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ZEROS OF THE DEDEKIND ZETA-FUNCTION
DISSERTATION

A Dissertation
presented in partial fulfillment of requirements
for the degree of Master of Science
in the Department of Mathematics
The University of Mississippi

by

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ABSTRACT

H. L. Montgomery proved a formula for sums over two sets of nontrivial zeros of the Riemann zeta-function. Assuming the Riemann Hypothesis, he used this formula and Fourier analysis to prove an estimate for the proportion of simple zeros of the Riemann zeta-function. We prove a generalization of his formula for the nontrivial zeros of the Dedekind zeta-function of a Galois number field, and use this formula and Fourier analysis to prove an estimate for the proportion of distinct zeros, assuming the Generalized Riemann Hypothesis.

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1 INTRODUCTION

For $s \in \mathbb{C}$, we let $s = \sigma + it$ where $\sigma, t \in \mathbb{R}$. The *Riemann zeta-function* is initially defined as a Dirichlet series over the positive integers and an Euler product over the primes:

$$\zeta(s) = \sum_{n=1}^{\infty} n^{-s} = \prod_{p \text{ prime}} (1 - p^{-s})^{-1} \quad (1.1)$$

for $\sigma > 1$. The equality, proved by Euler, follows from the Fundamental Theorem of Arithmetic. Riemann proved that $\zeta(s)$ can be continued analytically to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$. Riemann also proved that $\zeta(s)$ satisfies the *functional equation*

$$\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s) = \pi^{-\frac{1}{2}(1-s)} \Gamma\left(\frac{1}{2}\right) (1-s) \zeta(1-s). \quad (1.2)$$

From the poles of $\Gamma(s)$ at $s = 0, -1, -2, -3, \dots$, he observed that $\zeta(s)$ has simple zeros at $s = -2, -4, -6, \dots$. These are called the *trivial zeros* of the zeta function. He further noted that $\zeta(s)$ has infinitely many zeros in the *critical strip*, $0 \leq \sigma \leq 1$, which are known as the *non-trivial zeros* of $\zeta(s)$. We denote the nontrivial zeros of $\zeta(s)$ as $\rho = \beta + i\gamma$. From the functional equation, if ρ is a nontrivial zero then so is $1 - \rho$. Since $\overline{\zeta(s)} = \zeta(\bar{s})$, if ρ is a nontrivial zero then so is $\bar{\rho}$. From this, Riemann observed that the zeros are symmetric about the real axis and about the line $\sigma = \frac{1}{2}$. He made the following famous conjecture.

Riemann Hypothesis. All nontrivial zeros of $\zeta(s)$ in the critical strip are on the *critical line* $\sigma = \frac{1}{2}$.

Riemann introduced the zeta function as tool to study the prime numbers. Logarithmically differentiating the Euler product for $\zeta(s)$ we have

$$\frac{d}{ds} \log \zeta(s) = \frac{\zeta'(s)}{\zeta(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda(n)}{n^s}$$

for $\sigma > 1$, where the von Mangoldt function $\Lambda(n) = \log p$ if $n = p^k$ for a prime p and $k \in \mathbb{N}$ and $\Lambda(n) = 0$ otherwise. Using Riemann's ideas, in 1896, Hadamard and de la Vallée Poussin independently proved that

$$\sum_{n \leq x} \Lambda(n) \sim x$$

as $x \rightarrow \infty$ by carefully studying the zeros of $\zeta(s)$. The key is to show that $\zeta(s)$ has no nontrivial zeros on the line $\sigma = 1$ (so that $\frac{\zeta'(s)}{\zeta(s)}$ has no poles on the line $\sigma = 1$). This asymptotic formula is equivalent to:

Theorem 1.1 (Prime Number Theorem). *As $x \rightarrow \infty$, we have*

$$\sum_{p \leq x} 1 \sim \frac{x}{\log x}$$

where the sum runs over the primes p .

This theorem was originally conjectured by Gauss and Legendre. The properties of $\zeta(s)$ described above and the history of the Prime Number Theorem can be found in Davenport's book [Dav00].

Much effort has gone into studying the nontrivial zeros $\rho = \beta + i\gamma$ of $\zeta(s)$. It is known that

$$N(T) := \sum_{0 < \gamma \leq T} 1 = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T) \tag{1.3}$$

as $T \rightarrow \infty$. This was conjectured by Riemann and proved by von Mangoldt [Dav00]. Here the zeros are counted with multiplicity meaning that a zero with multiplicity m is counted m times in the sum. In a now famous paper, Montgomery [Mon73] studied the pair correlation

of the zeros of $\zeta(s)$ assuming the Riemann Hypothesis. We now describe Montgomery's theorem and a corollary. The main goal of my thesis will be to generalize Montgomery's results to the Dedekind zeta-function of a Galois number field.

1.1 Montgomery's Theorem.

We assume the Riemann Hypothesis in this section so that the nontrivial zeros can be written $\rho = \frac{1}{2} + i\gamma$. From (1.3), the average spacing between consecutive $\gamma \in (0, T]$ is

$$\approx \frac{\text{length}((0, T])}{\#\gamma \in (0, T]} \approx \frac{T}{\frac{T \log T}{2\pi}} = \frac{2\pi}{\log T}$$

as $T \rightarrow \infty$. So the sequence $\left\{ \gamma \frac{\log T}{2\pi} \right\}$ has average spacing equal to 1 as $T \rightarrow \infty$. With this in mind, Montgomery was interested in studying sums like

$$\sum_{0 < \gamma, \gamma' \leq T} R \left((\gamma - \gamma') \frac{\log T}{2\pi} \right)$$

where γ and γ' run over the imaginary parts of two sets of nontrivial zeros of $\zeta(s)$.

Montgomery defined the function

$$F(\alpha) = F(\alpha, T) = \frac{2\pi}{T \log T} \sum_{0 < \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma'), \quad \text{where } w(u) = \frac{4}{4 + u^2},$$

where α and $T \geq 2$ are real. Here γ and γ' run over the imaginary parts of two sets of nontrivial zeros of $\zeta(s)$. He was interested in this function because, for $R, \hat{R} \in L^1(\mathbb{R})$, one can show that

$$\sum_{0 < \gamma, \gamma' \leq T} R \left((\gamma - \gamma') \frac{\log T}{2\pi} \right) w(\gamma - \gamma') = \frac{T \log T}{2\pi} \int_{\mathbb{R}} F(\alpha) \hat{R}(\alpha) d\alpha. \quad (1.4)$$

Here \hat{R} is the Fourier transform of R defined as

$$\hat{R}(\alpha) = \int_{\mathbb{R}} R(u) e^{-2\pi i u \alpha} du.$$

Using the definition of the Fourier transform, we will prove the analogue of (1.4) for the zeros of the Dedekind zeta-function in Chapter 2 and the same proof can be used to prove (1.4).

Montgomery [Mon73] proved the following theorem about the function $F(\alpha)$.

Theorem 1.2. *Assume the Riemann Hypothesis. For real α and $T \geq 0$, we have that $F(\alpha)$ is real, $F(\alpha) \geq 0$, and $F(-\alpha) = -F(\alpha)$. For $\alpha \in [0, 1]$ we have*

$$F(\alpha) = \left(1 + o(1)\right) T^{-2\alpha} \log T + \alpha + o(1)$$

as $T \rightarrow \infty$.

We will prove an analogue of this theorem for the zeros of the Dedekind zeta-function. Originally Montgomery proved this theorem for $\alpha \in (0, 1)$ but it was later extended to $\alpha \in [0, 1]$ by Goldston and Montgomery [GM87]. Julia Mueller [Mue83] was the first to observe that $F(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$. We will give a modification of her proof for the zeros of the Dedekind zeta-function in Chapter 2.

The importance of Montgomery's theorem is that we can now estimate the right-hand side of (1.4) for a function $R \in L^1(\mathbb{R})$ with $\text{supp}(\hat{R}) \subseteq [-1, 1]$. With these conditions, for most *nice* functions R , Montgomery's theorem implies that

$$\sum_{0 < \gamma, \gamma' \leq T} R\left((\gamma - \gamma') \frac{\log T}{2\pi}\right) w(\gamma - \gamma') = \frac{T \log T}{2\pi} \left(\hat{R}(0) + \int_{-1}^1 |\alpha| \hat{R}(\alpha) d\alpha + o(1) \right). \quad (1.5)$$

We will prove a similar formula for the zeros the Dedekind zeta-function.

We state one of the several important corollaries that Montgomery derived from his theorem and (1.5). Let

$$N^s(T) = \#\{0 < \gamma \leq T : \rho = \frac{1}{2} + i\gamma \text{ is a simple zero of } \zeta(s)\}.$$

Choosing the Fourier pair

$$R(u) = \left(\frac{\sin \pi u}{\pi u}\right)^2, \quad \hat{R}(\alpha) = \max(1 - |\alpha|, 0)$$

in (1.5), he proved the following estimate for $N^s(T)$ which shows that asymptotically at least two thirds of the nontrivial zeros of $\zeta(s)$ are simple.

Corollary 1.3. *Assume the Riemann Hypothesis. Then*

$$N^s(T) \geq \left(\frac{2}{3} + o(1)\right) N(T)$$

as $T \rightarrow \infty$.

It was later observed by Montgomery and Taylor [Mon75] and Cheer and Goldston [CG93] that the constant $2/3$ can be very slightly improved using a more complicated choice of Fourier pair R and \hat{R} . We prove a generalization of Corollary 1.3 for the Dedekind zeta-function of a Galois number field K over \mathbb{Q} that applies to distinct zeros instead of simple zeros. The answer will depend on the degree $[K : \mathbb{Q}]$ of the number field.

1.2 Properties of the Dedekind zeta-function

Let K be a number field (a finite extension of \mathbb{Q}) where $m = [K : \mathbb{Q}]$ the degree of K . We let \mathcal{O}_K be the ring of integers of K . The *Dedekind zeta-function* is initially defined as a Dirichlet series over nonzero ideals I in \mathcal{O}_K and an Euler product over the prime ideals

P in \mathcal{O}_K :

$$\zeta_K(s) = \sum_{\substack{I \subset \mathcal{O}_K \\ I \neq 0}} \frac{1}{N(I)^s} = \prod_{P \subset \mathcal{O}_K} \left(1 - \frac{1}{N(P)^s}\right)^{-1}$$

for $\sigma > 1$. The equality follows from the fact that \mathcal{O}_K is a PID, so each $I \in \mathcal{O}_K$ can be written uniquely as $I = P_1^{\ell_1} P_2^{\ell_2} \cdots P_k^{\ell_k}$ for prime ideals P_1, \dots, P_j and $\ell_i \in \mathbb{N}$. Hecke proved that $\zeta_K(s)$ can be continued analytically to $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$, and he calculated the residue (which depends on the algebraic properties of K and is known as the *class number formula* for K). We can also write $\zeta_K(s)$ as Dirichlet series over the integers:

$$\zeta_K(s) = \sum_{n=1}^{\infty} \frac{r_K(n)}{n^s}$$

where $r_K(n) = \#\{I \subset \mathcal{O}_K \mid N(I) = n\}$, is the number of ideals in \mathcal{O}_K with norm n . It is known that $0 \leq r_K(n) \leq d_m(n)$ with $d_m(n)$ is the number of ways to write n as the product of $m = [K : \mathbb{Q}]$ positive integers.

Hecke also proved that $\zeta_K(s)$ satisfies the *functional equation*: there exist $r_1, r_2 \in \mathbb{N}$ with $r_1 + 2r_2 = m$ such that

$$\pi^{-m\frac{s}{2}} \zeta_K(s) \Gamma\left(\frac{s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{s+1}{2}\right)^{r_2} = \pi^{-m\frac{1-s}{2}} \zeta_K(1-s) \Gamma\left(\frac{1-s}{2}\right)^{r_1+r_2} \Gamma\left(\frac{1-s+1}{2}\right)^{r_2}. \quad (1.6)$$

Here r_1 is the number of real embeddings of K and r_2 the number of pairs of complex embeddings so that $m = r_1 + 2r_2$. From the poles of $\Gamma(s)$ at $s = 0, -1, -2, -3, \dots$, it can be seen that $\zeta_K(s)$ has a zero at $s = 0$ of order $r_1 + r_2 - 1$, zeros at $s = -2, -4, -6, \dots$ of order $r_1 + r_2$, and zeros at $s = -1, -3, -5, \dots$ of order r_2 . These are called the *trivial zeros* of $\zeta_K(s)$. The Dedekind zeta-function also has infinitely many zeros in the *critical strip*, $0 \leq \sigma \leq 1$, which are known as the *non-trivial zeros* of $\zeta_K(s)$. We will denote the nontrivial zeros of $\zeta_K(s)$ as $\rho = \beta + i\gamma$. From the functional equation, if ρ is a nontrivial zero then so is $1 - \rho$. Since $\overline{\zeta_K(s)} = \zeta_K(\bar{s})$, if ρ is a nontrivial zero then so is $\bar{\rho}$. Therefore the nontrivial

zeros are symmetric about the real axis and about the line $\sigma = \frac{1}{2}$. The analogue of the Riemann Hypothesis is believed to hold for $\zeta_K(s)$.

Generalized Riemann Hypothesis. All nontrivial zeros of $\zeta_K(s)$ in the critical strip are on the *critical line* $\sigma = \frac{1}{2}$.

Logarithmically differentiating the Euler product, we write

$$\frac{d}{ds} \log \zeta_K(s) = \frac{\zeta'_K(s)}{\zeta_K(s)} = - \sum_{n=1}^{\infty} \frac{\Lambda_K(n)}{n^s},$$

where $\Lambda_K(n)$ is a generalization of the von Mangoldt function. It follows from the Euler product that $\Lambda_K(n) = 0$ unless n is a prime power and also that

$$0 \leq \Lambda_K(n) \leq m\Lambda(n)$$

for all $n \in \mathbb{N}$. At $s = 1$, $\zeta_K(s)$ has a complicated residue but $\frac{\zeta'_K(s)}{\zeta_K(s)}$ has a simple pole at $s = 1$ with residue -1 . Landau used this and the fact that $\zeta_K(s)$ has no nontrivial zeros on the line $\sigma = 1$ to prove that

$$\sum_{n \leq x} \Lambda_K(n) \sim x,$$

as $x \rightarrow \infty$. This asymptotic formula is equivalent to:

Theorem 1.4 (Landau's Prime Ideal Theorem). *As $x \rightarrow \infty$, we have*

$$\sum_{\substack{P \subset \mathcal{O}_K \\ N(P) \leq x}} 1 \sim \frac{x}{\log x}$$

where the sum runs over the prime ideals P in \mathcal{O}_K with norm N less than or equal to x .

The above properties of the Dedekind zeta-function can be found in Narkiewicz's book [Nar04]. In this thesis, we are interested in studying the nontrivial zeros $\rho = \beta + i\gamma$ of

$\zeta_K(s)$. It is known that [IK04]

$$N_K(T) := \sum_{0 < \gamma \leq T} 1 = \frac{mT}{2\pi} \log \frac{T}{2\pi} + O_K(T) \quad (1.7)$$

as $T \rightarrow \infty$ where $m = [K : \mathbb{Q}]$. Assuming the Generalized Riemann Hypothesis so that the nontrivial zeros can be written $\rho = \frac{1}{2} + i\gamma$, from (1.7) we see that the average spacing between consecutive $\gamma \in (0, T]$ is

$$\approx \frac{\text{length}((0, T])}{\#\ \gamma \in (0, T]} \approx \frac{T}{\frac{mT \log T}{2\pi}} = \frac{2\pi}{m \log T}$$

as $T \rightarrow \infty$. So the sequence $\left\{ \gamma \frac{m \log T}{2\pi} \right\}$ has average spacing equal to 1 as $T \rightarrow \infty$. Following Montgomery, we study sums like

$$\sum_{0 < \gamma, \gamma' \leq T} R\left((\gamma - \gamma') \frac{m \log T}{2\pi} \right)$$

where R is a function and γ and γ' run over the imaginary parts of the nontrivial zeros of $\zeta_K(s)$. For this reason, we make the following definition.

Definition 1.5. *Let K be a number field with $m = [K : \mathbb{Q}]$. For any $\alpha \in \mathbb{R}$ and $T \geq 2$ we define*

$$F_K(\alpha) = \frac{2\pi}{mT \log T} \sum_{0 \leq \gamma, \gamma' \leq T} T^{i\alpha(\gamma - \gamma')} w(\gamma - \gamma')$$

where $w(u) = \frac{4}{4 + u^2}$ and γ, γ' run over the ordinates of the nontrivial zeros of $\zeta_K(s)$.

We now state some basic properties of $F_K(\alpha)$.

Proposition 1.6. *Let K be a number field. Then we have*

1. $F_K(\alpha)$ is even which means that $F_K(-\alpha) = F_K(\alpha)$.
2. $F_K(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$.

3. If $f, \hat{f} \in L^1(\mathbb{R})$ then

$$\sum_{0 \leq \gamma, \gamma' \leq T} f\left((\gamma - \gamma') \frac{m \log T}{2\pi}\right) w(\gamma - \gamma') = \frac{mT \log T}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) F_K(\alpha) d\alpha$$

where $w(u)$ is the weight function in Definition 1.5 and $\hat{f}(u) = \int_{-\infty}^{\infty} \hat{f}(x) e^{-2\pi i x u} dx$ denotes the Fourier transform of f .

If K is a Galois number field over \mathbb{Q} , then our analogue of Montgomery's Theorem (Theorem 1.2) for the nontrivial zeros of $\zeta_K(s)$ is:

Theorem 1.7. *Let K be a Galois number field over \mathbb{Q} with degree $m = [K : \mathbb{Q}]$, and assume the Generalized Riemann Hypothesis for $\zeta_K(s)$. For real α and $T \geq 0$, we have that $F_K(\alpha)$ is real, $F_K(\alpha) \geq 0$, and $F_K(-\alpha) = -F_K(\alpha)$. For $\alpha \in (-\frac{1}{m}, \frac{1}{m})$ we have*

$$F_K(\alpha) = m T^{-2m|\alpha|} \log T + m|\alpha| + o(1)$$

as $T \rightarrow \infty$.

We now state a corollary about the proportion of distinct nontrivial zeros of $\zeta_K(s)$. We count zeros in our sums with multiplicity, meaning that if a zero has multiplicity ℓ then it appears ℓ times in our sequence. We will let μ_γ be the multiplicity of a $\frac{1}{2} + i\gamma$ of $\zeta_K(s)$. Recall that

$$N_K(T) = \sum_{0 < \gamma \leq T} 1 \sim \frac{mT \log T}{2\pi}$$

and the number of *distinct zeros* of $\zeta_K(s)$ with $0 < \gamma \leq T$ is given by

$$N_K^d(T) = \sum_{\substack{0 < \gamma \leq T \\ \gamma \text{ distinct}}} 1 = \sum_{0 < \gamma \leq T} \frac{1}{\mu_\gamma} = \#\{0 < \gamma \leq T : \zeta_K(\frac{1}{2} + i\gamma) = 0\}.$$

We want to use Theorem 1.7 to count the proportion of distinct zeros of $\zeta_K(s)$ by comparing the ratio of $N_K^d(T)$ to $N_K(T)$. To do this, we define another sum

$$N_K^*(T) = \sum_{0 < \gamma \leq T} \mu_\gamma$$

and we notice that Cauchy's inequality implies that

$$\begin{aligned} N_K(T)^2 &= \left(\sum_{0 < \gamma \leq T} 1 \right)^2 \\ &= \left(\sum_{0 < \gamma \leq T} \frac{1}{\sqrt{\mu_\gamma}} \sqrt{\mu_\gamma} \right)^2 \\ &\leq \sum_{0 < \gamma \leq T} \frac{1}{\mu_\gamma} \sum_{0 < \gamma \leq T} \mu_\gamma \\ &= N_K^d(T) \cdot N_K^*(T). \end{aligned}$$

Therefore

$$N_K^d(T) \geq \frac{N_K(T)^2}{N_K^*(T)} \tag{1.8}$$

and so an upper bound for $N_K^*(T)$ gives a lower bound for $N_K^d(T)$.

Corollary 1.8. *Let K be a Galois number field over \mathbb{Q} with degree $m = [K : \mathbb{Q}]$, and assume the Generalized Riemann Hypothesis for $\zeta_K(s)$. Then, as $T \rightarrow \infty$,*

$$N_K^*(T) \leq \left(m + \frac{1}{3} + o(1) \right) N_K(T)$$

and therefore

$$N_K^d(T) \geq \left(\frac{3}{3m+1} + o(1) \right) N_K(T).$$

This shows that, assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, at least a proportion of $\frac{3}{3m+1}$ of the nontrivial zeros are distinct.

2 PROOF OF PROPOSITION 1.6

In this chapter, we use Fourier analysis to prove Proposition 1.6.

Proof of Proposition 1.6, part 1. We want to show that $F_k(-\alpha) = F_k(\alpha)$ for all real numbers α . We know that

$$F_k(\alpha) = \frac{2\pi}{mT \log(T)} \sum_{0 \leq \gamma, \gamma' \leq T} T^{im\alpha(\gamma - \gamma')} w(\gamma - \gamma').$$

Therefore

$$\begin{aligned} F_k(-\alpha) &= \frac{2\pi}{mT \log(T)} \sum_{0 \leq \gamma, \gamma' \leq T} T^{-im\alpha(\gamma - \gamma')} w(\gamma - \gamma') \\ &= \frac{2\pi}{mT \log(T)} \sum_{0 \leq \gamma, \gamma' \leq T} T^{im\alpha(\gamma' - \gamma)} w(\gamma' - \gamma) \\ &= F_k(\alpha) \end{aligned}$$

since $w(u)$ is even. □

To prove Proposition 1.6, part 2, we first need some lemmas.

Lemma 2.1. *If $g(u) = e^{-2|u|}$ for $u \in \mathbb{R}$, then $\hat{g}(x) = \frac{4}{4+4\pi^2x^2} = w(2\pi x)$ where $w(u)$ is the function in Definition 1.5.*

Proof. If $g(u) = e^{-2|u|}$, then for $x \in \mathbb{R}$ we have

$$\begin{aligned}
\hat{g}(x) &= \int_{-\infty}^{\infty} e^{-2|u|} e^{-2\pi i u x} du \\
&= \int_0^{\infty} e^{-2u} e^{-2\pi i u x} du + \int_{-\infty}^0 e^{2u} e^{-2\pi i u x} du \\
&= \int_0^{\infty} e^{-u(2+2\pi i x)} du + \int_{-\infty}^0 e^{u(2-2\pi i x)} du \\
&= \frac{1}{2+2\pi i x} - \frac{1}{2-2\pi i x} \\
&= \frac{4}{4+4\pi^2 x^2} \\
&= w(2\pi x),
\end{aligned}$$

as claimed. □

Before stating the next lemma, we define a function related to $F_K(\alpha)$:

$$F_K(X, T) = \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} w(\gamma - \gamma'), \quad (2.1)$$

where $X > 0$, $T \geq 2$, and γ, γ' run over the ordinates of two sets of nontrivial zeros of $\zeta_K(s)$.

Lemma 2.2. *We have*

$$F_K(X, T) = \int_{-\infty}^{\infty} \left| \sum_{0 < \gamma \leq T} X^{i\gamma} e^{i\gamma u} \right|^2 e^{-2|u|} du,$$

so therefore $F_K(X, T) \geq 0$.

Proof. Expanding the square

$$\left| \sum_{0 < \gamma \leq T} X^{i\gamma} e^{i\gamma u} \right|^2 = \left(\sum_{0 < \gamma \leq T} X^{i\gamma} e^{i\gamma u} \right) \left(\sum_{0 < \gamma' \leq T} X^{-i\gamma'} e^{-i\gamma' u} \right) = \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} e^{i(\gamma - \gamma')u}.$$

Therefore, by Lemma 2.1, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \left| \sum_{0 < \gamma \leq T} X^{i\gamma} e^{i\gamma u} \right|^2 e^{-2|u|} du &= \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} \int_{-\infty}^{\infty} e^{i(\gamma - \gamma')u} e^{-2|u|} du \\
&= \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} \int_{-\infty}^{\infty} e^{-i2\pi(\frac{\gamma' - \gamma}{2\pi})u} e^{-2|u|} du \\
&= \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} w(\gamma' - \gamma) \\
&= \sum_{0 < \gamma, \gamma' \leq T} X^{i(\gamma - \gamma')} w(\gamma - \gamma') \\
&= F_K(X, T),
\end{aligned}$$

since $w(u)$ is even. □

Proof of Proposition 1.6, part 2. Notice that Lemma 2.2 implies

$$F_K(\alpha) = \frac{mT \log T}{2\pi} F_K(T^{m\alpha}, T) \geq 0.$$

Hence $F_K(\alpha) \geq 0$ for all $\alpha \in \mathbb{R}$, as claimed. □

Proof of Proposition 1.6, part 3. Since $f \in L^1(\mathbb{R})$, by the Fourier inversion theorem we have

$$\begin{aligned}
\sum_{0 \leq \gamma, \gamma' \leq T} f\left((\gamma - \gamma') \frac{m \log T}{2\pi}\right) w(\gamma - \gamma') &= \sum_{0 \leq \gamma, \gamma' \leq T} \left(\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{2\pi i (\gamma - \gamma') \frac{m \log T}{2\pi} \alpha} d\alpha \right) w(\gamma - \gamma') \\
&= \sum_{0 \leq \gamma, \gamma' \leq T} \left(\int_{-\infty}^{\infty} \hat{f}(\alpha) e^{(\log T) m i \alpha (\gamma - \gamma')} d\alpha \right) w(\gamma - \gamma') \\
&= \sum_{0 \leq \gamma, \gamma' \leq T} \left(\int_{-\infty}^{\infty} \hat{f}(\alpha) T^{i m \alpha (\gamma - \gamma')} d\alpha \right) w(\gamma - \gamma') \\
&= \int_{-\infty}^{\infty} \hat{f}(\alpha) \left(\sum_{0 \leq \gamma, \gamma' \leq T} T^{i m \alpha (\gamma - \gamma')} w(\gamma - \gamma') \right) d\alpha \\
&= \frac{mT \log(T)}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) F_K(\alpha) d\alpha
\end{aligned}$$

as claimed. □

This completes the proof of Proposition 1.6.

3 PROOF OF COROLLARY 1.8

In this chapter, we use Theorem 1.7 and Fourier analysis to prove Corollary 1.8. We postpone the proof of Theorem 1.7 until a later chapter. We begin by stating and proving some lemmas.

Assuming the Generalized Riemann Hypothesis for $\zeta_K(s)$, Theorem 1.7 states that

$$F_K(\alpha) = \left(1 + o(1)\right) m T^{-2m|\alpha|} \log T + m|\alpha| + o(1)$$

as $T \rightarrow \infty$ if K be a Galois number field over \mathbb{Q} with degree $m = [K : \mathbb{Q}]$ and $\alpha \in \left(-\frac{1}{m}, \frac{1}{m}\right)$. Recall that the Dirac delta function, $\delta_0(u)$, satisfies $\int_{\mathbb{R}} f(u)\delta_0(u) du = f(0)$ for all nice functions f . The function $mT^{-2m|\alpha|} \log T$ in Theorem 1.7 acts like the Dirac delta function as we see in the following lemma. This implies that $F_K(\alpha) \approx m|\alpha| + \delta_0(\alpha)$ for $\alpha \in \left(-\frac{1}{m}, \frac{1}{m}\right)$.

Lemma 3.1. *Let g be a even function with $g^{(n)}$ bounded for $n = 0, 1, 2$. Then*

$$\int_{-\infty}^{\infty} g(\alpha) \left(mT^{-2m|\alpha|} \log T\right) d\alpha = g(0) + O\left(\frac{1}{\log T}\right),$$

as $T \rightarrow \infty$.

Proof. We prove the lemma using integration by parts. Since g is even

$$\begin{aligned}
\int_{-\infty}^{\infty} g(\alpha) \left(mT^{-2m|\alpha|} \log T \right) d\alpha &= 2m \log T \int_0^{\infty} g(\alpha) T^{-2m\alpha} d\alpha \\
&= 2m \log T \int_0^{\infty} g(\alpha) e^{-2m\alpha \log T} d\alpha \\
&= \left[-g(\alpha) e^{-2m\alpha \log T} \right]_0^{\infty} + \int_0^{\infty} g'(\alpha) e^{-2m\alpha \log T} d\alpha \\
&= g(0) + \left[-\frac{g'(\alpha) e^{-2m\alpha \log T}}{2m \log T} \right]_0^{\infty} + \int_0^{\infty} \frac{g''(\alpha) e^{-2m\alpha \log T}}{2m \log T} d\alpha \\
&= g(0) + \left(\frac{g'(0)}{2m \log T} + \int_0^{\infty} \frac{g''(\alpha) e^{-2m\alpha \log T}}{2m \log T} d\alpha \right) \\
&= g(0) + O_K \left(\frac{1}{\log T} \right),
\end{aligned}$$

as claimed. □

Lemma 3.2. *Let K be a Galois number field over \mathbb{Q} with degree $m = [K : \mathbb{Q}]$, and assume the Generalized Riemann Hypothesis for $\zeta_K(s)$. Then for even functions $f \in L^1(\mathbb{R})$ with $\text{supp}(\hat{f}) \subset (-\frac{1}{m}, \frac{1}{m})$, we have*

$$\sum_{0 < \gamma, \gamma' \leq T} f\left((\gamma - \gamma') \frac{m \log T}{2\pi} \right) w(\gamma - \gamma') = \frac{mT \log T}{2\pi} \left(\hat{f}(0) + 2m \int_0^{1/m} \alpha \hat{f}(\alpha) d\alpha + o(1) \right)$$

as $T \rightarrow \infty$.

Proof. We first use Proposition 1.6, Part 3 and that $\text{supp}(\hat{f}) \subset (-\frac{1}{m}, \frac{1}{m})$ to see that

$$\begin{aligned}
\sum_{0 \leq \gamma, \gamma' \leq T} f\left((\gamma - \gamma') \frac{m \log T}{2\pi} \right) w(\gamma - \gamma') &= \frac{mT \log T}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\alpha) F_K(\alpha) d\alpha \\
&= \frac{mT \log T}{2\pi} \int_{-1/m}^{1/m} \hat{f}(\alpha) F_K(\alpha) d\alpha
\end{aligned} \tag{3.1}$$

Now we use Theorem 1.7 and Lemma 3.1, to deduce that

$$\begin{aligned}
\int_{-1/m}^{1/m} \hat{f}(\alpha) F_K(\alpha) d\alpha &= \int_{-1/m}^{1/m} \hat{f}(\alpha) (mT^{-2m|\alpha|} \log T + m|\alpha| + o(1)) d\alpha \\
&= \hat{f}(0) + \int_{-1/m}^{1/m} m|\alpha| \hat{f}(\alpha) d\alpha + o(1) \\
&= \hat{f}(0) + 2m \int_0^{1/m} \alpha \hat{f}(\alpha) d\alpha + o(1)
\end{aligned} \tag{3.2}$$

since f (and \hat{f}) is even. Combining equations (3.1) and (3.2), we deduce the theorem. \square

Lemma 3.3. *If we let $f(u) = \left(\frac{\sin(\pi u)}{\pi u}\right)^2$ then $\hat{f}(v) = \max(1 - |v|, 0)$*

Proof. If we let $\hat{f}(v) = \max(1 - |v|, 0) = \begin{cases} 1 - |v|, & |v| \leq 1, \\ 0, & \text{otherwise,} \end{cases}$

then by the Fourier inversion theorem $f(u) = \int_{-\infty}^{\infty} \hat{f}(v) e^{2\pi i u v} dv$. So,

$$\begin{aligned}
f(u) &= \int_{-1}^1 (1 - |v|) e^{2\pi i u v} dv = \int_{-1}^0 (1 + v) e^{2\pi i u v} + \int_0^1 (1 - v) e^{2\pi i u v} \\
&= \frac{2}{4\pi^2 u^2} - \left(\frac{e^{2\pi i u} + e^{-2\pi i u}}{4\pi^2 u^2} \right) \\
&= \left(\frac{\sin(\pi u)}{\pi u} \right)^2,
\end{aligned}$$

as claimed. To see this, note that

$$\begin{aligned}
\left(\frac{\sin(\pi u)}{\pi u} \right)^2 &= \frac{\left(\frac{e^{\pi i u} - e^{-\pi i u}}{2i} \right)^2}{(\pi u)^2} \\
&= \left(\frac{e^{2\pi i u} - 2e^{\pi i u - \pi i u} + e^{-2\pi i u}}{4\pi^2 u^2} \right) \\
&= \frac{2}{4\pi^2 u^2} - \left(\frac{e^{2\pi i u} + e^{-2\pi i u}}{4\pi^2 u^2} \right).
\end{aligned}$$

□

In order to apply Theorem 1.7, we want a function whose Fourier transform is supported on $(\frac{-1}{m}, \frac{1}{m})$. Let $f(u) = \left(\frac{\sin(\pi u)}{\pi u}\right)^2$, be the function from the previous lemma. If we let $\hat{h}(v) = \beta \hat{f}(\beta v)$, then $\text{supp}(\hat{h}) \subseteq [\frac{-1}{\beta}, \frac{1}{\beta}]$.

What is h ? By the Fourier inversion theorem

$$\begin{aligned} h(u) &= \int_{\mathbb{R}} \hat{h}(v) e^{2\pi i u v} dv \\ &= \beta \int_{-\frac{1}{\beta}}^{\frac{1}{\beta}} \hat{f}(\beta v) e^{2\pi i u v} dv \\ &= \frac{\beta}{\beta} \int_{-1}^1 \hat{f}(x) e^{2\pi i u (\frac{x}{\beta})} dx \\ &= f\left(\frac{u}{\beta}\right). \end{aligned}$$

Here we used the substitution $x = \beta v$ in the second integral. So we have proved the following lemma.

Lemma 3.4. *If $h(x) = \left(\frac{\sin \frac{\pi x}{\beta}}{\frac{\pi x}{\beta}}\right)^2$ then $\hat{h}(v) = \beta \max(1 - |\beta v|, 0)$.*

We can now prove Corollary 1.8.

Proof of Corollary 1.8. Observation: for any $h \in L^1(\mathbb{R})$ with $h(0) = 1$ and $h(x) \geq 0$ for all x , we have

$$N_K^*(T) = \sum_{0 < \gamma \leq T} \mu_\gamma \leq \sum_{0 < \gamma, \gamma' \leq T} h\left((\gamma - \gamma') \frac{m \log T}{2\pi}\right) w(\gamma - \gamma').$$

To see this inequality, note that $h(0)w(0) = 1$, there are μ_γ terms with $\gamma = \gamma'$, and the other terms are positive. We estimate the sum on the right-hand side using Theorem 1.7 and the Fourier pair in Lemma 3.4 with $\beta > m$. Note that, for this choice of h , we have $h(0) = 1$,

$h(x) \geq 0$ for all x , and $\text{supp}(\hat{h}) \subseteq [-\frac{1}{\beta}, \frac{1}{\beta}] \subset (-\frac{1}{m}, \frac{1}{m})$. Since $\hat{h}(\alpha) = \beta \max(1 - |\beta\alpha|, 0)$

$$\begin{aligned}
\sum_{0 < \gamma, \gamma' \leq T} h\left((\gamma - \gamma') \frac{m \log T}{2\pi}\right) w(\gamma - \gamma') &= \frac{\beta T \log T}{2\pi} \left(\hat{h}(0) + 2\beta \int_0^{1/\beta} \alpha \hat{h}(\alpha) d\alpha + o(1) \right) \\
&= \frac{\beta T \log T}{2\pi} \left(\beta + 2\beta^2 \int_0^{1/\beta} \alpha (1 - \beta\alpha) d\alpha + o(1) \right) \\
&= \frac{\beta T \log T}{2\pi} \left(\beta + 2\beta^2 \int_0^{1/\beta} (\alpha - \beta\alpha^2) d\alpha + o(1) \right) \\
&= \frac{\beta T \log T}{2\pi} \left(\beta + 2\beta^2 \left(\frac{1}{2\beta^2} - \beta \frac{1}{3\beta^3} \right) + o(1) \right) \\
&= \frac{\beta T \log T}{2\pi} \left(\beta + \frac{1}{3} + o(1) \right) \\
&= \left(\beta + \frac{1}{3} + o(1) \right) N_K(T).
\end{aligned}$$

Therefore, letting $\beta \rightarrow m^+$, we have

$$N_K^*(T) \leq \left(m + \frac{1}{3} + o(1) \right) N_K(T).$$

This proves the first assertion in Corollary 1.8. To prove the second assertion, we note that the inequality (1.8) gives

$$N_K^d(T) \geq \frac{N_K(T)^2}{N_K^*(T)} \geq \left(\frac{1}{m + \frac{1}{3} + o(1)} \right) N_K(T) = \left(\frac{3}{3m + 1} + o(1) \right) N_K(T).$$

This completes the proof of Corollary 1.8. □

4 PAIR CORRELATION FOR THE SELBERG CLASS

In a now well known paper, Selberg [Sel92] introduced an axiomatic class of L -functions that he conjectured satisfied the Riemann Hypothesis. This is now called the *Selberg class*, we will denote it by \mathcal{S} . The Dedekind zeta-function of a number field is an element of \mathcal{S} .

In a subsequent paper, Murty and Perelli [MP99] proved a version of Montgomery's theorem for pairs of zeros of L -functions in \mathcal{S} in terms of the coefficients of the Dirichlet series of the logarithmic derivative of elements of \mathcal{S} . We use their work to prove our Theorem 1.7.

The *Selberg class* \mathcal{S} is defined by the following axioms.

1. (Dirichlet Series). Every $L \in \mathcal{S}$ has a Dirichlet series

$$L(s) = \sum_{n=1}^{\infty} \frac{a_L(n)}{n^s},$$

absolutely convergent for $\operatorname{Re}(s) = \sigma > 1$.

2. (Analytic continuation). There exists a (minimal) integer $m_L \geq 0$ such that $(s-1)^{m_L} L(s)$ is an entire function of finite order.
3. (Functional equation). $L \in \mathcal{S}$ satisfies a functional equation of type

$$\Phi(s) = w \overline{\Phi(1-s)},$$

where

$$\Phi(s) = N^s \prod_{j=1}^r \Gamma(\lambda_j s + \mu_j) L(s)$$

with $N > 0$, $\lambda_j > 0$, $\operatorname{Re}(\mu_j) \geq 0$, and $|w| = 1$. Here $\overline{L}(s) = \overline{L(\overline{s})}$.

4. (Ramanujan hypothesis). For every $\varepsilon > 0$, $a_L(n) \ll n^\varepsilon$.

5. (Euler product). $L \in \mathcal{S}$ satisfies

$$\log L(s) = \sum_{n=1}^{\infty} \frac{b_L(n)}{n^s},$$

where $b_L(n) = 0$ unless $n = p^m$ with $m \geq 1$, and $b_L(n) \ll n^\theta$ for some $\theta < \frac{1}{2}$.

In addition, we say the *degree* d_L of $L \in \mathcal{S}$ is

$$d_L = 2 \sum_{j=1}^r \lambda_j,$$

we write

$$-\frac{L'}{L}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^s}; \quad \Lambda_L(n) = b_L(n) \log n,$$

and we define

$$\psi_L(x) = \sum_{n \leq x} |\Lambda_L(n)|^2.$$

With this notation, Murty and Perelli [MP99] proved the following formula.

Proposition 4.1. *Let $L \in \mathcal{S}$ and assume that L satisfies the analogue of the Riemann Hypothesis. Let*

$$a_n(x) = \min \left(\left(\frac{n}{x} \right)^{\frac{1}{2}}, \left(\frac{x}{n} \right)^{\frac{3}{2}} \right) \quad \text{and} \quad w(x) = \frac{4}{4 + x^2}.$$

Then

$$\begin{aligned}
2\pi \sum_{|\gamma| \leq T} \sum_{|\gamma'| \leq T} x^{i(\gamma - \gamma')} w(\gamma - \gamma') &= 2d_L^2 \frac{T \log^2 T}{x^2} + \frac{2T}{x} \sum_{n=1}^{\infty} |\Lambda_L(n)|^2 a_n(x)^2 \\
&+ O\left(x \log^2 x + \frac{T \log T \log^{\frac{1}{2}} x}{x} + \left(\frac{T}{x}\right)^{\frac{1}{2}} \log T \log x + \log^3 T\right)
\end{aligned} \tag{4.1}$$

uniformly for $T \geq x > 1$ where γ, γ' run over the ordinates of two sets of nontrivial zeros of $L(s)$.

Proof. This is equation (30) in Murty and Perelli [MP99]. The proof follows Montgomery's original argument for the zeros of $\zeta(s)$ in [Mon73]. \square

5 PROOF OF THEOREM 1.7.

In this section we prove Theorem 1.7. The fact that $F_K(\alpha)$ is even, real-valued, and non-negative follows from Proposition 1.6. Moreover, it follows from the properties of $\zeta_K(s)$ in the introduction that $\zeta_K(s)$ is in the Selberg class. Moreover, the left-hand side of (4.1) equals $4\pi F_K(x, T)$ where $F_K(X, T)$ is the function defined in (2.1). Therefore Proposition 4.1 implies that

$$\begin{aligned}
 2\pi F_K(x, T) = & m^2 \frac{T \log^2 T}{x^2} + \frac{T}{x} \sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 \\
 & + O_K \left(x \log^2 x + \frac{T \log T \log^{\frac{1}{2}} x}{x} + \left(\frac{T}{x} \right)^{\frac{1}{2}} \log T \log x + \log^3 T \right)
 \end{aligned} \tag{5.1}$$

We prove Theorem 1.7 by estimating the sum on the right-hand side and then relating $F_K(x, T)$ to $F_K(\alpha)$. We do this using partial summation and the following lemma.

Lemma 5.1. *Let K be a Galois extension of \mathbb{Q} . Then*

$$\sum_{n \leq x} \Lambda_K(n)^2 = [K : \mathbb{Q}] x \log x + O_K(x)$$

as $x \rightarrow \infty$.

Proof. This follows from the proof of Lemma 5.2 of Milinovich and Turnage-Butterbaugh [MTB14]. □

Lemma 5.2. *Let K be a Galois extension of \mathbb{Q} and let $m = [K : \mathbb{Q}]$. Then, for $x \geq 1$, we have*

$$\sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 = mx \log x + O_K(x).$$

Since

$$a_n(x) = \min \left(\left(\frac{n}{x} \right)^{\frac{1}{2}}, \left(\frac{x}{n} \right)^{\frac{3}{2}} \right),$$

the sum

$$\sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 = \frac{1}{x} \sum_{n \leq x} n \Lambda_K(n)^2 + x^3 \sum_{n > x} \frac{\Lambda_K(n)^2}{n^3}.$$

Using Lemma 5.1 and partial summation, we show that

$$\sum_{n \leq x} n \Lambda_K(n)^2 = \frac{m}{2} x^2 \log x + O_K(x^2) \tag{5.2}$$

and that

$$\sum_{n > x} \frac{\Lambda_K(n)^2}{n^3} = \frac{m \log x}{2x^2} + O_K \left(\frac{1}{x^2} \right). \tag{5.3}$$

This implies that

$$\begin{aligned} \sum_{n=1}^{\infty} \Lambda_K(n)^2 a_n(x)^2 &= \frac{1}{x} \left(\frac{m}{2} x^2 \log x + O_K(x^2) \right) + x^3 \left(\frac{m \log x}{2x^2} + O_K \left(\frac{1}{x^2} \right) \right) \\ &= mx \log x + O_K(x), \end{aligned}$$

as stated in Lemma 5.2.

It remains to prove (5.2) and (5.3).

Proof of (5.2). Let $S(x) = \sum_{n \leq x} \Lambda_K^2(n) = mx \log x + O_K(x)$. Then by summation by parts, for all $j \in \mathbb{N}$ we have

$$\begin{aligned}
\sum_{n \leq x} \Lambda_K^2(n) n^j &= t^j S(t) \Big|_1^- - j \int_1^x S(t) t^{j-1} dt \\
&= x^j S(x) - j \int_1^x \left(m t^j \log t + O_K(t^j) \right) dt \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - j m \int_1^x t^j \log t dt + O_K \left(j \int_1^x (t^j) dt \right) \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - j m \left(\frac{t^{j+1}}{j+1} \log t \Big|_1^x - \int_1^x \frac{t^j}{j+1} dt \right) + O_K \left(\frac{j x^{j+1}}{j+1} \right) \\
&= m x^{j+1} \log x + O_K(x^{j+1}) - \frac{j m}{j+1} x^{j+1} \log x + O_K \left(\frac{j m x^{j+1}}{(j+1)^2} \right) + O_K(x^{j+1}) \\
&= m x^{j+1} \left(1 - \frac{j}{j+1} \right) \log x + O_K(x^{j+1}) \\
&= m x^{j+1} \left(\frac{1}{j+1} \right) \log x + O_K(x^{j+1}) \\
&= \frac{m}{j+1} x^{j+1} \log x + O_K(x^{j+1}).
\end{aligned}$$

Thus, for $j = 1$, we have

$$\frac{m}{2} x^2 \log x + O_K(x^2),$$

which proves (5.2). □

Proof of (5.3). Again let $S(x) = \sum_{n \leq x} \Lambda_K^2(n) = mx \log x + O_K(x)$. Then by summation by parts, for all $j \geq 2$ we have

$$\begin{aligned}
\sum_{n > x} \Lambda_K^2(n) \frac{1}{n^j} &= \frac{1}{t^j} S(t) \Big|_{x^-}^{\infty} + j \int_x^{\infty} S(t) \frac{1}{t^{j+1}} dt \\
&= \left(\frac{m}{t^{j-1}} \log t + O_K\left(\frac{1}{t^{j-1}}\right) \right) \Big|_{x^-}^{\infty} + j \int_x^{\infty} \left(\frac{m}{t^j} \log t + O_K\left(\frac{1}{t^j}\right) \right) dt \\
&= \frac{m \log x}{x^{j-1}} + O_K\left(\frac{1}{x^{j-1}}\right) + jm \int_x^{\infty} \frac{\log t}{t^j} dt + O_K\left(j \int_x^{\infty} \frac{1}{t^j} dt\right) \\
&= \frac{m \log x}{x^{j-1}} + O_K\left(\frac{1}{x^{j-1}}\right) + jm \left(\frac{1}{1-j} \frac{\log t}{t^{j-1}} \Big|_x^{\infty} + \frac{1}{j-1} \int_x^{\infty} \frac{1}{t^j} dt \right) \\
&\quad + O_K\left(\frac{j}{(j-1)x^{j-1}}\right) \\
&= \frac{m \log x}{x^{j-1}} + O_K\left(\frac{1}{x^{j-1}}\right) + \frac{jm \log x}{j-1 x^{j-1}} + O_K\left(\frac{jm}{(j-1)^2 x^{j-1}}\right) \\
&\quad + O_K\left(\frac{j}{(j-1)x^{j-1}}\right) \\
&= \frac{m}{x^{j-1}} \left(1 - \frac{j}{j-1}\right) \log x + O_K\left(\frac{1}{x^{j-1}}\right) \\
&= \frac{m}{x^{j-1}} \left(\frac{1}{j-1}\right) \log x + O_K\left(\frac{1}{x^{j-1}}\right) \\
&= \frac{m}{j-1} \left(\frac{\log x}{x^{j-1}}\right) + O_K\left(\frac{1}{x^{j-1}}\right).
\end{aligned}$$

Thus, for $j = 3$, we have

$$\frac{m \log x}{2 x^2} + O_K\left(\frac{1}{x^2}\right).$$

This completes the proof of (5.3). □

Combining (5.1) and Lemma 5.2, we have

$$\begin{aligned}
2\pi F_K(x, T) &= m^2 \frac{T \log^2 T}{x^2} + mT \log x \\
&\quad + O_K\left(T + x \log^2 x + \frac{T \log T \log^{\frac{1}{2}} x}{x} + \left(\frac{T}{x}\right)^{\frac{1}{2}} \log T \log x\right).
\end{aligned}$$

Setting $x = T^{m\alpha}$ for $\alpha \geq 0$ and then dividing by $mT \log T$, we derive that

$$F_K(\alpha) = mT^{-2m\alpha} \log T + m\alpha + O_K \left(\frac{1}{\log T} + \alpha^2 T^{m\alpha-1} \log T + \sqrt{\alpha} T^{-m\alpha} \sqrt{\log T} + \alpha T^{-\frac{1}{2}-\frac{1}{2}m\alpha} \log T \right).$$

If we assume that $0 \leq \alpha < \frac{1}{m}$, then all the error terms go to zero as $T \rightarrow \infty$. Since $F_K(-\alpha) = F_K(\alpha)$, we have shown that

$$F_K(\alpha) = mT^{-2m|\alpha|} \log T + m|\alpha| + o(1)$$

for $\alpha \in (-\frac{1}{m}, \frac{1}{m})$. This proves Theorem 1.7.

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