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Zeros of the Dedekind Zeta-Function

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ZEROS OF THE DEDEKIND ZETA-FUNCTION

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ABSTRACT

H. L. Montgomery proved a formula for sums over two sets of nontrivial zeros of the Riemann zeta-function. Assuming the Riemann Hypothesis, he used this formula and Fourier analysis to prove an estimate for the proportion of simple zeros of the Riemann zeta-function. We prove a generalization of his formula for the nontrivial zeros of the Dedekind zeta-function of a Galois number field, and use this formula and Fourier analysis to prove an estimate for the proportion of distinct zeros, assuming the Generalized Riemann Hypothesis.

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TABLE OF CONTENTS

1 INTRODUCTION

For $s \in \mathbb{C}$, we let $s = \sigma + it$ where $\sigma, t \in \mathbb{R}$. The *Riemann zeta-function* is initially defined as a Dirichlet series over the positive integers and an Euler product over the primes:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \tag{1.1}$$

for $\sigma > 1$. The equality, proved by Euler, follows from the Fundamental Theorem of Arithmetic. Riemann proved that $\zeta(s)$ can be continued analytically to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. Riemann also proved that $\zeta(s)$ satisfies the *functional equation*

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{1}{2}(1-s)}\Gamma(\tfrac{1}{2})(1-s))\zeta(1-s). \tag{1.2}$$

From the poles of $\Gamma(s)$ at $s = 0, -1, -2, -3, \ldots$, he observed that $\zeta(s)$ has simple zeros at $s = -2, -4, -6, \cdots$. These are called the *trivial zeros* of the zeta function. He further noted that $\zeta(s)$ has infinitely many zeros in the *critical strip*, $0 \le \sigma \le 1$, which are known as the *non-trivial zeros* of $\zeta(s)$. We denote the nontrivial zeros of $\zeta(s)$ as $\rho = \beta + i\gamma$. From the functional equation, if ρ is a nontrivial zero then so is $1 - \rho$. Since $\overline{\zeta(s)} = \zeta(\bar{s})$, if ρ is a nontrivial zero then so is $\bar{\rho}$. From this, Riemann observed that the zeros are symmetric about the real axis and about the line $\sigma = \frac{1}{2}$. He made the following famous conjecture.

Riemann Hypothesis. All nontrivial zeros of $\zeta(s)$ in the critical strip are on the *critical line* $\sigma = \frac{1}{2}$.

Riemann introduced the zeta function as tool to study the prime numbers. Logarithmically differentiating the Euler product for $\zeta(s)$ we have

$$\frac{d}{ds}\log\zeta(s) = \frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}$$

for $\sigma > 1$, where the von Mangoldt function $\Lambda(n) = \log p$ if $n = p^k$ for a prime p and $k \in \mathbb{N}$ and $\Lambda(n) = 0$ otherwise. Using Riemann's ideas, in 1896, Hadamard and de la Vallée Poussin independently proved that

$$\sum_{n\le x}\Lambda(n) \sim x$$

as $x \to \infty$ by carefully studying the zeros of $\zeta(s)$. The key is to show that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ (so that $\frac{\zeta'}{\zeta}(s)$ has no poles on the line $\sigma = 1$). This asymptotic formula is equivalent to:

Theorem 1.1 (Prime Number Theorem)**.** *As $x \to \infty$, we have*

$$\sum_{p\le x} 1 \sim \frac{x}{\log x}$$

where the sum runs over the primes p.

This theorem was originally conjectured by Gauss and Legendre. The properties of $\zeta(s)$ described above and the history of the Prime Number Theorem can be found in Davenport's book [Dav00].

Much effort has gone into studying the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. It is known that

$$N(T) := \sum_{0<\gamma\le T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{1.3}$$

as $T \to \infty$. This was conjectured by Riemann and proved by von Mangoldt [Dav00]. Here the zeros are counted with multiplicity meaning that a zero with multiplicity m is counted m times in the sum. In a now famous paper, Montgomery [Mon73] studied the pair correlation of the zeros of $\zeta(s)$ assuming the Riemann Hypothesis. We now describe Montgomery's theorem and a corollary. The main goal of my thesis will be to generalize Montgomery's results to the Dedekind zeta-function of a Galois number field.

1.1 Montgomery's Theorem.

We assume the Riemann Hypothesis in this section so that the nontrivial zeros can be written $\rho = \frac{1}{2} + i\gamma$. From (1.3), the average spacing between consecutive $\gamma \in (0, T]$ is

$$\approx \frac{\text{length}\big((0,T]\big)}{\#\ \gamma \in (0,T]} \approx \frac{T}{\frac{T\log T}{2\pi}} = \frac{2\pi}{\log T}$$

as $T \to \infty$. So the sequence $\left\{\gamma \frac{\log T}{2\pi}\right\}$ has average spacing equal to 1 as $T \to \infty$. With this in mind, Montgomery was interested in studying sums like

$$\sum_{0<\gamma,\gamma'\leq T} R\left((\gamma-\gamma')\frac{\log T}{2\pi}\right)$$

where γ and γ' run over the imaginary parts of two sets of nontrivial zeros of $\zeta(s)$.

Montgomery defined the function

$$F(\alpha) = F(\alpha, T) = \frac{2\pi}{T\log T}\sum_{0<\gamma,\gamma'\leq T} T^{i\alpha(\gamma-\gamma')}w(\gamma-\gamma'), \quad \text{where} \quad w(u) = \frac{4}{4+u^2},$$

where α and $T \geq 2$ are real. Here γ and γ' run over the imaginary parts of two sets of nontrivial zeros of $\zeta(s)$. He was interested in this function because, for $R, \hat{R} \in L^1(\mathbb{R})$, one can show that

$$\sum_{0<\gamma,\gamma'\leq T} R\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) w(\gamma-\gamma') = \frac{T\log T}{2\pi}\int_{\mathbb{R}} F(\alpha)\,\hat{R}(\alpha)d\alpha. \tag{1.4}$$

Here $\hat{R}$ is the Fourier transform of R defined as

$$\hat{R}(\alpha) = \int_{\mathbb{R}} R(u)\, e^{-2\pi i u \alpha} du.$$

Using the definition of the Fourier transform, we will prove the analogue of (1.4) for the zeros of the Dedekind zeta-function in Chapter 2 and the same proof can be used to prove (1.4).

Montgomery [Mon73] proved the following theorem about the function $F(\alpha)$.

Theorem 1.2. *Assume the Riemann Hypothesis. For real α and $T \geq 0$, we have that $F(\alpha)$ is real, $F(\alpha) \geq 0$, and $F(-\alpha) = -F(\alpha)$. For $\alpha \in [0, 1]$ we have*

$$F(\alpha) = \Big(1 + o(1)\Big) T^{-2\alpha} \log T + \alpha + o(1)$$

as $T \to \infty$.

We will prove an analogue of this theorem for the zeros of the Dedekind zeta-function. Originally Montgomery proved this theorem for $\alpha \in (0, 1)$ but it was later extended to $\alpha \in [0, 1]$ by Goldston and Montgomery [GM87]. Julia Mueller [Mue83] was the first to observe that $F(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$. We will give a modification of her proof for the zeros of the Dedekind zeta-function in Chapter 2.

The importance of Montgomery's theorem is that we can now estimate the right-hand side of (1.4) for a function $R \in L^1(\mathbb{R})$ with $\mathrm{supp}(\hat{R}) \subseteq [-1, 1]$. With these conditions, for most *nice* functions R, Montgomery's theorem implies that

$$\sum_{0 < \gamma, \gamma' \leq T} R\left((\gamma - \gamma')\frac{\log T}{2\pi}\right) w(\gamma - \gamma') = \frac{T \log T}{2\pi} \left(\hat{R}(0) + \int_{-1}^{1} |\alpha| \hat{R}(\alpha) d\alpha + o(1)\right). \quad (1.5)$$

We will prove a similar formula for the zeros the Dedekind zeta-function.

We state one of the several important corollaries that Montgomery derived from his theorem and (1.5). Let

$$N^s(T) = \#\big\{0 < \gamma \le T : \rho = \tfrac{1}{2} + i\gamma \text{ is a simple zero of } \zeta(s)\big\}.$$

Choosing the Fourier pair

$$R(u) = \left(\frac{\sin \pi u}{\pi u}\right)^2, \quad \hat{R}(\alpha) = \max(1 - |\alpha|, 0)$$

in (1.5), he proved the following estimate for $N^s(T)$ which shows that asymptotically at least two thirds of the nontrivial zeros of $\zeta(s)$ are simple.

Corollary 1.3. *Assume the Riemann Hypothesis. Then*

$$N^s(T) \ge \left(\frac{2}{3} + o(1)\right) N(T)$$

as $T \to \infty$.

It was later observed by Montgomery and Taylor [Mon75] and Cheer and Goldston [CG93] that the constant 2/3 can be very slightly improved using a more complicated choice of Fourier pair R and $\hat{R}$. We prove a generalization of Corollary 1.3 for the Dedekind zeta-function of a Galois number field K over $\mathbb{Q}$ that applies to distinct zeros instead of simple zeros. The answer will depend on the degree $[K : \mathbb{Q}]$ of the number field.

1.2 Properties of the Dedekind zeta-function

Let K be a number field (a finite extension of $\mathbb{Q}$) where $m = [K : Q]$ the degree of K. We let $\mathcal{O}_K$ be the ring of integers of K. The *Dedekind zeta-function* is initially defined as a Dirichlet series over nonzero ideals I in $\mathcal{O}_K$ and an Euler product over the prime ideals

P in $\mathcal{O}_K$:

$$\zeta_K(s) = \sum_{\substack{I \subset \mathcal{O}_K \\ I \neq 0}} \frac{1}{N(I)^s} = \prod_{P \subset \mathcal{O}_K} \left(1 - \frac{1}{N(P)^s}\right)^{-1}$$

for $\sigma > 1$. The equality follows from the fact that $\mathcal{O}_K$ is a PID, so each $I \in \mathcal{O}_K$ can be written uniquely as $I = P_1^{\ell_1} P_2^{\ell_2} \cdots P_k^{\ell_k}$ for prime ideals $P_1, ..., P_j$ and $\ell_i \in \mathbb{N}$. Hecke proved that $\zeta_K(s)$ can be continued analytically to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$, and he calculated the residue (which depends on the algebraic properties of K and is known as the *class number formula* for K). We can also write $\zeta_K(s)$ as Dirichlet series over the integers:

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{r_k(n)}{n^s}$$

where $r_K(n) = \#\{I \subset \mathcal{O}_K \mid N(I) = n\}$, is the number of ideals in $\mathcal{O}_K$ with norm n. It is known that $0 \leqslant r_K(n) \leqslant d_m(n)$ with $d_m(n)$ is the number of ways to write n as the product of $m = [K : \mathbb{Q}]$ positive integers.

Hecke also proved that $\zeta_K(s)$ satisfies the *functional equation:* there exist $r_1, r_2 \in \mathbb{N}$ with $r_1 + 2r_2 = m$ such that

$$\pi^{-m\frac{s}{2}} \zeta_K(s) \Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{s+1}{2}\right)^{r_2} = \pi^{-m\frac{1-s}{2}} \zeta_K(1-s) \Gamma\left(\frac{1-s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{1-s+1}{2}\right)^{r_2}. \tag{1.6}$$

Here r_1 is the number of real embeddings of K and r_2 the number of pairs of complex embeddings so that $m = r_1 + 2r_2$. From the poles of $\Gamma(s)$ at $s = 0, -1, -2, -3, \ldots$, it can be seen that $\zeta_K(s)$ has a zero at $s = 0$ of order $r_1 + r_2 - 1$, zeros at $s = -2, -4, -6, \cdots$ of order $r_1 + r_2$, and zeros at $s = -1, -3, -5, \ldots$ of order r_2. These are called the *trivial zeros* of $\zeta_K(s)$. The Dedekind zeta-function also has infinitely many zeros in the *critical strip*, $0 \leq \sigma \leq 1$, which are known as the *non-trivial zeros* of $\zeta_K(s)$. We will denote the nontrivial zeros of $\zeta_K(s)$ as $\rho = \beta + i\gamma$. From the functional equation, if ρ is a nontrivial zero then so is $1 - \rho$. Since $\overline{\zeta_K(s)} = \zeta_K(\bar{s})$, if ρ is a nontrivial zero then so is $\bar{\rho}$. Therefore the nontrivial zeros are symmetric about the real axis and about the line $\sigma = \frac{1}{2}$. The analogue of the Riemann Hypothesis is believed to hold for $\zeta_K(s)$.

Generalized Riemann Hypothesis. All nontrivial zeros of $\zeta_K(s)$ in the critical strip are on the *critical line* $\sigma = \frac{1}{2}$.

Logarithmically differentiating the Euler product, we write

$$\frac{d}{ds}\log \zeta_K(s) = \frac{\zeta_K'}{\zeta_K}(s) = -\sum_{n=1}^{\infty}\frac{\Lambda_K(n)}{n^s},$$

where $\Lambda_K(n)$ is a generalization of the von Mangoldt function. It follows from the Euler product that $\Lambda_K(n) = 0$ unless n is a prime power and also that

$$0 \leq \Lambda_K(n) \leq m\Lambda(n)$$

for all $n \in \mathbb{N}$. At $s = 1$, $\zeta_K(s)$ has a complicated residue but $\frac{\zeta_K'(s)}{\zeta_K(s)}$ has a simple pole at $s = 1$ with residue -1. Landau used this and the fact that $\zeta_K(s)$ has no nontrivial zeros on the line $\sigma = 1$ to prove that

$$\sum_{n \leq x} \Lambda_K(n) \sim x,$$

as $x \to \infty$. This asymptotic formula is equivalent to:

Theorem 1.4 (Landau's Prime Ideal Theorem)**.** *As $x \to \infty$, we have*

$$\sum_{\substack{P \subset \mathcal{O}_K \\ N(P) \leq x}} 1 \sim \frac{x}{\log x}$$

where the sum runs over the prime ideals P in $\mathcal{O}_K$ with norm N less than or equal to x.

The above properties of the Dedekind zeta-function can be found in Narkiewicz's book [Nar04]. In this thesis, we are interested in studying the nontrivial zeros $\rho = \beta + i\gamma$ of

$\zeta_K(s)$. It is known that [IK04]

$$N_K(T) := \sum_{0<\gamma\le T} 1 = \frac{mT}{2\pi}\log\frac{T}{2\pi} + O_K(T) \tag{1.7}$$

as $T \to \infty$ where $m = [K : \mathbb{Q}]$. Assuming the Generalized Riemann Hypothesis so that the nontrivial zeros can be written $\rho = \frac{1}{2} + i\gamma$, from (1.7) we see that the average spacing between consecutive $\gamma \in (0, T]$ is

$$\approx \frac{\text{length}\big((0,T]\big)}{\#\ \gamma \in (0,T]} \approx \frac{T}{\frac{mT\log T}{2\pi}} = \frac{2\pi}{m\log T}$$

as $T \to \infty$. So the sequence $\left\{\gamma\frac{m\log T}{2\pi}\right\}$ has average spacing equal to 1 as $T \to \infty$. Following Montgomery, we study sums like

$$\sum_{0<\gamma,\gamma'\le T} R\left((\gamma-\gamma')\frac{m\log T}{2\pi}\right)$$

where R is a function and γ and γ' run over the imaginary parts of the nontrivial zeros of $\zeta_K(s)$. For this reason, we make the following definition.

Definition 1.5. *Let K be a number field with $m = [K : \mathbb{Q}]$. For any $\alpha \in \mathbb{R}$ and $T \ge 2$ we define*

$$F_K(\alpha) = \frac{2\pi}{mT\log T}\sum_{0\le\gamma,\gamma'\le T} T^{im\alpha(\gamma-\gamma')}w(\gamma-\gamma')$$

where $w(u) = \dfrac{4}{4+u^2}$ and γ, γ' run over the ordinates of the nontrivial zeros of $\zeta_K(s)$.

We now state some basic properties of $F_K(\alpha)$.

Proposition 1.6. *Let K be a number field. Then we have*

1. *$F_K(\alpha)$ is even which means that $F_k(-\alpha) = F_k(\alpha)$.*

2. *$F_K(\alpha) \ge 0$ for all $\alpha \in \mathbb{R}$.*

3. If $f, \hat{f} \in L^1(\mathbb{R})$ then

$$\sum_{0 \le \gamma, \gamma' \le T} f\Big((\gamma - \gamma')\frac{m \log T}{2\pi}\Big) w(\gamma - \gamma') = \frac{mT \log T}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) F_K(\alpha)\ d\alpha$$

where $w(u)$ is the weight function in Definition 1.5 and $\hat{f}(u) = \int_{-\infty}^{\infty} \hat{f}(x) e^{-2\pi i x u} dx$ denotes the Fourier transform of f.

If K is a Galois number field over $\mathbb{Q}$, then our analogue of Montgomery's Theorem (Theorem 1.2) for the nontrivial zeros of $\zeta_K(s)$ is:

Theorem 1.7. *Let K be a Galois number field over $\mathbb{Q}$ with degree $m = [K : \mathbb{Q}]$, and assume the Generalized Riemann Hypothesis for $\zeta_K(s)$. For real α and $T \ge 0$, we have that $F_K(\alpha)$ is real, $F_K(\alpha) \ge 0$, and $F_K(-\alpha) = -F_K(\alpha)$. For $\alpha \in (-\frac{1}{m}, \frac{1}{m})$ we have*

$$F_K(\alpha) = m\, T^{-2m|\alpha|} \log T + m|\alpha| + o(1)$$

as $T \to \infty$.

We now state a corollary about the proportion of distinct nontrivial zeros of $\zeta_K(s)$. We count zeros in our sums with multiplicity, meaning that if a zero has multiplicity ℓ then it appears ℓ times in our sequence. We will let μ_γ be the multiplicity of a $\frac{1}{2} + i\gamma$ of $\zeta_K(s)$. Recall that

$$N_K(T) = \sum_{0 < \gamma \le T} 1 \sim \frac{mT \log T}{2\pi}$$

and the number of *distinct zeros* of $\zeta_K(s)$ with $0 < \gamma \le T$ is given by

$$N_K^d(T) = \sum_{\substack{0 < \gamma \le T \\ \gamma \text{ distinct}}} 1 = \sum_{0 < \gamma \le T} \frac{1}{\mu_\gamma} = \#\{0 < \gamma \le T : \zeta_K(\tfrac{1}{2} + i\gamma) = 0\}.$$

We want to use Theorem 1.7 to count the proportion of distinct zeros of $\zeta_K(s)$ by comparing the ratio of $N_K^d(T)$ to $N_K(T)$. To do this, we define another sum

$$N_K^*(T) = \sum_{0<\gamma\leq T} \mu_\gamma$$

and we notice that Cauchy's inequality implies that

$$\begin{aligned} N_K(T)^2 &= \left(\sum_{0<\gamma\leq T} 1 \right)^2 \\ &= \left(\sum_{0<\gamma\leq T} \frac{1}{\sqrt{\mu_\gamma}} \sqrt{\mu_\gamma} \right)^2 \\ &\leq \sum_{0<\gamma\leq T} \frac{1}{\mu_\gamma} \sum_{0<\gamma\leq T} \mu_\gamma \\ &= N_K^d(T) \cdot N_K^*(T). \end{aligned}$$

Therefore

$$N_K^d(T) \geq \frac{N_K(T)^2}{N_K^*(T)} \tag{1.8}$$

and so an upper bound for $N_K^*(T)$ gives a lower bound for $N_K^d(T)$.

Corollary 1.8. *Let K be a Galois number field over $\mathbb{Q}$ with degree $m = [K:\mathbb{Q}]$, and assume the Generalized Riemann Hypothesis for $\zeta_K(s)$. Then, as $T \to \infty$,*

$$N_K^*(T) \leq \left(m + \frac{1}{3} + o(1) \right) N_K(T)$$

and therefore

$$N_K^d(T) \geq \left(\frac{3}{3m+1} + o(1) \right) N_K(T).$$

This shows that, assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, at least a proportion of $\frac{3}{3m+1}$ of the nontrivial zeros are distinct.

2 PROOF OF PROPOSITION 1.6

In this chapter, we use Fourier analysis to prove Proposition 1.6.

Proof of Proposition 1.6, part 1. We want to show that $F_k(-\alpha) = F_k(\alpha)$ for all real numbers α. We know that

$$F_k(\alpha) = \frac{2\pi}{mT\log(T)} \sum_{0\le\gamma,\gamma'\le T} T^{im\alpha(\gamma-\gamma')} w(\gamma-\gamma').$$

Therefore

$$\begin{aligned} F_k(-\alpha) &= \frac{2\pi}{mT\log(T)} \sum_{0\le\gamma,\gamma'\le T} T^{-im\alpha(\gamma-\gamma')} w(\gamma-\gamma') \\ &= \frac{2\pi}{mT\log(T)} \sum_{0\le\gamma,\gamma'\le T} T^{im\alpha(\gamma'-\gamma)} w(\gamma'-\gamma) \\ &= F_k(\alpha) \end{aligned}$$

since $w(u)$ is even. $\square$

To prove Proposition 1.6, part 2, we first need some lemmas.

Lemma 2.1. *If $g(u) = e^{-2|u|}$ for $u \in \mathbb{R}$, then $\hat{g}(x) = \frac{4}{4+4\pi^2x^2} = w(2\pi x)$ where $w(u)$ is the function in Definition 1.5.*

Proof. If $g(u) = e^{-2|u|}$, then for $x \in \mathbb{R}$ we have

$$\begin{aligned}
\hat{g}(x) &= \int_{-\infty}^{\infty} e^{-2|u|} e^{-2\pi iux} du \\
&= \int_{0}^{\infty} e^{-2u} e^{-2\pi iux} du + \int_{-\infty}^{o} e^{2u} e^{-2\pi iux} du \\
&= \int_{0}^{\infty} e^{-u(2+2\pi iux)} du + \int_{-\infty}^{o} e^{u(2-2\pi iux)} du \\
&= \frac{1}{2+2\pi ix} - \frac{1}{2-2\pi ix} \\
&= \frac{4}{4+4\pi^2 x^2} \\
&= w(2\pi x),
\end{aligned}$$

as claimed. □

Before stating the next lemma, we define a function related to $F_K(\alpha)$:

$$F_K(X,T) = \sum_{0<\gamma,\gamma'\leq T} X^{i(\gamma-\gamma')} w(\gamma - \gamma'), \tag{2.1}$$

where $X > 0$, $T \geq 2$, and γ, γ' run over the ordinates of two sets of nontrivial zeros of $\zeta_K(s)$.

Lemma 2.2. *We have*

$$F_K(X,T) = \int_{-\infty}^{\infty} \left| \sum_{0<\gamma\leq T} X^{i\gamma} e^{i\gamma u} \right|^2 e^{-2|u|} du,$$

so therefore $F_K(X,T) \geq 0$.

Proof. Expanding the square

$$\left| \sum_{0<\gamma\leq T} X^{i\gamma} e^{i\gamma u} \right|^2 = \left(\sum_{0<\gamma\leq T} X^{i\gamma} e^{i\gamma u} \right) \left(\sum_{0<\gamma'\leq T} X^{-i\gamma'} e^{-i\gamma' u} \right) = \sum_{0<\gamma,\gamma'\leq T} X^{i(\gamma-\gamma')} e^{i(\gamma-\gamma')u}.$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left| \sum_{0<\gamma\le T} X^{i\gamma} e^{i\gamma u} \right|^2 e^{-2|u|} du &= \sum_{0<\gamma,\gamma'\le T} X^{i(\gamma-\gamma')} \int_{-\infty}^{\infty} e^{i(\gamma-\gamma')u} e^{-2|u|} du \\
&= \sum_{0<\gamma,\gamma'\le T} X^{i(\gamma-\gamma')} \int_{-\infty}^{\infty} e^{-i2\pi(\frac{\gamma'-\gamma}{2\pi})u} e^{-2|u|} du \\
&= \sum_{0<\gamma,\gamma'\le T} X^{i(\gamma-\gamma')} w(\gamma'-\gamma) \\
&= \sum_{0<\gamma,\gamma'\le T} X^{i(\gamma-\gamma')} w(\gamma-\gamma') \\
&= F_K(X,T),
\end{aligned}$$

since $w(u)$ is even. □

Proof of Proposition 1.6, part 2. Notice that Lemma 2.2 implies

$$F_K(\alpha) = \frac{mT\log T}{2\pi} F_K(T^{m\alpha}, T) \ge 0.$$

Hence $F_K(\alpha) \ge 0$ for all $\alpha \in \mathbb{R}$, as claimed. □

Proof of Proposition 1.6, part 3. Since $f \in L^1(\mathbb{R})$, by the Fourier inversion theorem we have

$$\begin{aligned}
\sum_{0\le\gamma,\gamma'\le T} f\left((\gamma-\gamma')\frac{m\log T}{2\pi}\right) w(\gamma-\gamma') &= \sum_{0\le\gamma,\gamma'\le T} \left(\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{2\pi i(\gamma-\gamma')\frac{m\log T}{2\pi}\alpha} d\alpha\right) w(\gamma-\gamma') \\
&= \sum_{0\le\gamma,\gamma'\le T} \left(\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{(\log T)^{mi\alpha(\gamma-\gamma')}} d\alpha\right) w(\gamma-\gamma') \\
&= \sum_{0\le\gamma,\gamma'\le T} \left(\int_{-\infty}^{\infty} \hat{f}(\alpha) T^{im\alpha(\gamma-\gamma')} d\alpha\right) w(\gamma-\gamma') \\
&= \int_{-\infty}^{\infty} \hat{f}(\alpha) \left(\sum_{0\le\gamma,\gamma'\le T} T^{im\alpha(\gamma-\gamma')} w(\gamma-\gamma')\right) d\alpha \\
&= \frac{mT\log(T)}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) F_K(\alpha) d\alpha
\end{aligned}$$

as claimed. $\square$

This completes the proof of Proposition 1.6.

3 PROOF OF COROLLARY 1.8

In this chapter, we use Theorem 1.7 and Fourier analysis to prove Corollary 1.8. We postpone the proof of Theorem 1.7 until a later chapter. We begin by stating and proving some lemmas.

Assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, Theorem 1.7 states that

$$F_K(\alpha) = \Big(1 + o(1)\Big)\, m\, T^{-2m|\alpha|} \log T + m|\alpha| + o(1)$$

as $T \to \infty$ if K be a Galois number field over $\mathbb{Q}$ with degree $m = [K : \mathbb{Q}]$ and $\alpha \in (-\frac{1}{m}, \frac{1}{m})$. Recall that the Dirac delta function, $\delta_0(u)$, satisfies $\int\limits_{\mathbb{R}} f(u)\delta_0(u)\ du = f(0)$ for all nice functions f. The function $mT^{-2m|\alpha|} \log T$ in Theorem 1.7 acts like the Dirac delta function as we see in the following lemma. This implies that $F_K(\alpha) \approx m|\alpha| + \delta_0(\alpha)$ for $\alpha \in \big(-\frac{1}{m}, \frac{1}{m}\big)$.

Lemma 3.1. *Let g be a even function with $g^{(n)}$ bounded for $n = 0, 1, 2$. Then*

$$\int_{-\infty}^{\infty} g(\alpha)\Big(mT^{-2m|\alpha|} \log T\Big) d\alpha = g(0) + O\Big(\frac{1}{\log T}\Big),$$

as $T \to \infty$.

Proof. We prove the lemma using integration by parts. Since g is even

$$\begin{aligned}
\int_{-\infty}^{\infty} g(\alpha)\Big(mT^{-2m|\alpha|}\log T\Big)d\alpha &= 2m\log T\int_0^\infty g(\alpha)T^{-2m\alpha}d\alpha \\
&= 2m\log T\int_0^\infty g(\alpha)e^{-2m\alpha\log T}d\alpha \\
&= \Big[-g(\alpha)e^{-2m\alpha\log T}\Big]\Big|_0^\infty + \int_0^\infty g'(\alpha)e^{-2m\alpha\log T}d\alpha \\
&= g(0) + \left[-\frac{g'(\alpha)e^{-2m\alpha\log T}}{2m\log T}\right]\Bigg|_0^\infty + \int_0^\infty \frac{g''(\alpha)e^{-2m\alpha\log T}}{2m\log T}d\alpha \\
&= g(0) + \left(\frac{g'(0)}{2m\log T} + \int_0^\infty \frac{g''(\alpha)e^{-2m\alpha\log T}}{2m\log T}d\alpha\right) \\
&= g(0) + O_K\left(\frac{1}{\log T}\right),
\end{aligned}$$

as claimed. □

Lemma 3.2. *Let K be a Galois number field over $\mathbb{Q}$ with degree $m = [K:\mathbb{Q}]$, and assume the Generalized Riemann Hypothesis for $\zeta_K(s)$. Then for even functions $f \in L^1(\mathbb{R})$ with supp$(\hat{f}) \subset (-\frac{1}{m}, \frac{1}{m})$, we have*

$$\sum_{0<\gamma,\gamma'\le T} f\Big((\gamma-\gamma')\frac{m\log T}{2\pi}\Big)w(\gamma-\gamma') = \frac{mT\log T}{2\pi}\left(\hat{f}(0) + 2m\int_0^{1/m}\alpha\,\hat{f}(\alpha)d\alpha + o(1)\right)$$

as $T \to \infty$.

Proof. We first use Proposition 1.6, Part 3 and that supp$(\hat{f}) \subset (-\frac{1}{m}, \frac{1}{m})$ to see that

$$\begin{aligned}
\sum_{0\le\gamma,\gamma'\le T} f\Big((\gamma-\gamma')\frac{m\log T}{2\pi}\Big)w(\gamma-\gamma') &= \frac{mT\log T}{2\pi}\int_{-\infty}^{\infty}\hat{f}(\alpha)F_K(\alpha)\,d\alpha \\
&= \frac{mT\log T}{2\pi}\int_{-1/m}^{1/m}\hat{f}(\alpha)F_K(\alpha)\,d\alpha
\end{aligned} \tag{3.1}$$

Now we use Theorem 1.7 and Lemma 3.1, to deduce that

$$\begin{aligned}
\int_{-1/m}^{1/m} \hat{f}(\alpha)F_K(\alpha)\ d\alpha &= \int_{-1/m}^{1/m} \hat{f}(\alpha)\left(mT^{-2m|\alpha|}\log T + m|\alpha| + o(1)\right)\ d\alpha \\
&= \hat{f}(0) + \int_{-1/m}^{1/m} m|\alpha|\hat{f}(\alpha)\ d\alpha + o(1) \qquad (3.2) \\
&= \hat{f}(0) + 2m\int_{0}^{1/m} \alpha\hat{f}(\alpha)\ d\alpha + o(1)
\end{aligned}$$

since f (and $\hat{f}$) is even. Combining equations (3.1) and (3.2), we deduce the theorem. $\square$

Lemma 3.3. *If we let* $f(u) = \left(\frac{sin(\pi u)}{\pi u}\right)^2$ *then* $\hat{f}(v) = \max(1-|v|, 0)$

Proof. If we let $\hat{f}(v) = \max(1-|v|,0) = \begin{cases} 1-|v|, & |v| \le 1, \\ 0, & otherwise, \end{cases}$

then by the Fourier inversion theorem $f(u) = \int_{-\infty}^{\infty} \hat{f}(v)e^{2\pi iuv}dv$. So,

$$\begin{aligned}
f(u) = \int_{-1}^{1}(1-|v|)e^{2\pi iuv}dv = \int_{-1}^{0}(1+v)e^{2\pi iuv} &+ \int_{0}^{1}(1-v)e^{2\pi iuv} \\
&= \frac{2}{4\pi^2u^2} - \left(\frac{e^{2\pi iu}+e^{-2\pi iu}}{4\pi^2u^2}\right) \\
&= \left(\frac{\sin(\pi u)}{\pi u}\right)^2,
\end{aligned}$$

as claimed. To see this, note that

$$\begin{aligned}
\left(\frac{\sin(\pi u)}{\pi u}\right)^2 &= \frac{\left(\frac{e^{\pi iu}-e^{-\pi iu}}{2i}\right)^2}{(\pi u)^2} \\
&= \left(\frac{e^{2\pi iu} - 2e^{\pi iu-\pi iu} + e^{-2\pi iu}}{4\pi^2u^2}\right) \\
&= \frac{2}{4\pi^2u^2} - \left(\frac{e^{2\pi iu}+e^{-2\pi iu}}{4\pi^2u^2}\right).
\end{aligned}$$

$\square$

In order to apply Theorem 1.7, we want a function whose Fourier transform is supported on $(\frac{-1}{m}, \frac{1}{m})$. Let $f(u) = \left(\frac{\sin(\pi u)}{\pi u}\right)^2$, be the function from the previous lemma. If we let $\hat{h}(v) = \beta \hat{f}(\beta v)$, then $\text{supp}(\hat{h}) \subseteq [\frac{-1}{\beta}, \frac{1}{\beta}]$.

What is h? By the Fourier inversion theorem

$$\begin{aligned} h(u) &= \int_{\mathbb{R}} \hat{h}(v) e^{2\pi i u v} dv \\ &= \beta \int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \hat{f}(\beta v) e^{2\pi i u v} dv \\ &= \frac{\beta}{\beta} \int_{-1}^{1} \hat{f}(x) e^{2\pi i u (\frac{x}{\beta})} dx \\ &= f\left(\frac{u}{\beta}\right). \end{aligned}$$

Here we used the substitution $x = \beta v$ in the second integral. So we have proved the following lemma.

Lemma 3.4. *If* $h(x) = \left(\frac{\sin \frac{\pi x}{\beta}}{\frac{\pi x}{\beta}}\right)^2$ *then* $\hat{h}(v) = \beta \max(1 - |\beta v|, 0)$.

We can now prove Corollary 1.8.

Proof of Corollary 1.8. Observation: for any $h \in L^1(\mathbb{R})$ with $h(0) = 1$ and $h(x) \geq 0$ for all x, we have

$$N_K^*(T) = \sum_{0<\gamma\leq T} \mu_\gamma \leq \sum_{0<\gamma,\gamma'\leq T} h\Big((\gamma - \gamma')\frac{m \log T}{2\pi}\Big) w(\gamma - \gamma').$$

To see this inequality, note that $h(0)w(0) = 1$, there are μ_γ terms with $\gamma = \gamma'$, and the other terms are positive. We estimate the sum on the right-hand side using Theorem 1.7 and the Fourier pair in Lemma 3.4 with $\beta > m$. Note that, for this choice of h, we have $h(0) = 1$,

$h(x) \geq 0$ for all x, and $\mathrm{supp}(\hat{h}) \subseteq [\frac{-1}{\beta}, \frac{1}{\beta}] \subset (\frac{-1}{m}, \frac{1}{m})$. Since $\hat{h}(\alpha) = \beta \max(1 - |\beta\alpha|, 0)$

$$\begin{aligned}
\sum_{0<\gamma,\gamma'\leq T} h\Big((\gamma-\gamma')\frac{m\log T}{2\pi}\Big) w(\gamma-\gamma') &= \frac{\beta T\log T}{2\pi}\left(\hat{h}(0) + 2\beta\int_0^{1/\beta} \alpha\,\hat{h}(\alpha)d\alpha + o(1)\right)\\
&= \frac{\beta T\log T}{2\pi}\left(\beta + 2\beta^2\int_0^{1/\beta} \alpha\,(1-\beta\alpha)d\alpha + o(1)\right)\\
&= \frac{\beta T\log T}{2\pi}\left(\beta + 2\beta^2\int_0^{1/\beta} (\alpha-\beta\alpha^2)d\alpha + o(1)\right)\\
&= \frac{\beta T\log T}{2\pi}\left(\beta + 2\beta^2\Big(\frac{1}{2\beta^2} - \beta\frac{1}{3\beta^3}\Big) + o(1)\right)\\
&= \frac{\beta T\log T}{2\pi}\left(\beta + \frac{1}{3} + o(1)\right)\\
&= \left(\beta + \frac{1}{3} + o(1)\right) N_K(T).
\end{aligned}$$

Therefore, letting $\beta \to m^+$, we have

$$N_K^*(T) \leq \left(m + \frac{1}{3} + o(1)\right) N_K(T).$$

This proves the first assertion in Corollary 1.8. To prove the second assertion, we note that the inequality (1.8) gives

$$N_K^d(T) \geq \frac{N_K(T)^2}{N_K^*(T)} \geq \Big(\frac{1}{m+\frac{1}{3}+o(1)}\Big)N_K(T) = \left(\frac{3}{3m+1} + o(1)\right) N_K(T).$$

This completes the proof of Corollary 1.8. $\square$

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4 PAIR CORRELATION FOR THE SELBERG CLASS

In a now well known paper, Selberg [Sel92] introduced an axiomatic class of L-functions that he conjectured satisfied the Riemann Hypothesis. This is now called the *Selberg class*, we will denote it by $\mathcal{S}$. The Dedekind zeta-function of a number field is an element of $\mathcal{S}$.

In a subsequent paper, Murty and Perelli [MP99] proved a version of Montgomery's theorem for pairs of zeros of L-functions in $\mathcal{S}$ in terms of the coefficients of the Dirichlet series of the logarithmic derivative of elements of $\mathcal{S}$. We use their work to prove our Theorem 1.7.

The *Selberg class* $\mathcal{S}$ is defined by the following axioms.

1. (Dirichlet Series). Every $L \in \mathcal{S}$ has a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s},$$

absolutely convergent for $Re(s) = \sigma > 1$.

2. (Analytic continuation). There exists a (minimal) integer $m_L \geq 0$ such that $(s-1)^{m_L} L(s)$ is an entire function of finite order.

3. (Functional equation). $L \in \mathcal{S}$ satisfies a functional equation of type

$$\Phi(s) = w\,\overline{\Phi}(1-s),$$

where

$$\Phi(s) = N^s \prod_{j=1}^{r} \Gamma(\lambda_j\, s + \mu_j)\, L(s)$$

with $N > 0, \lambda_j > 0, Re(\mu_j) \geq 0$, and $|w| = 1$. Here $\overline{L}(s) = \overline{L(\overline{s})}$.

4. (Ramanujan hypothesis). For every $\varepsilon > 0, a_L(n) \ll n^{\varepsilon}$.

5. (Euler product). $L \in \mathcal{S}$ satisfies

$$\log\, L(s) = \sum_{n=1}^{\infty} \frac{b_L(n)}{n^s},$$

where $b_L(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_L(n) \ll n^{\theta}$ for some $\theta < \frac{1}{2}$.

.

In addition, we say the *degree* d_L of $L \in \mathcal{S}$ is

$$d_L = 2 \sum_{j=1}^{r} \lambda_j,$$

we write

$$-\frac{L'}{L}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^s}; \qquad \Lambda_L(n) = b_L(n) \log n,$$

and we define

$$\psi_L(x) = \sum_{n \leq x} |\Lambda_L(n)|^2.$$

With this notation, Murty and Perelli [MP99] proved the following formula.

Proposition 4.1. *Let $L \in \mathcal{S}$ and assume that L satisfies the analogue of the Riemann Hypothesis. Let*

$$a_n(x) = \min\left(\left(\frac{n}{x}\right)^{\frac{1}{2}}, \left(\frac{x}{n}\right)^{\frac{3}{2}} \right) \quad \textit{and} \quad w(x) = \frac{4}{4 + x^2}.$$

Then

$$2\pi \sum_{|\gamma|\le T} \sum_{|\gamma'|\le T} x^{i(\gamma-\gamma')} w(\gamma-\gamma') \;=\; 2\, d_L^2\, \frac{T\log^2 T}{x^2} + \frac{2T}{x} \sum_{n=1}^{\infty} |\Lambda_L(n)|^2\, a_n(x)^2 + O\left(x\, \log^2 x + \frac{T\, \log T\, \log^{\frac{1}{2}} x}{x} + \left(\frac{T}{x}\right)^{\frac{1}{2}} \log T\, \log x + \log^3 T \right) \tag{4.1}$$

uniformly for $T \ge x > 1$ where γ, γ' run over the ordinates of two sets of nontrivial zeros of $L(s)$.

Proof. This is equation (30) in Murty and Perelli [MP99]. The proof follows Montgomery's original argument for the zeros of $\zeta(s)$ in [Mon73]. $\square$

5 PROOF OF THEOREM 1.7.

In this section we prove Theorem 1.7. The fact that $F_K(\alpha)$ is even, real-valued, and non-negative follows from Proposition 1.6. Moreover, it follows from the properties of $\zeta_K(s)$ in the introduction that $\zeta_K(s)$ is in the Selberg class. Moreover, the left-hand side of (4.1) equals $4\pi F_K(x,T)$ where $F_K(X,T)$ is the function defined in (2.1). Therefore Proposition 4.1 implies that

$$\begin{aligned} 2\pi F_K(x,T) = &= m^2 \frac{T\log^2 T}{x^2} + \frac{T}{x}\sum_{n=1}^{\infty} \Lambda_K(n)^2 \, a_n(x)^2 \\ &+ O_K\left(x \log^2 x + \frac{T \log T \log^{\frac{1}{2}} x}{x} + \left(\frac{T}{x}\right)^{\frac{1}{2}} \log T \log x + \log^3 T \right) \end{aligned} \tag{5.1}$$

We prove Theorem 1.7 by estimating the sum on the right-hand side and then relating $F_K(x,T)$ to $F_K(\alpha)$. We do this using partial summation and the following lemma.

Lemma 5.1. *Let K be a Galois extension of $\mathbb{Q}$. Then*

$$\sum_{n\le x} \Lambda_K(n)^2 = [K:\mathbb{Q}]\, x\log x + O_K(x)$$

as $x \to \infty$.

Proof. This follows from the proof of Lemma 5.2 of Milinovich and Turnage-Butterbaugh [MTB14]. $\square$

Lemma 5.2. *Let K be a Galois extension of $\mathbb{Q}$ and let $m = [K : \mathbb{Q}]$. Then, for $x \geq 1$, we have*

$$\sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 = mx \log x + O_K(x).$$

Since

$$a_n(x) = \min\left(\left(\frac{n}{x}\right)^{\frac{1}{2}}, \left(\frac{x}{n}\right)^{\frac{3}{2}}\right),$$

the sum

$$\sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 = \frac{1}{x}\sum_{n\leq x} n\, \Lambda_K(n)^2 + x^3 \sum_{n>x} \frac{\Lambda_K(n)^2}{n^3}.$$

Using Lemma 5.1 and partial summation, we show that

$$\sum_{n\leq x} n\, \Lambda_K(n)^2 = \frac{m}{2} x^2 \log x + O_K(x^2) \tag{5.2}$$

and that

$$\sum_{n>x} \frac{\Lambda_K(n)^2}{n^3} = \frac{m \log x}{2x^2} + O_K\left(\frac{1}{x^2}\right). \tag{5.3}$$

This implies that

$$\begin{aligned}\sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 &= \frac{1}{x}\left(\frac{m}{2}x^2 \log x + O_K(x^2)\right) + x^3\left(\frac{m\log x}{2x^2} + O_K\left(\frac{1}{x^2}\right)\right)\\ &= mx\log x + O_K(x),\end{aligned}$$

as stated in Lemma 5.2.

It remains to prove (5.2) and (5.3).

Proof of (5.2). Let $S(x) = \sum_{n \le x} \Lambda_K^2(n) = mx \log x + O_K(x)$. Then by summation by parts, for all $j \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{n\le x}^{\infty} \Lambda_K^2(n) n^j &= t^j S(t)\Big|_{1^-}^{x} - j \int_1^x S(t) t^{j-1} dt \\
&= x^j\, S(x) - j \int_1^x \Big(m t^j \log t + O_K(t^j) \Big) dt \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - jm \int_1^x t^j \log t dt + O_K\left(j \int_1^x (t^j) dt \right) \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - jm \left(\frac{t^{j+1}}{j+1} \log t \Big|_1^x - \int_1^x \frac{t^j}{j+1} dt \right) + O_K\left(\frac{j x^{j+1}}{j+1} \right) \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - \frac{jm}{j+1} x^{j+1} \log x + O_K\left(\frac{jm x^{j+1}}{(j+1)^2} \right) + O_K(x^{j+1}) \\
&= m x^{j+1} \left(1 - \frac{j}{j+1} \right) \log x + O_K(x^{j+1}) \\
&= m x^{j+1} \left(\frac{1}{j+1} \right) \log x + O_K(x^{j+1}) \\
&= \frac{m}{j+1} x^{j+1} \log x + O_K(x^{j+1}).
\end{aligned}$$

Thus, for $j = 1$, we have

$$\frac{m}{2} x^2 \log x + O_K(x^2),$$

which proves (5.2). $\square$

Proof of (5.3). Again let $S(x) = \sum_{n\le x} \Lambda_K^2(n) = mx\log x + O_K(x)$. Then by summation by parts, for all $j \ge 2$ we have

$$\begin{aligned}
\sum_{n>x}^{\infty} \Lambda_K^2(n)\frac{1}{n^j} &= \frac{1}{t^j}S(t)\Big|_{x^-}^{\infty} + j\int_x^{\infty} S(t)\frac{1}{t^{j+1}}dt \\
&= \left(\frac{m}{t^{j-1}}\log t + O_K\Big(\frac{1}{t^{j-1}}\Big)\right)\Bigg|_{x^-}^{\infty} + j\int_x^{\infty}\Big(\frac{m}{t^j}\log t + O_K\Big(\frac{1}{t^j}\Big)\Big)dt \\
&= \frac{m\log x}{x^{j-1}} + O_K\Big(\frac{1}{x^{j-1}}\Big) + jm\int_x^{\infty}\frac{\log t}{t^j}dt + O_K\left(j\int_x^{\infty}\frac{1}{t^j}dt\right) \\
&= \frac{m\log x}{x^{j-1}} + O_K\Big(\frac{1}{x^{j-1}}\Big) + jm\left(\frac{1}{1-j}\frac{\log t}{t^{j-1}}\Big|_x^{\infty} + \frac{1}{j-1}\int_x^{\infty}\frac{1}{t^j}dt\right) \\
&\quad + O_K\left(\frac{j}{(j-1)x^{j-1}}\right) \\
&= \frac{m\log x}{x^{j-1}} + O_K\Big(\frac{1}{x^{j-1}}\Big) + \frac{jm}{j-1}\frac{\log x}{x^{j-1}} + O_K\left(\frac{jm}{(j-1)^2x^{j-1}}\right) \\
&\quad + O_K\left(\frac{j}{(j-1)x^{j-1}}\right) \\
&= \frac{m}{x^{j-1}}\left(1-\frac{j}{j-1}\right)\log x + O_K\Big(\frac{1}{x^{j-1}}\Big) \\
&= \frac{m}{x^{j-1}}\left(\frac{1}{j-1}\right)\log x + O_K\Big(\frac{1}{x^{j-1}}\Big) \\
&= \frac{m}{j-1}\left(\frac{\log x}{x^{j-1}}\right) + O_K\Big(\frac{1}{x^{j-1}}\Big).
\end{aligned}$$

Thus, for $j=3$, we have

$$\frac{m}{2}\frac{\log x}{x^2} + O_K\left(\frac{1}{x^2}\right).$$

This completes the proof of (5.3). $\square$

Combining (5.1) and Lemma 5.2, we have

$$\begin{aligned}
2\pi F_K(x,T) = \; &= m^2\frac{T\log^2 T}{x^2} + mT\log x \\
&\quad + O_K\left(T + x\,\log^2 x + \frac{T\,\log T\,\log^{\frac{1}{2}} x}{x} + \Big(\frac{T}{x}\Big)^{\frac{1}{2}}\log T\,\log x\right).
\end{aligned}$$

Setting $x = T^{m\alpha}$ for $\alpha \geq 0$ and then dividing by $mT \log T$, we derive that

$$\begin{aligned} F_K(\alpha) \;&=\; m\,T^{-2m\alpha}\log T + m\alpha \\ &\quad + O_K\left(\frac{1}{\log T} + \alpha^2 T^{m\alpha-1}\log T + \sqrt{\alpha}\,T^{-m\alpha}\sqrt{\log T} + \alpha T^{-\frac{1}{2}-\frac{1}{2}m\alpha}\log T\right). \end{aligned}$$

If we assume that $0 \leq \alpha < \frac{1}{m}$, then all the error terms go to zero as $T \to \infty$. Since $F_K(-\alpha) = F_K(\alpha)$, we have shown that

$$F_K(\alpha) = m\,T^{-2m|\alpha|}\log T + m|\alpha| + o(1)$$

for $\alpha \in (-\frac{1}{m}, \frac{1}{m})$. This proves Theorem 1.7.

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VITA

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