1$ for some $k$, let $x_k^* = (\overline{b_i^{(k)}})_i$. Then $\| x_k^* \|_{\ell_{p_k}} = \left( \| (b_i)_i \|_{\ell_{\ell'}} \right)_{p_k} \leq 1$. Without loss of generality, assume that $p_1 = \cdots = p_k = 1$ and $p_{k+1} > 1, \cdots, p_n > 1$. Since $\sum_{i=k+1}^{n} \frac{s}{p_i} = \sum_{i=1}^{n} \frac{s}{p_i} = 1$, it follows that

$$
\left| \sum_{i=1}^{m} a_i b_i \right| = \left| \sum_{i=1}^{m} a_i \cdot 1 \cdots 1 \cdot \overline{b_i^{(k+1)}} \cdots \overline{b_i^{(n)}} \right| = \left| \sum_{i=1}^{m} a_i x_i^*(e_i) \cdots x_n^*(e_i) \right| \leq \| u \|_{\ell_s}.
$$

Thus

$$
\| (a_i)^m_{i=1} \|_{\ell_{\ell'}} = \sup \left\{ \left| \sum_{i=1}^{m} a_i b_i \right| : (b_i)_i \in B_{\ell_{\ell'}}^+ \right\} \leq \| u \|_{\ell_s}.
$$

Therefore, $\| u \|_{\ell_s} = \| (a_i)^m_{i=1} \|_{\ell_{\ell'}}$ and hence, $\Delta(\ell_{p_1} \otimes_{|\cdot|} \cdots \otimes_{|\cdot|} \ell_{p_n})$ is isometrically lattice isomorphic to $\ell_{\ell'}$.

Case 2: $s \leq 1$.

Take any positive element $u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \Delta(\ell_{p_1} \otimes_{|\cdot|} \cdots \otimes_{|\cdot|} \ell_{p_n})$, where $a_i \geq 0$ for $i = 1, \cdots, m$. Take $q_k \geq p_k'$ for $k = 1, \cdots, n$ such that $\frac{1}{q_1} + \cdots + \frac{1}{q_n} = 1$. By
Hölder inequality,

$$
\|u\|_{|\cdot|} = \sup \left\{ \sum_{i=1}^{m} a_i x_i^*(e_i) \cdots x_n^*(e_i) : x_k^* \in B_{\ell_{p_k}^+}, 1 \leq k \leq n \right\}
\leq \| (a_i)_{i=1}^{m} \|_{c_0} \cdot \sup \left\{ \left\| (b_i^{(1)})_{i, \ell_{q_1}} \cdots (b_i^{(n)})_{i, \ell_{q_n}} : (b_i^{(k)})_i \in B_{\ell_{p_k}^+}, 1 \leq k \leq n \right\}
\leq \| (a_i)_{i=1}^{m} \|_{c_0}. \]

On the other hand, for each fixed $j$, $1 \leq j \leq m$. Let $x_k^* = e_j \in B_{\ell_{p_k}^+}$ for $k = 1, \ldots, n$. Then

$$
\|u\|_{|\cdot|} \geq \left| \sum_{i=1}^{m} a_i x_i^*(e_i) \cdots x_n^*(e_i) \right| = a_j,
$$

which implies that $\|u\|_{|\cdot|} \geq \| (a_i)_{i=1}^{m} \|_{c_0}$. Therefore, $\|u\|_{|\cdot|} = \| (a_i)_{i=1}^{m} \|_{c_0}$ and hence, $\Delta(\ell_{p_1} \hat{\otimes} |\cdot| \cdots \hat{\otimes} |\cdot| \ell_{p_n})$ is isometrically lattice isomorphic to $c_0$.

**Remark 4.19.** In Theorem 4.18, if some (or all) of $p_i$’s are $\infty$, then $s \leq 1$. In this case, we consider $\ell_{p_i} = c_0$ if $p_i = \infty$, and consider $\ell_p = c_0$ if $p = \infty$. The Theorem 4.18 is still true. Thus we have the first special case, if at least one of $p_i$’s is $\infty$, then $\Delta(\ell_{p_1} \hat{\otimes} |\cdot| \cdots \hat{\otimes} |\cdot| \ell_{p_n})$ is isometrically lattice isomorphic to $c_0$. The second special case is that if all $p_i$’s are 1, then $\Delta(\ell_1 \hat{\otimes} |\cdot| \cdots \hat{\otimes} |\cdot| \ell_1)$ is isometrically lattice isomorphic to $\ell_1$.

Next we will use the main diagonal of $\ell_{p_1} \hat{\otimes} |\cdot| \cdots \hat{\otimes} |\cdot| \ell_{p_n}$ to characterize its reflexivity as well as being a KB-space.

**Theorem 4.20.** Let $1 < p_1, \ldots, p_n < \infty$. Then the following statements are equivalent:

(i) $\ell_{p_1} \hat{\otimes} |\cdot| \cdots \hat{\otimes} |\cdot| \ell_{p_n}$ is reflexive.

(ii) $\ell_{p_1} \hat{\otimes} |\cdot| \cdots \hat{\otimes} |\cdot| \ell_{p_n}$ is a KB-space.

(iii) $1/p_1' + \cdots + 1/p_n' < 1$. 

42
Proof. If $1/p'_1 + \cdots + 1/p'_n < 1$, then $\ell_{p'_1} \hat{\otimes} |x| \cdots \hat{\otimes} |x| \ell_{p'_n}$ is reflexive. Lemma 4.17 implies that $\ell_{p_1} \hat{\otimes} |x| \cdots \hat{\otimes} |x| \ell_{p_n}$ is reflexive and (iii) $\Rightarrow$ (i) follows. It is trivial that (i) $\Rightarrow$ (ii). Now suppose (ii) holds. Then by \cite[p.93, Theorem 2.4.14]{36}, $\ell$ does not contain any sublattice isomorphic to $c_0$. Theorem 4.18 implies that $1/p'_1 + \cdots + 1/p'_n < 1$ and (iii) follows.

\[ \square \]

Note that $\ell_{p_1} \hat{\otimes} |x| \cdots \hat{\otimes} |x| \ell_{p_n} = \ell_{p_1} \hat{\otimes} |x| (\ell_{p_2} \hat{\otimes} |x| \cdots \hat{\otimes} |x| \ell_{p_n})$. By induction, \cite[Theorems 6.7, 6.8]{9} yield the following corollary.

**Corollary 4.21.** Let $1 < p_1, \ldots, p_n < \infty$. Then $\ell_{p_1} \hat{\otimes} |x| \cdots \hat{\otimes} |x| \ell_{p_n}$ contains any sublattice neither isomorphic to $\ell_1$ nor isomorphic to $\ell_\infty$.

Note that Lemma 4.16 implies that $\ell_{p_1} \hat{\otimes} |x| \cdots \hat{\otimes} |x| \ell_{p_n}$ is $\sigma$-Dedekind complete. Corollary 4.21 and \cite[p.87, Corollary 2.4.3]{36} yield the following corollary.

**Corollary 4.22.** Let $1 < p_1, \ldots, p_n < \infty$. Then $\ell_{p_1} \hat{\otimes} |x| \cdots \hat{\otimes} |x| \ell_{p_n}$ is order continuous.

For Banach lattices $E_1, \ldots, E_n, F$ with $F$ Dedekind complete, let $K^r(E_1, \ldots, E_n; F)$ denote the sublattice of $L^r(E_1, \ldots, E_n; F)$ generated by all positive compact $n$-linear operators from $E_1 \times \cdots \times E_n$ to $F$. It is easy to see that $K^r(E_1, E_2, \ldots, E_n; F) = K^r(E_1; K^r(E_2, \ldots, E_n; F))$. By \cite[Proposition 2.1 and Theorem 5.2]{9}, for any $1 < p < \infty$, $K^r(\ell_p; F)$ is isometrically lattice isomorphic to $\ell_p \hat{\otimes} |x| F$. By induction, it follows that for any $1 < p_1, \ldots, p_n < \infty$, $K^r(\ell_{p_1}, \ldots, \ell_{p_n}; F)$ is isometrically lattice isomorphic to $\ell_{p'_1} \hat{\otimes} |x| \cdots \hat{\otimes} |x| \ell_{p'_n} \hat{\otimes} |x| F$. Thus Theorem 4.20 and Corollaries 4.21, 4.22 yield the following theorem.

**Theorem 4.23.** Let $1 < p_1, \ldots, p_n, q < \infty$.

(i) $K^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ contains any sublattice neither isomorphic to $\ell_1$ nor isomorphic to $\ell_\infty$.

(ii) $K^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is order continuous.

(iii) The following statements are equivalent:

(a) $K^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is reflexive.

43
(b) $\mathcal{K}^r(\ell_{p_1}, \ldots, \ell_{p_n}; \ell_q)$ is a KB-space.

(c) $1/p_1 + \cdots + 1/p_n < 1/q$.

Pitt [41] proved that every continuous linear operator from $\ell_p$ to $\ell_q$ is compact if and only if $p > q$. Chen and Wickstead [12] proved that if every positive linear operator from $\ell_p$ to $\ell_q$ is compact then we still have $p > q$. For the $n$-linear operator case, Alencar and Floret [43] proved that every continuous $n$-linear operator from $\ell_{p_1} \times \cdots \times \ell_{p_n}$ to $\ell_q$ is compact if and only if $1/p_1 + \cdots + 1/p_n < 1/q$ (also see [15]). Next we will give the same characterization for the positive compact $n$-linear operators on $\ell_{p_1} \times \cdots \times \ell_{p_n}$ as follows.

**Theorem 4.24.** Let $1 \leq p_1, \cdots, p_n, q < \infty$. Then every positive $n$-linear operator from $\ell_{p_1} \times \cdots \times \ell_{p_n}$ to $\ell_q$ is compact if and only if $1/p_1 + \cdots + 1/p_n < 1/q$.

**Proof.** It follows from [12, Theorem 4.9] that the theorem is true for $n = 1$. Now suppose that the theorem is true for $n - 1$. To show the theorem is true for $n$, we only need to show that $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ implies that $1/p_1 + \cdots + 1/p_n < 1/q$.

Now suppose that $\mathcal{K}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q) = \mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$. Then $\mathcal{K}^r(\ell_{p_1}; \ell_q) = \mathcal{L}^r(\ell_{p_1}; \ell_q)$ and hence, $p_1 > q \geq 1$. Also then $\mathcal{K}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q) = \mathcal{L}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)$. By induction hypothesis, $1/p_2 + \cdots + 1/p_n < 1/q$, which, by Corollaries 4.11 and 4.13, implies that $\mathcal{L}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)$ is a KB-space. Note that

\[
\mathcal{K}^r(\ell_{p_1}; \mathcal{L}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)) = \mathcal{K}^r(\ell_{p_1}; \mathcal{K}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q))
\]

\[
= \mathcal{K}^r(\ell_{p_1}, \ell_{p_2} \cdots, \ell_{p_n}; \ell_q)
\]

\[
= \mathcal{L}^r(\ell_{p_1}, \ell_{p_2} \cdots, \ell_{p_n}; \ell_q)
\]

\[
= \mathcal{L}^r(\ell_{p_1}; \mathcal{L}^r(\ell_{p_2}, \cdots, \ell_{p_n}; \ell_q)).
\]

It follows from Lemma 4.12 that $\mathcal{L}^r(\ell_{p_1}, \cdots, \ell_{p_n}; \ell_q)$ is a KB-space. Thus Corollaries 4.11 and 4.13 imply that $1/p_1 + \cdots + 1/p_n < 1/q$. \qed
From Corollary 4.11 and Theorems 4.23, 4.24, we have the following interesting results. For any $1 < p_1, \ldots, p_n, q < \infty$, $K^r(\ell_{p_1}, \ldots, \ell_{p_n}; \ell_q)$ is always order continuous, but $L^r(\ell_{p_1}, \ldots, \ell_{p_n}; \ell_q)$ is order continuous if and only if $L^r(\ell_{p_1}, \ldots, \ell_{p_n}; \ell_q) = K^r(\ell_{p_1}, \ldots, \ell_{p_n}; \ell_q)$.

For instance, if $1/p_1 + \cdots + 1/p_n \geq 1/q$ then $K^r(\ell_{p_1}, \ldots, \ell_{p_n}; \ell_q)$ is order continuous but $L^r(\ell_{p_1}, \ldots, \ell_{p_n}; \ell_q)$ is not order continuous.
5 DIAGONALS OF TENSOR PRODUCTS OF BANACH LATTICES WITH BASES

5.1 Diagonals of Projective Tensor Products of Banach Lattices

In this section we assume that $X$ is a Banach space with a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. Gelbaum and Lamadrid [19] showed that $\{e_i \otimes e_j : (i, j) \in \mathbb{N}^2\}$ with the square order is a basis of $\hat{\otimes}_{2,\pi} X$ (it is not necessarily an unconditional basis). For instance, Kwapien and Pelczynski [49] showed that $\{e_i \otimes e_j : (i, j) \in \mathbb{N}^2\}$ is not an unconditional basis of $\hat{\otimes}_{2,\pi} \ell_2$. In general, Grecu and Ryan [20] established that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n\}$ with the order defined in [20] is a basis of $\hat{\otimes}_{n,\pi} X$. They also showed that $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \cdots, i_n) \in \mathbb{N}^n, i_1 \geq \cdots \geq i_n\}$ with the order defined in [20] is a basis of $\hat{\otimes}_{n,s,\pi} X$.

Definition 5.1. Let $\Delta(\hat{\otimes}_{n,\pi} X)$ (resp. $\Delta(\hat{\otimes}_{n,s,\pi} X)$) denote the main diagonal space of $\hat{\otimes}_{n,\pi} X$ (resp. $\hat{\otimes}_{n,s,\pi} X$), that is, the closed subspace spanned in $\hat{\otimes}_{n,\pi} X$ (resp. in $\hat{\otimes}_{n,s,\pi} X$) by the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$.

Holub [25] proved that the tensor diagonal $\{e_i \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\hat{\otimes}_{2,\pi} X)$ and Sanchez [16] generalized the Holub’s result. He showed that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\hat{\otimes}_{n,\pi} X)$.

Bu and Buskes [5] proved that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\hat{\otimes}_{n,s,\pi} X)$ and Bu, Buskes, Popov, Tcaciuc, and Troitsky [8] showed that the tensor basis $\{e_i \otimes e_j : (i, j) \in \mathbb{N}^2\}$ with any order is a 1-unconditional basis of $\hat{\otimes}_{2,|\pi|} E$. 

46
By a Banach lattice with a Schauder basis we mean a Banach lattice in which the unit vectors form a basis and the order is defined coordinatewise. It follows that such a Schauder basis is 1-unconditional. Conversely, every Banach space with a 1-unconditional basis is a Banach lattice with the order defined coordinatewise. In what follows, \( E \) is a Banach lattice with a basis \( \{ e_i : i \in \mathbb{N} \} \). As a special case of [8, Lemma 22], the set \( \{ e_i \otimes e_j : (i, j) \in \mathbb{N}^2 \} \) with any order is a (1-unconditional) basis of \( \hat{\otimes}_{2,|\pi|} E \).

In a similar way, Bu and Buskes [5] also proved that the following lemma.

**Lemma.** The tensor basis \( \{ e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n \} \) with any order is a (1-unconditional) basis of \( \hat{\otimes}_{n,|\pi|} E \), and the tensor basis \( \{ e_{i_1} \otimes s \cdots \otimes s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geq \cdots \geq i_n \} \) with any order is a (1-unconditional) basis of \( \hat{\otimes}_{n,s,|\pi|} E \).

**Definition 5.2.** Let \( \Delta(\hat{\otimes}_{n,|\pi|} E) \) (resp. \( \Delta(\hat{\otimes}_{n,s,|\pi|} E) \)) denote the main diagonal space of \( \hat{\otimes}_{n,|\pi|} E \) (resp. \( \hat{\otimes}_{n,s,|\pi|} E \)), that is, the closed subspace spanned in \( \hat{\otimes}_{n,|\pi|} E \) (resp. in \( \hat{\otimes}_{n,s,|\pi|} E \)) by the tensor diagonal \( \{ e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \). It follows from the above lemma that \( \{ e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \) is a 1-unconditional basis of both \( \Delta(\hat{\otimes}_{n,|\pi|} E) \) and \( \Delta(\hat{\otimes}_{n,s,|\pi|} E) \).

In [5], Bu and Buskes proved that all four main diagonal spaces \( \Delta(\hat{\otimes}_{n,\pi} E) \), \( \Delta(\hat{\otimes}_{n,s,\pi} E) \), \( \Delta(\hat{\otimes}_{n,|\pi|} E) \), and \( \Delta(\hat{\otimes}_{n,s,|\pi|} E) \) are pairwise isometrically isomorphic.

It is natural question to ask whether this result holds for all four main diagonal spaces such as \( \Delta(\hat{\otimes}_{n,e} E) \), \( \Delta(\hat{\otimes}_{n,s,e} E) \), \( \Delta(\hat{\otimes}_{n,|e|} E) \), and \( \Delta(\hat{\otimes}_{n,s,|e|} E) \).
5.2 Diagonals of Injective Tensor Products of Banach Lattices

* All of results in this section have been proved in my paper “Diagonals of Injective Tensor Products of Banach Lattices with Bases”. This is joint work with Qingying Bu and Donghai Ji. (See [27].)

In this section we assume that $X$ is a Banach space with a 1-unconditional basis $\{e_i : i \in \mathbb{N}\}$. Gelbaum and Lamadrid [19] showed that $\{e_i \otimes e_j : (i, j) \in \mathbb{N}^2\}$ with the square order is a basis of $\tilde{\otimes}_2 X$. In general, Grecu and Ryan [20] showed that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ with the order defined in [20] is a basis of $\tilde{\otimes}_n X$. They also showed that $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geq \cdots \geq i_n\}$ with the order defined in [20] is a basis of $\tilde{\otimes}_n,s X$.

Let $\Delta(\tilde{\otimes}_n X)$ (resp. $\Delta(\tilde{\otimes}_n,s X)$) denote the main diagonal space of $\tilde{\otimes}_n X$ (resp. $\tilde{\otimes}_n,s X$), that is, the closed subspace spanned in $\tilde{\otimes}_n X$ (resp. in $\tilde{\otimes}_n,s X$) by the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$. It is known that $\{e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ and $\{e_{i_1} \otimes_s \cdots \otimes_s e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geq \cdots \geq i_n\}$, respectively, is not necessarily an unconditional basis of $\tilde{\otimes}_n X$ and $\tilde{\otimes}_n,s X$ (see, e.g., [49, 40, 48, 13, 39, 11]). Next we will use the following Rademacher averaging formula (see, e.g., [45, Lemma 2.22]) to show that the tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is an unconditional basis of both $\Delta(\tilde{\otimes}_n X)$ and $\Delta(\tilde{\otimes}_n,s X)$, and their diagonal projections are contractive.

**Lemma 5.3.** The tensor diagonal $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basis of $\Delta(\tilde{\otimes}_n X)$ and the projection $Q : \tilde{\otimes}_n X \to \Delta(\tilde{\otimes}_n X)$ defined by

$$Q(e_{i_1} \otimes \cdots \otimes e_{i_n}) = \begin{cases} e_{i_1} \otimes \cdots \otimes e_{i_n}, & \text{if } i_1 = \cdots = i_n, \\ 0, & \text{otherwise.} \end{cases}$$

is bounded with $\|Q\| \leq 1$. 

48
Proof. First we adopt Holub’s proof of Theorem 3.12 in [25] to prove that \( \{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \) is a 1-unconditional basic sequence of \( \tilde{\otimes}_{n,\epsilon} X \). Let \( I : X \rightarrow X \) be the identity operator and for \( \theta_i = \pm 1 \) \( (i \in \mathbb{N}) \), define \( T : X \rightarrow X \) by \( T(\sum_{i=1}^{\infty} a_i e_i) = \sum_{i=1}^{\infty} \theta_i a_i e_i \) for every \( x = \sum_{i=1}^{\infty} a_i e_i \in X \). Then \( \|T\| \leq 1 \). Now for any \( m \in \mathbb{N} \), by Lemma 3.10,

\[
\left\| \sum_{i=1}^{m} \theta_i a_i e_i \otimes \cdots \otimes e_i \right\|_{\epsilon} = \left\| \sum_{i=1}^{m} a_i T(e_i) \otimes I(e_i) \otimes \cdots \otimes I(e_i) \right\|_{\epsilon} \leq \left\| \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \right\|_{\epsilon}.
\]

Thus \( \{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \) is a 1-unconditional basic sequence of \( \tilde{\otimes}_{n,\epsilon} X \).

Next we show that \( Q \) is well-defined and bounded with \( \|Q\| \leq 1 \). Take any \( u = \sum_{i_1,\ldots,i_n} b_{i_1,\ldots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_n} \in \tilde{\otimes}_{n,\epsilon} X \). For every \( p, q \in \mathbb{N} \) with \( p < q \), let

\[
u_{p,q} = \sum_{i_1,\ldots,i_n = p}^{q} b_{i_1,\ldots,i_n} e_{i_1} \otimes \cdots \otimes e_{i_n}.
\]

Then there exist \( x_{j,k} = \sum_{i=1}^{\infty} a_{i,j,k} e_i \in X, k = 1, \ldots, m \) and \( j = 1, \ldots, n \) such that

\[
u_{p,q} = \sum_{k=1}^{m} x_{1,k} \otimes \cdots \otimes x_{n,k}.
\]

Thus

\[
b_{i,\ldots,i} = \sum_{k=1}^{m} a_{i,1,k} \cdots a_{i,n,k}, \quad p \leq i \leq q.
\]
By Rademacher averaging,

\[
\left\| \sum_{i=p}^{q} b_{i} \cdots i e_{i} \otimes \cdots \otimes e_{i} \right\|_{\epsilon} = \left\| \sum_{k=1}^{m} \sum_{i=p}^{q} a_{i,1,k} \cdots a_{i,n,k} e_{i} \otimes \cdots \otimes e_{i} \right\|_{\epsilon}
\]

\[
= \left\| \sum_{k=1}^{m} \sum_{i=p}^{q} (a_{i,1,k} e_{i}) \otimes \cdots \otimes (a_{i,n,k} e_{i}) \right\|_{\epsilon}
\]

\[
\leq \left\| \sum_{k=1}^{m} \int_{0}^{1} \left( \sum_{i=p}^{q} a_{i,1,k} r_{i}(t) e_{i} \right) \otimes \cdots \otimes \left( \sum_{i=p}^{q} a_{i,n,k} r_{i}(t) e_{i} \right) dt \right\|_{\epsilon}
\]

\[
\leq \int_{0}^{1} \left\| T_{t}(x_{1,k}) \right\| \otimes \cdots \otimes \left\| T_{t}(x_{n,k}) \right\| dt
\]

where \( T_{t} : X \to X \) is defined by \( T_{t}(x) = \sum_{i=1}^{\infty} a_{i} r_{i}(t) e_{i} \) for every \( x = \sum_{i=1}^{\infty} a_{i} e_{i} \in X \) and every \( t \in [0,1] \). Therefore, for every \( p, q \in \mathbb{N} \) with \( p < q \),

\[
\left\| \sum_{i=p}^{q} b_{i} \cdots i e_{i} \otimes \cdots \otimes e_{i} \right\|_{\epsilon} \leq \left\| \sum_{i_{1},\cdots,i_{n}=p}^{q} b_{i_{1},\cdots,i_{n}} e_{i_{1}} \otimes \cdots \otimes e_{i_{n}} \right\|_{\epsilon},
\]

which implies that \( Q \) is well-defined and bounded with \( \| Q \| \leq 1 \).

**Definition 5.4.** An \( n \)-homogeneous polynomial \( P : E \to Y \) is called *orthogonally additive* if \( P(x + y) = P(x) + P(y) \) whenever \( x, y \in E \) with \( x \perp y \).
Lemma 5.5. The tensor diagonal \(\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}\) is a 1-unconditional basis of \(\Delta(\hat{\otimes}_{n,s,t}X)\) and the projection \(Q_s : \hat{\otimes}_{n,s,t}X \to \Delta(\hat{\otimes}_{n,s,t}X)\) defined by

\[
Q_s(e_{i_1} \otimes_s \cdots \otimes_s e_{i_n}) = \begin{cases} 
    e_{i_1} \otimes_s \cdots \otimes_s e_{i_n}, & \text{if } i_1 = \cdots = i_n, \\
    0, & \text{otherwise.}
\end{cases}
\]

is bounded with \(\|Q_s\| \leq 1\).

Proof. For any \(m \in \mathbb{N}\), let \(u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i\), and let \(P_u : X^* \to \mathbb{R}\) be the \(n\)-homogeneous polynomial defined in (2.5), that is,

\[
P_u(x^*) = \sum_{i=1}^{m} a_i \left(x^*(e_i)\right)^n, \quad \forall \ x^* \in X^*.
\]  

(5.1)

For every \(a \in \mathbb{R}\) and every \(p > 0\) we define \(a^p = \text{sign}(a) \cdot |a|^p\). Note that \(X\) has a 1-unconditional basis \(\{e_i : i \in \mathbb{N}\}\). It can be a Banach lattice with the order defined coordinatewise. Also note that \(X^*\) is a sequence space via \(x^* \leftrightarrow (x^*(e_1), x^*(e_2), \ldots)\) for every \(x^* \in E^*\). Thus for every \(x^*, y^* \in X^*\), from functional calculation, \((x^p + y^p)^{\frac{1}{p}}\) is defined (coordinatewise) to be an element of \(X^*\) (see, e.g., [34, Section 1.d]), that is,

\[
(x^p + y^p)^{\frac{1}{p}}(e_i) = (x^*(e_i))^p + (y^*(e_i))^p^{\frac{1}{p}}, \quad i = 1, 2, \ldots.
\]

If, moreover, \(x^* \perp y^*\), then \((x^p + y^p)^{\frac{1}{p}} = x^* + y^*\) by [3] Lemma 3]. Thus

\[
(x^*(e_i) + y^*(e_i))^p = x^*(e_i)^p + y^*(e_i)^p, \quad i = 1, 2, \ldots.
\]  

(5.2)

It follows from (5.1) and (5.2) that \(P_u(x^* + y^*) = P_u(x^*) + P_u(y^*)\) for every \(x^*, y^* \in X^*\) with \(x^* \perp y^*\). Thus \(P_u\) is orthogonally additive (see [3, 51, 58, 26, 33]). Let \(T_u : X^* \times \cdots \times X^* \to \mathbb{R}\)
be the $n$-linear operator defined in (3.9), that is,

$$T_u(x_1^*, \ldots, x_n^*) = \sum_{i=1}^{m} a_i \cdot x_1^*(e_i) \cdots x_n^*(e_i), \quad \forall x_1^*, \ldots, x_n^* \in X^*.$$ 

Then $T_u$ is the symmetric $n$-linear operator associated to $P_u$. It follows from [5, Theorem 5.4] that $\|T_u\| = \|P_u\|$ and hence, $\|u\|_{s,\epsilon} = \|P_u\| = \|T_u\| = \|u\|_\epsilon$.

Now let $\theta_i = \pm 1 (i \in \mathbb{N})$. Then by (3.11) and Lemma 5.3,

$$\left\| \sum_{i=1}^{m} \theta_i a_i e_1 \otimes \cdots \otimes e_i \right\|_{s,\epsilon} \leq \left\| \sum_{i=1}^{m} a_i e_1 \otimes \cdots \otimes e_i \right\|_{\epsilon} = \|u\|_{\epsilon} \leq \left\| u \right\|_{s,\epsilon} = \left\| \sum_{i=1}^{m} a_i e_1 \otimes \cdots \otimes e_i \right\|_{s,\epsilon}.$$ 

Thus $\{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N}\}$ is a 1-unconditional basic sequence of $\tilde{\otimes}_{n,s,\epsilon} X$.

Similarly to the proof of Lemma 5.3, we can use the Rademacher averaging formula to show that $Q_s$ is well-defined and bounded with $\|Q_s\| \leq 1$.

By a Banach lattice with a Schauder basis we mean a Banach lattice in which the unit vectors form a basis and the order is defined coordinatewise. It follows that such a Schauder basis is 1-unconditional. Conversely, every Banach space with a 1-unconditional basis is a Banach lattice with the order defined coordinatewise. In what follows $E_1, \ldots, E_n$ are Banach lattices with (1-unconditional) basis $\{e_i^{(1)} : i \in \mathbb{N}\}, \ldots, \{e_i^{(n)} : i \in \mathbb{N}\}$, respectively.

**Theorem 5.6.** The tensor basis $\{e_i^{(1)} \otimes \cdots \otimes e_i^{(n)} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}$ with any order is a (1-unconditional) basis of $E_1 \tilde{\otimes}_{\epsilon_1} \cdots \tilde{\otimes}_{\epsilon_n} E_n$.

**Proof.** First, we will show that $(e_i^{(1)} \otimes \cdots \otimes e_i^{(n)}) \perp (e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)})$ provided $(i_1, \ldots, i_n) \neq (k_1, \ldots, k_n)$. Note that $E_1 \tilde{\otimes}_{\epsilon_1} \cdots \tilde{\otimes}_{\epsilon_n} E_n$ can be considered as a closed sublattice of $L^r(E_1^*, \ldots, E_n^*; \mathbb{R})$. 52
It suffices to show that
\[
\langle (e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}) \wedge (e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)}), (x_1^*, \ldots, x_n^*) \rangle = 0
\]
for every \( x_1^* \in E_{1}^+, \ldots, x_n^* \in E_{n}^+ \). Let
\[
\alpha_1 = x_1^*(e_{k_1}^{(1)}), \ldots, \alpha_n = x_n^*(e_{k_n}^{(n)}) \quad \text{and} \quad u_{1,1}^* = \alpha_1 f_{k_1}^{(1)}, \ldots, u_{n,1}^* = \alpha_n f_{k_n}^{(n)},
\]
where \( \{f_{i}^{(1)} : i \in \mathbb{N}\}, \ldots, \{f_{i}^{(n)} : i \in \mathbb{N}\} \) are, respectively, the appropriate bi-orthogonal functionals on \( E_{1}^*, \ldots, E_{n}^* \). Then for every \( x = \sum_{i=1}^{\infty} b_i e_{i}^{(1)} \in E_{1}^+ \),
\[
u_{1,1}^*(x) = \alpha_1 f_{k_1}^{(1)}(x) = x_1^*(e_{k_1}^{(1)}) b_{k_1} \leq \sum_{i=1}^{\infty} b_i x_i^*(e_{i}^{(1)}) = x_i^*(x).
\]
It follows that \( u_{1,1}^* \leq x_1^* \) and similarly, \( u_{2,1}^* \leq x_2^*, \ldots, u_{n,1}^* \leq x_n^* \).

Let \( u_{1,2}^* = x_1^* - u_{1,1}^*, \ldots, u_{n,2}^* = x_n^* - u_{n,1}^* \). Then \( (u_{1,1}^*, u_{1,2}^*), \ldots, (u_{n,1}^*, u_{n,2}^*) \) are partitions of \( x_1^*, \ldots, x_n^* \) respectively.

By [6, Proposition 2.2],
\[
\langle (e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}) \wedge (e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)}), (x_1^*, \ldots, x_n^*) \rangle \leq \sum_{j_1, \ldots, j_n=1}^{2} \langle e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}(u_{1,j_1}^*, \ldots, u_{n,j_n}^*) \rangle \wedge \langle e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)}(u_{1,j_1}^*, \ldots, u_{n,j_n}^*) \rangle.
\]
If all \( j_m \)'s are 1, then the general term
\[
\langle e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}(u_{1,j_1}^*, \ldots, u_{n,j_n}^*) \rangle \wedge \langle e_{k_1}^{(1)} \otimes \cdots \otimes e_{k_n}^{(n)}(u_{1,j_1}^*, \ldots, u_{n,j_n}^*) \rangle \leq \langle e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)}(u_{1,1}^*, \ldots, u_{n,1}^*) \rangle = u_{1,1}^*(e_{i_1}^{(1)}) \cdots u_{n,1}^*(e_{i_n}^{(n)}) = \alpha_1 f_{k_1}^{(1)}(e_{i_1}^{(1)}) \cdots \alpha_n f_{k_n}^{(n)}(e_{i_n}^{(n)}) = 0
\]
53
since \((i_1, \ldots, i_n) \neq (k_1, \ldots, k_n)\). If at least one of \(j_m\)'s is 2, say \(j_1 = 2\), then the general term

\[
\langle e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}, (u^*_1, j_1, \ldots, u^*_n, j_n) \rangle 
\leq \langle e^{(1)}_{k_1} \otimes e^{(2)}_{k_2} \otimes \cdots \otimes e^{(n)}_{k_n}, (u^*_1, 2, u^*_2, j_2, \ldots, u^*_n, j_n) \rangle
\]

\[
= u^*_{1,2}(e^{(1)}_{k_1}) \cdot u^*_{2,j_2}(e^{(2)}_{k_2}) \cdots u^*_{n,j_n}(e^{(n)}_{k_n})
\]

\[
= \left( \alpha_1 - \alpha_1 f^{(1)}_{k_1}(e^{(1)}_{k_1}) \right) \cdot u^*_{2,j_2}(e^{(2)}_{k_2}) \cdots u^*_{n,j_n}(e^{(n)}_{k_n}) = 0.
\]

Therefore,

\[
\langle (e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}) \wedge (e^{(1)}_{k_1} \otimes \cdots \otimes e^{(n)}_{k_n}), (x^*_1, \ldots, x^*_n) \rangle = 0
\]

and hence, \((e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n}) \perp (e^{(1)}_{k_1} \otimes \cdots \otimes e^{(n)}_{k_n})\).

Being a disjoint sequence in a Banach lattice, \(\{e^{(1)}_{i_1} \otimes \cdots \otimes e^{(n)}_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n\}\) is a 1-unconditional basic sequence of \(E_1 \otimes |\cdot| \cdots \otimes |\cdot| E_n\). It is left to show that its span is dense in \(E_1 \otimes |\cdot| \cdots \otimes |\cdot| E_n\).

Take any \(x_i \in E_i\) with \(\|x_i\| \leq 1\) for \(i = 1, \ldots, n\). Given any \(\sigma \in (0, 1)\), we can find basis projections \(P_i\) on \(E_i\), respectively, such that

\[
y_i = P_i(x_i) \quad \text{and} \quad \|y_i - x_i\| \leq \sigma, \quad i = 1, \ldots, n.
\]

Then

\[
\|x_1 \otimes x_2 - y_1 \otimes y_2\|_{|\cdot|} = \|x_1 \otimes (x_2 - y_2) + (x_1 - y_1) \otimes y_2\|_{|\cdot|}
\]

\[
\leq \|x_1\| \cdot \|x_2 - y_2\| + \|x_1 - y_1\| \cdot \|y_2\| \leq 2\sigma,
\]

and similarly,

\[
\|x_1 \otimes \cdots \otimes x_n - y_1 \otimes \cdots \otimes y_n\|_{|\cdot|} \leq n\sigma.
\]
Since \( y_1 \otimes \cdots \otimes y_n \) is in the span \( \{ e_{i_1}^{(1)} \otimes \cdots \otimes e_{i_n}^{(n)} : (i_1, \ldots, i_n) \in \mathbb{N}^n \} \), it follows that \( x_1 \otimes \cdots \otimes x_n \) is in the closure of this span. Thus this span is dense in \( E_1 \otimes \cdots \otimes E_n \) and hence, dense in \( E_1 \hat{\otimes}_\varepsilon \cdots \hat{\otimes}_\varepsilon E_n \).

In what follows \( E \) is a Banach lattice with a (1-unconditional) basis \( \{ e_i : i \in \mathbb{N} \} \).

The following consequence comes straightforward from the previous theorem.

**Corollary 5.7.** The tensor basis \( \{ e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n \} \) with any order is a (1-unconditional) basis of \( \hat{\otimes}_{\varepsilon} E \), and the tensor basis \( \{ e_{i_1} \otimes \cdots \otimes e_{i_n} : (i_1, \ldots, i_n) \in \mathbb{N}^n, i_1 \geq \cdots \geq i_n \} \) with any order is a (1-unconditional) basis of \( \hat{\otimes}_{\varepsilon} E \).

**Definition 5.8.** Let \( \Delta(\hat{\otimes}_{\varepsilon} E) \) (resp. \( \Delta(\hat{\otimes}_{\varepsilon} E) \)) denote the main diagonal space of \( \hat{\otimes}_{\varepsilon} E \) (resp. \( \hat{\otimes}_{\varepsilon} E \)), that is, the closed subspace spanned in \( \hat{\otimes}_{\varepsilon} E \) (resp. in \( \hat{\otimes}_{\varepsilon} E \)) by the tensor diagonal \( \{ e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \).

It follows from the above corollary that \( \{ e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \) is a (1-unconditional) basis of both \( \Delta(\hat{\otimes}_{\varepsilon} E) \) and \( \Delta(\hat{\otimes}_{\varepsilon} E) \).

Recall that we already have other two main diagonal spaces \( \Delta(\hat{\otimes}_{\varepsilon} E) \) and \( \Delta(\hat{\otimes}_{\varepsilon} E) \) introduced at the beginning of this section. Next we will show that all four main diagonal spaces are pairwise isometrically isomorphic. First we need the Hölder inequality (3.2) and then a lemma.

**Lemma 5.9.** For \( x_1^* \ldots, x_n^* \in E^{*+} \), define \( x^*: E \to \mathbb{R} \) by

\[
x^*(x) = \sum_{i=1}^{\infty} a_i \left( x_1^*(e_i) \right)^{\frac{1}{n}} \cdots \left( x_n^*(e_i) \right)^{\frac{1}{n}}
\]

for every \( x = \sum_{i=1}^{\infty} a_i e_i \in E \). Then \( x^* \in E^{*+} \) with \( \|x^*\| \leq \|x_1^*\|^{\frac{1}{n}} \cdots \|x_n^*\|^{\frac{1}{n}} \).

55
Proof. For any \( m \in \mathbb{N} \), by Hölder inequality,

\[
\sum_{i=1}^{m} \left| a_i x_i^* (e_i) \right|^\frac{1}{n} \cdots \left| x_n^* (e_i) \right|^\frac{1}{n} \leq \left( \sum_{i=1}^{m} |a_i x_i^* (e_i)| \right)^{\frac{1}{n}} \cdots \left( \sum_{i=1}^{m} |a_i x_n^* (e_i)| \right)^{\frac{1}{n}} = \left( x_1^* \left( \sum_{i=1}^{m} \pm a_i e_i \right) \right)^{\frac{1}{n}} \cdots \left( x_n^* \left( \sum_{i=1}^{m} \pm a_i e_i \right) \right)^{\frac{1}{n}} \leq \left\| x_1^* \right\|^{\frac{1}{n}} \cdots \left\| x_n^* \right\|^{\frac{1}{n}} \cdot \left\| \sum_{i=1}^{m} a_i e_i \right\|.
\]

Thus \( x^* \) is well-defined and \( x^* \in E^+ \) with \( \|x^*\| \leq \left\| x_1^* \right\|^{\frac{1}{n}} \cdots \left\| x_n^* \right\|^{\frac{1}{n}} \). \( \square \)

**Theorem 5.10.** All four main diagonal spaces \( \Delta(\Diamond_{n,\varepsilon} E) \), \( \Delta(\Diamond_{n,s,\varepsilon} E) \), \( \Delta(\Diamond_{n,|\varepsilon|} E) \), and \( \Delta(\Diamond_{n,s,|\varepsilon|} E) \) are pairwise isometrically isomorphic.

**Proof.** First we show that \( \Delta(\Diamond_{n,|\varepsilon|} E) \) is isometrically isomorphic to \( \Delta(\Diamond_{n,s,|\varepsilon|} E) \). Since \( \{e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \) is a basis of both \( \Delta(\Diamond_{n,|\varepsilon|} E) \) and \( \Delta(\Diamond_{n,s,|\varepsilon|} E) \), it suffices to show that \( \|u\|_{|\varepsilon|} = \|u\|_{s,|\varepsilon|} \) for every \( u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \otimes_n E \). Without loss of generality, we assume that \( u \geq 0 \), that is, \( a_i \geq 0 \) for \( i = 1, \ldots, m \). For any \( \sigma > 0 \) there exist \( x_1^*, \ldots, x_n^* \in B_{E^+} \) such that

\[
\|u\|_{|\varepsilon|} \leq \sum_{i=1}^{m} a_i x_i^* (e_i) \cdots x_n^* (e_i) + \sigma.
\]

Let \( x^* \) be defined in Lemma 5.9. Then \( x^* \in B_{E^+} \) and for each \( i \in \mathbb{N} \),

\[
x^* (e_i) = (x_i^* (e_i))^{\frac{1}{n}} \cdots (x_n^* (e_i))^{\frac{1}{n}}.
\]
Thus
\[ \| u \|_{s,|\epsilon|} \geq \sum_{i=1}^{m} a_i \left( x^*(e_i) \right)^n = \sum_{i=1}^{m} a_i x^*_i(e_i) \cdots x^*_n(e_i) \geq \| u \|_{|\epsilon|} - \sigma. \]

It follows that \( \| u \|_{s,|\epsilon|} \geq \| u \|_{|\epsilon|} \). From their definitions it is trivial that \( \| u \|_{s,|\epsilon|} \leq \| u \|_{|\epsilon|} \) and hence, \( \| u \|_{s,|\epsilon|} = \| u \|_{|\epsilon|} \).

Secondly, we show that \( \Delta (\hat{\otimes}_n,|\epsilon|E) \) is isometrically isomorphic to \( \Delta (\hat{\otimes}_n,|\epsilon|E) \). Since \( \{ e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \) is a basis of both \( \Delta (\hat{\otimes}_n,|\epsilon|E) \) and \( \Delta (\hat{\otimes}_n,|\epsilon|E) \), it suffices to show that \( \| u \|_{\epsilon} = \| u \|_{|\epsilon|} \) for every \( u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \hat{\otimes}_n E \). By Lemma 5.3,

\[
\| u \|_{\epsilon} = \| u \|_{|\epsilon|} = \left\| \sum_{i=1}^{m} |a_i| e_i \otimes \cdots \otimes e_i \right\|_{|\epsilon|} \\
= \left\| \sum_{i=1}^{m} |a_i| e_i \otimes \cdots \otimes e_i \right\|_{\epsilon} = \left\| \sum_{i=1}^{m} \pm a_i e_i \otimes \cdots \otimes e_i \right\|_{\epsilon} \\
\leq \left\| \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \right\|_{\epsilon} = \| u \|_{\epsilon}.
\]

On the other hand, it follows from their definitions that \( \| u \|_{\epsilon} \leq \| u \|_{|\epsilon|} \) and hence, \( \| u \|_{\epsilon} = \| u \|_{|\epsilon|} \).

Finally we show that \( \Delta (\hat{\otimes}_n,|\epsilon|E) \) is isometrically isomorphic to \( \Delta (\hat{\otimes}_n,s,|\epsilon|E) \). Since \( \{ e_i \otimes \cdots \otimes e_i : i \in \mathbb{N} \} \) is a basis of both \( \Delta (\hat{\otimes}_n,|\epsilon|E) \) and \( \Delta (\hat{\otimes}_n,s,|\epsilon|E) \), it suffices to show that \( \| u \|_{\epsilon} = \| u \|_{s,|\epsilon|} \) for every \( u = \sum_{i=1}^{m} a_i e_i \otimes \cdots \otimes e_i \in \hat{\otimes}_n E \), which was proved in the proof of Lemma 5.5. \( \square \)


VITA

The author was born on September 26, 1975 in Pusan, South Korea. He attended at Peniel high school in Pusan and graduated in 1994. He then enrolled at the Kosin University in Pusan, South Korea where he graduated *summa cum laude* with a B.A. in Mathematics in 1998. In the spring of 2000, he earned an M.S. in Mathematics from the Pusan National University. There he wrote a thesis in abstract algebra under the supervision of Professor Chan Huh. After graduation, he spent 5 semesters as a lecturer at the Kosin University and performed his military service in the Korea army as an artillery officer between 2001–2004.

After finishing his military service, he decided to study abroad and enrolled at the University of Alabama in 2007. In the summer of 2010, he earned an M.A. in Mathematics from the University of Alabama in Huntsville. In the fall of 2010, he began his graduate studies at the University of Mississippi in functional analysis under the supervision of Professor Qingying Bu. He is the recipient of the university’s 2012, 2013, and 2014 Summer Research Grant and the Dissertation Fellowship for the Spring 2015.