A Weighted Version of Erdős-Kac Theorem

Unique Subedi

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Approved by

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Advisor: Rizwanur Khan

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Advisor: Micah B. Milinovich

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Reader: Ayla Gafni

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Reader: Lê Thái Hoàng

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Abstract

Let $\omega(n)$ denote the number of distinct prime factors of a natural number n. A celebrated result of Erdős and Kac states that $\omega(n)$ as a Gaussian distribution. In this thesis, we establish a weighted version of Erdős-Kac Theorem. Specifically, we show that the Gaussian limiting distribution is preserved, but shifted, when $\omega(n)$ is weighted by the $k-$fold divisor function $\tau_k(n)$. We establish this result by computing all positive integral moments of $\omega(n)$ weighted by $\tau_k(n)$.

We also provide a proof of the classical identity of $\zeta(2n)$ for $n \in \mathbb{N}$ using Dirichlet's kernel.

To my parents for their love, encouragement, and unwavering support.

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Contents

Chapter 1

Introduction

"It is evident that the primes are randomly distributed but, unfortunately, we don't know what 'random' means."

– R.C. Vaughan

1.1 Theorem of Hardy & Ramanujan

Let $\omega(n)$ denote the number of distinct prime factors of a natural number n, that is

$$\omega(n) := \sum_{p|n} 1.$$

In a remarkable work in 1917, Hardy and Ramanujan [9] showed that $\omega(n)$ is of size $\log \log n$ for almost all n. More precisely, they proved that for every $\epsilon > 0$, the proportion of integers $n \leq x$ for which the inequality

$$(1 - \epsilon) \log \log n \leq \omega(n) \leq (1 + \epsilon) \log \log n$$

fails goes to 0 as $x \to \infty$. We say that the normal order of $\omega(n)$ is $\log \log n$.

The original proof by Hardy and Ramanujan is based on the inequality for

$$\pi_\nu(x) := \#\{n \le x \mid \omega(n) = \nu\}.$$

They established that there exist constants $c_0, c_1 > 0$ such that

$$\pi_\nu(x) < c_0 \frac{x}{\log x} \frac{(\log\log x + c_1)^{\nu-1}}{(\nu-1)!}$$

uniformly for all $x \ge 2$ and $\nu \ge 0$. Hardy and Ramanujan went on to establish another quantitative estimate of $\omega(n)$ in the same paper. Let $\xi(n) \to \infty$ as $n \to \infty$. Then, irrespective of rate of divergence of $\xi(n)$, we have

$$\log\log n - \xi(n)\sqrt{\log\log n} < \omega(n) < \log\log n + \xi(n)\sqrt{\log\log n}$$

for almost all n.

1.2 Theorem of Erdős & Kac

In 1934, Turán [20] provided a new proof of Hardy-Ramanujan Theorem based on the estimate

$$\frac{1}{x}\sum_{n\le x}(\omega(n) - \log\log n)^2 \ll \log\log x. \tag{1.2.1}$$

On top of being simple, Turán's proof readily extends to a large class of additive functions. Furthermore, his work is essentially the first step towards the use of probabilistic methods in number theory.

For historical context, Turán's argument is similar to that of Chebyshev's in his proof of *Law of Large Numbers* in probability theory. Turán, however, was unaware of Chebyshev's work at that time. It was Mark Kac, a probabilist, who first noticed this similarity and wrote to Turán asking if he can establish similar estimates of

$$\frac{1}{x}\sum_{n\le x}(\omega(n)-\log\log n)^m \tag{1.2.2}$$

for all $m \in \mathbb{N}$. However, it was not until 1954 when (1.2.2) was finally evaluated by Halberstam, the details of which will be discussed in the next section. In a lecture in 1939, Kac conjecturally pointed out that the functions

$$\delta(p) := \begin{cases} 1, & \text{if } p|n \\ 0, & \text{if } p\nmid n \end{cases}$$

are probabilistically independent for distinct values of p, and thus the theory of sum of independent random variables can be applied to study the distribution of

$$\omega(n) = \sum_p \delta(p).$$

Paul Erdős was in the audience, and immediately afterwards, Erdős and Kac [6] presented a celebrated result on the distribution of $\omega(n)$.

Theorem (Erdős - Kac, 1940)**.** *For any $\alpha \in \mathbb{R}$,*

$$\frac{1}{x}\sum_{\substack{n\le x\\ \omega(n)-\log\log x\le\alpha\sqrt{\log\log x}}} 1 \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-t^2/2}\,dt \tag{1.2.3}$$

as $x\to\infty$.

In other words, Erdős-Kac Theorem implies that the distribution of

$$\frac{\omega(n)-\log\log x}{\sqrt{\log\log x}} \tag{1.2.4}$$

is asymptotically Gaussian with mean 0 and variance 1. In 1958, Rényi and Turán [13] provided a quantitative version of the theorem, showing that $O(1/\sqrt{\log\log x})$ is the best possible uniform rate of convergence in (1.2.3).

Erdős-Kac Theorem is actually a version of Central Limit Theorem for $\omega(n)$. To observe this, notice that the average of $\omega(n)$ as n ranges over the integers below x is

$$\frac{1}{x}\sum_{n\leq x}\omega(n) = \frac{1}{x}\sum_{p\leq x}\sum_{\substack{n\leq x\\ p|n}} 1 = \frac{1}{x}\sum_{n\leq x}\left\lfloor\frac{x}{p}\right\rfloor = \log\log x + O(1). \tag{1.2.5}$$

Combining (1.2.5) and (1.2.1), we infer (non-rigorously) that as n ranges over the integers below x, the average size of $\omega(n)$ is roughly $\log\log x$ with a typical standard deviation of $\sqrt{\log\log x}$. We want to note that $\log\log x$ and $\log\log n$ are interchangeable here as they are close for almost all $n \leq x$. Thus, (1.2.4) is analogous to the well known normalization in Central Limit Theorem of probability theory. In fact, Erdős and Kac's original proof uses the Central Limit Theorem and Brun's sieve.

There are now many proofs of Erdős-Kac Theorem. For instance, simplifying previous work of Sathe, an argument of Selberg [16] can be used to provide a different proof. For $k \in \mathbb{N}$, define $\pi_k(x)$ as the number of integers $n \leq x$ with $\omega(n) = k$. In a series of papers in 1953 and 1954, Sathe [14, 15] proved an asymptotic estimate

$$\pi_k(x) = (1 + o(1))\frac{x}{\log x}\frac{(\log\log x)^{k-1}}{(k-1)!}, \tag{1.2.6}$$

as $x \to \infty$, uniformly for $1 \leq k < (e-\delta)\log\log x$ where $0 < \delta < e$ is fixed. Sathe's proof, based on the induction, was very involved and complicated. Selberg simplified Sathe's proof by establishing (1.2.6) from asymptotic estimates for the sum

$$\sum_{n\leq x} z^{\omega(n)}, \quad z \in \mathbb{C},$$

uniform for z in a specified range. Selberg's ideas were further developed by Delange [3], and this technique is now known as the Selberg–Delange method.

A different approach of proof that is relevant to the main result of this thesis uses a technique known as the method of moments. Halberstam [8] established asymptotic formulae for the 'central moments'

$$\sum_{n\leq x}(\omega(n) - \log\log x)^m \tag{1.2.7}$$

for each $m \in \mathbb{N}$, and showed that they agree with the moments of a Gaussian distribution. Since the Gaussian distribution is completely characterized by its moments (see [2, Chapter 30]), this allowed Halberstam to deduce a new proof of Erdős-Kac Theorem. The proof in [8] is very technical and involved. Billingsley [1] simplified the proof by avoiding some of Halberstam's heavy calculations with the use of further probability theory. Granville and Soundararajan [7] provided another elegant yet simple method to compute the moments (1.2.7). Moreover, they obtained an asymptotic estimate for the sums in (1.2.7) that holds uniformly for all natural numbers $m \leq (\log\log x)^{\frac{1}{3}}$. Moreover, this problem has also been studied using deeper probabilistic ideas. For instance, two relatively recent proofs using Stein's method, a tool from modern probability theory, were provided by Harper [10].

Throughout the years, many possible generalizations of the Erdős-Kac Theorem have been studied. For instance, Elliot [4, 5] established an Erdős-Kac type theorem with respect to weighted measure $\tau_2(n)^\alpha$, where $\tau_2(n) = \sum_{d|n} 1$ is the divisor function. Another generalization of Erdős-Kac Theorem over Gaussian field in short intervals was provided by Liu and Yang [11]. However, all these generalizations are based on the characteristic functions and uses Selberg-Delange method.

1.3 Weighted Erdős-Kac Theorem

In this article, we study the generalization of Erdős-Kac Theorem using the method of moments. In particular, motivated by Granville and Soundararajan's work [7], we evaluate the central moments of $\omega(n)$ with respect to weighted measure $\tau_k(n)$ where

$$\tau_k(n) = \sum_{n_1 \ldots n_k = n} 1$$

is the $k-$divisor function. Our work provides a new proof of Elliot's result for the case $\alpha = 1$, and generalizes that result to $\tau_k(n)$. In Proposition 3, we show that, as n ranges over the integers below x, the mean of $\omega(n)$ with respect to weighted measure $\tau_k(n)$ is $\sim k \log\log x$. Thus, in a joint work with Rizwanur Khan and Micah Milinovich, we establish the following asymptotic estimate for the weighted moments.

Theorem 1 (Weighted Moments)**.** *For fixed $k, m \in \mathbb{N}$, we have*

$$\frac{\sum_{n\le x}(\omega(n) - k\log\log x)^m \tau_k(n)}{\sum_{n\le x}\tau_k(n)} = \begin{cases} (m-1)!!(k\log\log x)^{m/2} + O\big((\log\log x)^{\frac{m-1}{2}}\big) & \text{if } m \text{ is even,} \\ O\big((\log\log x)^{\frac{m-1}{2}}\big) & \text{if } m \text{ is odd,} \end{cases} \tag{1.3.1}$$

where $(m-1)!!$ denotes the product of all odd integers up to and including $(m-1)$.

Notice that the quantity on right hand side of (1.3.1) are the moments of Gaussian distribution. Since Gaussian distribution is completely characterized by its moments, Theorem 1 implies the following weighted version of Erdős-Kac Theorem:

$$\forall \alpha \in \mathbb{R},\ \Big(\sum_{n\le x}\tau_k(n)\Big)^{-1} \sum_{\substack{n\le x \\ \omega(n)-k\log\log x \le \alpha\sqrt{k\log\log x}}} \tau_k(n) \longrightarrow \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\alpha} e^{-t^2/2}\,dt \quad \text{as} \quad x\to\infty. \tag{1.3.2}$$

Before proceeding with the actual proof, let us provide a heuristic for why (1.3.2) holds. Recall a well-known inequality

$$k^{\omega(n)} \leq \tau_k(n) \leq k^{\Omega(n)},$$

where $\Omega(n)$ counts prime factors of n with multiplicity, that is $\Omega(n) = \sum_{p^\alpha || n} \alpha$. As similar result holds if we replace $\omega(n)$ in Erdős-Kac Theorem by $\Omega(n)$, $\tau_k(n)$ is essentially an exponential of a Gaussian random variable. So, roughly speaking, a Gaussian $\omega(n)$ is being tilted by its exponential $\tau_k(n)$ in (1.3.2). Thus, we can view (1.3.2) as a manifestation of Girsanov's Theorem, which implies that if we tilt a Gaussian random variable with an exponential of itself, then the resulting random variable is also Gaussian with related mean and variance. This phenomenon can simply be proved by completing the square. Consider a Gaussian random variable X with mean 0 and variance 1 such that the distribution function of X is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{\frac{-x^2}{2}} dx.$$

If we weight the measure dx by e^x, the distribution function becomes

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{\frac{-x^2}{2}} e^x dx = \frac{\sqrt{e}}{\sqrt{2\pi}} \int_{-\infty}^{\alpha} e^{\frac{-(x-1)^2}{2}} dx,$$

where the equality is obtained by completing the square. Thus, the resulting distribution in this weighted space is still Gaussian but with shifted mean and variance.

We prove Theorem 1 in Chapter 3. Let us now discuss the second result of this thesis.

1.4 Dirichlet's kernel & $\zeta(2n)$

Dirichlet's kernel is defined by

$$D_n(x) := \sum_{k=-n}^{n} e^{ikx} = 1 + 2\sum_{k=1}^{n} \cos kx = \frac{\sin((n+1/2)x)}{\sin(x/2)}, \tag{1.4.1}$$

where the second and third equalities follow readily by the application of Euler's identity. Let $\zeta(s)$ denotes the Riemann zeta function. Then, the series representation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges absolutely for $s \in \mathbb{C}$ when $\mathrm{Re}(s) > 1$. For $n \in \mathbb{N}$, we define Bernoulli polynomials $B_n(x)$ by the generating function

$$\frac{ze^{xz}}{e^z - 1} = \sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} \tag{1.4.2}$$

for $|z| < 2\pi$. We call $B_n(0)$ the n^{th} Bernoulli number, which henceforth will be denoted as B_n.

In the second part of this thesis, we use Dirichlet's kernel to prove the classical result

$$\zeta(2n) = \sum_{\ell=1}^{\infty} \frac{1}{\ell^{2n}} = (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!} \tag{1.4.3}$$

for $n \in \mathbb{N}$.

Our work is motivated by the elegant calculation of Stark [17], which uses Dirichlet's kernel to give a quick proof of the identity

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

Stark establishes this identity by evaluating the integral

$$\int_0^\pi xD_{2m-1}(x)dx,$$

for $m \in \mathbb{N}$ in two different ways. On one hand, he evaluates this integral by using the definition of the Dirichlet kernel as the sum of cosines in (1.4.1). On the other hand, he evaluates the same integral by expressing $D_{2m-1}(x)$ as a ratio of sines. Upon letting $m \to \infty$, he derives the identity

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8},$$

which immediately gives the identity for $\zeta(2)$.

1.5 Notations and conventions

The sets $\mathbb{Z}, \mathbb{R},$ and $\mathbb{C}$ denote integers, real numbers, and complex numbers respectively. The set $\mathbb{N}$ denotes the set of positive integers, and its elements will be referred to as natural numbers.

We will also use Vinogradov's notation and big-O notation.

- $f(x) = O(g(x))$ implies that there exists a constant $c > 0$ such that $|f(x)| \leq cg(x)$ for all $x \geq a$.

- $f \ll g$ is equivalent to $f = O(g)$.

- $f \asymp g$ is also equivalent to $f \ll g$ and $g \ll f$.

- $f(x) \sim g(x)$ holds if and only if $\lim_{x\to\infty} \frac{f(x)}{g(x)} \to 1$.

Unless otherwise specificied, the implied constants in all asymptotic estimates are absolute.

If we have an asymptotic relation

$$f(x) = g(x) + O(h(x)),$$

we will refer to $g(x)$ as the "main term" and $O(h(x))$ as the "error term" in the estimation of $f(x)$.

Chapter 2

Preliminaries

"A technicality I am prepared to hide wildy behind."

– Jim Butcher, Storm Front

In this chapter, we present two necessary results on partial sum of $k-$divisor function. Using these results, we compute the average of $\omega(n)$ with respect to weighted measure $\tau_k(n)$. This computation of average allows us to reduce Theorem 1 to a technical proposition.

2.1 Partial sums involving $\tau_k(n)$

A well known result, due to Voronoi and Landau (see [19, Theorem 12.2]), states that

$$\sum_{n\leq x}\tau_k(n) = \operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)\Big) + O\Big(x^{\frac{k-1}{k+1}+\epsilon}\Big). \tag{2.1.1}$$

The leading order term of the residue is

$$\frac{x(\log x)^{k-1}}{(k-1)!}. \tag{2.1.2}$$

We prove a slightly more general result than (2.1.1) but with a weaker error term. This suffices for our application, as we only require a power savings in x/a.

Lemma 2. *For $a \in \mathbb{N}$ and $x \geq a$, we have*

$$\sum_{\substack{n \leq x \\ a|n}} \tau_k(n) = \operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,a)\Big) + O\Big(\big(\frac{x}{a}\big)^{\frac{k+3}{k+6}+\epsilon}\tau_k(a)M^{\omega(a)}\Big) \tag{2.1.3}$$

where

$$F(s,a) := \prod_{p^{\upsilon_p}||a}\Big(1-\big(1-\frac{1}{p^s}\big)^k\sum_{m=0}^{\upsilon_p-1}\frac{\tau_k(p^m)}{p^{ms}}\Big). \tag{2.1.4}$$

$$M := \Big(\frac{\sqrt{2}+1}{\sqrt{2}-1}\Big)^6 \tag{2.1.5}$$

Remark. *The Laurent series expansion of $\frac{x^s}{s}\zeta^k(s)F(s,a)$ around $s=1$ is*

$$x\cdot\Big(\frac{1}{s-1}+\gamma+\dots\Big)^k\cdot\Big(1+(s-1)\log x+\dots+\frac{((s-1)\log x)^{k-1}}{(k-1)!}+\dots\Big)$$
$$\cdot\,(1-(s-1)+(s-1)^2+\dots)\cdot(F(1,a)+(s-1)\frac{d^c}{ds^c}\Big|_{s=1}F(s,a)+\dots).$$

The leading term of $\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,a)\Big)$ is

$$F(1,a)\,\frac{x(\log x)^{k-1}}{(k-1)!}, \tag{2.1.6}$$

and its next largest terms are

$$\frac{x(\log x)^{k-1-c}}{(k-1-c)!\;c!}\frac{d^c}{ds^c}\Big|_{s=1}F(s,a) \text{ for } 1 \leq c \leq k-1. \tag{2.1.7}$$

Proof of Lemma 2. Our proof is based on the well-known Dirichlet series of $\zeta^k(s)$ and its corresponding Euler product,

$$\zeta^k(s) = \sum_{n=1}^{\infty} \frac{\tau_k(n)}{n^s} = \prod_p \sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}}. \tag{2.1.8}$$

The sum over m inside the product further evaluates to

$$\sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}} = \left(\frac{p^s}{p^s - 1}\right)^k. \tag{2.1.9}$$

Proof of (2.1.8) and (2.1.9) can be found in [19, Chapter 1]. We will estimate the sum $\sum_{\substack{n\le x\\ a|n}} \tau_k(n)$ by studying the Dirichlet's series

$$\sum_{\substack{n=1\\ a|n}}^{\infty} \frac{\tau_k(n)}{n^s} = \frac{1}{a^s} \sum_{b=1}^{\infty} \frac{\tau_k(ab)}{b^s},$$

where the equality follows by making the substitution $n = ab$. The corresponding Euler product of this Dirichlet's series is

$$\sum_{\substack{n=1\\ a|n}}^{\infty} \frac{\tau_k(n)}{n^s} = \frac{1}{a^s}\left(\prod_{p^{\upsilon_p}||a} \sum_{m=0}^{\infty} \frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}}\right)\left(\prod_{p\nmid a} \sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}}\right).$$

Completing the product to obtain Dirichlet's series of $\zeta^k(s)$ stated in (2.1.8), we get the equality

$$\sum_{\substack{n=1\\ a|n}}^{\infty} \frac{\tau_k(n)}{n^s} = \zeta^k(s)\frac{1}{a^s} \prod_{p^{\upsilon_p}||a} \left(\sum_{m=0}^{\infty} \frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}} \Big/ \sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}}\right). \tag{2.1.10}$$

We can rearrange the sum in the numerator as

$$\sum_{m=0}^{\infty} \frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}} = p^{s\upsilon_p} \sum_{m=\upsilon_p}^{\infty} \frac{\tau_k(p^m)}{p^{ms}} = p^{s\upsilon_p}\left(\sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}} - \sum_{m=0}^{\upsilon_p - 1} \frac{\tau_k(p^m)}{p^{ms}}\right).$$

This rearrangement followed by the use of (2.1.9) reduces (2.1.10) to

$$\sum_{\substack{n=1\\ a|n}}^{\infty} \frac{\tau_k(n)}{n^s} = \zeta^k(s) \prod_{p^{\upsilon_p}||a} \Big(1 - \big(1 - \frac{1}{p^s}\big)^k \sum_{m=0}^{\upsilon_p - 1} \frac{\tau_k(p^m)}{p^{ms}}\Big).$$

With the definition of $F(s,a)$ stated in (2.1.4), we can write

$$\sum_{\substack{n=1\\ a|n}}^{\infty} \frac{\tau_k(n)}{n^s} = \zeta^k(s) F(s,a).$$

Therefore, an application of Perron's formula (see [18, Part II. §2.1]) gives

$$\sum_{\substack{n\leq x\\ a|n}} \tau_k(n) = \frac{1}{2\pi i}\int_{c-iT}^{c+iT} x^s \zeta(s)^k F(s,a)\frac{ds}{s} \quad + \quad O\left(\sum_{\substack{n=1\\ a|n}} \Big(\frac{x}{n}\Big)^c \frac{\tau_k(n)}{T|\log\frac{x}{n}|}\right), \tag{2.1.11}$$

where we choose c and T such that $c > 1$ and T is large. First, let us handle the error term in (2.1.11). Making the substitution $n = ab$ and using inequality $\tau_k(ab) \leq \tau_k(a)\tau_k(b)$, we can write the error term as

$$\Big(\frac{x}{a}\Big)^c \frac{\tau_k(a)}{T} \sum_{b=1}^{\infty} \frac{\tau_k(b)}{b^c |\log\big(\frac{x/a}{b}\big)|}.$$

We estimate this error by splitting the range of sum into two pieces:
When $b < \frac{x/a}{2}$ or $b > \frac{3x/a}{2}$, the size of error is

$$\Big(\frac{x}{a}\Big)^c \frac{\tau_k(a)}{T} \sum_{\substack{b<\frac{x/a}{2}\\ b>\frac{3x/a}{2}}} \frac{\tau_k(b)}{b^c |\log\big(\frac{x/a}{b}\big)|} \ll \Big(\frac{x}{a}\Big)^c \frac{\tau_k(a)}{T} \sum_{b=1}^{\infty} \frac{\tau_k(b)}{b^c} = \Big(\frac{x}{a}\Big)^c \frac{\zeta^k(c)\tau_k(a)}{T} \ll \Big(\frac{x}{a}\Big)^c \frac{\tau_k(a)}{T(c-1)^k}.$$

When $\frac{x/a}{2} \leq b \leq \frac{3x/a}{2}$, we have $b^c \asymp (x/a)^c$ and $\tau_k(b) \ll b^{\epsilon/2} \ll (x/a)^{\epsilon/2}$. So, the size of error is

$$\left(\frac{x}{a}\right)^c \tau_k(a) \sum_{\frac{x/a}{2}<b<\frac{3x/a}{2}} \frac{\tau_k(b)}{b^c T |\log\left(\frac{x/a}{b}\right)|} \ll \frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{\epsilon/2} \sum_{\frac{x/a}{2}<b<\frac{3x/a}{2}} \frac{1}{|\log\left(\frac{x/a}{b}\right)|}. \tag{2.1.12}$$

To evaluate the rightmost sum above, we note that $b = [\frac{x}{a}]+\nu$, where $-0.5x/a \leq \nu \leq 0.5x/a$. As usual, $[\frac{x}{a}]$ and $\{\frac{x}{a}\}$ denote the integer part and the fractional part of $\frac{x}{a}$ respectively. So, we have

$$\Big|\log\frac{x/a}{b}\Big| = \Big|\log\frac{[x/a]+\{x/a\}}{[x/a]+\nu}\Big| = \Big|\log\Big(1-\frac{\nu-\{x/a\}}{[x/a]+\nu}\Big)\Big| \asymp \frac{|\nu|a}{x}.$$

Thus, (2.1.12) is

$$\ll \frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{\epsilon/2} \sum_{|\nu|\leq 0.5x/a} \frac{x}{a|\nu|} \ll \frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{1+\epsilon/2} \log\left(\frac{x}{a}\right) \ll \frac{\tau_k(a)}{T}\left(\frac{x}{a}\right)^{1+\epsilon},$$

using the estimate $\log\left(\frac{x}{a}\right) \ll \left(\frac{x}{a}\right)^{\epsilon/2}$.

Next, we evaluate the integral in (2.1.11) by moving the line of integration around the rectangle $c-iT$, $\frac{1}{2}-iT$, $\frac{1}{2}+iT$, $c+iT$. The residue collected at $s=1$ gives the main term in (2.1.3). The remaining two horizontal and one vertical integrals contribute to the error term that we need to bound. To bound these integrals, we use the following well known bound of $\zeta(s)$ on the half line:

$$\zeta\big(\frac{1}{2}+it\big) = O(t^{\frac{1}{6}}). \tag{2.1.13}$$

Then, by Phragmén–Lindelöf principle, we have

$$\zeta(\sigma+it) = O(t^{\frac{1}{6}(c-\sigma)/(c-\frac{1}{2})}), \tag{2.1.14}$$

uniformly in our rectangle, i.e, $\sigma \in [\frac{1}{2}, c]$ and $t \in [-T, T]$. We also need an estimate of

$F(s,a)$ in this region. Notice that $F(s,a)$ is everything except $\zeta^k(s)$ on the right hand side of (2.1.10). Thus, we have

$$|F(s,a)| = \frac{1}{a^\sigma} \prod_{p^{\upsilon_p}||a} \left| \Big(\sum_{m=0}^{\infty} \frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}} \Big/ \sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}} \Big) \right|.$$

Using the inequality $\tau_k(p^{m+\upsilon_p}) \leq \tau_k(p^m)\tau_k(p^{\upsilon_p})$ and (2.1.9), we get an upper-bound

$$\Big| \sum_{m=0}^{\infty} \frac{\tau_k(p^{m+\upsilon_p})}{p^{ms}} \Big| \leq \tau_k(p^{\upsilon_p}) \sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{m\sigma}} = \tau_k(p^{\upsilon_p}) \Big(\frac{p^\sigma}{p^\sigma - 1} \Big)^k.$$

The use identity (2.1.9) once again gives an upper- bound of remaining sum as well:

$$\Big| \Big(\sum_{m=0}^{\infty} \frac{\tau_k(p^m)}{p^{ms}} \Big)^{-1} \Big| = \Big| \Big(\frac{p^s - 1}{p^s} \Big)^k \Big| \leq \Big(\frac{p^\sigma + 1}{p^\sigma} \Big)^k.$$

Notice that $\big((p^\sigma+1)/(p^\sigma-1)\big)^k \leq \big((\sqrt{2}+1)/(\sqrt{2}-1)\big)^k = M^{k/6}$ on $\sigma \in [1/2, c]$, where M is defined on (2.1.5). So, combining these inequalities yields

$$|F(s,a)| \leq \frac{1}{a^\sigma} \prod_{p^{\upsilon_p}||a} \tau_k(p^{\upsilon_p}) \Big(\frac{p^\sigma + 1}{p^\sigma - 1} \Big)^k \leq \frac{\tau_k(a) M^{\frac{k}{6}\omega(a)}}{a^\sigma}. \tag{2.1.15}$$

Thus, using the estimates of $\zeta(s)$ and $F(s,a)$ stated in (2.1.13), (2.1.14), and (2.1.15) respectively, the horizontal integral is

$$\begin{aligned} \int_{\frac{1}{2}+iT}^{c+iT} x^s \zeta^k(s) F(s,a) \frac{ds}{s} &= O\left(\tau_k(a) \int_{\frac{1}{2}}^{c} \Big(\frac{x}{a}\Big)^\sigma T^{\frac{1}{6}(c-\sigma)/(c-\frac{1}{2})-1} d\sigma \right) \\ &= O\Big(T^{-1} \Big(\frac{x}{a}\Big)^c \tau_k(a) M^{\frac{k}{6}\omega(a)} \Big) + O\Big(T^{\frac{k}{6}-1} \Big(\frac{x}{a}\Big)^{\frac{1}{2}} \tau_k(a) M^{\frac{k}{6}\omega(a)} \Big). \end{aligned}$$

The same bound exists for another horizontal integral from $c - iT$ to $\frac{1}{2} - iT$.

Similarly, the vertical integral is

$$\int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} x^s F(s,a)\zeta^k(s)\frac{ds}{s} = O\Big(\tau_k(a)M^{\frac{k}{6}\omega(a)}\int_{-T}^{T}\big(\frac{x}{a}\big)^{\frac{1}{2}}\,|\zeta\big(\frac{1}{2}+it\big)|\,\frac{dt}{|t|}\Big)$$
$$= O\Big(\tau_k(a)M^{\frac{k}{6}\omega(a)}\big(\frac{x}{a}\big)^{\frac{1}{2}}\int_{-T}^{T}|t|^{\frac{k}{6}-1}dt\Big)$$
$$= O\Big(\big(\frac{x}{a}\big)^{\frac{1}{2}}T^{\frac{k}{6}}\tau_k(a)M^{\frac{k}{6}\omega(a)}\Big).$$

Choosing $c = 1+\epsilon$, the total error is

$$\ll \big(\frac{x}{a}\big)^{1+\epsilon}\frac{\tau_k(a)}{T} + \big(\frac{x}{a}\big)^{\frac{1}{2}}T^{\frac{k}{6}}\tau_k(a)M^{\frac{k}{6}\omega(a)} + \big(\frac{x}{a}\big)^{\frac{1}{2}}T^{\frac{k}{6}-1}\tau_k(a)M^{\frac{k}{6}\omega(a)}.$$

Clearly, the second term above is bigger than the third term. And apart from $\epsilon's$, the first and second terms are equal if we take

$$T = \Big(\frac{x}{a}\Big)^{\frac{3}{k+6}}M^{\frac{-k}{k+6}\omega(a)},$$

which gives the desired estimate of the error,

$$O\Big(\Big(\frac{x}{a}\Big)^{\frac{k+3}{k+6}+\epsilon}\tau_k(a)M^{\omega(a)}\Big).$$

□

2.2 Average of $\omega(n)$ weighted by $\tau_k(n)$

Using these results on partial sum of $k-$divisor function, we compute average order of $\omega(n)$ with respect to the weighted measure $\tau_k(n)$.

Proposition 3 (Average)**.** *As n ranges over the integers below x, the average of $\omega(n)$ with* *respect to weighted measure $\tau_k(n)$ is $k \log\log x + O(1)$.*

Proof. The average of $\omega(n)$ with respect to weighted measure $\tau_k(n)$ is

$$\frac{\sum_{n\le x}\omega(n)\tau_k(n)}{\sum_{n\le x}\tau_k(n)}.$$

Notice that

$$\sum_{n\le x}\omega(n)\tau_k(n) = \sum_{n\le x}\Big(\sum_{p|n}1\Big)\tau_k(n) = \sum_{p\le x}\sum_{\substack{n\le x\\ p|n}}\tau_k(n).$$

Thus, by Lemma 2, the average reduces to

$$\Big(\sum_{n\le x}\tau_k(n)\Big)^{-1}\sum_{p\le x}\left(\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,p)\Big) + O\Big(\Big(\frac{x}{p}\Big)^{\frac{k+3}{k+6}+\epsilon}\tau_k(p)M^{\omega(p)}\Big)\right). \tag{2.2.1}$$

Using the estimates (2.1.1) (2.1.2) and employing the leading expression of residue stated in (2.1.6), we deduce that the main term in (2.2.1) is

$$\sum_{p\le x}F(1,p) = k\log\log x + O(1), \tag{2.2.2}$$

where the equality follows by the use of well-known Mertern's estimate as

$$F(1,p) = 1-(1-\frac{1}{p})^k = 1-\sum_{\ell=0}^{k}\binom{k}{\ell}\Big(\frac{-1}{p}\Big)^k = \frac{k}{p}+O\Big(\frac{1}{p^2}\Big).$$

Now to complete the proof, it suffices to show that the remainder terms in (2.2.1) is $O(1)$. Since k is fixed, we have $\tau_k(p) = k = O(1)$ and $M^{\omega(p)} = M = O(1)$. Thus, using (2.1.1) and (2.1.2), the contribution of error term in (2.2.1) is

$$\ll \frac{1}{x(\log x)^{k-1}}\sum_{p\le x}\Big(\frac{x}{p}\Big)^{\frac{k+3}{k+6}+\epsilon} \ll 1.$$

Next, we handle the remaining contributions of residue. In view of (2.1.7), these contributions are proportional to

$$\frac{1}{x(\log x)^{k-1}} \sum_{p \le x} \left(x \, (\log x)^{k-1-c} \, \frac{d^c}{ds^c}\Big|_{s=1} F(s,p) \right)$$

for $1 \le c \le k-1$. Writing $F(s,p) = \exp\{\log\big(F(s,p)\big)\}$, a simple computation shows that

$$\frac{d^c}{ds^c}\Big|_{s=1} F(s,p) \ll \frac{(\log p)^c}{p}.$$

Now, using a well-known estimate,

$$\sum_{p \le x} \frac{(\log p)^c}{p} \ll (\log x)^c,$$

the remaining contributions of the residue are

$$\ll \frac{1}{(\log x)^c} \sum_{p \le x} \frac{(\log p)^c}{p} \ll 1.$$

This completes our proof. $\square$

Chapter 3

Weighted Moments of $\omega(n)$

"And I knew exactly what to do. But in a much more real sense, I had no idea what to do."

– Michael Scott, The Office

In this chapter, we prove Theorem 1, which establishes an asymptotic formulae for the moments of $\omega(n)$ weighted by $\tau_k(n)$.

3.1 Reduction of Theorem 1

We deduce Theorem 1 from the following technical proposition.

Proposition 4. *Define*

$$f_p(n) = \begin{cases} -F(1,p), & \textit{if } p \nmid n. \\ 1 - F(1,p), & \textit{if } p|n. \end{cases}$$

Let $z \geq 10^{10}$ be a real number. For $k, m \in \mathbb{N}$, we have

$$\frac{\sum_{n\leq x}\Big(\sum_{p\leq z} f_p(n)\Big)^m \tau_k(n)}{\sum_{n\leq x}\tau_k(n)} = \begin{cases} (m-1)!!\ \ (k\log\log z)^{m/2} + O\big((\log\log z)^{\frac{m-1}{2}}\big) & \textit{if } m \textit{ is even,} \\ O\big((\log\log z)^{\frac{m-1}{2}}\big) & \textit{if } m \textit{ is odd,} \end{cases} \tag{3.1.1}$$

where $(m-1)!!$ denotes the product of all odd integers up to and including $(m-1)$.

3.1.1 Deduction of Theorem 1 from Proposition 4

Let $z = x^{\frac{1}{(k+6)m}}$. Recall that $k\log\log x = \sum_{p\leq x} F(1,p) + O(1)$. So for $n \leq x$, we have

$$\begin{aligned} \omega(n) - k\log\log x &= \sum_{p|n} 1 - \Big(\sum_{p\leq x} F(1,p) + O(1)\Big) \\ &= \sum_{\substack{p\leq z\\ p|n}} 1 + \sum_{\substack{p>z\\ p|n}} 1 - \sum_{p\leq z} F(1,p) - \sum_{z<p\leq x} F(1,p) + O(1) \\ &= \sum_{p\leq z} f_p(n) + \sum_{\substack{p>z\\ p|n}} 1 - \sum_{z<p\leq x} F(1,p) + O(1). \end{aligned}$$

Notice that $\sum_{\substack{p>z\\ p|n}} 1 = O(1)$ because x can have at most $(k+6)m$ prime divisors between z and x. Furthermore, $\sum_{z<p\leq x} F(1,p) = k\log\log x - k\log\log z + O(1) = O(1)$, thus yielding

$$\omega(n) - k\log\log x = \sum_{p\leq z} f_p(n) + O(1).$$

Now the binomial expansion gives

$$\frac{\sum_{n\le x}\Big(\omega(n)-k\log\log x\Big)^m\tau_k(n)}{\sum_{n\le x}\tau_k(n)}=\frac{\sum_{n\le x}\Big(\sum_{p\le z}f_p(n)\Big)^m\tau_k(n)}{\sum_{n\le x}\tau_k(n)} + O\left(\frac{\sum_{n\le x}\Big|\sum_{p\le z}f_p(n)\Big|^{m-1}\tau_k(n)}{\sum_{n\le x}\tau_k(n)}\right). \tag{3.1.2}$$

Now to deduce Theorem 1 from Proposition 4, it suffices to show that the size of error term in (3.1.2) is $\ll (\log\log z)^{\frac{m-1}{2}}$ (the size of error term in (3.1.1)).

Suppose $m-1$ is even. Then, using (3.1.1), we obtain that the error term in (3.1.2) is

$$\ll (\log\log z)^{\frac{m-1}{2}}.$$

Suppose $m-1$ is odd. Then, Cauchy-Schwarz inequality gives

$$\sum_{n\le x}\Big|\sum_{p\le z}f_p(n)\Big|^{m-1}\tau_k(n)\le\Big(\sum_{n\le x}\Big(\sum_{p\le z}f_p(n)\Big)^{m-2}\tau_k(n)\Big)^{\frac{1}{2}}\Big(\sum_{n\le x}\Big(\sum_{p\le z}f_p(n)\Big)^{m}\tau_k(n)\Big)^{\frac{1}{2}}.$$

Notice that both terms above can be handled using (3.1.1), and thus we obtain that the error term in (3.1.2) to be

$$\ll (\log\log z)^{\frac{m-1}{2}}.$$

Therefore, we have now established that Proposition 4 implies Theorem 1.

3.2 Proof of Proposition 4

We now present the proof of Proposition 4. For $r\in\mathbb{N}$, define $f_r(n):=\prod_{p^\alpha||r}f_p(n)^\alpha$. Then, we can write

$$\sum_{n\le x}\Big(\sum_{p\le z}f_p(n)\Big)^m\tau_k(n)=\sum_{p_1,\ldots,p_m\le z}\sum_{n\le x}f_{p_1\cdots\cdots p_m}(n)\tau_k(n),$$

which allows us to write the left hand side of (3.1.1) as

$$\sum_{p_1,\ldots,p_m\le z}\frac{\sum_{n\le x} f_{p_1,\ldots,p_m}(n)\tau_k(n)}{\sum_{n\le x}\tau_k(n)}. \tag{3.2.1}$$

Let us consider more generally $\sum_{n\le x} f_r(n)\tau_k(n)$. Since there are m p_i's in $f_{p_1,\ldots,p_m}(n)$ and each $p_i \le z = x^{\frac{1}{(k+6)m}}$, we only need to consider

$$r \le x^{\frac{1}{k+6}}.$$

Suppose R be the square-free part of r, that is $R = \prod_{p^\alpha\|r} p$. Notice that if $d = (n,R)$, then $f_r(n) = f_r(d)$. Furthermore, we observe that

$$\begin{aligned}
\sum_{n\le x} f_r(n)\tau_k(n) &= \sum_{a|R} f_r(a)\sum_{\substack{n\le x\\(n,R)=d}}\tau_k(n) = \sum_{a|R} f_r(a)\sum_{\substack{n\le x\\ d|n\\(R/d,n/d)=1}}\tau_k(n)\\
&= \sum_{ab|R} f_r(a)\mu(b)\sum_{\substack{n\le x\\ ab|n}}\tau_k(n)\\
&= \sum_{ab|R} f_r(a)\mu(b)\left(\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,ab)\Big) + O\Big(\big(\frac{x}{ab}\big)^{\frac{k+3}{k+6}+\epsilon}\tau_k(ab)M^{\omega(ab)}\Big)\right),
\end{aligned}$$

where the last equality follows from Lemma 2.

Notice that $M^{\omega(ab)} \le \tau_{\lceil M\rceil}(ab) \le (ab)^{\epsilon/2}$ and $\tau_k(a) \ll (ab)^{\epsilon/2}$. Thus, the error term above is $O(x^{\frac{k+3}{k+6}+\epsilon})$. Since we are summing this error over $r \le x^{\frac{1}{k+6}}$, the total contribution of the error is

$$\ll \sum_{r\le x^{\frac{1}{k+6}}} x^{\frac{k+3}{k+6}+\epsilon} \ll x^{\frac{k+4}{k+6}+\epsilon}.$$

Thus, using the estimates (2.1.1) and (2.1.2), we deduce that

$$\frac{\sum_{n\le x} f_r(n)\tau_k(n)}{\sum_{n\le x}\tau_k(n)} = \frac{(k-1)!}{x(\log x)^{k-1}}\sum_{ab|R} f_r(a)\mu(b)\left(\operatorname*{Res}_{s=1}\Big(\frac{x^s}{s}\zeta^k(s)F(s,ab)\Big)\right) + O(x^{\frac{-2}{k+6}+\epsilon}). \tag{3.2.2}$$

Employing the leading expression of residue stated in (2.1.6), the main term in (3.2.2) is $\sum_{ab|R} f_r(a)\mu(b)F(1,ab)$. For convenience, let us give this main term a name:

$$G(r) := \sum_{ab|R} f_r(a)\mu(b)F(1,ab).$$

Clearly, $G(r)$ is multiplicative, i.e,

$$G(r) = \prod_{p^\alpha || r} G(p^\alpha).$$

Notice that for any prime p, we have

$$\begin{aligned} G(p^\alpha) &= (\text{contribution of } ab=1) + (\text{contribution of } a=p, b=1) + (\text{contribution of } a=1, b=p)\\ &= (-F(1,p))^\alpha + (1-F(1,p))^\alpha F(1,p) - (-F(1,p))^\alpha F(1,p)\\ &= (-F(1,p))^\alpha(1-F(1,p)) + (1-F(1,p))^\alpha F(1,p). \end{aligned}$$

When $\alpha = 1$, we get $G(p) = 0$. And for $\alpha = 2$, we have

$$G(p^2) = \big(F(1,p)\big)^2 + (1-F(1,p))^2F(1,p) - \big(-F(1,p)\big)^2F(1,p) = \big(F(1,p)\big)\big(1-F(1,p)\big) \ge 0,$$

as $0 < F(1,p) < 1$. Thus, $G(r)$ is only supported on square-full integers. For any $\alpha \ge 2$, we get

$$G(p^\alpha) = \frac{k}{p} + O\Big(\frac{1}{p^2}\Big), \tag{3.2.3}$$

by using the estimate $F(1,p) = \frac{k}{p} + O(\frac{1}{p^2})$.

3.2.1 Main term of (3.2.1)

We now evaluate the sum (3.2.1). Since $G(r)$ is only supported square-full integers, the main term in (3.2.1) is

$$\sum_{\substack{p_1\ldots p_m\le z\\ p_1\ldots p_m \text{ square-full}}} G(p_1\ldots p_m).$$

Suppose $q_1 < q_2 < \ldots q_t$ be the distinct primes in $p_1p_2\ldots p_m$ such that $p_1\ldots p_m = q_1^{\alpha_1}\ldots q_t^{\alpha_t}$. Since $p_1p_2\ldots p_m$ is square-full, we have $t \le m/2$. Thus, (3.2.1) can be expressed as

$$\sum_{t\le m/2}\sum_{q_1<q_2<\ldots<q_t\le z}\sum_{\substack{\alpha_1,\ldots,\alpha_t\ge 1\\ \sum\alpha_i=m}}\frac{m!}{\alpha_1!\ldots\alpha_t!}G(q_1^{\alpha_1})\ldots G(q_t^{\alpha_t}).$$

When m is even, we have a term $t = m/2$ (where all $\alpha_i = 2$) that yields Gaussian moments. Using the estimate (3.2.3), this term contributes

$$\frac{m!}{2^{m/2}(m/2)!}\sum_{q_1<q_2<\ldots<q_{m/2}\le z}\prod_{i=1}^{m/2}\left(\frac{k}{q_i}+O\Big(\frac{1}{q_i^2}\Big)\right).$$

Dropping the condition that primes q_i's need to be distinct, we note that the sum is clearly bounded from above by $\Big(\sum_{q\le z}\big(k/q+O(1/q)\big)\Big)^{m/2} = k\log\log z+O(1)$. If we fix $q_1,\ldots,q_{m/2-1}$, then the sum over $q_{m/2}$ is at least $\sum_{\pi_{m/2}\le q\le z}\big(k/q+O(1/q)\big)$, where π_n denotes the n-th prime. Proceeding in the similar manner by fixing other combinations of q_is, we note that the sum over q's is bounded from below by $\sum_{\pi_{m/2}\le q\le z}\big(k/q+O(1/q)\big)^{m/2}$. Therefore, the contribution of term with $t = m/2$ is

$$(m-1)!!\,(k\log\log z)^{m/2} + O\Big((\log\log z)^{\frac{m}{2}-1}\Big), \tag{3.2.4}$$

where we have defined $(m-1)!! := (m!)/(2^{m/2}(m/2)!)$. We can notice that $(m-1)!!$ is the product of all integers upto and including $m-1$.

When $t < m/2$, we have $G(q_i^{\alpha_i}) \ll 1/q_i$. Thus, the contribution of such terms are

$$\ll \sum_{q_1<q_2<\ldots<q_t\le z} \frac{1}{q_1\ldots q_t} \ll \Big(\sum_{q\le z}\frac{1}{q}\Big)^t \ll (\log\log z)^t \ll (\log\log z)^{\frac{m}{2}-1}. \tag{3.2.5}$$

3.2.2 Remainder terms in (3.2.1)

Now we have to handle the remainder terms in (3.2.1), which are the contributions of non-leading terms of the residue and the error term in (3.2.2). The contribution of the error term is

$$\ll \sum_{\substack{p_1\ldots p_m\le z\\ p_1\ldots p_m \text{ square-full}}} x^{\frac{-2}{k+6}+\epsilon} \ll \pi(z)^m\, x^{\frac{-2}{k+6}+\epsilon}.$$

Since $(\pi(z)^m) \ll z^m = x^{\frac{1}{k+6}}$, the contribution of error term above is

$$\ll x^{\frac{-1}{k+6}+\epsilon} \ll 1, \tag{3.2.6}$$

as the exponent of x, apart from ϵ, is negative for all finite values of k and ϵ can be taken arbitrarily small.

Next we handle the remaining contributions of the residue. Before proceeding further, let us note that, in view of (2.1.7), the next largest contribution of residue in (3.2.2) are proportional to

$$\frac{1}{(\log x)^c}\sum_{ab|R} f_r(a)\mu(b)\frac{d^c}{ds^c}\Big|_{s=1}F(s,ab), \tag{3.2.7}$$

for $1\le c\le k-1$. Define

$$G(s,r) := \sum_{ab|R} f_r(a)\mu(b)F(s,ab),$$

such that $G(1,r) = G(r)$. Therefore, the contribution of (3.2.7) to (3.2.1) is proportional to

$$\frac{1}{(\log x)^c} \sum_{p_1,p_2,\ldots,p_m \leq z} \frac{d^c}{ds^c}\Big|_{s=1} G(s, p_1 p_2 \ldots p_m).$$

For $q_1 < q_2 < \ldots < q_t$ distinct primes in $p_1, p_2, \ldots, p_m$, suppose $p_1 \ldots p_m = q_1^{\alpha_1} \ldots q_t^{\alpha_t}$. Then, the expression above is equal to

$$\frac{1}{(\log x)^c} \sum_{t \leq m} \sum_{q_1<q_2<\ldots<q_t \leq z} \sum_{\substack{\alpha_1,\ldots,\alpha_t \geq 1 \\ \sum \alpha_i = m}} \frac{m!}{\alpha_1! \ldots \alpha_t!} \frac{d^c}{ds^c}\Big|_{s=1} G(s, q_1^{\alpha_1}) \ldots G(s, q_t^{\alpha_t}),$$

which, using product rule for differentiation, can be expressed as

$$\frac{1}{(\log x)^c} \sum_{t \leq m} \sum_{q_1<q_2<\ldots<q_t \leq z} \sum_{\substack{\alpha_1,\ldots,\alpha_t \geq 1 \\ \sum \alpha_i = m}} \frac{m!}{\alpha_1! \ldots \alpha_t!} \sum_{\beta_1+\beta_2+\ldots+\beta_t = c} \frac{c!}{\beta_1! \ldots \beta_t!} \prod_{i=1}^{t} \frac{d^{\beta_i}}{ds^{\beta_i}}\Big|_{s=1} G(s, q_i^{\alpha_i}) \tag{3.2.8}$$

Here, the sum over β_is counts all possible ways $G(s, q_i^{\alpha_i})$'s can be differentiated using product rule. Some β_is can be 0, which represents $G(s, q_i^{\alpha_i})$ that are not differentiated. If we have a case where $\beta_i = 0$ and $\alpha_i = 1$, then the whole sum collapses to 0 because we will have a factor $G(1, q_i) = 0$ in the product. Therefore, every occurrence of $G(s, q_i)$ needs to be differentiated.

Before proceeding further, let us note that for any prime q, we have $\frac{d^{\beta_i}}{ds^{\beta_i}}\Big|_{s=1} F(s,q) \ll \frac{(\log q)_i^{\beta}}{q}$, which in turn implies that $\frac{d^{\beta_i}}{ds^{\beta_i}}\Big|_{s=1} G(s, q^{\alpha}) \ll \frac{(\log q)^{\beta_i}}{q}$. If we plug this estimation of $(G(s, q_i^{\alpha_i}))^{(\beta_i)}\Big|_{s=1}$ in (3.2.8), we get that the contribution of all $G(s, q_i^{\alpha_i})$ that are differentiated at least once amounts to $O\Big(\Big(\sum_{q \leq z} \frac{\log q}{q}\Big)^c\Big)$. And the contribution of all $G(s, q_i^{\alpha_i})$ that remain undifferentiated is $O\Big(\Big(\sum_{q \leq z} \frac{1}{q}\Big)^{t-c}\Big)$. Since all $G(s, q_i^{\alpha_i})$ that remain undifferentiated have $\alpha_i \geq 2$, we must have $t - c \leq \frac{m-1}{2}$. Therefore, the contribution of (3.2.8) is

$$\ll \frac{1}{(\log x)^c} \sum_{t \leq m} \Big(\sum_{q \leq z} \frac{\log q}{q}\Big)^c \Big(\sum_{q \leq z} \frac{1}{q}\Big)^{t-c} \ll \Big(\sum_{q \leq z} \frac{1}{q}\Big)^{\frac{m-1}{2}} \ll (\log \log z)^{\frac{m-1}{2}}. \tag{3.2.9}$$

Proposition 4 follows after combining (3.2.4), (3.2.5), (3.2.6), and (3.2.9).

3.2.3 Example

We provide an example that portrays our technique of handling remainder terms in Proposition 4. Let us assume a case where there are three distinct primes q_1, q_2, q_3 with $\alpha_1 = 1$, $\alpha_2 = 2$ and $\alpha_3 = 3$. Since $q_1 q_2^2 q_3^3$ is not square-full, this term only contributes to the error in the computation of the sixth moment. We show that the contribution of this term is smaller than the size of the main term of the sixth moment, i.e., $(k \log\log z)^3$

Suppose $c = 1$. $G(s, q_1)$ has to be differentiated, otherwise the contribution collapses to 0. The size of error in this case is

$$\frac{1}{\log x} \sum_{q_1, q_2, q_3 \le z} \frac{\log q_1}{q_1 q_2 q_3} \ll \Big(\sum_{q \le z} \frac{1}{q} \Big)^2 \ll (\log\log z)^2.$$

Suppose $c = 2$, that is

$$\frac{d^2}{ds^2}\Big|_{s=1} G(s, q_1)\, G(s, q_2^2)\, G(s, q_3^3). \tag{3.2.10}$$

Using product rule for differentiation in (3.2.10), we get two different types of terms: one where $G(s, q_1)$ is differentiated both of the times and another where $G(s, q_1)$ is differentiated once and either one of $G(s, q_2^2)$ or $G(s, q_3^3)$ is differentiated once. In the case where $G(s, q_1)$ is differentiated both of the times, we have

$$G(s, q_2^{\alpha_2})\, G(s, q_3^{\alpha_3}) \frac{d^2}{ds^2}\Big|_{s=1} G(s, q_1) \ll \frac{(\log q_1)^2}{q_1} \frac{1}{q_2} \frac{1}{q_3},$$

thus making the size of contribution

$$\ll \frac{1}{(\log x)^2} \sum_{q_1,q_2,q_3 \le z} \frac{(\log q_1)^2}{q_1 q_2 q_3} \ll \sum_{q \le z} \frac{1}{q} \ll \log\log z.$$

Now consider the case where both $G(s, q_1)$ and $G(s, q_2^2)$ are differentiated once. Since

$$G(s, q_3^3) \left.\frac{d}{ds}\right|_{s=1} \{G(s, q_1)\} \left.\frac{d}{ds}\right|_{s=1} \{G(s, q_2^2)\} \ll \frac{\log q_1}{q_1} \frac{\log q_2}{q_2} \frac{1}{q_3},$$

the contribution of such term is

$$\ll \frac{1}{(\log x)^2} \sum_{q_1,q_2,q_3 \le z} \frac{\log q_1 \ \log q_2}{q_1 q_2 q_3} \ll \sum_{q \le z} \frac{1}{q} \ll \log\log z.$$

Similar estimates can be obtained for the case where both $G(s, q_1)$ and $G(s, q_3^3)$ are differentiated once. Furthermore, the size of contribution of such term is $\ll 1$ when $c \ge 3$.

Chapter 4

Evaluating $\zeta(2n)$ using Dirichlet's Kernel

"It gets easier."

– Jogging Baboon, BoJack Horseman

In this chapter, we prove the classical identity of $\zeta(2n)$ stated in (1.4.3) using Dirichlet's kernel. In particular, generalizing the idea of Stark [17] discussed in Chapter 1, we prove (1.4.3) by evaluating the integral

$$\int_0^1 B_{2n}(t)\, D_{2m}(\pi t)\, dt, \tag{4.0.1}$$

for $m, n \in \mathbb{N}$, in two different ways. On one hand, we evaluate this integral by using the definition of the Dirichlet kernel as the sum of cosines in (1.4.1). On the other hand, we evaluate (4.0.1) by expressing $D_{2m}(\pi t)$ as a ratio of sines. The formula for $\zeta(2n)$ in (1.4.3) will follow from these two calculations upon letting $m \to \infty$.

After discovering our proof, we became aware of the paper [21] that uses trigonometric functions similar to the definition of Dirichlet's kernel in (1.4.1) to evaluate $\zeta(2n)$. However, the use of Dirichlet's kernel simplifies the proof in [21], and it illustrates another connection between Dirichlet's kernel and the Bernoulli numbers.

4.1 Properties of Bernoulli polynomials and Bernoulli numbers

We now state the properties of Bernoulli polynomials and Bernoulli numbers necessary for our proof. For $n \in \mathbb{N}$, we recall the well-known identities

$$B_n'(t) = n\, B_{n-1}(t) \tag{4.1.1}$$

and

$$\int_0^1 B_n(x)\, dx = 0. \tag{4.1.2}$$

Also recall that $B_1(0) = -B_1(1) = \frac{1}{2}$, $B_{2n}(0) = B_{2n}(1)$ for $n \geq 1$, and $B_{2n-1}(0) = B_{2n-1}(1) = 0$ for $n \geq 2$. Standard properties of Bernoulli polynomials and numbers can be found in [12, Appendix B].

The following integral is a key component of our proof of (1.4.3).

Lemma 5. *For $k, n \in \mathbb{N}$, we have*

$$I_n^{(k)} := \int_0^1 B_{2n}(t) \cos(k\pi t)\, dt = \begin{cases} \dfrac{(-1)^{n-1}(2n)!}{k^{2n}\pi^{2n}}, & \text{for } k \text{ even,} \\ 0, & \text{for } k \text{ odd.} \end{cases}$$

Proof. Integrating by parts twice using (4.1.1), we obtain

$$I_n^{(k)} = \frac{1}{k\pi} B_{2n}(t)\sin(k\pi t)\Big|_0^1 + \frac{2n}{k^2\pi^2} B_{2n-1}(t)\cos(k\pi t)\Big|_0^1 - \frac{2n(2n-1)}{k^2\pi^2}\int_0^1 B_{2(n-1)}(t)\cos(k\pi t)\, dt. \tag{4.1.3}$$

The first term vanishes as $\sin(k\pi t) = 0$ at both endpoints. When $n = 1$ the second term gives

$$\frac{2}{k^2\pi^2} B_1(t)\cos(k\pi t)\Big|_0^1 = \frac{1+(-1)^k}{k^2\pi^2}$$

since $B_1(0) = -B_1(1) = \frac{1}{2}$, while the third term equals 0 since $B_0(t) = 1$. This gives

$$I_1^{(k)} := \int_0^1 B_2(t)\cos(k\pi t)\, dt = \begin{cases} \dfrac{2}{k^2\pi^2}, & \text{for } k \text{ even,} \\ 0, & \text{for } k \text{ odd.} \end{cases}$$

For $n \geq 2$ we use the fact that $B_{2n-1}(0) = B_{2n-1}(1) = 0$ to see that the second term in (4.1.3) vanishes giving the recursive relation

$$I_n^{(k)} = -\frac{2n(2n-1)}{k^2\pi^2} I_{n-1}^{(k)}.$$

For k fixed, the lemma now follows by induction on n. □

4.2 Evaluating the integral with $D_{2m}(t)$ as a sum of cosines

Using the second representation for the Dirichlet kernel in (1.4.1) and then interchanging the order of integration and summation, we derive that

$$\begin{aligned}
\int_0^1 B_{2n}(t)\, D_{2m}(\pi t)\, dt &= \int_0^1 B_{2n}(t)\left(1 + 2\sum_{k=1}^{2m}\cos(k\pi t)\right) dt \\
&= \int_0^1 B_{2n}(t)\, dt + 2\sum_{k=1}^{2m}\int_0^1 B_{2n}(t)\cos(k\pi t)\, dt \\
&= 2\sum_{\substack{k=1 \\ k \text{ even}}}^{2m}\int_0^1 B_{2n}(t)\cos(k\pi t)\, dt.
\end{aligned}$$

By (4.1.2), the integral of the Bernoulli polynomial vanishes and by Lemma 5 the terms with k odd are zero. In the remaining sum over even k, we make the substitution $k = 2\ell$ and again use Lemma 5 to find that

$$\begin{aligned}\int_0^1 B_{2n}(t)\, D_{2m}(\pi t)\, dt &= 2\sum_{\ell=1}^{m}\int_0^1 B_{2n}(t)\cos(2\ell\pi t)\, dt \\ &= 2\frac{(-1)^{n-1}(2n)!}{(2\pi)^{2n}}\sum_{\ell=1}^{m}\frac{1}{\ell^{2n}}.\end{aligned} \tag{4.2.1}$$

4.3 Evaluating the integral with $D_{2m}(t)$ as a ratio of sines

For fixed n, our goal is to show that

$$\int_0^1 B_{2n}(t)\, D_{2m}(\pi t)\, dt = B_{2n} + O\Big(\frac{1}{m}\Big). \tag{4.3.1}$$

Since $B_{2n}(0) = B_{2n}$, it follows that

$$B_{2n}(t) = B_{2n} + \big(B_{2n}(t) - B_{2n}(0)\big) = B_{2n} + t\, P_n(t)$$

for some polynomial $P_n(t)$. The second representation for the Dirichlet kernel in (1.4.1) implies that

$$\int_0^1 D_{2m}(\pi t)\, dt = \int_0^1 \Big(1 + 2\sum_{k=1}^{2m}\cos \pi k t\Big)\, dt = 1.$$

Hence, using the definition of $P_n(t)$ and the third representation for $D_m(t)$ in (1.4.1), we derive that

$$\begin{aligned}\int_0^1 B_{2n}(t)\, D_{2m}(\pi t)\, dt &= B_{2n}\int_0^1 D_{2m}(\pi t)\, dt + \int_0^1 t\, P_n(t)\, D_{2m}(\pi t)\, dt\\ &= B_{2n} + \int_{0^+}^1 t\, P_n(t)\, \frac{\sin\left((4m+1)\pi t/2\right)}{\sin\left(\pi t/2\right)}\, dt,\end{aligned}$$

where 0^+ indicates the right-hand limit as we approach 0. Integrating by parts, we find that

$$\begin{aligned}&\int_{0^+}^1 t\, P_n(t)\, \frac{\sin\left((4m+1)\pi t/2\right)}{\sin\left(\pi t/2\right)}\, dt\\ &\quad= \left(\frac{t\, P_n(t)}{\sin\left(\pi t/2\right)}\right) \frac{2\cos\left((4m+1)\pi t/2\right)}{\pi(4m+1)}\bigg|_{0^+}^1 - \int_{0^+}^1 \frac{d}{dt}\left\{\frac{t\, P_n(t)}{\sin\left(\pi t/2\right)}\right\} \frac{2\cos\left((4m+1)\pi t/2\right)}{\pi(4m+1)}\, dt.\end{aligned}$$

Letting $f(t) = t/\sin(\pi t/2)$, this equals

$$f(t)\, P_n(t)\, \frac{2\cos\left((4m+1)\pi t/2\right)}{\pi(4m+1)}\bigg|_{0^+}^1 - \int_{0^+}^1 \left(f'(t)\, P_n(t) + f(t)\, P_n'(t)\right) \frac{2\cos\left((4m+1)\pi t/2\right)}{\pi(4m+1)}\, dt.$$

A standard calculus exercise shows that

$$\frac{2}{\pi} < f(t) \le 1 \quad \text{and} \quad 0 < f'(t) \le 1$$

for $0 < t \le 1$. Thus, recalling that n fixed, we conclude that

$$\int_{0^+}^1 t\, P_n(t)\, \frac{\sin\left((4m+1)\pi t/2\right)}{\sin\left(\pi t/2\right)}\, dt = O\Big(\frac{1}{m}\Big).$$

Combining estimates, we have proved (4.3.1).

4.4 Finishing the proof

Equating the expressions in (4.2.1) and (4.3.1), we have shown that

$$2\frac{(-1)^{n-1}(2n)!}{(2\pi)^{2n}}\sum_{\ell=1}^{m}\frac{1}{\ell^{2n}} = B_{2n} + O\Big(\frac{1}{m}\Big).$$

Letting $m \to \infty$, we now see that

$$\sum_{\ell=1}^{\infty}\frac{1}{\ell^{2n}} = (-1)^{n-1}\frac{(2\pi)^{2n}B_{2n}}{2(2n)!}$$

for every $n \in \mathbb{N}$.

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