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ZEROS OF THE DEDEKIND ZETA-FUNCTION  
DISSERTATION

A Dissertation  
presented in partial fulfillment of requirements  
for the degree of Master of Science  
in the Department of Mathematics  
The University of Mississippi

by

MASHAEL ALSHARIF

December 2018

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## ABSTRACT

H. L. Montgomery proved a formula for sums over two sets of nontrivial zeros of the Riemann zeta-function. Assuming the Riemann Hypothesis, he used this formula and Fourier analysis to prove an estimate for the proportion of simple zeros of the Riemann zeta-function. We prove a generalization of his formula for the nontrivial zeros of the Dedekind zeta-function of a Galois number field, and use this formula and Fourier analysis to prove an estimate for the proportion of distinct zeros, assuming the Generalized Riemann Hypothesis.

## ACKNOWLEDGEMENTS

Foremost, I would like to express my sincere gratitude to my advisor Prof. Micah Baruch Milinovich for the continuous support of my master study and research, for his patience, motivation, enthusiasm, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my master study.

Besides my adviser, I would like to thank my family: my parents whose love, guidance, and support are with me in whatever I pursue. I wish to thank my loving and supportive husband, Fahad, and my wonderful son, Faisal, who provide unending inspiration.

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## 1 INTRODUCTION

For  $s \in \mathbb{C}$ , we let  $s = \sigma + it$  where  $\sigma, t \in \mathbb{R}$ . The *Riemann zeta-function* is initially defined as a Dirichlet series over the positive integers and an Euler product over the primes:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \quad (1.1)$$

for  $\sigma > 1$ . The equality, proved by Euler, follows from the Fundamental Theorem of Arithmetic. Riemann proved that  $\zeta(s)$  can be continued analytically to  $\mathbb{C} \setminus \{1\}$  with a simple pole at  $s = 1$ . Riemann also proved that  $\zeta(s)$  satisfies the *functional equation*

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1}{2}(1-s)} \Gamma\left(\frac{1}{2}\right) (1-s) \zeta(1-s). \quad (1.2)$$

From the poles of  $\Gamma(s)$  at  $s = 0, -1, -2, -3, \dots$ , he observed that  $\zeta(s)$  has simple zeros at  $s = -2, -4, -6, \dots$ . These are called the *trivial zeros* of the zeta function. He further noted that  $\zeta(s)$  has infinitely many zeros in the *critical strip*,  $0 \leq \sigma \leq 1$ , which are known as the *non-trivial zeros* of  $\zeta(s)$ . We denote the nontrivial zeros of  $\zeta(s)$  as  $\rho = \beta + i\gamma$ . From the functional equation, if  $\rho$  is a nontrivial zero then so is  $1 - \rho$ . Since  $\overline{\zeta(s)} = \zeta(\bar{s})$ , if  $\rho$  is a nontrivial zero then so is  $\bar{\rho}$ . From this, Riemann observed that the zeros are symmetric about the real axis and about the line  $\sigma = \frac{1}{2}$ . He made the following famous conjecture.

**Riemann Hypothesis.** All nontrivial zeros of  $\zeta(s)$  in the critical strip are on the *critical line*  $\sigma = \frac{1}{2}$ .

Riemann introduced the zeta function as tool to study the prime numbers. Logarithmically differentiating the Euler product for  $\zeta(s)$  we have

$$\frac{d}{ds} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

for  $\sigma > 1$ , where the von Mangoldt function  $\Lambda(n) = \log p$  if  $n = p^k$  for a prime  $p$  and  $k \in \mathbb{N}$  and  $\Lambda(n) = 0$  otherwise. Using Riemann's ideas, in 1896, Hadamard and de la Vallée Poussin independently proved that

$$\sum_{n \leq x} \Lambda(n) \sim x$$

as  $x \rightarrow \infty$  by carefully studying the zeros of  $\zeta(s)$ . The key is to show that  $\zeta(s)$  has no nontrivial zeros on the line  $\sigma = 1$  (so that  $\frac{\zeta'(s)}{\zeta(s)}$  has no poles on the line  $\sigma = 1$ ). This asymptotic formula is equivalent to:

**Theorem 1.1** (Prime Number Theorem). *As  $x \rightarrow \infty$ , we have*

$$\sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

where the sum runs over the primes  $p$ .

This theorem was originally conjectured by Gauss and Legendre. The properties of  $\zeta(s)$  described above and the history of the Prime Number Theorem can be found in Davenport's book [Dav00].

Much effort has gone into studying the nontrivial zeros  $\rho = \beta + i\gamma$  of  $\zeta(s)$ . It is known that

$$N(T) := \sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{1.3}$$

as  $T \rightarrow \infty$ . This was conjectured by Riemann and proved by von Mangoldt [Dav00]. Here the zeros are counted with multiplicity meaning that a zero with multiplicity  $m$  is counted  $m$  times in the sum. In a now famous paper, Montgomery [Mon73] studied the pair correlation



of the zeros of  $\zeta(s)$  assuming the Riemann Hypothesis. We now describe Montgomery's theorem and a corollary. The main goal of my thesis will be to generalize Montgomery's results to the Dedekind zeta-function of a Galois number field.

### 1.1 Montgomery's Theorem.

We assume the Riemann Hypothesis in this section so that the nontrivial zeros can be written  $\rho = \frac{1}{2} + i\gamma$ . From (1.3), the average spacing between consecutive  $\gamma \in (0, T]$  is

$$\approx \frac{\text{length}((0, T])}{\#\gamma \in (0, T]} \approx \frac{T}{\frac{T \log T}{2\pi}} = \frac{2\pi}{\log T}$$

as  $T \rightarrow \infty$ . So the sequence  $\left\{ \gamma \frac{\log T}{2\pi} \right\}$  has average spacing equal to 1 as  $T \rightarrow \infty$ . With this in mind, Montgomery was interested in studying sums like

$$\sum_{0 < \gamma, \gamma' \leq T} R \left( (\gamma - \gamma') \frac{\log T}{2\pi} \right)$$

where  $\gamma$  and  $\gamma'$  run over the imaginary parts of two sets of nontrivial zeros of  $\zeta(s)$ .

Montgomery defined the function

$$F(\alpha) = F(\alpha, T) = \frac{2\pi}{T \log T} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'), \quad \text{where } w(u) = \frac{4}{4 + u^2},$$

where  $\alpha$  and  $T \geq 2$  are real. Here  $\gamma$  and  $\gamma'$  run over the imaginary parts of two sets of nontrivial zeros of  $\zeta(s)$ . He was interested in this function because, for  $R, \hat{R} \in L^1(\mathbb{R})$ , one can show that

$$\sum_{0 < \gamma, \gamma' \leq T} R \left( (\gamma - \gamma') \frac{\log T}{2\pi} \right) w(\gamma - \gamma') = \frac{T \log T}{2\pi} \int_{\mathbb{R}} F(\alpha) \hat{R}(\alpha) d\alpha. \quad (1.4)$$

Here  $\hat{R}$  is the Fourier transform of  $R$  defined as

$$\hat{R}(\alpha) = \int_{\mathbb{R}} R(u) e^{-2\pi i u \alpha} du.$$

Using the definition of the Fourier transform, we will prove the analogue of (1.4) for the zeros of the Dedekind zeta-function in Chapter 2 and the same proof can be used to prove (1.4).

Montgomery [Mon73] proved the following theorem about the function  $F(\alpha)$ .

**Theorem 1.2.** *Assume the Riemann Hypothesis. For real  $\alpha$  and  $T \geq 0$ , we have that  $F(\alpha)$  is real,  $F(\alpha) \geq 0$ , and  $F(-\alpha) = -F(\alpha)$ . For  $\alpha \in [0, 1]$  we have*

$$F(\alpha) = \left(1 + o(1)\right) T^{-2\alpha} \log T + \alpha + o(1)$$

as  $T \rightarrow \infty$ .

We will prove an analogue of this theorem for the zeros of the Dedekind zeta-function. Originally Montgomery proved this theorem for  $\alpha \in (0, 1)$  but it was later extended to  $\alpha \in [0, 1]$  by Goldston and Montgomery [GM87]. Julia Mueller [Mue83] was the first to observe that  $F(\alpha) \geq 0$  for all  $\alpha \in \mathbb{R}$ . We will give a modification of her proof for the zeros of the Dedekind zeta-function in Chapter 2.

The importance of Montgomery's theorem is that we can now estimate the right-hand side of (1.4) for a function  $R \in L^1(\mathbb{R})$  with  $\text{supp}(\hat{R}) \subseteq [-1, 1]$ . With these conditions, for most *nice* functions  $R$ , Montgomery's theorem implies that

$$\sum_{0 < \gamma, \gamma' \leq T} R\left((\gamma - \gamma') \frac{\log T}{2\pi}\right) w(\gamma - \gamma') = \frac{T \log T}{2\pi} \left( \hat{R}(0) + \int_{-1}^1 |\alpha| \hat{R}(\alpha) d\alpha + o(1) \right). \quad (1.5)$$

We will prove a similar formula for the zeros the Dedekind zeta-function.

We state one of the several important corollaries that Montgomery derived from his theorem and (1.5). Let

$$N^s(T) = \#\{0 < \gamma \leq T : \rho = \frac{1}{2} + i\gamma \text{ is a simple zero of } \zeta(s)\}.$$

Choosing the Fourier pair

$$R(u) = \left(\frac{\sin \pi u}{\pi u}\right)^2, \quad \hat{R}(\alpha) = \max(1 - |\alpha|, 0)$$

in (1.5), he proved the following estimate for  $N^s(T)$  which shows that asymptotically at least two thirds of the nontrivial zeros of  $\zeta(s)$  are simple.

**Corollary 1.3.** *Assume the Riemann Hypothesis. Then*

$$N^s(T) \geq \left(\frac{2}{3} + o(1)\right) N(T)$$

as  $T \rightarrow \infty$ .

It was later observed by Montgomery and Taylor [Mon75] and Cheer and Goldston [CG93] that the constant  $2/3$  can be very slightly improved using a more complicated choice of Fourier pair  $R$  and  $\hat{R}$ . We prove a generalization of Corollary 1.3 for the Dedekind zeta-function of a Galois number field  $K$  over  $\mathbb{Q}$  that applies to distinct zeros instead of simple zeros. The answer will depend on the degree  $[K : \mathbb{Q}]$  of the number field.

## 1.2 Properties of the Dedekind zeta-function

Let  $K$  be a number field (a finite extension of  $\mathbb{Q}$ ) where  $m = [K : \mathbb{Q}]$  the degree of  $K$ . We let  $\mathcal{O}_K$  be the ring of integers of  $K$ . The *Dedekind zeta-function* is initially defined as a Dirichlet series over nonzero ideals  $I$  in  $\mathcal{O}_K$  and an Euler product over the prime ideals

$P$  in  $\mathcal{O}_K$ :

$$\zeta_K(s) = \sum_{\substack{I \subset \mathcal{O}_K \\ I \neq 0}} \frac{1}{N(I)^s} = \prod_{P \subset \mathcal{O}_K} \left(1 - \frac{1}{N(P)^s}\right)^{-1}$$

for  $\sigma > 1$ . The equality follows from the fact that  $\mathcal{O}_K$  is a PID, so each  $I \in \mathcal{O}_K$  can be written uniquely as  $I = P_1^{\ell_1} P_2^{\ell_2} \cdots P_k^{\ell_k}$  for prime ideals  $P_1, \dots, P_j$  and  $\ell_i \in \mathbb{N}$ . Hecke proved that  $\zeta_K(s)$  can be continued analytically to  $\mathbb{C} \setminus \{1\}$  with a simple pole at  $s = 1$ , and he calculated the residue (which depends on the algebraic properties of  $K$  and is known as the *class number formula* for  $K$ ). We can also write  $\zeta_K(s)$  as Dirichlet series over the integers:

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{r_K(n)}{n^s}$$

where  $r_K(n) = \#\{I \subset \mathcal{O}_K \mid N(I) = n\}$ , is the number of ideals in  $\mathcal{O}_K$  with norm  $n$ . It is known that  $0 \leq r_K(n) \leq d_m(n)$  with  $d_m(n)$  is the number of ways to write  $n$  as the product of  $m = [K : \mathbb{Q}]$  positive integers.

Hecke also proved that  $\zeta_K(s)$  satisfies the *functional equation*: there exist  $r_1, r_2 \in \mathbb{N}$  with  $r_1 + 2r_2 = m$  such that

$$\pi^{-m\frac{s}{2}} \zeta_K(s) \Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{s+1}{2}\right)^{r_2} = \pi^{-m\frac{1-s}{2}} \zeta_K(1-s) \Gamma\left(\frac{1-s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{1-s+1}{2}\right)^{r_2}. \quad (1.6)$$

Here  $r_1$  is the number of real embeddings of  $K$  and  $r_2$  the number of pairs of complex embeddings so that  $m = r_1 + 2r_2$ . From the poles of  $\Gamma(s)$  at  $s = 0, -1, -2, -3, \dots$ , it can be seen that  $\zeta_K(s)$  has a zero at  $s = 0$  of order  $r_1 + r_2 - 1$ , zeros at  $s = -2, -4, -6, \dots$  of order  $r_1 + r_2$ , and zeros at  $s = -1, -3, -5, \dots$  of order  $r_2$ . These are called the *trivial zeros* of  $\zeta_K(s)$ . The Dedekind zeta-function also has infinitely many zeros in the *critical strip*,  $0 \leq \sigma \leq 1$ , which are known as the *non-trivial zeros* of  $\zeta_K(s)$ . We will denote the nontrivial zeros of  $\zeta_K(s)$  as  $\rho = \beta + i\gamma$ . From the functional equation, if  $\rho$  is a nontrivial zero then so is  $1 - \rho$ . Since  $\overline{\zeta_K(s)} = \zeta_K(\bar{s})$ , if  $\rho$  is a nontrivial zero then so is  $\bar{\rho}$ . Therefore the nontrivial

zeros are symmetric about the real axis and about the line  $\sigma = \frac{1}{2}$ . The analogue of the Riemann Hypothesis is believed to hold for  $\zeta_K(s)$ .

**Generalized Riemann Hypothesis.** All nontrivial zeros of  $\zeta_K(s)$  in the critical strip are on the *critical line*  $\sigma = \frac{1}{2}$ .

Logarithmically differentiating the Euler product, we write

$$\frac{d}{ds} \log \zeta_K(s) = \frac{\zeta'_K(s)}{\zeta_K(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda_K(n)}{n^s},$$

where  $\Lambda_K(n)$  is a generalization of the von Mangoldt function. It follows from the Euler product that  $\Lambda_K(n) = 0$  unless  $n$  is a prime power and also that

$$0 \leq \Lambda_K(n) \leq m\Lambda(n)$$

for all  $n \in \mathbb{N}$ . At  $s = 1$ ,  $\zeta_K(s)$  has a complicated residue but  $\frac{\zeta'_K(s)}{\zeta_K(s)}$  has a simple pole at  $s = 1$  with residue  $-1$ . Landau used this and the fact that  $\zeta_K(s)$  has no nontrivial zeros on the line  $\sigma = 1$  to prove that

$$\sum_{n \leq x} \Lambda_K(n) \sim x,$$

as  $x \rightarrow \infty$ . This asymptotic formula is equivalent to:

**Theorem 1.4** (Landau's Prime Ideal Theorem). *As  $x \rightarrow \infty$ , we have*

$$\sum_{\substack{P \subset \mathcal{O}_K \\ N(P) \leq x}} 1 \sim \frac{x}{\log x}$$

where the sum runs over the prime ideals  $P$  in  $\mathcal{O}_K$  with norm  $N$  less than or equal to  $x$ .

The above properties of the Dedekind zeta-function can be found in Narkiewicz's book [Nar04]. In this thesis, we are interested in studying the nontrivial zeros  $\rho = \beta + i\gamma$  of

$\zeta_K(s)$ . It is known that [IK04]

$$N_K(T) := \sum_{0 < \gamma \leq T} 1 = \frac{mT}{2\pi} \log \frac{T}{2\pi} + O_K(T) \quad (1.7)$$

as  $T \rightarrow \infty$  where  $m = [K : \mathbb{Q}]$ . Assuming the Generalized Riemann Hypothesis so that the nontrivial zeros can be written  $\rho = \frac{1}{2} + i\gamma$ , from (1.7) we see that the average spacing between consecutive  $\gamma \in (0, T]$  is

$$\approx \frac{\text{length}((0, T])}{\#\gamma \in (0, T]} \approx \frac{T}{\frac{mT \log T}{2\pi}} = \frac{2\pi}{m \log T}$$

as  $T \rightarrow \infty$ . So the sequence  $\left\{ \gamma \frac{m \log T}{2\pi} \right\}$  has average spacing equal to 1 as  $T \rightarrow \infty$ . Following Montgomery, we study sums like

$$\sum_{0 < \gamma, \gamma' \leq T} R\left( (\gamma - \gamma') \frac{m \log T}{2\pi} \right)$$

where  $R$  is a function and  $\gamma$  and  $\gamma'$  run over the imaginary parts of the nontrivial zeros of  $\zeta_K(s)$ . For this reason, we make the following definition.

**Definition 1.5.** *Let  $K$  be a number field with  $m = [K : \mathbb{Q}]$ . For any  $\alpha \in \mathbb{R}$  and  $T \geq 2$  we define*

$$F_K(\alpha) = \frac{2\pi}{mT \log T} \sum_{0 \leq \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma')$$

where  $w(u) = \frac{4}{4 + u^2}$  and  $\gamma, \gamma'$  run over the ordinates of the nontrivial zeros of  $\zeta_K(s)$ .

We now state some basic properties of  $F_K(\alpha)$ .

**Proposition 1.6.** *Let  $K$  be a number field. Then we have*

1.  $F_K(\alpha)$  is even which means that  $F_K(-\alpha) = F_K(\alpha)$ .
2.  $F_K(\alpha) \geq 0$  for all  $\alpha \in \mathbb{R}$ .

3. If  $f, \hat{f} \in L^1(\mathbb{R})$  then

$$\sum_{0 \leq \gamma, \gamma' \leq T} f\left((\gamma - \gamma') \frac{m \log T}{2\pi}\right) w(\gamma - \gamma') = \frac{mT \log T}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) F_K(\alpha) d\alpha$$

where  $w(u)$  is the weight function in Definition 1.5 and  $\hat{f}(u) = \int_{-\infty}^{\infty} \hat{f}(x) e^{-2\pi i x u} dx$  denotes the Fourier transform of  $f$ .

If  $K$  is a Galois number field over  $\mathbb{Q}$ , then our analogue of Montgomery's Theorem (Theorem 1.2) for the nontrivial zeros of  $\zeta_K(s)$  is:

**Theorem 1.7.** *Let  $K$  be a Galois number field over  $\mathbb{Q}$  with degree  $m = [K : \mathbb{Q}]$ , and assume the Generalized Riemann Hypothesis for  $\zeta_K(s)$ . For real  $\alpha$  and  $T \geq 0$ , we have that  $F_K(\alpha)$  is real,  $F_K(\alpha) \geq 0$ , and  $F_K(-\alpha) = -F_K(\alpha)$ . For  $\alpha \in (-\frac{1}{m}, \frac{1}{m})$  we have*

$$F_K(\alpha) = m T^{-2m|\alpha|} \log T + m|\alpha| + o(1)$$

as  $T \rightarrow \infty$ .

We now state a corollary about the proportion of distinct nontrivial zeros of  $\zeta_K(s)$ . We count zeros in our sums with multiplicity, meaning that if a zero has multiplicity  $\ell$  then it appears  $\ell$  times in our sequence. We will let  $\mu_\gamma$  be the multiplicity of a  $\frac{1}{2} + i\gamma$  of  $\zeta_K(s)$ . Recall that

$$N_K(T) = \sum_{0 < \gamma \leq T} 1 \sim \frac{mT \log T}{2\pi}$$

and the number of *distinct zeros* of  $\zeta_K(s)$  with  $0 < \gamma \leq T$  is given by

$$N_K^d(T) = \sum_{\substack{0 < \gamma \leq T \\ \gamma \text{ distinct}}} 1 = \sum_{0 < \gamma \leq T} \frac{1}{\mu_\gamma} = \#\{0 < \gamma \leq T : \zeta_K(\frac{1}{2} + i\gamma) = 0\}.$$

We want to use Theorem 1.7 to count the proportion of distinct zeros of  $\zeta_K(s)$  by comparing the ratio of  $N_K^d(T)$  to  $N_K(T)$ . To do this, we define another sum

$$N_K^*(T) = \sum_{0 < \gamma \leq T} \mu_\gamma$$

and we notice that Cauchy's inequality implies that

$$\begin{aligned} N_K(T)^2 &= \left( \sum_{0 < \gamma \leq T} 1 \right)^2 \\ &= \left( \sum_{0 < \gamma \leq T} \frac{1}{\sqrt{\mu_\gamma}} \sqrt{\mu_\gamma} \right)^2 \\ &\leq \sum_{0 < \gamma \leq T} \frac{1}{\mu_\gamma} \sum_{0 < \gamma \leq T} \mu_\gamma \\ &= N_K^d(T) \cdot N_K^*(T). \end{aligned}$$

Therefore

$$N_K^d(T) \geq \frac{N_K(T)^2}{N_K^*(T)} \tag{1.8}$$

and so an upper bound for  $N_K^*(T)$  gives a lower bound for  $N_K^d(T)$ .

**Corollary 1.8.** *Let  $K$  be a Galois number field over  $\mathbb{Q}$  with degree  $m = [K : \mathbb{Q}]$ , and assume the Generalized Riemann Hypothesis for  $\zeta_K(s)$ . Then, as  $T \rightarrow \infty$ ,*

$$N_K^*(T) \leq \left( m + \frac{1}{3} + o(1) \right) N_K(T)$$

and therefore

$$N_K^d(T) \geq \left( \frac{3}{3m+1} + o(1) \right) N_K(T).$$

This shows that, assuming the Generalized Riemann Hypothesis for  $\zeta_K(s)$ , at least a proportion of  $\frac{3}{3m+1}$  of the nontrivial zeros are distinct.



## 2 PROOF OF PROPOSITION 1.6

In this chapter, we use Fourier analysis to prove Proposition 1.6.

*Proof of Proposition 1.6, part 1.* We want to show that  $F_k(-\alpha) = F_k(\alpha)$  for all real numbers  $\alpha$ . We know that

$$F_k(\alpha) = \frac{2\pi}{mT \log(T)} \sum_{0 \leq \gamma, \gamma' \leq T} T^{im\alpha(\gamma - \gamma')} w(\gamma - \gamma').$$

Therefore

$$\begin{aligned} F_k(-\alpha) &= \frac{2\pi}{mT \log(T)} \sum_{0 \leq \gamma, \gamma' \leq T} T^{-im\alpha(\gamma - \gamma')} w(\gamma - \gamma') \\ &= \frac{2\pi}{mT \log(T)} \sum_{0 \leq \gamma, \gamma' \leq T} T^{im\alpha(\gamma' - \gamma)} w(\gamma' - \gamma) \\ &= F_k(\alpha) \end{aligned}$$

since  $w(u)$  is even. □

To prove Proposition 1.6, part 2, we first need some lemmas.

**Lemma 2.1.** *If  $g(u) = e^{-2|u|}$  for  $u \in \mathbb{R}$ , then  $\hat{g}(x) = \frac{4}{4+4\pi^2x^2} = w(2\pi x)$  where  $w(u)$  is the function in Definition 1.5.*

*Proof.* If  $g(u) = e^{-2|u|}$ , then for  $x \in \mathbb{R}$  we have

$$\begin{aligned}
\hat{g}(x) &= \int_{-\infty}^{\infty} e^{-2|u|} e^{-2\pi i u x} du \\
&= \int_0^{\infty} e^{-2u} e^{-2\pi i u x} du + \int_{-\infty}^0 e^{2u} e^{-2\pi i u x} du \\
&= \int_0^{\infty} e^{-u(2+2\pi i x)} du + \int_{-\infty}^0 e^{u(2-2\pi i x)} du \\
&= \frac{1}{2+2\pi i x} - \frac{1}{2-2\pi i x} \\
&= \frac{4}{4+4\pi^2 x^2} \\
&= w(2\pi x),
\end{aligned}$$

as claimed. □

Before stating the next lemma, we define a function related to  $F_K(\alpha)$ :

$$F_K(X, T) = \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} w(\gamma - \gamma'), \quad (2.1)$$

where  $X > 0$ ,  $T \geq 2$ , and  $\gamma, \gamma'$  run over the ordinates of two sets of nontrivial zeros of  $\zeta_K(s)$ .

**Lemma 2.2.** *We have*

$$F_K(X, T) = \int_{-\infty}^{\infty} \left| \sum_{0 < \gamma \leq T} X^{i\gamma} e^{i\gamma u} \right|^2 e^{-2|u|} du,$$

so therefore  $F_K(X, T) \geq 0$ .

*Proof.* Expanding the square

$$\left| \sum_{0 < \gamma \leq T} X^{i\gamma} e^{i\gamma u} \right|^2 = \left( \sum_{0 < \gamma \leq T} X^{i\gamma} e^{i\gamma u} \right) \left( \sum_{0 < \gamma' \leq T} X^{-i\gamma'} e^{-i\gamma' u} \right) = \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} e^{i(\gamma - \gamma')u}.$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left| \sum_{0 < \gamma \leq T} X^{i\gamma} e^{i\gamma u} \right|^2 e^{-2|u|} du &= \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} \int_{-\infty}^{\infty} e^{i(\gamma - \gamma')u} e^{-2|u|} du \\
&= \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} \int_{-\infty}^{\infty} e^{-i2\pi(\frac{\gamma' - \gamma}{2\pi})u} e^{-2|u|} du \\
&= \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} w(\gamma' - \gamma) \\
&= \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} w(\gamma - \gamma') \\
&= F_K(X, T),
\end{aligned}$$

since  $w(u)$  is even. □

*Proof of Proposition 1.6, part 2.* Notice that Lemma 2.2 implies

$$F_K(\alpha) = \frac{mT \log T}{2\pi} F_K(T^{m\alpha}, T) \geq 0.$$

Hence  $F_K(\alpha) \geq 0$  for all  $\alpha \in \mathbb{R}$ , as claimed. □

*Proof of Proposition 1.6, part 3.* Since  $f \in L^1(\mathbb{R})$ , by the Fourier inversion theorem we have

$$\begin{aligned}
\sum_{0 \leq \gamma, \gamma' \leq T} f\left((\gamma - \gamma') \frac{m \log T}{2\pi}\right) w(\gamma - \gamma') &= \sum_{0 \leq \gamma, \gamma' \leq T} \left( \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{2\pi i (\gamma - \gamma') \frac{m \log T}{2\pi} \alpha} d\alpha \right) w(\gamma - \gamma') \\
&= \sum_{0 \leq \gamma, \gamma' \leq T} \left( \int_{-\infty}^{\infty} \hat{f}(\alpha) e^{(\log T) m i \alpha (\gamma - \gamma')} d\alpha \right) w(\gamma - \gamma') \\
&= \sum_{0 \leq \gamma, \gamma' \leq T} \left( \int_{-\infty}^{\infty} \hat{f}(\alpha) T^{i m \alpha (\gamma - \gamma')} d\alpha \right) w(\gamma - \gamma') \\
&= \int_{-\infty}^{\infty} \hat{f}(\alpha) \left( \sum_{0 \leq \gamma, \gamma' \leq T} T^{i m \alpha (\gamma - \gamma')} w(\gamma - \gamma') \right) d\alpha \\
&= \frac{mT \log(T)}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) F_K(\alpha) d\alpha
\end{aligned}$$

as claimed. □

This completes the proof of Proposition 1.6.

### 3 PROOF OF COROLLARY 1.8

In this chapter, we use Theorem 1.7 and Fourier analysis to prove Corollary 1.8. We postpone the proof of Theorem 1.7 until a later chapter. We begin by stating and proving some lemmas.

Assuming the Generalized Riemann Hypothesis for  $\zeta_K(s)$ , Theorem 1.7 states that

$$F_K(\alpha) = \left(1 + o(1)\right) m T^{-2m|\alpha|} \log T + m|\alpha| + o(1)$$

as  $T \rightarrow \infty$  if  $K$  be a Galois number field over  $\mathbb{Q}$  with degree  $m = [K : \mathbb{Q}]$  and  $\alpha \in \left(-\frac{1}{m}, \frac{1}{m}\right)$ . Recall that the Dirac delta function,  $\delta_0(u)$ , satisfies  $\int_{\mathbb{R}} f(u)\delta_0(u) du = f(0)$  for all nice functions  $f$ . The function  $mT^{-2m|\alpha|} \log T$  in Theorem 1.7 acts like the Dirac delta function as we see in the following lemma. This implies that  $F_K(\alpha) \approx m|\alpha| + \delta_0(\alpha)$  for  $\alpha \in \left(-\frac{1}{m}, \frac{1}{m}\right)$ .

**Lemma 3.1.** *Let  $g$  be a even function with  $g^{(n)}$  bounded for  $n = 0, 1, 2$ . Then*

$$\int_{-\infty}^{\infty} g(\alpha) \left(mT^{-2m|\alpha|} \log T\right) d\alpha = g(0) + O\left(\frac{1}{\log T}\right),$$

as  $T \rightarrow \infty$ .

*Proof.* We prove the lemma using integration by parts. Since  $g$  is even

$$\begin{aligned}
\int_{-\infty}^{\infty} g(\alpha) \left( mT^{-2m|\alpha|} \log T \right) d\alpha &= 2m \log T \int_0^{\infty} g(\alpha) T^{-2m\alpha} d\alpha \\
&= 2m \log T \int_0^{\infty} g(\alpha) e^{-2m\alpha \log T} d\alpha \\
&= \left[ -g(\alpha) e^{-2m\alpha \log T} \right]_0^{\infty} + \int_0^{\infty} g'(\alpha) e^{-2m\alpha \log T} d\alpha \\
&= g(0) + \left[ -\frac{g'(\alpha) e^{-2m\alpha \log T}}{2m \log T} \right]_0^{\infty} + \int_0^{\infty} \frac{g''(\alpha) e^{-2m\alpha \log T}}{2m \log T} d\alpha \\
&= g(0) + \left( \frac{g'(0)}{2m \log T} + \int_0^{\infty} \frac{g''(\alpha) e^{-2m\alpha \log T}}{2m \log T} d\alpha \right) \\
&= g(0) + O_K \left( \frac{1}{\log T} \right),
\end{aligned}$$

as claimed. □

**Lemma 3.2.** *Let  $K$  be a Galois number field over  $\mathbb{Q}$  with degree  $m = [K : \mathbb{Q}]$ , and assume the Generalized Riemann Hypothesis for  $\zeta_K(s)$ . Then for even functions  $f \in L^1(\mathbb{R})$  with  $\text{supp}(\hat{f}) \subset (-\frac{1}{m}, \frac{1}{m})$ , we have*

$$\sum_{0 < \gamma, \gamma' \leq T} f\left( (\gamma - \gamma') \frac{m \log T}{2\pi} \right) w(\gamma - \gamma') = \frac{mT \log T}{2\pi} \left( \hat{f}(0) + 2m \int_0^{1/m} \alpha \hat{f}(\alpha) d\alpha + o(1) \right)$$

as  $T \rightarrow \infty$ .

*Proof.* We first use Proposition 1.6, Part 3 and that  $\text{supp}(\hat{f}) \subset (-\frac{1}{m}, \frac{1}{m})$  to see that

$$\begin{aligned}
\sum_{0 \leq \gamma, \gamma' \leq T} f\left( (\gamma - \gamma') \frac{m \log T}{2\pi} \right) w(\gamma - \gamma') &= \frac{mT \log T}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) F_K(\alpha) d\alpha \\
&= \frac{mT \log T}{2\pi} \int_{-1/m}^{1/m} \hat{f}(\alpha) F_K(\alpha) d\alpha
\end{aligned} \tag{3.1}$$

Now we use Theorem 1.7 and Lemma 3.1, to deduce that

$$\begin{aligned}
\int_{-1/m}^{1/m} \hat{f}(\alpha) F_K(\alpha) d\alpha &= \int_{-1/m}^{1/m} \hat{f}(\alpha) (mT^{-2m|\alpha|} \log T + m|\alpha| + o(1)) d\alpha \\
&= \hat{f}(0) + \int_{-1/m}^{1/m} m|\alpha| \hat{f}(\alpha) d\alpha + o(1) \\
&= \hat{f}(0) + 2m \int_0^{1/m} \alpha \hat{f}(\alpha) d\alpha + o(1)
\end{aligned} \tag{3.2}$$

since  $f$  (and  $\hat{f}$ ) is even. Combining equations (3.1) and (3.2), we deduce the theorem.  $\square$

**Lemma 3.3.** *If we let  $f(u) = \left(\frac{\sin(\pi u)}{\pi u}\right)^2$  then  $\hat{f}(v) = \max(1 - |v|, 0)$*

*Proof.* If we let  $\hat{f}(v) = \max(1 - |v|, 0) = \begin{cases} 1 - |v|, & |v| \leq 1, \\ 0, & \text{otherwise,} \end{cases}$

then by the Fourier inversion theorem  $f(u) = \int_{-\infty}^{\infty} \hat{f}(v) e^{2\pi i u v} dv$ . So,

$$\begin{aligned}
f(u) &= \int_{-1}^1 (1 - |v|) e^{2\pi i u v} dv = \int_{-1}^0 (1 + v) e^{2\pi i u v} + \int_0^1 (1 - v) e^{2\pi i u v} \\
&= \frac{2}{4\pi^2 u^2} - \left( \frac{e^{2\pi i u} + e^{-2\pi i u}}{4\pi^2 u^2} \right) \\
&= \left( \frac{\sin(\pi u)}{\pi u} \right)^2,
\end{aligned}$$

as claimed. To see this, note that

$$\begin{aligned}
\left( \frac{\sin(\pi u)}{\pi u} \right)^2 &= \frac{\left( \frac{e^{\pi i u} - e^{-\pi i u}}{2i} \right)^2}{(\pi u)^2} \\
&= \left( \frac{e^{2\pi i u} - 2e^{\pi i u - \pi i u} + e^{-2\pi i u}}{4\pi^2 u^2} \right) \\
&= \frac{2}{4\pi^2 u^2} - \left( \frac{e^{2\pi i u} + e^{-2\pi i u}}{4\pi^2 u^2} \right).
\end{aligned}$$

□

In order to apply Theorem 1.7, we want a function whose Fourier transform is supported on  $(\frac{-1}{m}, \frac{1}{m})$ . Let  $f(u) = \left(\frac{\sin(\pi u)}{\pi u}\right)^2$ , be the function from the previous lemma. If we let  $\hat{h}(v) = \beta \hat{f}(\beta v)$ , then  $\text{supp}(\hat{h}) \subseteq [\frac{-1}{\beta}, \frac{1}{\beta}]$ .

What is  $h$ ? By the Fourier inversion theorem

$$\begin{aligned} h(u) &= \int_{\mathbb{R}} \hat{h}(v) e^{2\pi i u v} dv \\ &= \beta \int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \hat{f}(\beta v) e^{2\pi i u v} dv \\ &= \frac{\beta}{\beta} \int_{-1}^1 \hat{f}(x) e^{2\pi i u (\frac{x}{\beta})} dx \\ &= f\left(\frac{u}{\beta}\right). \end{aligned}$$

Here we used the substitution  $x = \beta v$  in the second integral. So we have proved the following lemma.

**Lemma 3.4.** *If  $h(x) = \left(\frac{\sin \frac{\pi x}{\beta}}{\frac{\pi x}{\beta}}\right)^2$  then  $\hat{h}(v) = \beta \max(1 - |\beta v|, 0)$ .*

We can now prove Corollary 1.8.

*Proof of Corollary 1.8.* Observation: for any  $h \in L^1(\mathbb{R})$  with  $h(0) = 1$  and  $h(x) \geq 0$  for all  $x$ , we have

$$N_K^*(T) = \sum_{0 < \gamma \leq T} \mu_\gamma \leq \sum_{0 < \gamma, \gamma' \leq T} h\left((\gamma - \gamma') \frac{m \log T}{2\pi}\right) w(\gamma - \gamma').$$

To see this inequality, note that  $h(0)w(0) = 1$ , there are  $\mu_\gamma$  terms with  $\gamma = \gamma'$ , and the other terms are positive. We estimate the sum on the right-hand side using Theorem 1.7 and the Fourier pair in Lemma 3.4 with  $\beta > m$ . Note that, for this choice of  $h$ , we have  $h(0) = 1$ ,



$h(x) \geq 0$  for all  $x$ , and  $\text{supp}(\hat{h}) \subseteq [-\frac{1}{\beta}, \frac{1}{\beta}] \subset (-\frac{1}{m}, \frac{1}{m})$ . Since  $\hat{h}(\alpha) = \beta \max(1 - |\beta\alpha|, 0)$

$$\begin{aligned}
\sum_{0 < \gamma, \gamma' \leq T} h\left((\gamma - \gamma') \frac{m \log T}{2\pi}\right) w(\gamma - \gamma') &= \frac{\beta T \log T}{2\pi} \left( \hat{h}(0) + 2\beta \int_0^{1/\beta} \alpha \hat{h}(\alpha) d\alpha + o(1) \right) \\
&= \frac{\beta T \log T}{2\pi} \left( \beta + 2\beta^2 \int_0^{1/\beta} \alpha (1 - \beta\alpha) d\alpha + o(1) \right) \\
&= \frac{\beta T \log T}{2\pi} \left( \beta + 2\beta^2 \int_0^{1/\beta} (\alpha - \beta\alpha^2) d\alpha + o(1) \right) \\
&= \frac{\beta T \log T}{2\pi} \left( \beta + 2\beta^2 \left( \frac{1}{2\beta^2} - \beta \frac{1}{3\beta^3} \right) + o(1) \right) \\
&= \frac{\beta T \log T}{2\pi} \left( \beta + \frac{1}{3} + o(1) \right) \\
&= \left( \beta + \frac{1}{3} + o(1) \right) N_K(T).
\end{aligned}$$

Therefore, letting  $\beta \rightarrow m^+$ , we have

$$N_K^*(T) \leq \left( m + \frac{1}{3} + o(1) \right) N_K(T).$$

This proves the first assertion in Corollary 1.8. To prove the second assertion, we note that the inequality (1.8) gives

$$N_K^d(T) \geq \frac{N_K(T)^2}{N_K^*(T)} \geq \left( \frac{1}{m + \frac{1}{3} + o(1)} \right) N_K(T) = \left( \frac{3}{3m + 1} + o(1) \right) N_K(T).$$

This completes the proof of Corollary 1.8. □

## 4 PAIR CORRELATION FOR THE SELBERG CLASS

In a now well known paper, Selberg [Sel92] introduced an axiomatic class of  $L$ -functions that he conjectured satisfied the Riemann Hypothesis. This is now called the *Selberg class*, we will denote it by  $\mathcal{S}$ . The Dedekind zeta-function of a number field is an element of  $\mathcal{S}$ .

In a subsequent paper, Murty and Perelli [MP99] proved a version of Montgomery's theorem for pairs of zeros of  $L$ -functions in  $\mathcal{S}$  in terms of the coefficients of the Dirichlet series of the logarithmic derivative of elements of  $\mathcal{S}$ . We use their work to prove our Theorem 1.7.

The *Selberg class*  $\mathcal{S}$  is defined by the following axioms.

1. (Dirichlet Series). Every  $L \in \mathcal{S}$  has a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s},$$

absolutely convergent for  $\operatorname{Re}(s) = \sigma > 1$ .

2. (Analytic continuation). There exists a (minimal) integer  $m_L \geq 0$  such that  $(s-1)^{m_L} L(s)$  is an entire function of finite order.
3. (Functional equation).  $L \in \mathcal{S}$  satisfies a functional equation of type

$$\Phi(s) = w \overline{\Phi}(1-s),$$

where

$$\Phi(s) = N^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) L(s)$$

with  $N > 0$ ,  $\lambda_j > 0$ ,  $\operatorname{Re}(\mu_j) \geq 0$ , and  $|w| = 1$ . Here  $\overline{L}(s) = \overline{L(\overline{s})}$ .

4. (Ramanujan hypothesis). For every  $\varepsilon > 0$ ,  $a_L(n) \ll n^\varepsilon$ .

5. (Euler product).  $L \in \mathcal{S}$  satisfies

$$\log L(s) = \sum_{n=1}^{\infty} \frac{b_L(n)}{n^s},$$

where  $b_L(n) = 0$  unless  $n = p^m$  with  $m \geq 1$ , and  $b_L(n) \ll n^\theta$  for some  $\theta < \frac{1}{2}$ .

In addition, we say the *degree*  $d_L$  of  $L \in \mathcal{S}$  is

$$d_L = 2 \sum_{j=1}^r \lambda_j,$$

we write

$$-\frac{L'}{L}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^s}; \quad \Lambda_L(n) = b_L(n) \log n,$$

and we define

$$\psi_L(x) = \sum_{n \leq x} |\Lambda_L(n)|^2.$$

With this notation, Murty and Perelli [MP99] proved the following formula.

**Proposition 4.1.** *Let  $L \in \mathcal{S}$  and assume that  $L$  satisfies the analogue of the Riemann Hypothesis. Let*

$$a_n(x) = \min \left( \left( \frac{n}{x} \right)^{\frac{1}{2}}, \left( \frac{x}{n} \right)^{\frac{3}{2}} \right) \quad \text{and} \quad w(x) = \frac{4}{4 + x^2}.$$

Then

$$\begin{aligned}
2\pi \sum_{|\gamma| \leq T} \sum_{|\gamma'| \leq T} x^{i(\gamma - \gamma')} w(\gamma - \gamma') &= 2d_L^2 \frac{T \log^2 T}{x^2} + \frac{2T}{x} \sum_{n=1}^{\infty} |\Lambda_L(n)|^2 a_n(x)^2 \\
&+ O\left(x \log^2 x + \frac{T \log T \log^{\frac{1}{2}} x}{x} + \left(\frac{T}{x}\right)^{\frac{1}{2}} \log T \log x + \log^3 T\right)
\end{aligned} \tag{4.1}$$

uniformly for  $T \geq x > 1$  where  $\gamma, \gamma'$  run over the ordinates of two sets of nontrivial zeros of  $L(s)$ .

*Proof.* This is equation (30) in Murty and Perelli [MP99]. The proof follows Montgomery's original argument for the zeros of  $\zeta(s)$  in [Mon73].  $\square$

## 5 PROOF OF THEOREM 1.7.

In this section we prove Theorem 1.7. The fact that  $F_K(\alpha)$  is even, real-valued, and non-negative follows from Proposition 1.6. Moreover, it follows from the properties of  $\zeta_K(s)$  in the introduction that  $\zeta_K(s)$  is in the Selberg class. Moreover, the left-hand side of (4.1) equals  $4\pi F_K(x, T)$  where  $F_K(X, T)$  is the function defined in (2.1). Therefore Proposition 4.1 implies that

$$\begin{aligned}
 2\pi F_K(x, T) = & m^2 \frac{T \log^2 T}{x^2} + \frac{T}{x} \sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 \\
 & + O_K \left( x \log^2 x + \frac{T \log T \log^{\frac{1}{2}} x}{x} + \left( \frac{T}{x} \right)^{\frac{1}{2}} \log T \log x + \log^3 T \right)
 \end{aligned} \tag{5.1}$$

We prove Theorem 1.7 by estimating the sum on the right-hand side and then relating  $F_K(x, T)$  to  $F_K(\alpha)$ . We do this using partial summation and the following lemma.

**Lemma 5.1.** *Let  $K$  be a Galois extension of  $\mathbb{Q}$ . Then*

$$\sum_{n \leq x} \Lambda_K(n)^2 = [K : \mathbb{Q}] x \log x + O_K(x)$$

as  $x \rightarrow \infty$ .

*Proof.* This follows from the proof of Lemma 5.2 of Milinovich and Turnage-Butterbaugh [MTB14]. □

**Lemma 5.2.** *Let  $K$  be a Galois extension of  $\mathbb{Q}$  and let  $m = [K : \mathbb{Q}]$ . Then, for  $x \geq 1$ , we have*

$$\sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 = mx \log x + O_K(x).$$

Since

$$a_n(x) = \min \left( \left( \frac{n}{x} \right)^{\frac{1}{2}}, \left( \frac{x}{n} \right)^{\frac{3}{2}} \right),$$

the sum

$$\sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 = \frac{1}{x} \sum_{n \leq x} n \Lambda_K(n)^2 + x^3 \sum_{n > x} \frac{\Lambda_K(n)^2}{n^3}.$$

Using Lemma 5.1 and partial summation, we show that

$$\sum_{n \leq x} n \Lambda_K(n)^2 = \frac{m}{2} x^2 \log x + O_K(x^2) \tag{5.2}$$

and that

$$\sum_{n > x} \frac{\Lambda_K(n)^2}{n^3} = \frac{m \log x}{2x^2} + O_K \left( \frac{1}{x^2} \right). \tag{5.3}$$

This implies that

$$\begin{aligned} \sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 &= \frac{1}{x} \left( \frac{m}{2} x^2 \log x + O_K(x^2) \right) + x^3 \left( \frac{m \log x}{2x^2} + O_K \left( \frac{1}{x^2} \right) \right) \\ &= mx \log x + O_K(x), \end{aligned}$$

as stated in Lemma 5.2.

It remains to prove (5.2) and (5.3).

*Proof of (5.2).* Let  $S(x) = \sum_{n \leq x} \Lambda_K^2(n) = mx \log x + O_K(x)$ . Then by summation by parts, for all  $j \in \mathbb{N}$  we have

$$\begin{aligned}
\sum_{n \leq x} \Lambda_K^2(n) n^j &= t^j S(t) \Big|_1^- - j \int_1^x S(t) t^{j-1} dt \\
&= x^j S(x) - j \int_1^x \left( m t^j \log t + O_K(t^j) \right) dt \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - j m \int_1^x t^j \log t dt + O_K \left( j \int_1^x (t^j) dt \right) \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - j m \left( \frac{t^{j+1}}{j+1} \log t \Big|_1^x - \int_1^x \frac{t^j}{j+1} dt \right) + O_K \left( \frac{j x^{j+1}}{j+1} \right) \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - \frac{j m}{j+1} x^{j+1} \log x + O_K \left( \frac{j m x^{j+1}}{(j+1)^2} \right) + O_K(x^{j+1}) \\
&= m x^{j+1} \left( 1 - \frac{j}{j+1} \right) \log x + O_K(x^{j+1}) \\
&= m x^{j+1} \left( \frac{1}{j+1} \right) \log x + O_K(x^{j+1}) \\
&= \frac{m}{j+1} x^{j+1} \log x + O_K(x^{j+1}).
\end{aligned}$$

Thus, for  $j = 1$ , we have

$$\frac{m}{2} x^2 \log x + O_K(x^2),$$

which proves (5.2). □

*Proof of (5.3).* Again let  $S(x) = \sum_{n \leq x} \Lambda_K^2(n) = mx \log x + O_K(x)$ . Then by summation by parts, for all  $j \geq 2$  we have

$$\begin{aligned}
\sum_{n > x} \Lambda_K^2(n) \frac{1}{n^j} &= \frac{1}{t^j} S(t) \Big|_{x^-}^{\infty} + j \int_x^{\infty} S(t) \frac{1}{t^{j+1}} dt \\
&= \left( \frac{m}{t^{j-1}} \log t + O_K\left(\frac{1}{t^{j-1}}\right) \right) \Big|_{x^-}^{\infty} + j \int_x^{\infty} \left( \frac{m}{t^j} \log t + O_K\left(\frac{1}{t^j}\right) \right) dt \\
&= \frac{m \log x}{x^{j-1}} + O_K\left(\frac{1}{x^{j-1}}\right) + jm \int_x^{\infty} \frac{\log t}{t^j} dt + O_K\left(j \int_x^{\infty} \frac{1}{t^j} dt\right) \\
&= \frac{m \log x}{x^{j-1}} + O_K\left(\frac{1}{x^{j-1}}\right) + jm \left( \frac{1}{1-j} \frac{\log t}{t^{j-1}} \Big|_x^{\infty} + \frac{1}{j-1} \int_x^{\infty} \frac{1}{t^j} dt \right) \\
&\quad + O_K\left(\frac{j}{(j-1)x^{j-1}}\right) \\
&= \frac{m \log x}{x^{j-1}} + O_K\left(\frac{1}{x^{j-1}}\right) + \frac{jm \log x}{j-1 x^{j-1}} + O_K\left(\frac{jm}{(j-1)^2 x^{j-1}}\right) \\
&\quad + O_K\left(\frac{j}{(j-1)x^{j-1}}\right) \\
&= \frac{m}{x^{j-1}} \left(1 - \frac{j}{j-1}\right) \log x + O_K\left(\frac{1}{x^{j-1}}\right) \\
&= \frac{m}{x^{j-1}} \left(\frac{1}{j-1}\right) \log x + O_K\left(\frac{1}{x^{j-1}}\right) \\
&= \frac{m}{j-1} \left(\frac{\log x}{x^{j-1}}\right) + O_K\left(\frac{1}{x^{j-1}}\right).
\end{aligned}$$

Thus, for  $j = 3$ , we have

$$\frac{m \log x}{2 x^2} + O_K\left(\frac{1}{x^2}\right).$$

This completes the proof of (5.3). □

Combining (5.1) and Lemma 5.2, we have

$$\begin{aligned}
2\pi F_K(x, T) &= m^2 \frac{T \log^2 T}{x^2} + mT \log x \\
&\quad + O_K\left(T + x \log^2 x + \frac{T \log T \log^{\frac{1}{2}} x}{x} + \left(\frac{T}{x}\right)^{\frac{1}{2}} \log T \log x\right).
\end{aligned}$$



Setting  $x = T^{m\alpha}$  for  $\alpha \geq 0$  and then dividing by  $mT \log T$ , we derive that

$$F_K(\alpha) = mT^{-2m\alpha} \log T + m\alpha + O_K \left( \frac{1}{\log T} + \alpha^2 T^{m\alpha-1} \log T + \sqrt{\alpha} T^{-m\alpha} \sqrt{\log T} + \alpha T^{-\frac{1}{2}-\frac{1}{2}m\alpha} \log T \right).$$

If we assume that  $0 \leq \alpha < \frac{1}{m}$ , then all the error terms go to zero as  $T \rightarrow \infty$ . Since  $F_K(-\alpha) = F_K(\alpha)$ , we have shown that

$$F_K(\alpha) = mT^{-2m|\alpha|} \log T + m|\alpha| + o(1)$$

for  $\alpha \in (-\frac{1}{m}, \frac{1}{m})$ . This proves Theorem 1.7.

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