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Essays on bubbles, production, and anti-bubble policy

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ESSAYS ON BUBBLES, PRODUCTION, AND ANTI-BUBBLE POLICY

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ABSTRACT

This dissertation provides two papers on asset market bubbles. The first chapter analyzes the welfare effect of anti-bubble policy in a macroeconomic model containing both an asset market and a goods market. Overall, this chapter shows that anti-bubble policy decreases the welfare of asset seller, increases the welfare of asset buyers, and has no effect on the welfare of the production side of the goods market. The second chapter provides examples of strong greater-fool bubbles with three states of the world and three periods. It provides examples of strong greater-fool bubbles at various levels of endowment durability, as well as an example with a one-period bond issued in period 2.

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CHAPTER 1

Flexible Asset Markets and Sluggish Goods Markets: Lessons from a Simple Macroeconomic Model

1.1 Introduction

Should monetary authorities attempt to suppress asset price bubbles, or even burst them outright? Around the time of the United States tech bubble, Bernanke and Gertler (1999) and Bernanke and Gertler (2001) examined this question in the context of a macroeconomic model with *exogenous* bubbles and warned that setting interest rates in response to asset price movements, i.e., asset price targeting, would increase volatility in output and inflation. Cecchetti et al. (2000) and Cecchetti et al. (2002) examine a similar model and find that asset price targeting is a necessary component of monetary authorities' optimal interest rate rule. However, Roubini (2006) criticized these models for their use of exogenous bubbles rather than *endogenous* bubbles.

Using models of endogenous bubbles like those in Samuelson (1958) and Tirole (1985), Gali (1977) examines the suppression of these bubbles through interest rate hikes in an overlapping generations (OLG) macroeconomic model, with sticky goods prices. However, these OLG bubbles require an infinite horizon because people must believe that these bubbles will last forever, at least in expected value. In contrast, real-world bubbles seem to be characterized by more short-run, finite-horizon speculative activity. Thus, the overpricing in Samuelson (1958), Tirole (1985), and Gali (1977) may be more descriptive of inter-generational saving rather than the sort of short-run, speculative fluctuation of concern to policymakers.

In greater-fool bubble models, like those of Allen et al. (1993), Conlon (2004),Conlon (2015), Holt (2018), and Liu and Conlon (2018), agents trade a bubble-prone asset even though they know that the bubble will only last a finite number of periods. These models may therefore better capture the sorts of short-run fluctuations relevant to counter-cyclical policy. Both Conlon (2015) and Holt (2018) examine the welfare implications of a monetary authority which bursts these greater-fool bubbles by releasing information about their over-pricing. However, unlike the welfare analysis in Gali (1977), Conlon (2015) and Holt (2018) focus on welfare in asset markets themselves and do not examine spillover effects on welfare in goods markets with sluggish goods prices.

We examine the welfare effects of anti-bubble policy in a finite-horizon, greater fool bubble model that includes a goods market where prices are sticky, as in Gali (1977),. This model captures the Keynesian notion of an economy with volatile asset markets and sluggish goods markets. Unlike Gali (1977), our bubble has a greater-fool flavor and can exist in a finite horizon environment, which makes it more like the speculative bubbles in Conlon (2015) and Holt (2018), rather than the OLG bubbles in Tirole (1985) and Samuelson (1958). Our bubble model should also capture the short-run nature of real world bubbles.

The long-term goal of our research is to examine the welfare effects of anti-bubble policy in a macroeconomic model containing both an asset market, where a bubble may exist, and a goods market, where prices of the goods are set through Fischer (1977)-style long-term price contracts. Thus, the full model will have an asset market with flexible prices and a goods market with sticky prices.

As a first step, we separate the asset market from the goods market and consider the welfare implications of anti-bubble policy in the asset market itself. In order to more easily incorporate a volatile asset market into the broader macroeconomic model, we examine an asset market that experiences a "semi-bubble," rather than a "strong bubble" like that in Allen et al. (1993). We say that an asset experiences a semi-bubble if it is overpriced even though *some* people know that it is overpriced (see Holt (2018) who calls this a "weak bubble"). By contrast, an asset experiences a strong bubble if it is overpriced, and *everyone* knows that it is overpriced (Allen et al. (1993)). Previous models, like Conlon (2015) and

Holt (2018), have examined the welfare impact of deflating overpriced assets in models with strong bubbles, where trading of the overpriced asset is motivated by gains to *cross-state consumption smoothing*. The present analysis, however, examines the welfare effects on the asset market of bursting a semi-bubble in a model where trading of the overpriced asset is motivated by *intertemporal* consumption smoothing, as in Liu and White (2018), instead of cross-state consumption smoothing. That is, trade in this paper is motivated by agents' desire to stabilize their consumption paths over time by transferring consumption from high-income periods to low-incomes periods.

As in Conlon (2015) and Holt (2018), the anti-bubble policy of the current paper takes the form of "pricking" bubbles by revealing information rather than "leaning against the wind" by using interest rate policy to suppress bubbles. There are two states of the world in our model, and the asset is overpriced in one of these states. The seller always knows whether or not the asset is overpriced. We also assume that whatever is known to the seller is also known to the central bank. Under an anti-bubble policy, if the asset is overpriced, then the central bank announces this information to the asset market.[1] If the asset is not overpriced, then the lack of an announcement from the central bank is an implicit endorsement of the asset's value. Note that our model of policy is simpler than a model where a central bank suppresses a bubble by raising interest rates – that is, by selling assets which compete with the return on the overpriced asset.[2]

Overall, I find that deflating the overpriced asset tends to hurt risk-averse asset sellers

[1]Note that the introduction of derivatives may also transmit information about whether the derivative's underlying asset is overpriced. For example, Fostel and Geanakoplos (2012) noticed that the fall in US homes prices in 2006 immediately followed the introduction of publicly traded CDS contracts. More recently, Hale et al. (2018) noticed that the crash in the price of Bitcoin coincided with the introduction of Bitcoin futures on the CME exchange. Thus, the information-revelation policy in this paper could be re-interpreted simply as a policy which encourages the development of markets which better transmit information from informed investors to others.

[2]Future research should examine this more realistic type of anti-bubble policy in finite-horizon models like ours. Gali (1977) finds that increasing interest rates in his OLG model in response to a bubble tends to exacerbate the bubble's growth, which leads him to question the welfare benefits of anti-bubble policy. More recently, Allen et al. (2018) examine interest rate policy in a risk-shifting model of loans with default costs. They find that ex ante welfare is improved when a central bank commits to raising interest rates in response to a developing bubble, even when the social costs of default are small.

in the market as long as their underlying endowment of the consumption good is constant, rather than correlated with the future return on the asset. On the other hand, in contrast to the results of Holt (2018), where trade is motivated by risk-sharing rather than intertemporal consumption smoothing, the current model finds that a policy of deflating overpriced assets helps asset buyers.[3] Specifically, risk-averse asset buyers are always helped, ex ante, by anti-bubble policy when they have decreasing absolute risk aversion (or DARA) preferences.

Ongoing work is examining the welfare implications for goods markets when anti-bubble policy enables information possessed by better-informed asset sellers to be transmitted to goods-price setters who must set prices in advance.

The next section presents the model of an asset market containing a semi-bubble. Section 3 examines the welfare impact of anti-bubble policy on risk-averse asset-sellers. Section 4 examines the welfare impacts of anti-bubble policy on risk-averse asset buyers. Section 5 discusses future work on models which include goods markets with prices set in advance.

1.2 A Weak Bubble with Intertemporal Consumption Smoothing

The model presented builds on assumptions similar to those from previous greater-fool bubble papers, namely Allen et al. (1993), Conlon (2004), Conlon (2015), Holt (2018), Zheng (2013), Liu and Conlon (2018), and more recently,Liu and White (2018). The asset market is Walrasian, and it lasts for two periods. There are two agents: the asset buyer and the asset seller. For ease of exposition, the seller will be given the female pronoun "she", and the buyer will be given the male pronoun "he". As in Liu and White (2018), agents consume a perishable consumption good in every period, which this paper will call "apples." There is no money in our model, but agents can trade shares of the asset X, which we call apple trees, in exchange for apples. Apple trees are the only asset in the model, and they are risky. An apple tree could pay one bushel of apples in the future per tree owned (i.e.,

[3]This result also contrasts with the three-period example inLiu and White (2018) where the ultimate greater-fool asset buyer can be hurt by anti-bubble policy, since it interferes with his intertemporal consumption smoothing.

per share), or it could pay nothing.

At the beginning of period 1, the seller possesses more information about the risky asset's value than the buyer does. She also possesses all of the available shares of the asset, S, while the buyer initially possesses no shares. There are two states of the world, called L and H. Thus, the state space is $\Omega = \{L, H\}$, and the prior probabilities for each state are π_L and π_H, respectively.

In state L, the asset pays nothing in period 2; in other words, its period-2 dividend is zero. In state H, the asset pays a period-2 dividend of 1 unit of the consumption good (a bushel of apples) for each share of the asset owned.[4] Figure 1 shows the period-1 information structure of both agents. In period 1, the seller knows the true state with certainty, so she

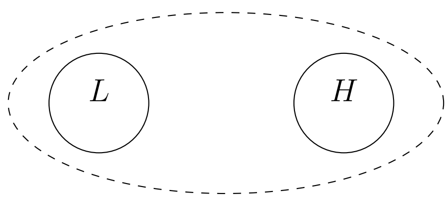

Figure 1.1: Period-1 Information Structure

can distinguish L from H. On the other hand, the buyer is unable to distinguish L from H in period 1.

Note that, the existence of asymmetric information is not pivotal for our results. Our results would still hold even if the asset seller does not know the true state of the world, and so, had the same information set as the asset buyer. Instead, our results are driven by uncertainty about the true state of the world as it pertains to the risky asset's return. However, since the central bank must know the true state of the world in order to implement anti-bubble policy, it makes some sense that some segments of the asset market are more informed than others (namely, sellers). Otherwise, we would be making the lofty assumption that the central bank knows more about the asset market than *every* asset-market participant.

[4]Note, the asset does not pay a dividend in period 1 in either state of the world.

For ease of notation, assume buyers and sellers have the same utility function over consumption of apples, $u(c)$, though nothing below depends on this. Thus, the lifetime utility for the buyer is $U^B = u(c_1^B) + \beta u(c_2^B)$, and the lifetime utility for the seller is $U^S = u(c_1^S) + \beta u(c_2^S)$. In addition to apples from the traded apple trees, agents receive perishable endowments of apples in each period from trees which cannot be traded. In period 1, the buyer receives an endowment of $e_1^B(L) = e_1^B(H) = e_1^B$ bushels of apples. The seller receives a period-1 endowment of $e_1^S(L) = e_1^S(H) = e_1^S$ bushels of apples. Furthermore, the buyer and the seller receive period-2 endowments of $e_2^B(L) = e_2^B(H) = e_2^B$ and $e_2^S(L) = e_2^S(H) = e_2^S$, respectively. Notice that agents' endowments in period t are the same in both states of the world L and H.[5]

In the equilibria that are considered below, we assume e_1^S is sufficiently small relative to e_2^S that the seller turns out to sell all the shares in her possession, S, in both states L and H of period 1. Thus, the seller sells the asset in state L of period 1 because she knows it is worthless (the dividend is zero in period 2), and she sells the asset in state H of period 1 to augment her small period-1 endowment of the consumption good, and so, smooth her consumption intertemporally over periods 1 and 2. Therefore, the asset seller could be said to have a large "liquidity demand" in period 1, state H (see Liu and White (2018)). The buyer, on the other hand, cannot distinguish state L from state H at the beginning of period 1. Also, because the seller will sell S shares in both states of the world, the buyer will remain unable to distinguish L from H in period 1, even after observing the seller's behavior. Thus, a lemons problem is present in period 1, but trade will nevertheless occur if we also assume that e_2^B is so small relative to e_1^B that the buyer is willing to risk purchasing a worthless asset in hopes of adding to his small period-2 endowment of the consumption good, and so, making his state H consumption, at least, intertemporally smoother.

[5]This assumption must be true for the buyer in period 1 in order for a semi-bubble to exist. Otherwise, he could distinguish L from H in period 1, and he would have no reason to purchase the asset in state L. However, period 1 endowments could differ for the seller while still allowing a semi-bubble in period 1. However, in order to focus on the role of intertemporal consumption smoothing in the analysis, the equal-endowments assumption will be maintained for both agents in both periods throughout the paper, though see footnote 7 below.

Finally, note the very extreme assumptions that we make to simplify the model. There is only one asset, and so no money, the consumption is entirely perishable, and there is no short-selling of the risky asset. Also, we choose parameters below so that sellers are always at their short-sale constraint in period 1, even when they know the asset is valuable. Thus, trading is very simple, and its only effect on period 2 welfare is in determining who gets the potential period-2 dividend. Future research should therefore incorporate money as well as other assets into the model. Note, however, that this will dramatically complicate the effect of period 1 policy on period 2 wealth and consumption.

1.2.1 The Buyer's Decision

1.2.1.1 The Buyer in the Absence of an Anti-Bubble Policy

In order for a weak bubble to exist in state L, period 1, the buyer must be willing to purchase the asset even though he knows it may be worthless. It is therefore assumed that e_1^B is sufficiently large relative to e_2^B so that the buyer has a strong motive to smooth his state-H consumption over periods 1 and 2 by giving up some consumption in period 1, to buy the asset, in order to consume the dividend in period 2. What's more, we make the buyer's gains from smoothing his state-H consumption over periods 1 and 2 so large that he is willing to purchase the asset even though the true state might be L.

In period 1, the buyer receives an endowment e_1^B, and he purchases X shares from the seller at a per-share price of P. In the absence of an anti-bubble policy, the buyer's period-1 consumption is the same across states L and H, so $c_1^B(L) = c_1^B(H) = c_1^B = e_1^B - PX$. In period 2, the buyer consumes e_1^B plus any dividends that he receives from the asset. Thus, his period-2 consumption in state L is simply $c_2^B(L) = e_2^B$, and his period-2 consumption in state H is $c_2^B(H) = e_2^B + X$, since the dividend is 1 unit of consumption for each share that is owned

The buyer chooses X so as to maximize his expected lifetime utility, given in (1.1).

$$E(U^B) = u(e_1^B - PX) + \pi_L \beta u(e_2^B) + \pi_H \beta u(e_2^B + X). \tag{1.1}$$

The buyer's first order condition (FOC) with respect to X determines his demand for the asset, and it therefore also helps to determine the buyer's equilibrium willingness-to-pay (WTP) for a share of the asset. We choose parameters such that, in equilibrium, the asset buyer purchases all of the available shares of the risky asset, so $X = S$. As a result, the buyer's WTP determines the equilibrium asset price, so $P = WTP_B$, where WTP_B is the solution to the the buyer's equilibrium FOC as shown in (1.2) below and B stands for "bubble." Since the buyer's FOC with respect to X is

$$u'(e_1^B - PX)P = \pi_H \beta u'(e_2^B + X),$$

the equilibrium asset price in the absence of anti-bubble policy, or P_{NP}, is determined by

$$u'(e_1^B - P_{NP}S)P_{NP} = \pi_H \beta u'(e_2^B + S) \tag{1.2}$$

where the subscript "NP"stands for "No Policy".

Under a bubble equilibrium, the buyer's ex-post welfare in state L is

$$u(e_1^B - P_{NP}S) + \beta u(e_2^B), \tag{1.3}$$

while his ex-post welfare in state H is

$$u(e_1^B - P_{NP}S) + \beta u(e_2^B + S). \tag{1.4}$$

The asset buyer's welfare is clearly higher in state H than it is in state L, since the asset pays a dividend in period 2 of state H. Ex ante, the asset buyer's expected welfare, given that he buys all S shares, is

$$EW^B_{NP} = u(e^B_1 - P_{NP}S) + \pi_L \beta u(e^B_2) + \pi_H \beta u(e^B_2 + S). \tag{1.5}$$

1.2.1.2 The Buyer in the Presence of an Anti-Bubble Policy

Under an anti-bubble policy, the central bank reveals whether the asset's dividend is zero, resulting in a "no-bubble equilibrium." As such, the policy enables the asset buyer to distinguish L from H. In state L, the buyer has no reason to purchase the asset since it pays no dividend, so his state L welfare under the policy is

$$u(e^B_1) + \beta u(e^B_2). \tag{1.6}$$

This is larger than his state-L welfare in the absence of a policy, since under the policy, the buyer does not waste resources in state L, purchasing an asset that will not pay a dividend in the future.

In state H under the policy, the buyer knows for sure that the asset will pay a dividend, so he continues to purchase all of the seller's shares. This means that the asset buyer's first order condition under the anti-bubble policy, after substituting in $X = S$, is

$$u'(e^B_1 - P_P S)P_P = \beta u'(e^B_2 + S), \tag{1.7}$$

where the subscript "P "stands for "Policy". Intuitively, the buyer's WTP in state H in the presence of anti-bubble policy, which is determined by (1.7), should be larger than WTP_B since anti-bubble policy enables the buyer to know with certainty that the asset will pay a dividend. That is, we should have $P_P > P_{NP}$, which is, in fact, what we find (see Lemma I below). The buyer's welfare in state H of the policy is then

$$u(e^B_1 - P_P S) + \beta u(e^B_2 + S), \tag{1.8}$$

which is less than his state-H welfare in the absence of a policy, as shown in (1.4), since P_P is larger than P_{NP}. The buyer's overall ex ante welfare under the policy is

$$EW_P^B = \pi_H \left(u(e_1^B - P_P S) + \beta u(e_2^B + S) \right) + \pi_L \left(u(e_1^B) + \beta u(e_2^B) \right). \tag{1.9}$$

In summary, the buyer will be hurt in state H by the anti-bubble policy, but helped in state L. Thus, it is not immediately clear whether the policy improves or harms the buyer's overall ex ante welfare.

1.2.1.3 Asset Price Comparisons Across Equilibria

As discussed in the previous subsection, it is intuitively clear that the state-H asset price under the policy, or P_P, should be larger than the asset price in the absence of policy, or P_{NP}. Anti-bubble policy eliminates any risk surrounding the asset's dividend. So if the true state is H, the buyer's WTP for the asset is larger under the policy than without the policy. The assumption of Walrasian markets ensures that the equilibrium asset price equals the buyer's WTP which leads to Lemma I.

Lemma I: $P_P > P_{NP}$.

Proof. First, subtract the FOC in (2) from the FOC in (7)

$$u'(e_1^B - P_P S)P_P - u'(e_1^B - P_{NP} S)P_{NP} = \pi_L \beta u'(e_2^B + S) > 0. \tag{1.10}$$

This must be positive since the marginal utility of consumption is positive.

Since the buyer is assumed to be risk-averse, $u'(c)$ is decreasing in c, which means that $u'(e_1^B - PS)P$ is increasing in P. This, in conjunction with (1.10), implies $P_P > P_{NP}$. $\square$

If the FOC (1.7) is used to substitute for $\beta u'(e_2^B + S)$ in the FOC (1.2), then a relationship between the policy and no-policy asset prices can be determined, as

$$u'(e_1^B - P_{NP} S)P_{NP} = \pi_H u'(e_1^B - P_P S)P_P. \tag{1.11}$$

Lemma II: $\pi_H P_P < P_{NP}$.

Proof. First, $P_P > P_{NP}$, by Lemma 1, which implies $e_1^B - P_P S < e_1^B - P_{NP} S$. Risk aversion, or $u'(c)$ decreasing in c, then implies $u'(e_1^B - P_P S) > u'(e_1^B - P_{NP} S)$. Finally, rewrite (1.11) as

$$\frac{\pi_H P_P}{P_{NP}} = \frac{u'(e_1^B - P_{NP} S)}{u'(e_1^B - P_P S)} < 1,$$

which implies Lemma II. □

Intuitively, $\pi_H P_P$ is the ex ante expected price that the buyer pays for the asset in a no-bubble equilibrium. Likewise, P_{NP} is the price of the asset in a bubble equilibrium. Thus, Lemma II says that the ex ante expected asset price under the policy, before the one knows whether the state is L or H, is less than the asset price in the absence of policy. This is because the buyer is poorer in period 1, state H of the no-bubble equilibrium relative to the bubble equilibrium, which reduces the price he is willing to pay, P_P, to the point that $\pi_H P_P < P_{NP}$.

1.2.2 The Seller's Decision

To ensure that the asset seller sells all her shares in both states of the world in the bubble equilibrium, it is assumed that e_2^S is very large relative to e_1^S. Thus, she sells the asset in state L in period 1 because she knows it is worthless, but she sells it in state H in period 1 as a means of transferring consumption from period 2 to period 1, in order to supplement her smaller period-1 endowment of the consumption good. That is, as explained above, the asset seller has a large liquidity demand in period 1, state H.

In the absence of a policy, the seller's period-1 consumption in both states L and H of the bubble equilibrium equal $c_1^S = e_1^S + P_{NP} S$. Since she sells all of her shares in period 1, the asset seller's period-2 consumption is simply $c_2^S = e_2^S$ for both L and H in the absence of a policy. She will always sell all her shares in state L since she knows the dividend will be zero. Furthermore, she will always sell all of her shares in state H as long as

$$u'(e_1^S + P_{NP}S)P_{NP} \geq \beta u'(e_2^S), \tag{1.12}$$

that is, if the period-1 marginal utility of selling an additional share is greater than or equal to the period-2 marginal utility of consuming the dividend when the seller sells her entire stock, S, of shares. Thus, while the buyer determines the equilibrium asset price, the seller determines the volume of shares traded (i.e., S). The seller's ex ante expected welfare in the absence of policy is

$$EW_{NP}^S = u(e_1^S + P_{NP}S) + \beta u(e_2^S). \tag{1.13}$$

In state L under the policy, the buyer knows the asset is worthless, so he does not purchase any shares from the seller. This means that the seller's state-L consumption in period 1 is $c_1^S(L) = e_1^S$. Since the asset pays no dividend, the seller's period-2 consumption in state-L remains $c_2^S(L) = e_2^S$. The seller's state-L welfare under the policy is therefore

$$u(e_1^S) + \beta u(e_2^S), \tag{1.14}$$

which is less than her state-L welfare in the absence of an anti-bubble policy.

In state H under the policy, the seller continues to sell all of her shares, S, in period 1 as long as

$$u'(e_1^S + P_P S)P_P \geq \beta u'(e_2^S). \tag{1.15}$$

Note that the condition in (1.12) does not necessarily imply the condition in (1.15), or visa-versa. However, we assume parameters so that short-sale constraints in (1.12) and (1.15) are jointly true.

Under the policy, the seller's period-2 consumption in state H continues to be $c_2^S(H) = e_2^S$. However, her period-1 consumption in state H, $c_1^S(H) = e_1^S + P_P S$, is larger than her period-1 consumption in state H without the policy since $P_P > P_{NP}$. Thus, the seller's state-H welfare under the policy,

$$u(e_1^S + P_P S) + \beta u(e_2^S), \tag{1.16}$$

is greater than her state-H welfare in the bubble equilibrium, since $P_P > P_{NP}$.

In summary, the asset seller's welfare is hurt by an anti-bubble policy in state L, but her welfare is improved by an anti-bubble policy in state H, since she gets a higher price. The seller's overall ex ante expected welfare under the policy is

$$EW_P^S = \pi_H \left(u(e_1^S + P_P S) + \beta u(e_2^S)\right) + \pi_L \left(u(e_1^S) + \beta u(e_2^S)\right). \tag{1.17}$$

As with the buyer, it is not immediately clear whether the policy improves or harms the asset seller's ex ante welfare. We consider this issue next.

1.3 Anti-Bubble Policy and the Asset Seller's Ex Ante Welfare

To understand how anti-bubble policy affects the seller's ex ante welfare, it's necessary to compare EW_P^S from (1.17) with EW_{NP}^S from (1.13). Define ΔEW^S as the difference in welfare across the two different policies. Subtracting (1.13) from (1.17) and simplifying gives

$$\begin{aligned}\Delta EW^S &= EW_P^S - EW_{NP}^S \\ &= \pi_H \left[u(e_1^S + P_P S) - u(e_1^S + P_{NP} S)\right] + \pi_L \left[u(e_1^S) - u(e_1^S + P_{NP} S)\right]\end{aligned} \tag{1.18}$$

which can be rewritten as the difference of two weighted integrals, or

$$E\Delta W^S = \pi_H \int\limits_{e_1^S + P_{NP} S}^{e_1^S + P_P S} u'(c)dc - \pi_L \int\limits_{e_1^S}^{e_1^S + P_{NP} S} u'(c)dc.$$

If $\Delta EW^S > 0$, then anti-bubble policy improves the seller's ex ante welfare, and if $\Delta EW^S < 0$, then anti-bubble hurts the seller's ex ante welfare.

A few point are in order in regards to (1.18). First, ΔEW^S is entirely dependent on the seller's period-1 consumption. Her utilities from period-2 consumption cancel out. This means that anti-bubble policy has no effect on the seller's period-2 welfare. This is intuitive since her period-2 consumption behavior is the same in both equilibria. Second, as previously discussed, anti-bubble policy helps the seller in state H by raising the asset price, so the first term on the right hand side of (1.18), $\pi_H \left[u(e_1^S + P_P S) - u(e_1^S + P_{NP} S)\right]$, is positive. However, anti-bubble policy hurts the seller in state L since it prevents her from exploiting the buyer, so the second term on the right hand side of (1.18), $\pi_L \left[u(e_1^S) - u(e_1^S + P_{NP} S)\right]$, is negative. Thus, by examining (1.18), it is not obvious whether the seller's period-1 ex ante welfare is helped or hurt by anti-bubble policy.

Figure 2 graphs the seller's period-1 marginal utility (MU_S) curve. Notice that it is downward-sloping indicating her risk-aversion. In the bubble equilibrium, the seller's period-1 consumption is $e_1^S + P_{NP} S$ in both states L and H. Thus, her period-1 utility in the bubble equilibrium is the area under her marginal utility curve from point B to the vertical axis.

Under the anti-bubble policy, the seller's state-L consumption is lowered to e_1^S. This means that her state L utility under the policy is the smaller area under MU_S from point A to the vertical axis. The seller's state-H consumption under the policy is raised to $e_1^S + P_P S$, with $P_P > P_{NP}$, which means her state-H utility is the larger area under MU_S from point C to the vertical axis.

Thus, in state L, anti-bubble policy causes the seller to lose the area under MU_S from B to A. This area equals

$$u(e_1^S) - u(e_1^S + P_{NP} S) = \int\limits_{e_1^S}^{e_1^S + P_{NP} S} u'(c)dc.$$

In state H, however, the anti-bubble policy causes the seller to gain the area under

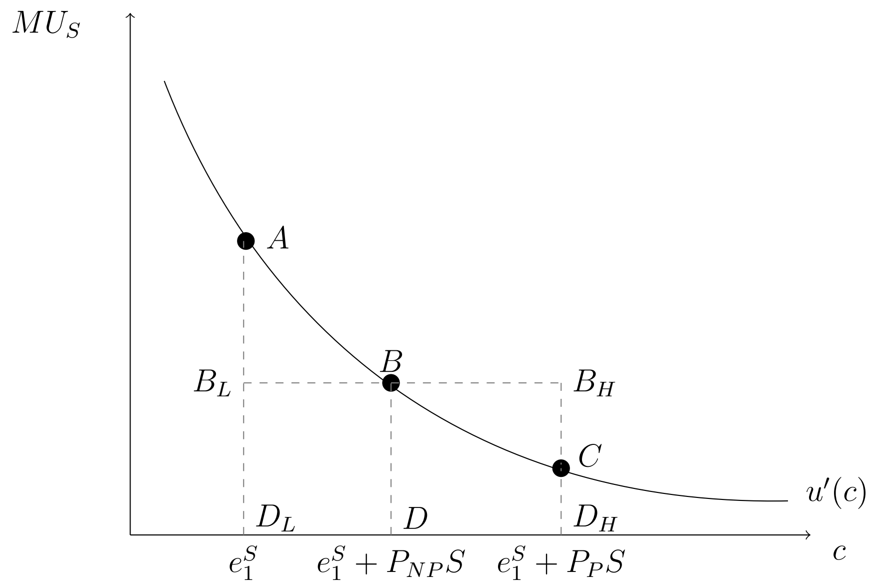

Figure 1.2: The Seller's period-1 Marginal Utility Curve (MU_S)

MU_S from B to C. This area equals

$$u(e_1^S + P_PS) - u(e_1^S + P_{NP}S) = \int_{e_1^S+P_{NP}S}^{e_1^S+P_PS} u'(c)dc.$$

To determine the overall effect on the seller's welfare, we build on Holt (2018) by first examining the impact of anti-bubble policy when changes in the seller's wealth are evaluated using marginal utilities for the seller which are held constant at $u'(e_1^S + P_{NP}S)$ (see point B in Figure 2). This produces an estimate of ΔEW^S which we'll call $\widetilde{\Delta EW^S}$. In state L, the policy causes the seller to lose

$$u'(e_1^S + P_{NP}S)P_{NP}S \tag{1.19}$$

when MU_S is held constant at point B. In Figure 2, this is the area of the rectangle B_LBDD_L. This is an underestimate of the true loss, which is the area of the curvi-linear trapezoid $ABDD_L$. Thus, the state-L loss component in (1.19) underestimates the true state-L loss from anti-bubble policy by the amount,

$$\int_{e_1^S}^{e_1^S+P_{NP}S} u'(c)dc - u'(e_1^S + P_{NP}S)P_{NP}S > 0. \tag{1.20}$$

This is the area of the curvi-linear triangle ABB_L.

In the state H, the policy causes the seller to gain

$$u'(e_1^S + P_{NP}S)(P_P - P_{NP})S \tag{1.21}$$

where again MU_S is held constant at point B. In Figure 2, this is the area of the rectangle BB_HD_HD. This is an overestimate of the seller's true gain, which is the area of the curvi-linear trapezoid BCD_HD. Thus, the state-H gain component of (21) overestimates the true gain from anti-bubble policy by the amount,

$$u'(e_1^S + P_{NP}S)(P_P - P_{NP})S - \int_{e_1^S+P_{NP}S}^{e_1^S+P_PS} u'(c)dc > 0. \tag{1.22}$$

This is the area of the curvi-linear triangle BB_HC.

The overall ex ante welfare impact, when MU_S is held constant at $u'(e_1^S + P_{NP}S)$, is then

$$\widetilde{\Delta EW^S} = -\pi_L u'(e_1^S + P_{NP}S)P_{NP}S + \pi_H u'(e_1^S + P_{NP}S)(P_P - P_{NP})S, \tag{1.23}$$

which can be simplified to

$$\widetilde{\Delta EW^S} = -u'(e_1^S + P_{NP}S)(P_{NP} - \pi_H P_P)S < 0,$$

where the inequality follows since $\pi_H P_P < P_{NP}$ by Lemma II. Thus, the seller's ex ante welfare is harmed by the policy when MU_S is held constant at $u'(e_1^S + P_{NP}S)$. The additional effect of the two curvi-linear triangles in (1.19) and (1.22), comprise the "Hirshleifer effect" of receiving additional information, which is the welfare loss to agents after becoming more informed (Hirshleifer (1971); Holt, 2018).[6] Combining $\widetilde{\Delta EW^S} < 0$ with this Hirshleifer effect leads to Proposition I.

Proposition I: The seller is always hurt by anti-bubble policy, so

$$E\Delta W^S = \pi_H \int_{e_1^S+P_{NP}S}^{e_1^S+P_PS} u'(c)dc - \pi_L \int_{e_1^S}^{e_1^S+P_{NP}S} u'(c)dc < 0.$$

Proof. We already know that $\widetilde{\Delta EW^S} < 0$. Subtracting π_L times the expression in (1.20) and π_H times the expression in (1.22) from $\widetilde{\Delta EW^S}$ gives ΔEW^S. This implies $\Delta EW^S < \widetilde{\Delta EW^S} < 0$.[7,8] □

1.4 Anti-Bubble Policy and the Asset Buyer's Ex Ante Welfare

1.4.1 The Change in the Buyer's Ex Ante Expected Welfare

As with the asset seller, define $E\Delta W^B$ as the difference in the buyer's ex ante welfare across the different equilibria. Specifically, $E\Delta W^B$ is defined as

$$\Delta EW^B = EW_P^B - EW_{NP}^B$$

or

$$\Delta EW^B = \pi_H u(e_1^B - P_PS) + \pi_L u(e_1^B) - u(e_1^B - P_{NP}S), \tag{1.24}$$

[6]Note that this isn't exactly a Hirshleifer effect since the seller already knows the true state of the world. However, since the information revelation doesn't actually affect the seller's behavior, but only the buyer's behavior, the result is essentially a Hirshleifer effect.

[7]This result may not hold if the seller's period-1 endowment differs between state L and state H. Specifically, if the state-H endowment is low relative to the state-L endowment, then the policy improves the asset's value as a hedge against the seller's endowment, so it may be possible for the seller to benefit from the policy.

[8]This is intuitive since the policy introduces randomness into asset price, while simultaneously decreasing its expectation. As a result, the seller's period-1 consumption falls in expectation, but also becomes random. This implies that the seller's ex ante utility must fall since she is risk averse and has local nonsatiation preferences over consumption. I would like to thank Feng Liu for this helpful insight.

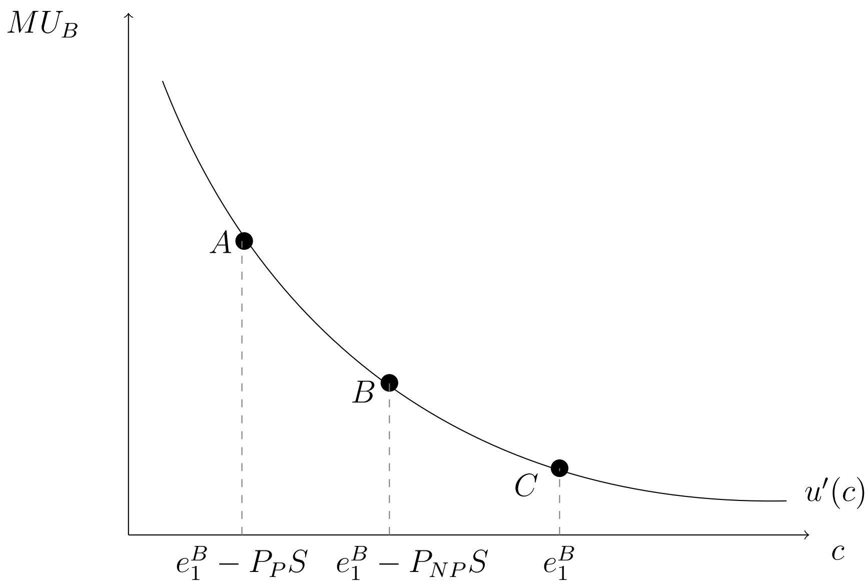

Figure 1.3: The Buyer's period-1 Marginal Utility Curve (MU_B)

which is found by subtracting EW_{NP}^B, in (1.5), from EW_P^B, in (1.9). If $\Delta EW^B > 0$, then the buyer benefits from an anti-bubble policy, and if $\Delta EW^B < 0$, then the buyer is hurt by an anti-bubble policy.

Notice that, as with the seller, the policy has no effect on the buyer's period-2 ex ante welfare, since the policy doesn't affect his period-2 consumption. This means that analyzing the policy's effect on the buyer is equivalent to determining whether the policy lowers or raises his period-1 expected utility. If $\pi_L = 1 - \pi_H$ is substituted into (1.24), then ΔEW^B can be expressed solely in terms of π_H, or

$$\Delta EW^B = \left[u(e_1^B) - u(e_1^B - P_{NP}S)\right] - \pi_H \left[u(e_1^B) - u(e_1^B - P_P S)\right]. \tag{1.25}$$

Examine (1.25) using the buyer's marginal utility curve, or $u'(c)$, in Figure 4. The difference inside the brackets of the first term in (1.25) is the area under $u'(c)$ from point C

to point B. In other words, it is the integral of $u'(c)$ from e_1^B to $e_1^B - P_{NP}S$, or

$$\int_{e_1^B - P_{NP}S}^{e_1^B} u'(c)\ dc = \left[u(e_1^B) - u(e_1^B - P_{NP}S)\right]. \tag{1.26}$$

Now examine the difference inside the brackets of the second term in (1.25). This is the area under $u'(c)$ from point C to point A. In other words, it is the integral of $u'(c)$ from from e_1^B to $e_1^B - P_PS$, or

$$\int_{e_1^B - P_PS}^{e_1^B} u'(c)\ dc = \left[u(e_1^B) - u(e_1^B - P_PS)\right]. \tag{1.27}$$

This means that ΔEW^B may be rewritten in integral form as

$$\Delta EW^B = \int_{e_1^B - P_{NP}S}^{e_1^B} u'(c)\ dc - \pi_H \int_{e_1^B - P_PS}^{e_1^B} u'(c)\ dc. \tag{1.28}$$

1.4.2 The Impact of Anti-Bubble Policy on the Buyer's Ex Ante Welfare

We prove that anti-bubble policy improves the asset buyer's ex ante expected utility in two environments: the case of linear marginal utility and the case of decreasing absolute risk aversion (DARA). The DARA case is the empirically relevant case since actual investors almost certainly have DARA preferences (see Cohn et al. (1975); Friend and Blume (1977)). However, we temporarily also consider the linear case since it uses a different strategy of proof that may provide interesting insights.

1.4.2.1 The Case of Linear Marginal Utility

Figure 4 shows the buyer's marginal utility (MU) curve when it is linear. The area of the trapezoid BCD_LD in Figure 4 is equivalent to the first term, i.e. (1.26), in (1.25) or

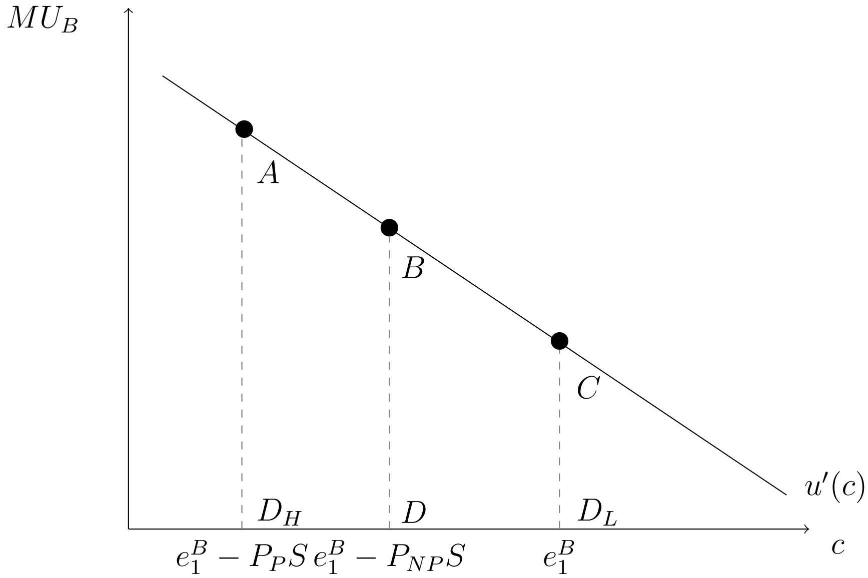

Figure 1.4: The Buyer's period-1 Linear Marginal Utility Curve (MU_B)

(1.28). Thus, in this linear MU case, (1.26) simply equals

$$\int_{e_1^B - P_{NP}S}^{e_1^B} u'(c)\ dc = \frac{1}{2}\left[u'(e_1^B - P_{NP}S) + u'(e_1^B)\right] P_{NP}S, \tag{1.29}$$

using the formula for the area of a trapezoid. Furthermore, π_H times the the area of the trapezoid ACD_LD_H in Figure 4 is equivalent to the second term in (1.25) or (1.28), i.e., π_H times (1.27). In other words, this second term can be rewritten, in the linear MU case, again using the formula for the area of a trapezoid, as

$$\pi_H \int_{e_1^B - P_P S}^{e_1^B} u'(c)\ dc = \frac{\pi_H}{2}\left[u'(e_1^B - P_P S) + u'(e_1^B)\right] P_P S. \tag{1.30}$$

Lastly, subtracting (1.30) from (1.29) gives ΔEW^B for the case of linear marginal utility,

$$\Delta EW^B = \frac{1}{2}\left[u'(e_1^B - P_{NP}S) + u'(e_1^B)\right] P_{NP}S - \frac{\pi_H}{2}\left[u'(e_1^B - P_P S) + u'(e_1^B)\right] P_P S. \tag{1.31}$$

This leads to Proposition II.

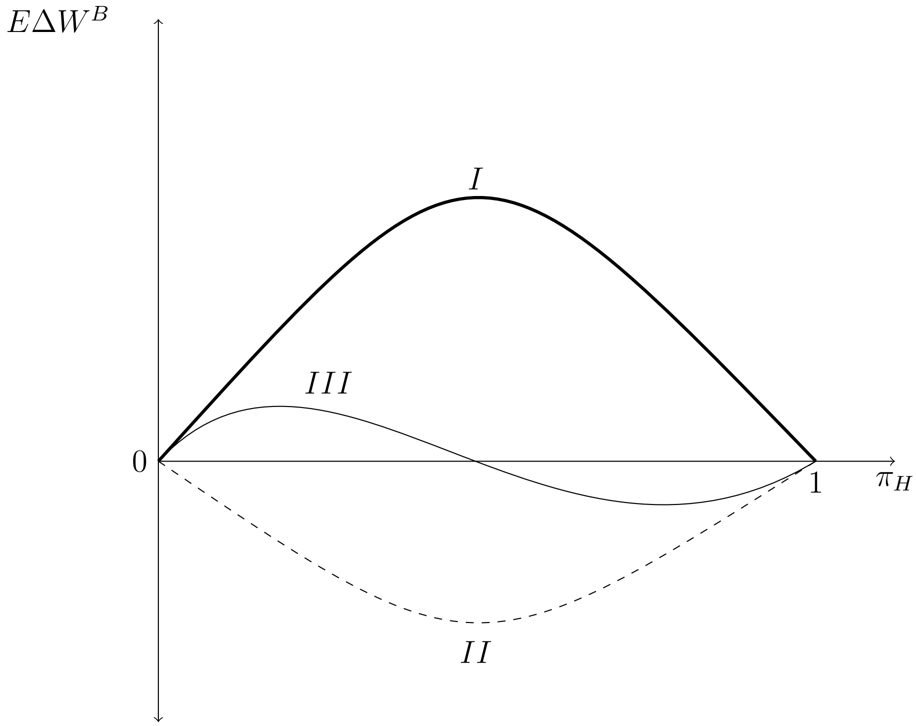

Figure 1.5: Possible Relationships between π_H and ΔEW^B

Proposition II: The asset buyer benefits from anti-bubble policy, so $\Delta EW^B > 0$, when $u'(c)$ is linear.

Proof. First, multiply (1.31) out to get

$$\Delta EW^B = \frac{1}{2}u'(e_1^B - P_{NP}S)P_{NP}S + \frac{1}{2}u'(e_1^B)P_{NP}S - \frac{\pi_H}{2}u'(e_1^B - P_PS)P_PS - \frac{\pi_H}{2}u'(e_1^B)P_PS.$$

Next, apply (1.11) to the third term, and cancel the first and third terms. This simplifies the above expression to

$$\Delta EW^B = \frac{1}{2}u'(e_1^B)P_{NP}S - \frac{\pi_H}{2}u'(e_1^B)P_PS = \frac{1}{2}u'(e_1^B)(P_{NP} - \pi_H P_P)S < 0,$$

where the inequality follows since $\pi_H P_P < P_{NP}$ by Lemma II. $\square$

1.4.2.2 The Case of Decreasing Absolute Risk Aversion (DARA)

Let $R(c)$ be the buyer's coefficient of absolute risk aversion when his consumption is c, so

$$R(c) = -\frac{u''(c)}{u'(c)}.$$

Because we assume that the buyer has decreasing absolute risk aversion, it must be true that

$$R'(c) < 0.$$

This means that $R(c)$ is increasing in P_{NP}.

In order to prove $\Delta EW^B > 0$ for the DARA case, we first determine the relationship between π_H and ΔEW^B. By examining ΔEW^B, from (1.25) above, at $\pi_H = 1$ and $\pi_H = 0$, we find that the relationship between π_H and ΔEW^B must be something like that shown in Figure 5. This is because when $\pi_H = 1$,

$$\Delta EW^B|_{\pi_H=1} = \left[u'(e_1^B) - u(e_1^B - P_P S)\right] - 1 \cdot \left[u(e_1^B) - u(e_1^B - P_P S)\right] = 0,$$

since $P_{NP} = P_P$ when $\pi_H = 1$. Similarly, when $\pi_H = 0$,

$$\Delta EW^B|_{\pi_H=0} = \left[u(e_1^B) - u(e_1^B - 0 \cdot S)\right] + 0 \cdot \left[u(e_1^B) - u(e_1^B - P_P S)\right] = 0,$$

since $P_{NP} = 0$ when $\pi_H = 0$.

Figure 5 graphs three hypothetically possible relationships between ΔEW^B and π_H. Consider first the relationship labeled I, which is the thick inverted U-shaped graph. If this is the true relationship between ΔEW^B and π_H, then $\Delta EW^B > 0$ would be true for all π_H. In other words, if relationship I were the true relation between ΔEW^B and π_H, then anti-bubble policy would always improve the buyer's expected welfare. This implies that if

$$\frac{d^2 \Delta EW^B}{d{\pi_H}^2} < 0 \quad \forall\, \pi_H \in [0,1],$$

then $\Delta EW^B > 0$ must also be true for any $\pi_H \in [0,1]$. Alternatively, if II (or the dashed U-shaped graph) is the true relationship between ΔEW^B and π_H, then the buyer's welfare is always worsened by anti-bubble policy. Lastly, if the true relationship is as in III, which is the thin solid curve between I and II, then it's not possible to place a definitive sign on ΔEW^B. We prove that I is the true relationship between ΔEW^B and π_H. First, we prove a lemma about the effect of π_H on P_{NP}.

Lemma III: P_{NP} is increasing in π_H

Proof. Take the derivative of the FOC in (1.2) with respect to π_H using implicit differentiation. This gives

$$u'\left(e_1^B - P_{NP}S\right)\frac{dP_{NP}}{d\pi_H} - u''\left(e_1^B - P_{NP}S\right)P_{NP}S\frac{dP_{NP}}{d\pi_H} = \beta u'\left(e_2^B + S\right). \tag{1.32}$$

Since marginal utility is always positive and $-u''(c) > 0$, equation (1.32) implies $\frac{dP_{NP}}{d\pi_H} > 0$. □

Lemma III should make intuitive sense. The buyer is willing to pay more for the risky asset when it is more likely that the asset will pay a dividend. A similar intuition applies in Lemma I. In fact, Lemma I is essentially a special case of Lemma III.

Equation (1.32) can be simplified using the definition of absolute risk aversion when the buyer consumes $e_1^B - P_{NP}S$. In other words, using

$$-u''\left(e_1^B - P_{NP}S\right) = R\left(e_1^B - P_{NP}S\right)\cdot u'\left(e_1^B - P_{NP}S\right),$$

(1.32) can be simplified to

$$u'\left(e_1^B - P_{NP}S\right) \cdot \frac{dP_{NP}}{d\pi_H} = \frac{\beta u'\left(e_2^B + S\right)}{1 + R\left(e_1^B - P_{NP}S\right) \cdot P_{NP}S}. \tag{1.33}$$

Lemma IV: For a buyer with DARA risk preferences,

$$\frac{d^2 \Delta EW^B}{d{\pi_H}^2} < 0 \quad \forall\, \pi_H \in (0,1).$$

Proof. First, differentiate (1.25) with respect to π_H and use $\frac{dP_P}{d\pi_H} = 0$. This gives

$$\frac{d\Delta EW^B}{d\pi_H} = u\left(e_1^B - P_P S\right) - u\left(e_1^B\right) + u'\left(e_1^B - P_{NP}S\right) \cdot \frac{dP_{NP}}{d\pi_H} \cdot S. \tag{1.34}$$

Next, substitute (1.33) into this, which gives

$$\frac{d\Delta EW^B}{d\pi_H} = u\left(e_1^B - P_P S\right) - u\left(e_1^B\right) + \frac{\beta u'\left(e_2^B + S\right)}{1 + R\left(e_1^B - P_{NP}S\right) \cdot P_{NP}S} \cdot S. \tag{1.35}$$

Finally, differentiate (1.35) with respect to π_H, again using $\frac{dP_P}{d\pi_H} = 0$. This gives

$$\frac{d^2 \Delta EW^B}{d{\pi_H}^2} = -\frac{\beta u'\left(e_2^B + S\right) \cdot \left[R\left(e_1^B - P_{NP}S\right) - R'\left(e_1^B - P_{NP}S\right) P_{NP}S\right]}{\left[1 + R\left(e_1^B - P_{NP}S\right) \cdot P_{NP}S\right]^2} \cdot \frac{dP_{NP}}{d\pi_H} \cdot S^2 < 0.$$

This is less than zero because $R'(e_1^B - P_{NP}S) < 0$ and because $\frac{dP_{NP}}{d\pi_H} > 0$ by Lemma III. $\square$

Note that the proof in Lemma IV implies that $\frac{d^2 \Delta EW^B}{d{\pi_H}^2} < 0$ is also true when the buyer has constant absolute risk aversion (or CARA), since $R'(c) = 0$ in this case.

Proposition III: When the asset buyer has DARA preferences, $\Delta EW^B > 0$ for $0 < \pi_H < 1$, so the policy benefits the buyer.

Proof. According to Lemma IV, ΔEW^B must be strictly concave with respect to π_H. Also, $\Delta EW^B = 0$ at the endpoints $\pi_H = 1$ and $\pi_H = 0$, which means $\Delta EW^B(\pi_H)$ is positive somewhere in between the endpoints. Thus, by the definition of strict concavity,

$$\Delta EW^B(\pi_H) > \pi_H \cdot \Delta EW^B(1) + (1 - \pi_H) \cdot \Delta EW^B(0) = 0$$

for $\pi_H \in (0, 1)$. $\square$

1.5 Addition of a Goods Market

In this section we incorporate a goods market, with price rigidities, into the above model. In the next section, we examine the ex ante welfare impact from anti-bubble policy to this goods market.

There are two types of goods: cars and apples. The asset buyer consumes cars and apples in period 2, but he consumes only apples in period 1. Thus, in the model presented so far, the perishable endowment good that the buyer consumed is equivalent to these apples.

Additionally, we introduce a third type of agent called workers. These workers provide differentiated labor, and each worker operates as a monopolist in the specific kind of labor they provide. The workers sell their labor to a market of competitive, constant-return-to-scale car producers. The car producers combine the labor from the workers to produce cars, which are then sold to the asset buyers. As shown in Figure 6, the producers will receive payment in the form of apples for the cars that they sell to the buyers, and the workers will receive payment in apples for the labor they provide to the car producers. Since the industry of car producers is competitive with constant returns to scale, their profits will be zero, so analyzing the welfare of the car production industry will involve examining only the welfare of the workers.

The complete macroeconomic model is intended to reflect Keynes's notion of an economy with a volatile asset market and a sluggish goods market. We therefore employ Fischer (1977)-style pricing in our car market to reflect a goods market with sticky prices (Fischer (1977)). Specifically, the monopolist workers set their wages, in terms of apples, in period 1, based on their expectations of the buyers' period-2 demand for cars. However, the buyer's

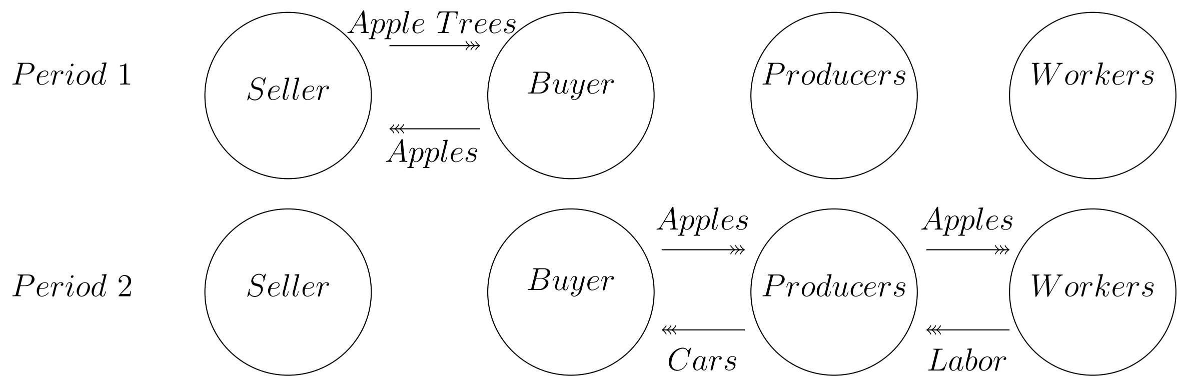

Figure 1.6: The Structure of the Macroeconomic Model

demand for cars in period 2 is uncertain, since it is contingent on the state of the world i.e., on whether or not the apple trees actually yield a positive dividend of apples in period 2. In the absence of policy, the workers must choose their period-2 wages in period 1 without access to the seller's information concerning the true state. Because car producers are competitive with constant-returns to scale, this means that the price of cars (in terms of apples) will be determined by the wages set in period 1, even though the production and consumption of cars does not occur until period 2. The question is then whether an anti-bubble policy, which reveals asset sellers' knowledge to wage-setting workers, improves welfare for both the workers and the buyers in the goods market.

1.5.1 The Buyers

The asset seller does not consume any cars, so the analysis of her behavior is unchanged from that already presented. However, the asset buyer does consume cars, and his consumption of cars depends on the state of the world as well as on the central bank's policy regime. In the absence of policy, the buyer's expected utility is

$$EU^B = u(e_1^B - PX) + \pi_H \beta \left[u(e_2^B + X - P_{NP}^c c_{NP}^H) + v(c_{NP}^H) \right] + \pi_L \beta \left[u(e_2^B - P_{NP}^c) + v(c_{NP}^L) \right], \tag{1.36}$$

where c_{NP}^H is the number of cars that the buyer consumes in state H, c_{NP}^L is the number of cars he consumes is state L, $v(c)$ is the utility that the buyer receives from cars, and P_{NP}^c is the price of cars in terms of apples.[9] Note that, in the absence of policy, workers are unable to distinguish L from H in period 1, so the wages they choose will be the same in both states of the world in period 2, because the workers choose their wages in period 1. Thus, since the producer market is competitive with constant returns to scale, the price of cars will be proportional the workers' wage in equilibrium. As a result, the price of cars will the be the same in both states of the bubble equilibrium. This is how we captures Keynesian price stickiness.

The buyer chooses his period-2 consumption of apples and cars as a function of his period-2 wealth for each state L and H. He then chooses the number of shares that he wishes to purchase in period 1, which ultimately will determine his period-2 wealth. Of course, as above, we choose parameters so that, in equilibrium, he will purchase all of the asset-seller's shares, so $X = S$.

In the absence of policy, the buyer's FOC for c^H in period 2, given his asset purchases, X, is

$$u'\left(e_2^B + X - P_{NP}^c c_{NP}^H\right) P_{NP}^c = v'(c_{NP}^H), \tag{1.37}$$

and his FOC for c_{NP}^L is

$$u'\left(e_2^B - P_{NP}^c c_{NP}^L\right) P_{NP}^c = v'(c_{NP}^L). \tag{1.38}$$

The FOC in (1.37) determines the buyer's demand for cars in state H, while the FOC in (1.38) determines his demand for cars in state L.

The buyer's FOC with respect to X is then

$$u'(e_1^B - PX)P = \pi_H \beta \left[u'(e_2^B + X - P_{NP}^c c_{NP}^H) \left(1 - P_{NP}^c \frac{dc_{NP}^H}{dX}\right) + v'(c_{NP}^H)\frac{dc_{NP}^H}{dX}\right].$$

By (1.37), the terms involving dc_{NP}^H/dX cancel, so this can be simplified to

$$u'(e_1^B - PX)P = \pi_H \beta u'(e_2^B + X - P_{NP}^c c_{NP}^H) \tag{1.39}$$

[9] In other words, the buyer pays P_{NP}^c apples in order to receive one car.

(this is essentially the Envelope Theorem). Equation (1.39) then determines the buyer's demand for the asset. Note that, since the asset does not pay a dividend in state L, c_{NP}^{L} is unaffected by the buyer's choice of X. Again, we choose parameters such that, in equilibrium, the buyer purchases all of the seller's shares of the asset, so $X = S$. This means (1.37) and (1.39), respectively, become

$$u'(e_2^B + S - P_{NP}^c c_{NP}^H) P_{NP}^c = v'(c_{NP}^H), \tag{1.40}$$

$$u'(e_1^B - P_{NP} S) P_{NP} = \pi_H \beta u'(e_2^B + S - P_{NP}^c c_{NP}^H). \tag{1.41}$$

Next, under an anti-bubble policy, the monopolist workers will adjust their wages to reflect the different demands for cars across states L and H. This will cause the price of cars in state L to differ from that in state H (presumably $P_L^c < P_H^c$). Thus, the buyer's demand for cars in state H of a no-bubble equilibrium is given by

$$u'(e_2^B + S - P_H^c c_P^H) P_H^c = v'(c_P^H), \tag{1.42}$$

where c_P^H is the buyer's state-H consumption of cars under an anti-bubble policy, and we again assume that the buyers purchases all S shares of the asset. Similarly, the buyer's demand for cars in state L of a no-bubble equilibrium is given by

$$u'(e_2^B - P_L^c c_P^L) P_L^c = v'(c_P^L), \tag{1.43}$$

where c_P^L is the buyer's state-L consumption of cars under an anti-bubble policy.

Under an anti-bubble policy, the buyer will not purchase the asset in state L since he knows that it will not pay a dividend. As a result, (1.43) is the only relevant FOC for the buyer in state L of the policy equilibrium. However, in state H the buyer will continue to purchase the asset in order to intertemporally smooth his consumption of apples and allow him to increase his consumption of cars in period 2. As in the model of a semi-bubble above, the state-H asset price under the policy will be higher than the no-policy price since the buyer has a higher WTP. The buyer's WTP, P_P, in state H of the no-bubble equilibrium is determined by

$$u'(e_1^B - P_P S)P_P = \beta u'(e_2^B + S - P_H^c c_P^H). \tag{1.44}$$

Because the asset price does not affect the asset's period-2 dividend, the buyer's period-2 welfare is uanffected by the period-1 asset price. As a result, we only need to focus on the buyer's period-2 utility when analyzing how anti-bubble policy affects his welfare in the goods market. Thus, in the context of the goods market, the buyer's welfare in the absence of anti-bubble policy is

$$EW_{NP}^{g,B} = \pi_H \left[u(e_2^B + S - P_{NP}^c c_{NP}^H) + v(c_{NP}^H)\right] + \pi_L \left[u(e_2^B - P_{NP}^c c_{NP}^L) + v(c_{NP}^L)\right],$$

while his welfare in the presence of anti-bubble policy is

$$EW_P^{g,B} = \pi_H \left[u(e_2^B + S - P_H^c c_P^H) + v(c_P^H)\right] + \pi_L \left[u(e_2^B - P_L^c c_P^L) + v(c_P^L)\right],$$

where g signifies "goods market." This means that

$$\Delta EW_g^B = EW_P^{g,B} - EW_{NP}^{g,B},$$

provides a measure of how anti-bubble policy affects buyers in the goods market. We can also break up ΔEW_g^B into the two components ΔEU_2^B and ΔEU_c^B, such that

$$\Delta EW_g^B = \Delta EU_2^B + \Delta EU_c^B \tag{1.45}$$

where

$$\Delta EU_2^B = \left[\pi_H u(e_2^B + S - P_H^c c_P^H) + \pi_L u(e_2^B - P_L^c c_P^L)\right] \\ - \left[\pi_H u(e_2^B + S - P_{NP}^c c_{NP}^H) + \pi_L u(e_2^B - P_{NP}^c c_{NP}^L)\right], \tag{1.46}$$

and

$$\Delta EU_c^B = \left[\pi_H v(c_P^H) + \pi_L v(c_P^L)\right] - \left[\pi_H v(c_{NP}^H) + \pi_L v(c_{NP}^L)\right]. \tag{1.47}$$

In general terms, ΔEU_2^B measures the impact of anti-bubble policy on the buyer's ex ante expected utility from consuming apples in period 2, and ΔEU_c^B measures the impact of anti-bubble policy on the buyer's ex ante expected utility from consuming cars. Thus, buyers benefit from anti-bubble policy if $\Delta EW_g^B > 0$, but they are hurt by anti-bubble policy if $\Delta EW_g^B < 0$. Our goal is to determine the sign of ΔEW_g^B under the most general conditions.

1.5.2 The Workers and Producers in the Goods Market

There is a unit mass continuum of workers who, as monopolists, sell their labor to a competitive industry of car producers. Each worker has the utility function

$$U^W = w_i L_i - C(L_i), \tag{1.48}$$

where is L_i is the amount of labor that worker i sells, w_i is the wage in apples that s/he receives, $w_i L_i$ is the number of apples s/he consumes, and $C(L_i)$ is the worker's utility cost of labor.

Car producers combine the labor of the workers to produce cars. The production technology for the car producers has constant returns to scale and is expressed as the CES production function

$$c = \left[\int_0^1 L_i^\gamma \; di\right]^{\frac{1}{\gamma}}, \tag{1.49}$$

which may be rewritten as

$$c^{\gamma} = \int_0^1 L_i^{\gamma}\ di. \tag{1.50}$$

Assume $0 < \gamma < 1$.

Given the wage w_i for the labor input of the i^{th} worker, producers choose each L_i so as to minimize their cost of production

$$\int_0^1 w_i L_i\ di$$

given their constraint (1.47). Thus, producers choose L_i to minimize the Lagrangian

$$\int_0^1 w_i L_i\ di + \lambda \left[c^{\gamma} - \int_0^1 L_i^{\gamma}\ di \right]. \tag{1.51}$$

The first order condition for L_i is

$$w_i - \lambda\gamma L_i^{\gamma-1} = 0$$

which gives

$$L_i = \left(\frac{\gamma\lambda}{w_i} \right)^{1/(1-\gamma)}. \tag{1.52}$$

Plugging this back into the production function in (1.47) gives, after some rearrangement,

$$\lambda = \frac{1}{\gamma} \left[\int_0^1 w_i^{-\gamma/(1-\gamma)}\ di \right]^{-(1-\gamma)/\gamma} c^{1-\gamma} = \frac{1}{\gamma} W c^{1-\gamma} \tag{1.53}$$

where

$$W = \left[\int_0^1 w_i^{-\gamma/(1-\gamma)}\ di \right]^{-(1-\gamma)/\gamma} \tag{1.54}$$

is a wage index. Plugging (1.50) into (1.49) gives the Hicksian labor demand for labor input

i

$$L_i = \left(\frac{W}{w_i}\right)^{1/(1-\gamma)} c. \tag{1.55}$$

Worker i's total consumption of apples, $w_i L_i$, is constrained by the demand for labor by car producers, which is ultimately determined by the demand for cars by asset buyers. This means that we can express worker i's apple consumption as

$$c_{2i}^{kA} = w_i L_i = w_i \left(\frac{W}{w_i}\right)^{1/(1-\gamma)} c^k = (w_i)^{-\gamma/1-\gamma} (W)^{1/(1-\gamma)} c^k \qquad k = L, H \tag{1.56}$$

where c_{2i}^{kA} is worker i's consumption of apples in period 2 in state $k = L, H$, and c^k is the total demand for cars by asset buyers in period 2, state $k = L, H$.

In the absence of policy, worker i must choose a monopoly wage w_i in period 1, before the demand for cars in known. Supplying labor L_i yields a disutility of $C(L_i)$ to worker i, as in (1.45) above. Worker i supplies L_i^H units of labor in state H and L_i^L units of labor in state L, as determined by the demand they face, at the wage w_i. Worker i therefore chooses the wage w_i to maximize his/her expected utility, or

$$EU^W = \pi_H \left(c_{2i}^{HA} - C(L_i^H)\right) + \pi_L \left(c_{2i}^{LA} - C(L_i^L)\right), \tag{1.57}$$

where L_i^H and L_i^L are determined by (1.52), given w_i and the demand for cars, c^H and c^L, respectively. Note that this requires w_i to be larger than $C'(L_i^k)$ for $k = L, H$ (see below).

In order to find the worker's FOC, differentiate (1.54) with respect to w_i and use

$$\frac{\partial L_i^k}{\partial w_i} = \frac{-1}{1-\gamma}\left(\frac{1}{w_i}\right) L_i^k \qquad \text{and} \qquad \frac{\partial c_{2i}^{kA}}{\partial w_i} = \frac{-\gamma}{1-\gamma} L_i^k \qquad \text{where} \qquad k = L, H$$

by (1.52) and (1.53), respectively. This gives

$$\pi_H \left[-\frac{\gamma}{1-\gamma} L_i^H + C'(L_i^H) \cdot \frac{1}{1-\gamma} \cdot \frac{1}{w_i} L_i^H\right] + \pi_L \left[-\frac{\gamma}{1-\gamma} L_i^L + C'(L_i^L) \cdot \frac{1}{1-\gamma} \cdot \frac{1}{w_i} L_i^L\right] = 0, \tag{1.58}$$

which yields

$$w_i = \frac{1}{\gamma}\left[\frac{\pi_L L_i^L}{\pi_L L_i^L + \pi_H L_i^H} C'(L_i^L) + \frac{\pi_H L_i^H}{\pi_L L_i^L + \pi_H L_i^H} C'(L_i^H)\right], \tag{1.59}$$

where is w_i is greater than $C'(L_i^k)$ for $k = L, H$ if γ is far enough below 1.

Next, we assume that all producers are identical, which means $w_i = w_j = w$ and $L_i^k = L_j^k = L^k$ for all $i, j \in [0,1]$ and $k = L, H$. The latter assumption implies $c^L = L^L$ and $c^H = L^H$, by (1.46). Furthermore, since the producer market is competitive, the price of a car in units of apples is $P^c = W = w$. As a result, the competitive equilibrium price of a car is given by

$$P_{NP}^c = \frac{1}{\gamma}\left[\frac{\pi_L c_{NP}^L}{\pi_L c_{NP}^L + \pi_H c_{NP}^H} C'(c_{NP}^L) + \frac{\pi_H c_{NP}^H}{\pi_L c_{NP}^L + \pi_H c_{NP}^H} C'(c_{NP}^H)\right]. \tag{1.60}$$

In the presence on an anti-bubble policy, the workers know the demand for cars with certainty when choosing their wage. Thus, there are two car prices in a no-bubble equilibrium. For instance, P_L^c is the price of a car in state L under the policy, which is given by

$$P_L^c = \frac{1}{\gamma} C'(c_P^L), \tag{1.61}$$

where c_P^L is the production and demand for cars in state L under the policy. Similarly, P_P^H is the price of a car in state H under the policy, which is given by

$$P_H^c = \frac{1}{\gamma} C'(c_P^H), \tag{1.62}$$

where c_P^H is the production and demand for cars in state H under the policy.

Examining the welfare impact of anti-bubble policy on the worker entails comparing the worker's expected utility in a bubble equilibrium with his expected utility in a no-bubble equilibrium. In the absence of policy, the worker's expected utility is

$$EU_{NP}^W = \pi_H\left(P_{NP}^c c_{NP}^H - C(c_{NP}^H)\right) + \pi_L\left(P_{NP}^c c_{NP}^L - C(c_{NP}^L)\right), \tag{1.63}$$

and his expected utility under the policy is

$$EU_P^W = \pi_H \left(P_H^c c_P^H - C(c_P^H) \right) + \pi_L \left(P_L^c c_P^L - C(c_P^L) \right). \tag{1.64}$$

Thus, we can measure the impact of anti-bubble policy on the worker by the difference between (1.61) and (1.60), or

$$E\Delta U^W = EU_P^W - EU_{NP}^W. \tag{1.65}$$

If $E\Delta U^W > 0$, then workers benefit from anti-bubble policy, and if $E\Delta U^W < 0$, then workers are hurt by anti-bubble policy. As with $E\Delta W^S$ for the asset seller and $E\Delta W^B$ for the buyer in Section 4 above, our goal is to determine the sign of $E\Delta U^W$ for the worker under the most general conditions possible.

1.6 A Conjecture Concerning Goods Prices Across Policy and No-Policy Equilibria

In a policy equilibrium, producers know the future demand for cars with certainty. In this equilibrium, a regime where the producer must set the period-2 car price in period 1 will have the same outcome as a regime where the car producer is able to set this car price in period 2. This means that a policy equilibrium is essentially equivalent to an economy where car prices are always flexible. As a result, we could alternatively refer to the no-policy equilibrium as a "sticky-price equilibrium," and we could refer to the policy equilibrium as a "flexible-price equilibrium." This implies that anti-bubble policy converts sticky prices into flexible prices.

We conjecture that when workers have increasing marginal utility cost of effort, the goods price in a no-policy equilibrium lies between the two state-contingent goods prices in a policy equilibrium, so

$$P_L^c < P_{NP}^c < P_H^c. \tag{1.66}$$

This should be somewhat intuitive, since price setters in a no-policy regime would presumably set P^c_{NP} according to some weighted average of P^c_L and P^c_H. The next two sections provide specifications of the model whereby we can easily demonstrate that this conjecture is true. The main insight from this conjecture, assuming it's true, is that it necessarily means that the volatility of the production and consumption of cars is greater in a no-policy equilibrium than it is in a policy equilibrium. In other words, this conjecture implies that sticky goods prices amplify the variance of car production that arises from asset-driven changes to demand.

Figure 7 graphs the "quasi-supply" and demand for cars. Since the demand for cars is contingent on the state of the world, there are two demand curves, D^c_L and D^c_H. For any given car price, the buyer's demand in state H, D^c_H, is larger than his demand in state L, D^c_L, since the buyer's state-H wealth is larger than his state-L wealth. The "quasi-supply" curve is "quasi" because the price on the curve is greater than the monopolist workers' marginal cost curve by the factor $\frac{1}{\gamma} > 1$. This means workers would still be willing to produce at levels where the car price is below the QS^c curve, such as at c^H_{NP}, as long as the price is greater than marginal cost. As Figure 7 shows, the production and consumption of cars is more volatile in no-policy or sticky-price equilibrium. Car production in state L of a no-policy equilibrium is lower than it would be in a policy equilibrium, while car production in state H of a no-policy equilibrium is larger than it would be in a policy equilibrium. In a Kenyesian sense, we could say that the existing car production volatility, which is driven by volatility in asset markets, is amplified when car prices are sticky.

1.6.1 Car Production

The quasi-supply curve QS^c shown in Figure 7 corresponds to the product of the mark up $1/\gamma$ and workers' marginal cost function $C'(c)$, so that, when the demand curve moves, QS^c doesn't move. In other words, QS^c in Figure 7 is simply a generalized version of the workers' FOC in a policy equilibrium like those shown in equations (1.61) and (1.62). This QS^c curve is not a supply curve, but rather a "quasi-supply curve," because the workers are monopolists and supply curves do not exist in monopoly-market equilibria. Thus, the

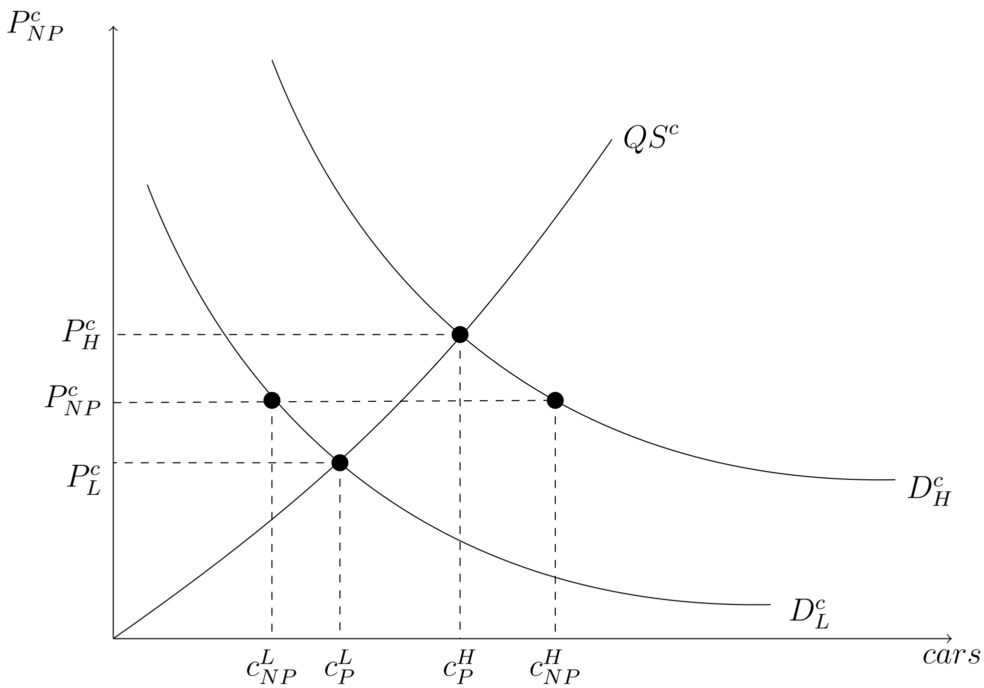

Figure 1.7: Genera Model of Demand and Quasi-Supply of Cars

car price $\tilde{P}^c$ that producers choose to charge for producing $\tilde{c}$ units of cars is determined according to the quasi-supply curve

$$\tilde{P}^c = \frac{1}{\gamma} C'(\tilde{c}). \tag{1.67}$$

If $C''(\tilde{c}) = 0$, so $C'(\tilde{c}) = \lambda$, say, with λ constant, then QS^c is a horizontal line, and, regardless of the state of the world and/or the bubble-policy equilibrium, the car price is simply $\tilde{P}^c = \frac{\lambda}{\gamma}$, where λ is a constant marginal (utility) cost of car production. An example of this is shown in Figure 8 below. On the other hand, if $C''(\tilde{c}) > 0$, then QS^c is upward-sloping like in Figure 7. If $C^{(3)}(\tilde{c}) = 0$, then QS^c is upward-sloping and linear, and if $C^{(3)}(\tilde{c}) > 0$, then QS^c is upward-sloping and convex.

1.6.2 The Demand for Cars

The buyer's period-2 FOC with respect to car consumption determines his demand for cars. So, for a given car price $\tilde{P}^c$, the buyer's demand for cars $\tilde{c}$ is determined by

$$u'\left(\tilde{A}-\tilde{P}^c\tilde{c}\right)\tilde{P}^c = v'\left(\tilde{c}\right), \tag{1.68}$$

where $\tilde{A}$ is the buyer's period-2 state-contingent wealth such that

$$\tilde{A} = \begin{cases} A_L = e_2^B\text{: in state L} \\ A_H = e_2^B + S\text{: in state H} \end{cases} \tag{1.69}$$

Note that the policy doesn't affect the demand curves for cars, since buyers' period-2 wealth, given by (1.69), is not affected by the policy. This is because we assume asset sellers are always short-sale constrained so that buyers always purchase all available shares of the risky asset. Thus, the only effect of policy on the goods market is whether sellers can use the information revealed to set prices. Equation (1.68) is a generalization of the state-and-equilibrium specific FOCs in (1.38), (1.40), (1.42), (1.43). The buyer's expected period-2 wealth is

$$E\left[\tilde{A}\right] = \pi_L e_2^B + \pi_H\left(e_2^B + S\right) = e_2^B + \pi_H S, \tag{1.70}$$

while the variance of his period-2 wealth is

$$Var\left[\tilde{A}\right] = \pi_L\pi_H S^2. \tag{1.71}$$

Holding $\tilde{A}$ constant, the derivative of (1.68) with respect to $\tilde{P}^c$ is,

$$\frac{d\tilde{c}}{d\tilde{P}^c} = \frac{u'\left(\tilde{A}-\tilde{P}^c\tilde{c}\right) - u''\left(\tilde{A}-\tilde{P}^c\tilde{c}\right)\cdot\tilde{P}^c\tilde{c}}{u''\left(\tilde{A}-\tilde{P}^c\tilde{c}\right)\cdot\left(\tilde{P}^c\right)^2 + v''\left(\tilde{c}\right)} < 0. \tag{1.72}$$

This is less than zero since $u''(\cdot) < 0$, $u'(\cdot) > 0$, and $v''(\cdot) < 0$. Thus, the buyer's demand curve for cars is downward slopping as expected. Now, holding $\tilde{P}^c$ constant, the derivative of (1.68) with respect to $\tilde{A}$ is

$$\frac{d\tilde{c}}{d\tilde{A}} = \frac{u''\left(\tilde{A} - \tilde{P}^c\tilde{c}\right) \cdot \left(\tilde{P}^c\right)^2 + v''(\tilde{c})}{u''\left(\tilde{A} - \tilde{P}^c\tilde{c}\right) \cdot \tilde{P}^c} > 0, \tag{1.73}$$

which is greater than zero because $u''(\cdot) < 0$ and $v''(\cdot) < 0$. Thus, for a constant car price, a positive wealth shock to the buyer will increase his demand for cars, as expected.

1.7 A Welfare Analysis of the Goods Market when Workers have Constant Marginal Utility-Costs

First, note that

$$E[\tilde{c}_{NP}] = \pi_L c_{NP}^L + \pi_H c_{NP}^H$$

is the buyer's expected consumption of cars in a no-policy equilibrium. This means that we can rewrite (1.60) as

$$P^c E[\tilde{c}_{NP}] = \frac{1}{\gamma}\left[\pi_L c_{NP}^L C'\left(c_{NP}^L\right) + \pi_H c_{NP}^H C'\left(c_{NP}^H\right)\right], \tag{1.74}$$

where $P^c E[\tilde{c}_{NP}]$ is the expected revenue that worker's receive from providing labor for car production in a no-policy equilibrium. In this section, we assume $C(\tilde{c}) = \lambda\tilde{c}$, where λ is a constant, so $C'(\tilde{c}) = \lambda$. Using (1.74), this implies that the car price in no-policy equilibrium is $P_{NP}^c = \frac{\lambda}{\gamma}$. Substituting $C'(\tilde{c}) = \lambda$ into the policy FOCs in (1.62) and (1.61), we find that the car prices for states L and H are also $P_L^c = P_H^c = P_{NP}^c = \frac{\lambda}{\gamma}$, so the equilibrium car price is unaffected by the state of the world as well as the central bank's policy on asset bubbles. As a result, the state-L production of cars is the same across the two policy equilibria, so $c_{NP}^L = c_P^L$. The same it true for the state-H production of cars, so $c_{NP}^H = c_P^H$. This is shown in Figure 8.

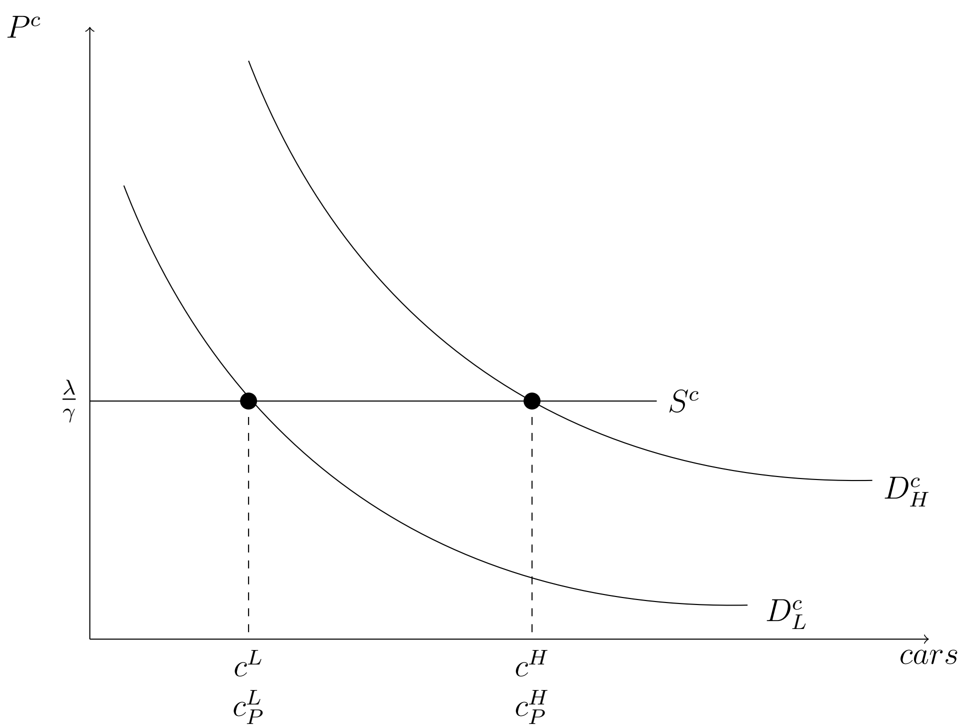

Figure 1.8: Demand and Quasi-Supply for Cars when $C'(\cdot) = \lambda$.

The result from Figure 8 implies that the worker's expected revenue in a no-policy equilibrium, or $E\left[\tilde{R}_{NP}\right] = P^c E\left[\tilde{c}_{NP}\right]$, is the same as his expected revenue in a policy equilibrium, or $E\left[\tilde{R}_P\right] = \pi_L P^c_L c^L_P + \pi_H P^c_H c^H_P$. The same is true for his expected total costs, so $E\left[C\left(\tilde{c}_{NP}\right)\right] = E\left[C\left(\tilde{c}_P\right)\right]$. Thus, if workers have constant marginal costs, then their ex ante expected welfare is unaffected by anti-bubble policy, so $E\Delta U^W = E^W_P - EU^W_{NP} = 0$. This is obvious, since price-setters don't use information from the asset market even if they have it, so they don't care whether the policy reveals this information.

1.8 A Welfare Analysis of the Goods Market with Logarithmic Utility and Quadratic Costs

In this section, we assume $u(\cdot) = v(\cdot) = \ln(\cdot)$. This is a Cobb-Douglas utility function, so the buyer always spends a constant proportion of his wealth on cars. Thus, when $\tilde{P}^c$ increases (decreases), $\tilde{c}$ decreases (increases), but $\tilde{P}^c\tilde{c}$ remains unchanged. We also assume

$$C\left(\tilde{c}\right) = \frac{\lambda}{2}\tilde{c}^2,$$

which means

$$C'(\tilde{c}) = \lambda \tilde{c}.$$

1.8.1 Consumption and Production in a No-Policy Equilibrium

In a no-policy equilibrium, the buyer's state-H FOC (1.40) becomes

$$\frac{P^c_{NP}}{e^B + S - P^c_{NP} c^H_{NP}} = \frac{1}{c^H_{NP}}, \tag{1.75}$$

and his state-L FOC (1.38) becomes

$$\frac{P^c_{NP}}{e^B - P^c_{NP} c^L_{NP}} = \frac{1}{c^L_{NP}}. \tag{1.76}$$

The buyer receives all of his wealth as apples, so we let apples be the numeraire good. Because the buyer has Cobb-Douglas utility, he spends a constant proportion of his wealth on cars, so for a given level of wealth $\tilde{A}$, the amount of apples spent on cars does not change. Thus, Equations (1.75) and (1.76), respectively, imply

$$P^c_{NP} c^H_{NP} = \frac{e^B_2 + S}{2} = \frac{A^H}{2} \tag{1.77}$$

in state H and

$$P^c_{NP} c^L_{NP} = \frac{e^B_2}{2} = \frac{A^L}{2} \tag{1.78}$$

in state L. More generally,

$$P^c_{NP} \tilde{c}_{NP} = \frac{\tilde{A}}{2}, \tag{1.79}$$

so half of the buyer's wealth is spent on car consumption, and the buyer's demand for cars can be expressed as

$$\tilde{c}_{NP} = \frac{\tilde{A}}{2P^c_{NP}}. \tag{1.80}$$

Thus, the workers' expected revenue in a no-policy equilibrium is

$$E\left[\tilde{R}_{NP}\right] = P^c_{NP} E\left[\tilde{c}_{NP}\right] = \frac{1}{2} E\left[\tilde{A}\right]. \tag{1.81}$$

The worker's FOC (1.60) becomes

$$P^c_{NP} = \frac{1}{\gamma}\left[\frac{\pi_L c^L_{NP}}{E\left[\tilde{c}_{NP}\right]} \cdot \lambda c^L_{NP} + \frac{\pi_H c^H_{NP}}{E\left[\tilde{c}_{NP}\right]} \cdot \lambda c^H_{NP}\right], \tag{1.82}$$

which may be written more concisely as

$$P^c_{NP} = \frac{\lambda}{\gamma}\left[\frac{E\left[\left(\tilde{c}_{NP}\right)^2\right]}{E\left[\tilde{c}_{NP}\right]}\right] = \frac{2}{\gamma}\left[\frac{E\left[C\left(\tilde{c}_{NP}\right)\right]}{E\left[\tilde{c}_{NP}\right]}\right]. \tag{1.83}$$

Equation (1.83) says that the car price in a no-policy equilibrium is a mark-up of $\frac{2}{\gamma}$ on the ratio of the expected cost over the expected consumption/production. Equation (1.83) can also be written as

$$P^c_{NP} E\left[\tilde{c}_{NP}\right] = \frac{\lambda}{\gamma} E\left[\left(\tilde{c}_{NP}\right)^2\right] = \frac{2}{\gamma} E\left[C\left(\tilde{c}_{NP}\right)\right], \tag{1.84}$$

so that expected revenue is greater than the expected cost by a factor of $\frac{2}{\gamma}$. Substituting (1.81) and (1.80) into (1.84) gives

$$\frac{1}{2} E\left[\tilde{A}\right] = \frac{\lambda}{\gamma} E\left[\left(\frac{\tilde{A}}{2P^c_{NP}}\right)^2\right],$$

which simplifies to

$$\left(P^c_{NP}\right)^2 = \frac{2\lambda}{4\gamma}\frac{E\left[\tilde{A}^2\right]}{E\left[\tilde{A}\right]},$$

and finally gives

$$P^c_{NP} = q \cdot \sqrt{\frac{E\left[\tilde{A}^2\right]}{E\left[\tilde{A}\right]}}, \tag{1.85}$$

where $q = \sqrt{\frac{\lambda}{2\gamma}}$. Substituting

$$E\left[\tilde{A}^2\right] = Var\left[\tilde{A}\right] + E\left[\tilde{A}\right]^2,$$

into (1.85) and canceling common factors gives

$$P_{NP}^c = q \cdot \sqrt{\frac{Var\left[\tilde{A}\right]}{E\left[\tilde{A}\right]} + E\left[\tilde{A}\right]}. \tag{1.86}$$

Finally, substituting $E\left[\tilde{A}\right] = e_2^B + \pi_H S$ and $Var\left[\tilde{A}\right] = \pi_L \pi_H S^2$, from (1.70) and (1.71) respectively, into (1.86) gives

$$P_{NP}^c = q \cdot \sqrt{\frac{\pi_L \pi_H S^2}{e_2^B + \pi_H S} + (e_2^B + \pi_H S)}.$$

This can be written more concisely as

$$P_{NP}^c = q \cdot \sqrt{e_2^B + (\pi_L \delta + \pi_H) S}, \tag{1.87}$$

where

$$\delta = \frac{\pi_H S}{e_2^B + \pi_H S} < 1,$$

which implies $(\pi_L \delta + \pi_H) < 1$, so

$$A^L = e_2^B < e_2^B + \pi_H S = E\left[\tilde{A}\right] < e_2^B + (\pi_L \delta + \pi_H) S < e_2^B + S = A^H. \tag{1.88}$$

The inequalities in (1.88) will be used in the next section to show that our conjecture in (1.66) is true for the present case.

1.8.2 Consumption and Production in a Policy Equilibrium

In a policy equilibrium (or no-bubble equilibrium), the buyer's state-H FOC (1.42) becomes

$$\frac{P_H^c}{e_2^B + S - P_H^c c_P^H} = \frac{1}{c_P^H}, \tag{1.89}$$

and his state-L FOC (1.43) becomes

$$\frac{P_L^c}{e_2^B - P_L^c c_P^L} = \frac{1}{c_P^L}. \tag{1.90}$$

The amount of the buyer's wealth that is spent on cars in state H is then

$$P_H^c c_P^H = \frac{e_2^B + S}{2} = \frac{A^H}{2}, \tag{1.91}$$

and the amount of his wealth spent on cars in state L is

$$P_L^c c_P^L = \frac{e_2^B}{2} = \frac{A^L}{2}. \tag{1.92}$$

Thus, comparing (1.91) and (1.92) to (1.77) and (1.78) shows that, for each state of the world, anti-bubble policy does not affect the amount of apples the buyer spends on cars. This means that workers' expected revenue in a policy equilibrium, $E\left[\tilde{R}_P\right] = \pi_L P_L^c c_P^L + \pi_H P_H^c c_P^H$, is the same as their expected revenue in a no-policy equilibrium, so

$$E\left[\tilde{R}_P\right] = E\left[\tilde{R}_{NP}\right] = \frac{1}{2} E\left[\tilde{A}\right]. \tag{1.93}$$

Next, applying $C'(\tilde{c}) = \lambda \tilde{c}$ to (1.67), the worker's FOC in state H is

$$P_H^c = \frac{\lambda}{\gamma} c_P^H. \tag{1.94}$$

Multiplying both sides of (1.94) by P_H^c, using (1.91) on the right-hand side, and solving for

P_H^c gives

$$P_H^c = q \cdot \sqrt{e_2^B + S} = q \cdot \sqrt{A^H}. \tag{1.95}$$

Similarly, multiplying both sides of

$$P_L^c = \frac{\lambda}{\gamma} c_P^L, \tag{1.96}$$

by P_L^c, using (1.92) on the right-hand side, and solving for P_L^c gives

$$P_L^c = q \cdot \sqrt{e_2^B} = q \cdot \sqrt{A^L}. \tag{1.97}$$

Applying (1.88) to (1.87), (1.97), and (1.95) implies that $P_L^c < P_{NP}^c < P_H^c$. Thus, when $u(\cdot) = v(\cdot) = \ln(\cdot)$ and $C(\tilde{c}) = \frac{\lambda}{2}(\tilde{c})^2$, our conjecture in (1.66) is true.

The ex ante expected car price in a policy equilibrium is

$$E\left[\tilde{P}_P\right] = \pi_L P_L^c + \pi_H P_H^c = q \cdot E\left[\sqrt{\tilde{A}}\right], \tag{1.98}$$

which is found using (1.95) and (1.97). Furthermore, by Jensen's Inequality, we know that

$$E\left[\sqrt{\tilde{A}}\right] < \sqrt{E\left[\tilde{A}\right]},$$

so because $E\left[\tilde{A}\right] = e_2^B + \pi_H S < e_2^B + (\pi_L \delta + \pi_H) S$, it must be the case that

$$E\left[\tilde{P}_P\right] < P_{NP}^c,$$

using (1.87) and (1.98). Thus, anti-bubble policy decreases the ex ante expected car price, which suggests that anti-bubble policy may increase the buyer's expected consumption of cars. We therefore conjecture that

$$E\left[\tilde{c}_P\right] > E\left[\tilde{c}_{NP}\right],$$

which we prove in Lemma VI.

1.8.3 The Worker's Ex Ante Expected Welfare

As already shown in (1.93), anti-bubble policy does not affect the the worker's expected revenue in the present case. It turns out that anti-bubble policy does not affect the worker's expected cost either. This is shown in Lemma V. It will therefore follow that anti-bubble policy doesn't affect workers' welfare, as is shown in Proposition IV.

Lemma V: If $u(\cdot) = v(\cdot) = \ln(\cdot)$ and $C(\tilde{c}) = \frac{\lambda}{2}(\tilde{c})^2$, then $E[C(\tilde{c}_{NP})] = E[C(\tilde{c}_P)]$.

Proof. Using (1.94), the state-H revenue in a policy equilibrium can be rewritten as

$$P_H^c c_P^H = \frac{\lambda}{\gamma}\left(c_P^H\right)^2 = \frac{2}{\gamma}C\left(c_P^H\right). \tag{1.99}$$

Similarly, using (1.96), the state-L revenue in a policy equilibrium can be rewritten as

$$P_L^c c_P^L = \frac{\lambda}{\gamma}\left(c_P^L\right)^2 = \frac{2}{\gamma}C\left(c_P^L\right). \tag{1.100}$$

Thus, the expected revenue in a policy equilibrium is

$$E\left[\tilde{R}_P\right] = \pi_L\frac{2}{\gamma}C\left(c_P^L\right) + \pi_H\frac{2}{\gamma}C\left(c_P^H\right) = \frac{2}{\gamma}E\left[C\left(\tilde{c}_P\right)\right]. \tag{1.101}$$

Combining this with (1.93) and (1.84) gives

$$\frac{2}{\gamma}E\left[C\left(\tilde{c}_P\right)\right] = E\left[\tilde{R}_P\right] = E\left[\tilde{R}_{NP}\right] = P_{NP}^c E\left[\tilde{c}_{NP}\right] = \frac{2}{\gamma}E\left[C\left(\tilde{c}_{NP}\right)\right], \tag{1.102}$$

which implies Lemma V. □

Proposition IV: If $u(\cdot) = v(\cdot) = \ln(\cdot)$ and $C(\tilde{c}) = \frac{\lambda}{2}(\tilde{c})^2$, then $E\Delta U^W = 0$.

Proof. Rewrite (1.63) and (1.64), respectively, as

$$EU_{NP}^{W} = E\left[\tilde{R}_{NP}\right] - E\left[C\left(\tilde{c}_{NP}\right)\right] \tag{1.103}$$

and

$$EU_{P}^{W} = E\left[\tilde{R}_{P}\right] - E\left[C\left(\tilde{c}_{P}\right)\right], \tag{1.104}$$

so, using Lemma V and (1.93), $E\Delta U^{W}$ is

$$E\Delta U^{W} = E\left[\tilde{R}_{P}\right] - E\left[\tilde{R}_{NP}\right] + E\left[C\left(\tilde{c}_{NP}\right)\right] - E\left[C\left(\tilde{c}_{P}\right)\right] = 0. \tag{1.105}$$

□

1.8.4 The Buyer's Ex Ante Expected Welfare

First, consider the buyer's period-2 consumption of apples. Because $P_{NP}^{c} c_{NP}^{H} = P_{H}^{c} c_{P}^{H}$ and $P_{NP}^{c} c_{NP}^{L} = P_{L}^{c} c_{P}^{L}$, anti-bubble policy does not impact the buyers' period-2 spending on cars, and so, does not impact their period-2 apple consumption. It therefore does not affect their period-2 ex ante expected utility of apple consumption. That is,

$$\pi_{L} u\left(e_{2}^{B} - P_{NP}^{c} c_{NP}^{L}\right) + \pi_{H} u\left(e_{2}^{B} + S - P_{NP}^{c} c_{NP}^{H}\right) = \pi_{L} u\left(e_{2}^{B} - P_{L}^{c} c_{P}^{L}\right) + \pi_{H} u\left(e_{2}^{B} + S - P_{H}^{c} c_{P}^{H}\right). \tag{1.106}$$

Thus, the effect of policy, ΔEW_{g}^{B}, depends solely on the buyers' expected utility from consumption of cars, which we call ΔEU_{c}^{B}, so

$$\Delta EW_{g}^{B} = \Delta EU_{c}^{B}, \tag{1.107}$$

where

$$\Delta EU_c^B = \left[\pi_L v\left(c_P^L\right) + \pi_H v\left(c_P^H\right)\right] - \left[\pi_L v\left(c_{NP}^L\right) + \pi_H v\left(c_{NP}^H\right)\right] = E\left[v\left(\tilde{c}_P\right)\right] - E\left[v\left(\tilde{c}_{NP}\right)\right]. \tag{1.108}$$

Lemma VI: If $u(\cdot) = v(\cdot) = \ln(\cdot)$ and $C(\tilde{c}) = \frac{\lambda}{2}(\tilde{c})^2$, then $E[\tilde{c}_P] > E[\tilde{c}_{NP}]$.

Proof. Let $G_{NP}^s(\tilde{c})$ be the formula for the secant line that runs from $C\left(c_{NP}^L\right)$ to $C\left(c_{NP}^H\right)$ on the worker's total cost curve (see Figure 9), so that

$$G_{NP}^s\left(E\left[\tilde{c}_{NP}\right]\right) = E\left[C\left(\tilde{c}_{NP}\right)\right]. \tag{1.109}$$

Likewise, let $G_P^s(\tilde{c})$ be the policy-equilibrium analog of $G_{NP}^s(\tilde{c})$, so that

$$G_P^s\left(E\left[\tilde{c}_P\right]\right) = E\left[C\left(\tilde{c}_P\right)\right]. \tag{1.110}$$

As shown in Figure 9, $G_{NP}^s(c) > G_P^s(c)$ for any c in the open interval (c_P^L, c_P^H), so

$$G_{NP}^s\left(E\left[\tilde{c}_P\right]\right) > G_P^s\left(E\left[\tilde{c}_P\right]\right).$$

Applying Lemma V, together with (1.109) and (1.110), gives

$$G_{NP}^s\left(E\left[\tilde{c}_P\right]\right) > G_P^s\left(E\left[\tilde{c}_P\right]\right) = G_{NP}^s\left(E\left[\tilde{c}_{NP}\right]\right),$$

which implies

$$E\left[\tilde{c}_P\right] > E\left[\tilde{c}_{NP}\right].$$

□

Thus, anti-bubble policy increases the buyer's expected car consumption. Intuitively, since $\tilde{c}_{NP}$ is more volatile and $C(\cdot)$ is convex, all else equal, one would expect $E[C(\tilde{c}_{NP})] > E[C(\tilde{c}_P)]$. Yet, as Lemma V shows, they are in fact equal, so it must be true that $E[\tilde{c}_{NP}] <$

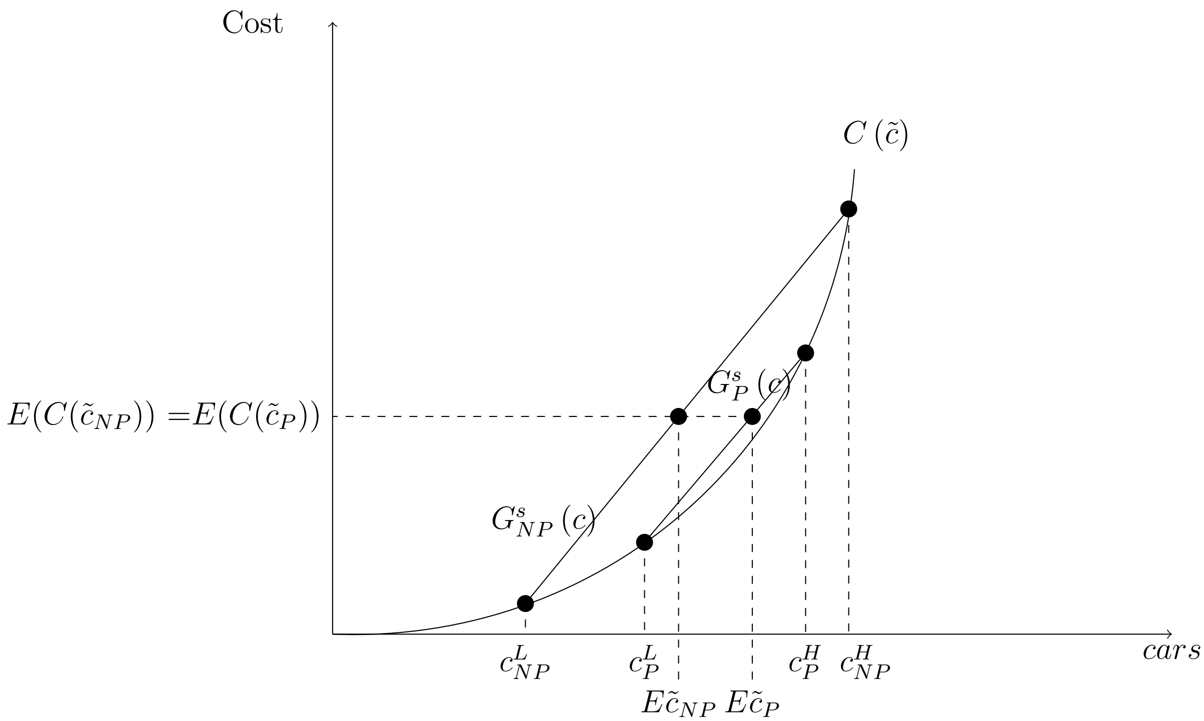

Figure 1.9: Expected Costs of Car Production

$E[\tilde{c}_P]$

Lemma VII: If $u(\cdot) = v(\cdot) = \ln(\cdot)$ and $C(\tilde{c}) = \frac{\lambda}{2}(\tilde{c})^2$, then $E[v(\tilde{c}_P)] > E[v(\tilde{c}_{NP})]$.

Proof. This is intuitively clear. Because $v(\cdot) = \ln(\cdot)$, the buyer is risk averse and also prefers more consumption over less. Thus, anti-bubble policy increases $E[v(\tilde{c})]$ because the buyer's car consumption is less volatile and larger in expected value in a policy equilibrium than it is in a no-policy equilibrium. This is easily seen is Figure 10.[10] □

Proposition V: If $u(\cdot) = v(\cdot) = \ln(\cdot)$ and $C(\tilde{c}) = \frac{\lambda}{2}\tilde{c}^2$, then $\Delta EW^B > 0$.

Proof. First, note that ΔEW^B can be broken up into three parts, in terms of apple consumption in period 1, apple consumption in period 2, and car production in period 2. This gives

$$\Delta EW^B = \Delta EU_1^B + \Delta EU_2^B + \Delta EU_c^B,$$

[10] I wish to thank Feng Liu for impressing on me the importance of analyzing ex ante expected welfare changes through a graphical method like that in Figure 10.

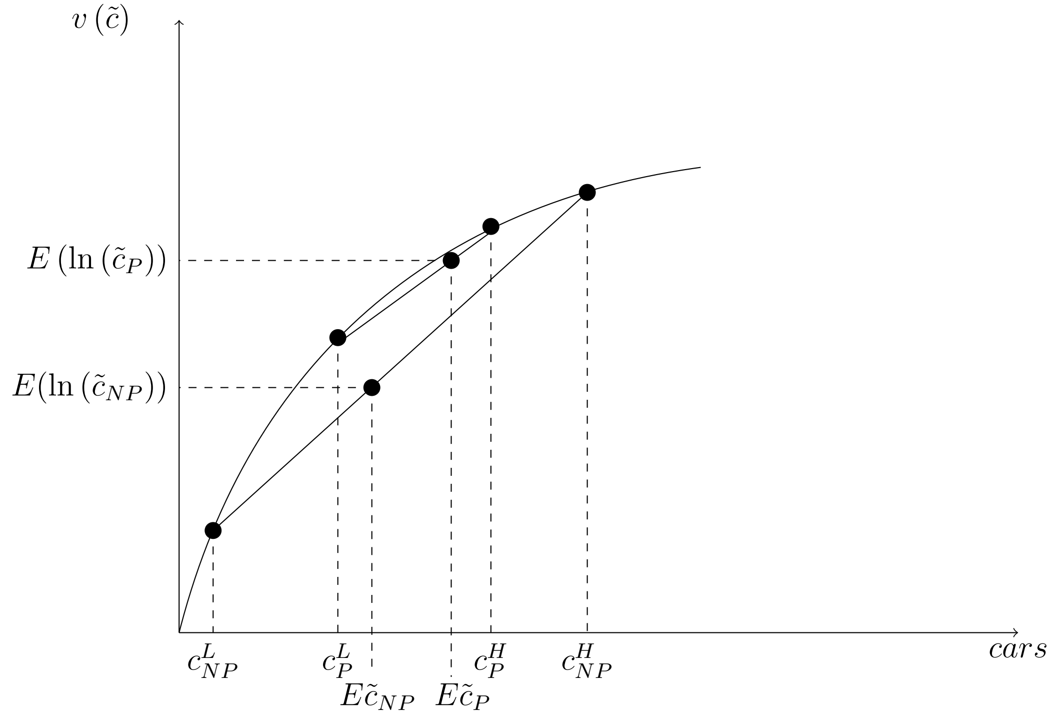

Figure 1.10: Expected Utility of Car Consumption

where

$$\Delta EU_1^B = \pi_H u\left(e_1^B - P_P S\right) + \pi_L u\left(e_1^B\right) - u\left(e_1^B - P_{NP} S\right)$$

$$\Delta EU_2^B = \left[\pi_H u\left(e_2^B + S - P_H^c c_P^H\right) + \pi_L u\left(e_2^B - P_L^c c_P^L\right)\right]$$

$$- \left[\pi_H u\left(e_2^B + S - P_{NP}^c c_{NP}^H\right) + \pi_L u\left(e_2^B - P_{NP}^c c_{NP}^L\right)\right]$$

$$\Delta EU_c^B = \left[\pi_L v\left(c_P^L\right) + \pi_H v\left(c_P^H\right)\right] - \left[\pi_L v\left(c_{NP}^L\right) + \pi_H v\left(c_{NP}^H\right)\right].$$

Proposition III implies that $\Delta EU_1^B > 0$ since $\ln(\cdot)$ is an example of a DARA utility function.[11] Equation (1.106) implies $\Delta EU_2^B = 0$. Finally, Lemma VII implies $\Delta EU_c^B > 0$. $\square$

[11]Recall that Proposition III relies on Lemma IV which shows that, in the absence of a goods (car) market in period 2, ΔEU_1^B is a concave function in π_H, i.e., when $v(\cdot) = 0$. To prove Lemma IV, we took the second derivative of (**??**) with respect to π_H and showed that it is less than zero. If a goods market is present in period 2, then $\pi_H \beta u'\left(e_2^B + S\right)$ from the right-hand side of (1.2) must be replaced with $\pi_H \beta u'\left(e_2^B + S - P_{NP}^c c_{NP}^H\right)$ in the FOC (1.41). However, because the buyer has Cobb-Douglas utility, then $P_{NP}^c c_{NP}^H = \frac{e_2^B + S}{2}$, so $\frac{d\ P_{NP}^c c_{NP}^H}{d\ \pi_H} = 0$. This implies that the second derivative of (1.2) must have the same sign as the second derivative of (1.41), which is negative.

1.9 Conclusion

In summary, we find that anti-bubble policy decreases the ex ante welfare of asset sellers because it decreases their expected consumption while increasing the volatility of their consumption across states of the world. On the other hand, asset buyers appear to benefit from anti-bubble policy. In the absence of a goods market, anti-bubble policy increases the asset buyers' ex ante expected welfare when they have DARA preferences, because the utility gains from a larger expected consumption outweighs the the utility loses from more volatile cross-state consumption. Asset buyers also benefit from anti-bubble policy, for certain utility and cost specifications, when a goods market is incorporated into the model. Specifically, for the logarithmic utility, quadratic cost case, anti-bubble policy improves the asset buyer's ex ante expected utility in the goods market. Finally, in both specifications of our goods market, anti-bubble policy does not have any effect on the ex ante expected welfare of workers/producers in the goods market.

CHAPTER 2

Three-State Rational Greater-Fool Bubbles

2.1 Introduction

This chapter provides the general conditions under which a strong bubble can exist in a three-state, three-period model with multi-period consumption. We first examines the simplest case of the model where only one asset is available to trade: the risky bubble asset. Next, we show that a strong bubble is still possible even when endowments are durable. Finally, we introduce a risk-free asset into our strong bubble model of a greater-fool bubble. In this section, we demonstrate that a strong bubble is not possible when a risk-free asset is traded in period 1. However, related work from Conlon et al. (2019) shows that a strong bubble can coexist with a risk-free asset in models with four states of the world.

2.2 A Simple Bubble

2.2.1 Setup and Equilibrium

This appendix considers the general case of a three-state, three-period strong bubble model, with only one asset available to trade, which is risky. In this general case, agent j faces the following expected utility maximization problem in his/her period-1 information set $I_{1,i}^j$,

$$\max \sum_{t=1}^{3} \beta^{t-1} E\left[U(C_t^j(\omega))\middle|\, \omega \in I_{1,i}^j\right]$$

$$s.t. \quad C_t^j(\omega) = \begin{cases} e_t^j(\omega) + p_t(\omega)X_t^j(\omega) \text{ for } t = 1, 2 \\ e_3^j(\omega) + p_3(\omega)X_3^j(\omega) + d(\omega)s_3^j(\omega) \text{ for } t = 3 \end{cases}$$

$$X_t^j(\omega) \text{ is measurable w.r.t. } \mathbb{P}_t^j$$

$$X_t^j(\omega) \leq s_{t-1}^j(\omega)$$

$$s_t^j(\omega) = s_{t-1}^j(\omega) - X_t^j(\omega)$$

$$s_0^j(\omega) = n_j \geq 0, \text{ with } n_j \text{ independent of } \omega.$$

Here ω is a typical state of the world, $j = E, F$ indicates Ellen and Frank, respectively, $\mathbb{P}_t^j$ is Agent j's period-t (possibly price-refined) information partition for $t = 1, 2, 3$, $e_t^j(\omega)$ is his/her endowment, $s_0^j(\omega) = n_j$ is his/her initial endowment of the risky asset, $s_t^j(\omega)$ is his/her risky-asset holdings at the end of period t, $d(\omega)$ is the risky asset's state-ω dividend, which is only paid in period 3, $X_t^j(\omega)$ is his/her net sales of the risky asset, $p_t(\omega)$ is the price of the risky asset, and $C_t^j(\omega)$ is Agent j's period-t consumption. All of these are defined in greater detail in the main paper.

We define $M_t^j(\omega) = \pi(\omega)\beta^{t-1}U'\left(C_t^j(\omega)\right)$ to be the discounted shadow price of consumption in period t, state ω. In other words, $M_t^j(\omega)$ is the *ex ante* expected marginal utility that Agent j attaches to an additional unit of consumption in state ω in period t. Similarly, $M_t^j(\{\omega_1, ..., \omega_k\})$ is the discounted shadow price of consumption in the collection of states $\{\omega_1, ..., \omega_k\}$, so $M_t^j(\{\omega_1, ..., \omega_k\}) = M_t^j(\omega_1) + M_t^j(\omega_2) + ... + M_t^j(\omega_k)$.

Note that these shadow prices depend on final consumption, and so, will depend endogenously on the equilibrium outcome.

Recall that Ellen's and Frank's information partitions are

$$\begin{aligned} \textbf{Ellen:} \quad & \text{Period 1} : \{\{b, L\}, \{H\}\} \\ & \text{Period 2} : \{\{b\}, \{L\}, \{H\}\} \\ \textbf{Frank:} \quad & \text{Period 1} : \{\{b\}, \{L, H\}\} \\ & \text{Period 2} : \{\{b\}, \{L, H\}\}, \end{aligned} \tag{2.1}$$

while the true state of the world is common knowledge to the agents by the beginning of period 3. Agents' endowments in period t are dependent on the state of the world ω, but they must conform to their information partitions. For instance, Ellen's period-1 endowment in state b must equal her period-1 endowment in state L, so $e_1^E(b) = e_1^E(L) = e_1^E(\{b, L\})$, by slight abuse of notation. The risky asset only pays a dividend in state H, period 3, so $d(H) = d > 0$ and $d(b) = d(L) = 0$.

In general, Agent j's period-t first-order condition (FOC) from his/her expected utility maximization problem, given his/her information set $I_{t,i}^j = \{\omega_1, ..., \omega_k\}$, is

$$M_t^j(I_{t,i}^j)p_t(\omega) \geq \sum_{\omega \in I_{t,i}^j} M_{t+1}^j(\omega)p_{t+1}(\omega). \tag{2.2}$$

If Agent j is a buyer or a holder of the risky asset in period t, then (2.2) necessarily holds as an equality. If (2.2) holds as a strict inequality, then Agent j must necessarily sell any shares s/he owns, and in fact, must be short-sale constrained. We use this in the proof of Proposition A.

Proposition A: Define $\bar{p}_1$ and $\bar{p}_2$ as solutions to

$$\bar{p}_2 = \frac{\pi(H)\beta U'\left(e_3^F(H) + (n_E + n_F)d\right)}{[\pi(L) + \pi(H)]\, U'\left(e_2^F(\{L, H\}) - (n_E + n_F)\bar{p}_2\right)} d = \frac{M_3^F(H)}{M_2^F(\{L, H\})} d \tag{2.3}$$

and

$$\bar{p}_1 = \frac{\beta U'\left(e_2^E(H) + (n_E + n_F)\bar{p}_2\right)}{U'\left(e_2^E(H) - n_F\bar{p}_1\right)} \bar{p}_2 = \frac{M_2^E(H)}{M_1^E(H)} \bar{p}_2. \tag{2.4}$$

Then in a three-state, three-period economy with the information structure described in (2.1), there is a strong bubble equilibrium, such that

$$\begin{aligned}
&p_1(\omega) = \bar{p}_1 \text{ for all } \omega \in \Omega, && X_1^F(\omega) = -X_1^E(\omega) = n_F \text{ for all } \omega \in \Omega,\\
&p_2(b) = 0, && X_2^E(b) = X_2^F(b) = 0,\\
&p_2(L) = p_2(H) = \bar{p}_2, && X_2^E(\omega) = -X_2^F(\omega) = n_E + n_F \text{ for } \omega = L, H,\\
&p_3(b) = p_3(L) = 0, p_3(H) = d, && X_3^E(\omega) = X_3^F(\omega) = 0 \text{ for all } \omega \in \Omega,
\end{aligned}$$

if and only if the following three conditions are met.[1]

$$\begin{aligned}
\text{Condition 1}: \; & \frac{M_2^E(L)}{M_1^E(\{b,L\})} = \frac{M_2^E(H)}{M_1^E(H)},\\
\text{Condition 2}: \; & \frac{M_2^E(H)}{M_1^E(H)} \geq \frac{M_2^F(\{L,H\})}{M_1^F(\{L,H\})},\\
\text{Condition 3}: \; & \frac{M_3^F(H)}{M_2^F(\{L,H\})} \geq \frac{M_3^E(H)}{M_2^E(H)}.
\end{aligned} \tag{2.5}$$

In the equilibrium, Condition 1 will imply that Ellen's period-1 willingness-to-pay (WTP) for the risky asset is the same in her cells $\{b, L\}$ and $\{H\}$. Condition 2 will imply that Ellen's period-1 WTP in $\{H\}$, and so also in $\{b, L\}$, is greater than or equal to Frank's in $\{L, H\}$. Note also that Frank's period-1 WTP is zero in state b since he knows the asset price will be zero in the next period. Thus, Ellen's WTP will be higher than Frank's in state b as well as in states L and H, and so in every state in period 1. As a result, Ellen will buy and hold the risky asset in period 1, whatever the state is. Condition 3 will imply that Frank's period-2 WTP for the asset in $\{L, H\}$ is greater than or equal to Ellen's in $\{H\}$. Thus, Ellen will sell the asset in state H in period 2. Ellen will also prefer to sell in state L, since she knows the asset price will be zero in the next period. Thus, Ellen will sell and Frank will buy the asset in both states L and H in period 2. In summary, if state b occurs, Ellen will be the fool who buys a worthless asset in period 1. If state L occurs, Ellen will buy the asset in period 1 and Frank, as a greater fool, will buy the asset back in period 2. Finally, if

[1] Recall that the shadow prices $M_t^j(\cdot)$ in (2.5) depend on the state-contingent consumption, and so, depend on the prices given in (2.3) and (2.4).

state H occurs, Ellen will buy the valuable asset to smooth her consumption between periods 1 and 2, and Frank will buy the asset back in period 2 to smooth his consumption between periods 2 and 3. There is no fool in this last case.

Proof of Sufficiency: We first show that (2.5) gives sufficient conditions for Proposition A's prices and net sales to satisfy the FOCs and equilibrium conditions. The SOCs hold automatically since markets are competitive and utility is concave.

Proof. Start with the last period, period 3. At this point, the true state of the world is common knowledge to both Ellen and Frank. As a result, the period-3 asset price simply equals its state-contingent dividend, so $p_3(b) = p_3(L) = 0$ and $p_3(H) = d$. Also, neither Ellen nor Frank has a motive to trade the asset, so $X_3^E(\omega) = X_3^F(\omega) = 0$ is optimal for all $\omega \in \Omega$.

Next, consider period 2. In state b, it is common knowledge that the asset is worthless, so the asset price equals zero, i.e., $p_2(b) = 0$, and there is no motive to trade, so $X_2^E(b) = X_2^F(b) = 0$ is optimal.

In states L and H, the risky asset's worth is not common knowledge, so trades of the risky asset may occur at a positive price. Assume Frank's period-2 FOC in states L and H holds as an equality. That is, Frank's FOC, given that he holds shares in the risky asset, is

$$M_2^F(\{L, H\})p_2(\{L, H\}) = M_3^F(H)d, \tag{2.6}$$

where by abuse of notation $p_2(\{L, H\})$ represents the common period-2 price in states L and H. This, along with (2.3), implies that the period-2 equilibrium prices in states L and H are $p_2(L) = p_2(H) = \bar{p}_2$.

Given the equilibrium price $\bar{p}_2$ and Condition 3, Ellen's FOC in state H holds automatically, since

$$\bar{p}_2 = \frac{M_3^F(H)}{M_2^F(\{L, H\})}d \geq \frac{M_3^E(H)}{M_2^E(H)}d.$$

As a result, Condition 3 implies that Ellen's period-2 FOC in state H given that she hold zeros shares of the risky asset at the end of this period is

$$M_2^E(H)p_2(H) \geq M_3^E(H)d, \tag{2.7}$$

while her state-L analog holds automatically since she know the asset price will be zero next period. Thus, in $\{L, H\}$, Frank bids the price up to $\bar{p}_2$, and Ellen is willing to sell all her asset shares at this price in state H as well as in state L. Ellen's and Frank's period-2 net sales of $X_2^E(\omega) = -X_2^F(\omega) = s_1^E(\omega)$ are therefore optimal for $\omega = L, H$. Also, assuming for the moment that

$$s_1^E(L) = s_1^E(H), \tag{2.8}$$

the quantity of shares traded does not differ across states L and H, so Frank cannot endogenously refine his period-2 cell $\{L, H\}$. Thus, the period-2 equilibrium prices and net sales are also optimal with respect to agents' price-refined information sets, since there is no actual price refinement.

Finally, consider period 1. Using (2.2), Ellen's FOC in her cell $\{b, L\}$ is

$$M_1^E(\{b, L\})p_1(\{b, L\}) \geq M_2^E(L)\bar{p}_2$$

, and Frank's FOC in his cell $\{L, H\}$ is

$$M_1^F(\{L, H\})p_1(\{L, H\}) \geq M_1^F(\{L, H\})\bar{p}_2. \tag{2.9}$$

Again, in equilibrium, at least one of the agents' FOCs must hold as an equality to determine the equilibrium price. Condition 2 in (2.5) implies that Ellen's period-1 WTP for the asset in state H is greater than or equal to Frank's WTP in $\{L, H\}$, while Condition 1 implies that Ellen's period-1 WTP in her cell $\{b, L\}$ is the same as her WTP in $\{H\}$. Thus, Conditions 1 and 2 together imply that Ellen's period-1 WTP is always greater than or equal to Frank's WTP in states L and H. It is also greater than Frank's WTP in state b, which is zero, since Frank knows the asset price will fall to zero in the next period. As a result, Ellen's FOCs must hold as equalities. That is,

$$M_1^E(\{b,L\})p_1(\{b,L\}) = M_2^E(L)\bar{p}_2, \tag{2.10}$$

and

$$M_1^E(H)p_1(H) = M_2^E(H)\bar{p}_2. \tag{2.11}$$

These two equalities, together with Condition 1, show the period-1 equilibrium prices are equal, so $p_1(b) = p_1(L) = p_1(H)$. Equations (2.11) and (2.4) further show that $p_1(b) = p_1(L) = p_1(H) = \bar{p}_1$. Given the equilibrium price $\bar{p}_1$ and Condition 2, Frank's period-1 FOC in $\{L,H\}$, (2.9), holds automatically, since

$$\bar{p}_1 = \frac{M_2^E(H)}{M_1^E(H)}\bar{p}_2 \geq \frac{M_1^F(\{L,H\})}{M_1^F(\{L,H\})}\bar{p}_2.$$

Thus, at price $\bar{p}_1$, Frank sells all his asset shares in every state of the world. Ellen's and Frank's period-1 net sales of $-X_1^E(\omega) = X_1^F(\omega) = s_0^F(\omega) = n_F$ are therefore optimal for all $\omega \in \Omega$. Since neither the quantity of shares traded nor the asset price differs across the three states, neither Ellen nor Frank can endogenously refine their period-1 information partitions, and so the equilibrium prices and net sales obtained above are also optimal with respect to agents' price-refined information partitions. Finally, note that $s_1^E(L) = s_1^E(H) = n_E + n_F$, so the assumption in (2.8) is met.

A strong bubble occurs in state b, since $p_1(b) = \bar{p}_1 > 0$, even though both Ellen and Frank know, in that state, that the asset will never pay a dividend. The conditions in (2.5) are therefore sufficient for the existence of a strong bubble in a three-state, three-period economy with the information structure given in (2.1). □

Proof of necessity: We prove that a strong bubble equilibrium requires the three conditions in (2.5).

Proof. Any strong bubble must occur in state b in the three-state, three-period economy, because state b is the only state where everyone knows the asset will not pay a dividend. Suppose then that the true state is b. In period 1, Frank knows for certain that the true state is b, and he also knows the true state will become common knowledge in the next period, so the asset price will fall to zero. Thus, Frank strictly prefers to sell all his asset shares in state b in period 1 at any positive price. Frank therefore must also sell all his shares in his cell $\{L, H\}$, since otherwise Ellen could distinguish state b from L and would not buy in state b, which would unravel the bubble. Similarly, Ellen must also buy the same amount of shares in state H as she does in state L, and at the same price, to ensure that Frank cannot endogenously refine his cell $\{L, H\}$. Otherwise, in state b, it would become common knowledge that the true state is b or L, not H. That is, it would become common knowledge that the asset is worthless. As a result, Ellen would again not buy the asset in state b at a positive price, which would again unravel the bubble. In her period-1 cell $\{b, L\}$, Ellen is not sure of the true state, but she knows the period-2 asset price will crash to zero if the state is b. Thus, Ellen is only willing to buy the overpriced asset in $\{b, L\}$ in period 1 if she expects to sell it in state L in period 2. Thus, Frank must be willing to buy the risky asset in state L in period 2.

Moreover, Frank is only willing to buy the asset in state L if he thinks it might be valuable, that is, only if he thinks the state might be H, where the asset will pay a dividend. Thus, Ellen must also be willing to sell in state H in period 2.

Thus, in a strong bubble equilibrium, Ellen must buy the risky asset in period 1 in every state, at the same price, and Frank must buy the risky asset in states L and H in period 2. Using (2.2), it follows that Frank's FOC in his period-2 cell $\{L, H\}$ must hold as an equality, as in (2.6). This gives $p_2(L) = p_2(H) = \bar{p}_2$. At the price $\bar{p}_2$, Ellen's FOC in her period-2 cell $\{H\}$, i.e., (2.7), together with (2.6), implies Condition 3 in (2.5).

Using (2.2) again, Ellen's FOCs in her period-1 cells $\{b, L\}$ and $\{H\}$ must also hold as equalities, since she buys in these states. Also, these two equalities must give the same price. Since $p_2(L) = p_2(H) = \bar{p}_2$, these two equalities produce the same price only if Condition 1 in (2.5) holds, as can be seen by comparing (2.10) and (2.11). The period-1 price is thus $p_1(b) = p_1(L) = p_1(H) = \bar{p}_1$. Finally, given $p_1(L) = p_1(\{L, H\})$, Frank's FOC in $\{L, H\}$, i.e., (2.9), together with (2.11), implies Condition 2 in (2.5). Therefore, the three conditions in (2.5) are necessary for a strong bubble equilibrium in the three-state, three-period economy. $\square$

Note that Condition 1 requires Ellen to have a period-1 consumption smoothing motive to buy the asset that is stronger in state L than it is in state H. Specifically, this consumption smoothing motive is sufficiently stronger in L that her WTP in $\{b, L\}$ is the same as her WTP in $\{H\}$, even though in $\{b, L\}$ she is not sure whether the true state is L, where she can resell the asset to Frank in period 2, or b, where she cannot. Condition 2 means that Ellen's period-1 consumption smoothing motive, to buy the asset, in her cell $\{H\}$ is stronger than Frank's period-1 consumption smoothing motive in his cell $\{L, H\}$. Similarly, Condition 3 means that Frank has a stronger consumption smoothing motive to buy the risky asset relative to Ellen in period 2 in state H, even though Frank is not sure whether the true state is H, where the asset pays a dividend, or L, where it does not.

2.2.2 Numerical Example A

First, consider the numerical example in the main paper. We can check whether the Conditions in Proposition A are met, which they should be since we already know that a strong bubble equilibrium exists in this example. Condition 1 holds for the numerical example in the main paper since

$$\frac{M_2^E(L)}{M_1^E(\{b, L\})} = \frac{1}{2} = \frac{M_2^E(H)}{M_1^E(H)},$$

and Condition 2 holds since

$$\frac{M_2^E(L)}{M_1^E(\{b,L\})} = \frac{1}{2} \geq \frac{1}{4} = \frac{M_2^F(\{L,H\})}{M_1^F(\{L,H\})}.$$

Finally, Condition 3 holds since

$$\frac{M_3^F(H)}{M_2^F(\{L,H\})} = \frac{1}{2} \geq \frac{5}{14} = \frac{M_3^E(H)}{M_2^E(H)}.$$

The asset prices for the strong bubble equilibrium in the main paper have simple, algebraic solutions because agents have logarithmic utility. For other utility functions, however, the solutions for asset prices may be difficult to solve algebraically.

Table 2.1: Agents' Endowments for Numerical Example A

State	b	L	H
Period 1	82	82	18
Period 2	20	1	12
Period 3	400	400	400

Ellen

State	b	L	H
Period 1	4	4	4
Period 2	8	20	20
Period 3	20	20	8

Frank

In general, however, the asset prices in a strong bubble equilibrium, p_1 and p_2, are relatively easy to solve recursively. We first find $\overline{p}_2$ using (2.3), which is one equation in one unknown. Once we have a value for $\overline{p}_2$, we can find $\overline{p}_1$ using (2.4), which is also one equation in one unknown, given $\overline{p}_2$. We can then use the values for $\overline{p}_1$ and $\overline{p}_2$ to find the state-and-time dependent consumption values for Ellen and Frank and check whether Conditions 1, 2 and 3 hold, to ensure that these prices represent a strong bubble equilibrium.

Consider a numerical example similar to the one in Liu and White (2019), where $\pi(\omega) = \frac{1}{3}$ for $\omega \in \Omega$, $\beta = 1$, $n_E = n_F = 1$, and $d = 4$. However, in contrast to the main paper's numerical example, suppose that (i) both Ellen and Frank have a utility function $U(c) = \frac{4}{3}C^{\frac{3}{4}}$ instead of logarithmic utility and (ii) agents' endowments are those shown in Table 2.1 rather than those in Tables 1 and 2 of Liu and White (2019)..

In this numerical example, the analog of (2.3) is

$$\overline{p}_2 = \frac{\frac{1}{3}\left[8+2(4)\right]^{-\frac{1}{4}}}{\frac{2}{3}\left[20-2\overline{p}_2\right]^{-\frac{1}{4}}} \cdot 4 = (20-2\overline{p}_2)^{\frac{1}{4}},$$

which is difficult to solve algebraically. However, using Newton's method, or any comparable numerical algorithm, it becomes clear that the solution is $\overline{p}_2 = 2$. Next, using $\overline{p}_2 = 2$, the analog of (2.4) is

$$\overline{p}_1 = \frac{\frac{1}{3}\left[12+2(2)\right]^{-\frac{1}{4}}}{\frac{1}{3}\left[18-\overline{p}_1\right]^{-\frac{1}{4}}} \cdot 2 = (18-\overline{p}_1)^{\frac{1}{4}},$$

which is also difficult to solve algebraically. However, again, using Netwon's Method, for example, we find $\overline{p}_1 = 2$.

Table 2.2: Agents' Consumption for Numerical Example A

State	b	L	H
Period 1	80	80	16
Period 2	20	5	16
Period 3	400	400	400

Ellen

State	b	L	H
Period 1	6	6	6
Period 2	8	16	16
Period 3	10	20	16

Frank

We then use $p_1 = 2$ and $p_2 = 2$ to calculate Ellen's and Frank's state-and-time dependent consumption, as shown in Table 2.2 respectively. Using these consumption tables, we can check to see if Conditions 1, 2, and 3 are met. If they are met, then we again have a strong bubble equilibrium. Starting with Conditions 1, we can see that

$$\frac{M_2^E(L)}{M_1^E(\{b,L\})} = \frac{1}{2} \cdot \frac{(5)^{-\frac{1}{4}}}{(80)^{-\frac{1}{4}}} = 1,$$

and

$$\frac{M_2^E(H)}{M_1^E(H)} = \frac{(16)^{-\frac{1}{4}}}{(16)^{-\frac{1}{4}}} = 1.$$

This ensures that the period-1 asset prices across all states of the world are equal, so Ellen's information doesn't leak to Frank, which therefore prevents an endogenous refining of Frank's period-1 cell $\{L, H\}$. Next, we see that Conditions 2 holds since

$$\frac{M_2^F(\{L,H\})}{M_1^F(\{L,H\})} = \frac{(16)^{-\frac{1}{4}}}{(6)^{-\frac{1}{4}}} = \left(\frac{3}{8}\right)^{\frac{1}{4}},$$

which is less than the above value of $\frac{M_2^E(H)}{M_1^E(H)} = 1$. Finally, Condition 3 holds since

$$\frac{M_2^E(H)}{M_1^E(H)} = \frac{(400)^{-\frac{1}{4}}}{(16)^{-\frac{1}{4}}} = 0.447,$$

which is less than

$$\frac{M_3^F(H)}{M_2^F(\{L,H\})} = \frac{\frac{1}{3}(16)^{-\frac{1}{4}}}{\frac{1}{3}(16)^{-\frac{1}{4}}} = 0.5.$$

2.3 A Bubble in the Case of a Durable Good

In both Appendix A and the main paper, we assume that endowments of the consumption good are completely perishable, and no riskless asset, e.g., money, exists. In those examples, there are no means available, other than through trades of the bubble asset, for agents to save portions of their endowment in order to intertemporally smooth their consumption. In this appendix, we show that a strong bubble can still exist even if agents have durable (i.e., storable) endowments. We do this by introducing a storage facility where agents can privately store portions of their durable endowment. In general terms, this private storage facility can be thought of as a bank that offers savings deposits. An agent can discreetly "deposit" portions of his/her endowment into this storage facility, and other agents are not able to observe the quantity of endowment placed into these storage facilities.

We begin this appendix by examining the utility-maximizing conditions that determine agents' savings and storage decisions. We then show that the strong bubble equilibrium in the main paper's numerical example remains unchanged when we allow endowments to be *partially* durable (i.e., partially perishable), because storage remains sufficiently unattractive. In general, a storage technology can be incorporated into any pre-existing three-state model of a strong bubble without changing the equilibrium prices and asset trades, if the return on storage is sufficiently low. This is because the storage technology can be specified to have such unappealing intertemporal consumption smoothing benefits that agents never use storage, and instead, decide to allocate all of their savings toward the bubble asset. Next, we present a numerical example of a strong bubble where endowments are *completely* durable (i.e., not perishable at all), and where storage is actually used in equilibrium. Specifically, Ellen uses storage in period 1 in states b and L, and Frank uses storagein period 2 in states L and H. Finally, we prove that, when storage technology is available, Ellen never uses storage in state H (in either period 1 or 2) and Frank never uses storage in period 1 in his cell $\{L, H\}$.

2.3.1 Utility-Maximizing Storage Conditions

Let $\delta \in [0, 1)$ be the consumption good's one-period depreciation rate, so one unit of consumption good placed in storage today will provide $1 - \delta$ units of consumption after one period. Thus, endowments are at least partially durable. Suppose, then, that in period 1, state ω, Agent j places $v_1^j(\omega)$ units of his/her period-1 endowment into storage. Agent j will then have $v_1^j(\omega)(1 - \delta)$ additional units of the consumption good at the beginning of period 2. Note also that no one will use storage in period 3, so $v_3^j(\omega) = 0$ for all $\omega \in \Omega$ and for $j = E, F$, because agents cannot consume after period 3. Thus, Agent j's state-ω consumption for periods 1, 2, and 3, respectively, are

$$\begin{aligned}
C_1^j(\omega) &= e_1^j(\omega) - v_1^j(\omega) + p_1(\omega)X_1^j(\omega) \\
C_2^j(\omega) &= e_2^j(\omega) + (1 - \delta)v_1^j(\omega) - v_2^j(\omega) + p_2(\omega)X_2^j(\omega) \\
C_3^j(\omega) &= e_3^j(\omega) + (1 - \delta)v_2^j(\omega) + p_3(\omega)X_3^j(\omega) + d(\omega)s_3^j(\omega) \, .
\end{aligned} \tag{2.12}$$

Agents are free to choose any nonnegative portion of their endowment to store, but they cannot store a negative amount, i.e., they cannot borrow through the storage technology.

Agent j chooses his/her level of storage, $v_t^j(\omega)$, according to the his/her FOC

$$M_t^j(I_{t,i}^j) \geq M_{t+1}^j(I_{t,i}^j)\,(1-\delta)\,, \tag{2.13}$$

in each of his/her period-t information sets $I_{t,i}^j$, and this holds as an equality if the agent stores a positive amount of the consumption good. Thus, Agent j places none of his/her endowment in storage, so $v_t^j(\omega) = 0$, if (2.13) holds as a strict inequality, while $v_t^j(\omega) > 0$ implies that (2.13) holds as an equality. This means Agent j's storage decision is always subject to the complementary slackness condition

$$v_t^j(\omega) \cdot \left[M_t^j(I_{t,i}^j) - M_{t+1}^j(I_{t,i}^j)\,(1-\delta)\right] = 0. \tag{2.14}$$

Alternatively, we can rewrite (2.13) as

$$1 \geq \frac{M_{t+1}^j(I_{t,i}^j)}{M_t^j(I_{t,i}^j)} \cdot (1-\delta)\,, \tag{2.15}$$

where the right-hand side of (2.15) is Agent j's period-t WTP in his/her cell $I_{t,i}^j$, in terms of the amount of period-t consumption he/she is willing to give up to receive $1-\delta$ units of consumption in period $t+1$. Again, Agent j places none of his/her endowment in storage, so $v_t^j(\omega) = 0$, if (2.15) is a strict inequality, while $v_t^j(\omega) > 0$ only if (2.15) holds as an equality. Intuitively, $v_t^j(\omega) = 0$ if Agent j's WTP for storage is less than the cost of storage, which is always one unit of the consumption good, but if $v_t^j(\omega) > 0$, then Agent j's WTP for storage is equal to the cost of storage.

The FOCs (2.7), (2.6), (2.9), (2.10), and (2.11) from Section 2.1 of course, continue to apply, but the marginal utilities used to determine $\overline{p}_1$ and $\overline{p}_2$ are now evaluated at the consumption levels in (2.12) instead of those from Section 2.2.

In addition, the three conditions in (2.5) are still necessary for an equilibrium, with consumptions again calculated using (2.12), since the necessity part of Proposition A still applies. However, these conditions are no longer sufficient if storage is possible, though they are sufficient for for $\overline{p}_2$ and $\overline{p}_1$, from (2.3) and (2.4), to satisfy the FOCs (2.7), (2.6), (2.9), (2.10), and (2.11), with (2.12) used for consumption.

Finally, note that, because storage is always private, a storage decision by one agent will not cause endogenous refinement of another agent's information partitions. This means, for instance, that Ellen can store different amounts of her period-1 endowment across her cells $\{b, L\}$ and $\{H\}$, without revealing to Frank whether the true states is L or H.

2.3.2 Numerical Example B1: Partially Durable Endowments

Consider the numerical example from Liu and White (2019). In this example, agents' endowments are completely perishable, which means $\delta = 1$. However, the strong bubble equilibrium in this numerical example continues to exist even if we allow agents' endowments to be partially durable, or $\delta < 1$, as long as δ is sufficiently close to one.

Table 2.3: Agents' Willingness-to-Pay for Storage for Numerical Example B1

State	b	L	H
Period 1	0.3	0.3	0.05
Period 2	0.1	0.5	0.036

Ellen

State	b	L	H
Period 1	0.021	0.025	0.025
Period 2	0.2	0.133	0.133

Frank

For example, suppose we allow $\delta = 0.9$ so that $1 - \delta = 0.1$. This slight adjustment in our parameters does not unravel the strong bubble equilibrium in the main paper's numerical example. The asset prices continue to be $p_1 = 1$ and $p_2 = 2$, and the pattern of trade of the risky-bubble asset remains $X_1^F(\omega) = -X_1^E(\omega) = 1 \ \forall\ \omega \in \Omega$ and $X_2^E(\omega) = -X_2^F(\omega) = 2$ for $\omega = L, H$. This is because the change in δ does not affect Ellen's or Frank's consumption, since neither agent chooses to use storage to smooth his/her consumption intertemporally, so $v_t^j(\omega) = 0$ for all $\omega \in \Omega$, for $t = 1, 2, 3$ and for $j = E, F$. This becomes more apparent once we use Ellen's and Frank's consumption values from Tables 5 and 6 in Liu and White (2019) to calculate their respective state-and-time dependent WTPs for storage, which are shown in Table 2.3. For each state and period, Ellen's and Frank's WTP for storage is less than the marginal cost of storage, which is one unit of the consumption good. As a result, $v_t^j(\omega) = 0$ $\forall\ \omega \in \Omega$ must be optimal for both Ellen and Frank in each period t in the main paper's example. If an agent's WTP were greater than one, then zero storage would be suboptimal since the agent could improve his/her expected utility by increasing storage until his/her WTP falls to one.

2.3.3 Numerical Example B2: Perfectly Durable Endowments

We now show that a strong bubble equilibrium can exist even if agents' endowments are completely durable, so $\delta = 0$. Like the numerical example in the main paper, we assume $n_E = n_F = 1$, $\pi(\omega) = \frac{1}{3}\ \forall\ \omega \in \Omega$, $U(C) = \ln C$, $\beta = 1$, $d(b) = d(L) = 0$, and $d(H) = d = 4$. However, in contrast to the main paper's numerical example, we assume that Ellen and Frank have the state-and-time dependent endowments shown in Table 2.4. respectively.

Table 2.4: Agents' Endowments for Numerical Example B2

State	b	L	H
Period 1	17	17	7
Period 2	12	8	8
Period 3	15	30	30

Ellen

State	b	L	H
Period 1	7	6	6
Period 2	10	40	40
Period 3	10	16	8

Frank

As in previous strong bubble equilibria, Ellen buys Frank's one share of the risky asset in period 1 in states b, L, and H, and Frank buys Ellen's two shares in period 2 in states L and H. The period-1 equilibrium asset price is $p_1 = 1\ \forall\ \omega \in \Omega$, and the period-2 equilibrium asset price is $p_2 = 2$ for $\omega = L, H$. Frank's period-1 gross return from the risky asset in his cell is riskless, at $\{L, H\}$ is $\frac{p_2}{p_1} = 2$, and Ellen's period-2 gross return from the risky asset in her cell $\{H\}$ is riskless, at $\frac{d}{p_1} = 2$. Both of these returns are greater than the gross return to storage of 1, so Frank doesn't use storage in period 1 in his cell $\{L, H\}$ and Ellen doesn't use storage in in period 2 in her cell $\{H\}$.

Ellen's and Frank's state-contingent storage for periods 1 and 2, or $v_1^j(\omega)$ and $v_2^j(\omega)$, are shown in Table 2.5. Notice that, as we would expect, Ellen does not save through

Table 2.5: Agents' Storage for Numerical Example B2

State	b	L	H
Period 1	2	2	0
Period 2	0	0	0

Ellen

State	b	L	H
Period 1	0	0	0
Period 2	0	10	10

Frank

storage in state H in period 2 while Frank does not save through storage in period 1 in his cell $\{L, H\}$. However, Ellen does store 2 units of her period-1 endowment in her cell $\{b, L\}$ while Frank stores 10 units of his period-2 endowment in his cell $\{L, H\}$.

Table 2.6: Agents' Consumption for Numerical Example B2

State	b	L	H
Period 1	14	14	6
Period 2	14	14	12
Period 3	15	30	30

Ellen

State	b	L	H
Period 1	8	7	7
Period 2	10	26	26
Period 3	10	26	26

Frank

Ellen's and Frank's state-and-time dependent consumption are shown in Table 2.6. Using these consumption values, we can check Conditions 1, 2, and 3 to make sure that the FOCs (2.7), (2.6), (2.9), (2.10), and (2.11) still hold with, $\overline{p}_2$ and $\overline{p}_1$ from (2.3) and (2.4). Condition 1 holds since

$$\frac{M_2^E(H)}{M_1^E(H)} = \frac{1/12}{1/6} = \frac{M_2^E(L)}{M_1^E(\{b, L\})} = \frac{(1/14)}{(1/14) + (1/14)} = \frac{1}{2},$$

and Condition 2 holds since

$$\frac{M_2^F(\{L, H\})}{M_1^F(\{L, H\})} = \frac{(1/26) + (1/26)}{(1/7) + (1/7)} = \frac{7}{26} \leq \frac{1}{2}.$$

Additionally, Condition 3 holds since

$$\frac{M_3^E(H)}{M_2^E(H)} = \frac{1/30}{1/12} = \frac{12}{30} \leq \frac{M_3^F(H)}{M_2^F(\{L, H\})} = \frac{(1/26)}{(1/26) + (1/26)} = \frac{1}{2}.$$

Thus, a strong bubble is present even though the consumption good is completely durable.

We use (2.3) and (2.4) to confirm that $p_1 = 1$ and $p_2 = 2$ are indeed the equilibrium asset price values. Beginning with period 2, its clear that $p_2 = 2$ is the equilibrium asset price for period 2, since

$$\overline{p}_2 = \frac{M_3^F(H)}{M_3^F(\{L,H\})} \cdot d = \frac{1}{2} \cdot 4 = 2.$$

Furthermore, $p_1 = 1$ is the correct equilibrium asset price for period 1, since

$$\overline{p}_1 = \frac{M_2^E(H)}{M_1^E(H)} \cdot \overline{p}_2 = \frac{1}{2} \cdot 2 = 1.$$

Of course, the FOCs (2.7), (2.6), (2.9), (2.10), and (2.11) can also be checked directly.

Table 2.7: Agents' Willingness-to-Pay for Storage for Numerical Example B2

State	b	L	H
Period 1	1	1	0.5
Period 2	0.93	0.47	0.4

Ellen

State	b	L	H
Period 1	0.8	0.27	0.27
Period 2	1	1	1

Frank

Finally, we can check agents' storage FOCs by calculating their respective state-and-time dependent WTPs for storage. Ellen's and Frank's WTPs for storage are calculated in Table 2.7. Notice that, when an agent stores a positive quantity of the consumption good, his/her WTP is always 1. In period 1, Ellen places 2 units of her endowment into storage in states b and L, so her WTP is 1 as shown in the top row of the left panel of Table 2.6. In period 2, Frank places 10 units of his endowment into storage in states L and H, so his WTP is 1 as shown in the bottom row of the right panel of Table 2.6. Furthermore, notice that agents do not use storage when their WTP is less than 1. For instance, Frank does not use storage in period 1 in any state of the world because his WTP is always less than 1. The same is true for Ellen in period 1, state H as well as in period 2, states b and L. However, notice that Frank's WTP is 1 in period 2 in state b even though he does not use storage. This is because a storage WTP equal to 1 is a *necessary* condition for positive storage but it is not a sufficient condition. Conversely, a storage WTP less than 1 is a sufficient condition for zero storage, but it is not a necessary condition.

2.3.4 Agents' General Storage Decisions

We now examine agents' storage decisions in the general context of any strong bubble equilibrium, given our three-state, three-period structure, and show that, in states where risky-bubble asset has a riskless positive payoff next period, the gross return to the risky-bubble asset exceeds the gross return to storage. This result allows us to determine the state and period where an agent never uses the storage technology. Proposition B1 examines Ellen's period-1 storage decision in state H.

Proposition B1: In any strong bubble equilibrium where storage technology is available, the gross return to the risky asset is always greater than the gross return to storage in period 1 in states L and H, so $\frac{p_2}{p_1} > 1 - \delta$. Furthermore, Ellen never uses storage in period 1 in state H, so $v_1^E(H) = 0$.

Proof. Compare the reciprocal of Condition 1 with the reciprocal of Ellen's period-1, state-H analog of (2.15). This gives

$$1 - \delta \leq \frac{M_1^E(\{b, L\})}{M_2^E(\{b, L\})} < \frac{M_1^E(\{b, L\})}{M_2^E(L)} = \frac{M_1^E(H)}{M_2^E(H)} = \frac{p_2}{p_1} \tag{2.16}$$

so $\frac{p_2}{p_1} > 1 - \delta$. In period 1 in state H, Ellen knows with certainty that $\frac{p_2}{p_1} > 1 - \delta$, so she devotes all of her savings to the risky-bubble asset to maximize the value of her savings next period. This means that none of her savings are allocated to storage, so $v_1^E(H) = 0$. □

Note that (2.16) implies $M_1^E(H) > M_2^E(H)(1-\delta)$, so by the complementary slackness condition in (2.14), Ellen's period-1 storage in state H must be $v_1^E(H) = 0$.

Intuitively, in her period-1 cell $\{b, L\}$, Ellen can use the risky-bubble asset to intetermporally smooth her consumption in state L but not in state b. In contrast, the storage technology allows Ellen to intertemporally smooth her consumption in *both* states b and L. Thus, Ellen requires a larger state-L return on the risky-bubble asset than on storage to compensate her for the loss on the risky asset in state b. Since the return to risky asset is the same across states L and H, the state-H return on the risky asset must also be greater than the return to storage.

Next, consider Frank's period-1 storage decision in his cell $\{L, H\}$. We examine this in Proposition B2.

Proposition B2: In any strong bubble equilibrium where storage technology is available, Frank will never use storage in period 1 in states L and H, so $v_1^F(\{L, H\}) = 0$.

Proof. In period 1 in his cell $\{L, H\}$, Frank knows with certainty that $\frac{p_2}{p_1} > 1 - \delta$, so he devotes all of his savings to the risky-bubble asset to maximize the value of his savings next period. This means that none of his savings are allocated to storage, so $v_1^F(\{L, H\} = 0$. □

Note that combining the inequality $\frac{p_2}{p_1} > 1 - \delta$ with the reciprocal of Condition 2 gives

$$1 - \delta < \frac{M_1^E(H)}{M_2^E(H)} \leq \frac{M_1^F(\{L, H\})}{M_2^F(\{L, H\})}.$$

This implies $M_1^F(\{L, H\}) > M_2^F(\{L, H\})\,(1 - \delta)$, so $v_1^F(\{L, H\}) = 0$ by the complementary slackness condition.

Intuitively, Frank has a relatively large liquidity demand in period 1 in his cell $\{L, H\}$, so he is not willing to save to earn a relatively large gross return of p_2/p_1. Since he is unwillilling to save to earn a larger gross return of p_2/p_1, he must therefore also be unwilling to store to earn a smaller gross return of $1 - \delta$.

Finally, in Proposition B3, we examine Ellen's period-2 storage decision in state H.

Proposition B3: In any strong bubble equilibrium where storage technology is available, the gross return to the risky asset is always greater than the gross return to storage in period 2 in state H, so $\frac{d}{p_2} > 1 - \delta$. Furthermore, Ellen never uses storage in period 2, state H, so $v_2^E(H) = 0$.

Proof. First, recall that Frank's period-2 storage decision in his cell $\{L, H\}$ is made according to his FOC in (2.6), i.e., $M_2^F(\{L, H\}) \geq M_3^F(\{L, H\})\,(1 - \delta)$. Combing this with the reciprocal of Condition 3 implies

$$1-\delta \leq \frac{M_2^F(\{L,H\})}{M_3^F(\{L,H\})} < \frac{M_2^F(\{L,H\})}{M_3^F(H)} = \frac{d}{p_2}, \tag{2.17}$$

so $\frac{d}{p_2} > 1-\delta$. Because Ellen knows $\frac{d}{p_2} > 1-\delta$ with certainty, she devotes all of her savings to the risky asset in order to maximize the value of her savings next period. As a result, she allocates none of her savings to storage, so $v_2^E(H) = 0$. $\square$

Intuitively, between periods 2 and 3, the risky-bubble asset allows Frank to smooth his intertemporal consumption in state H but not in state L. However, the storage technology allows Frank to smooth his intertemporal consumption in both states L and H. Thus, in state H, Frank requires a larger return on the risky-bubble asset than on the storage technology. Ellen knows the true state is H, so she knows for sure that her savings from the risky-asset would receive a gross return of $\frac{p_3(H)}{p_2(H)} = \frac{d}{p_2} > 1-\delta$. Since Ellen is unwilling to save to earn a relatively large gross return of d/p_2, she must also be unwilling to store to earn a smaller gross return of $1-\delta$.

2.4 A Risky-Bubble Asset with a Risk-Free Asset

We now examine strong bubble equilibria when risk-free assets are available. We assume that each agent j is endowed with $n_f^j(t)$ shares of the risk-free asset at the beginning of period t, so $n_f^T(t) = n_f^E(t) + n_f^F(t)$ is the total number of shares of the risk-free asset at the beginning of period t. With a probability of one, each share of the risk-free asset pays a dividend of one unit of the consumption good at some point in the future, depending on the risk-free asset's maturity structure. Each risk-free asset can mature in either one period or two periods. A risk-free asset purchased in period t that matures after one period will pay a dividend in period $t+1$, while one that matures after two periods will pay a dividend in period $t+2$. Since our strong bubble model has three periods, agents can have two separate endowments of the risk-free asset if it has a one-period maturity: one endowment received at the beginning of period 1 and another endowment received at the beginning of period 2. However, for shares of a risk-free asset that mature after two periods, agents can only receive their endowments at the beginning of period 1.[2] As a result, if the risk-free asset has a two-period maturity, then agents never receive a period-2 endowment of shares of the risk-free asset, so $n_f^j(2) = 0$.

2.4.1 Risk-Free Asset with a One-Period Maturity

Consider a risk-free asset that matures after one period. Let $q_t(\omega)$ be the period-t price of a share of the risk-free in state ω, and let $Y_t^j(\omega)$ be agent j's net sales of the risk-free asset over the course of period t. At the beginning of period t, Agent j receives an endowment of $n_f^j(t)$ shares of the risk-free asset for $t = 1, 2$. Finally, let $z_t^j(\omega)$ be the number of shares of the risk-free asset that agent j has held to maturity by the beginning of period $t+1$. Thus, if the risk-free asset has a maturity of one period, then Agent j's state-and-time dependent consumption is

$$\begin{aligned}
C_1^j(\omega) &= e_1^j(\omega) + q_1(\omega)Y_1^j(\omega) + z_0^j(\omega) + p_1(\omega)X_1^j(\omega) \\
C_2^j(\omega) &= e_2^j(\omega) + q_2(\omega)Y_2^j(\omega) + z_1^j(\omega) + p_2(\omega)X_2^j(\omega) \\
C_3^j(\omega) &= e_3^j(\omega) + q_3(\omega)Y_3^j(\omega) + z_2^j(\omega) + p_3(\omega)X_3^j(\omega) + d(\omega)s_3^j(\omega)
\end{aligned} \tag{2.18}$$

where $s_t^j(\omega) = s_{t-1}^j(\omega) - X_t^j(\omega)$ and $z_t^j(\omega) = n_f^j(t) - Y_t^j(\omega)$. As in section 2.2, the true state of the world is common knowledge by the beginning of period 3, and consumption ceases after the end of period 3. As a result, every period-3 asset price equals the its period-3 divided, so $q_3(\omega) = 1$ and $p_3(\omega) = d(\omega)$, and all trading stops by the beginning of period 3, so $Y_3^j(\omega) = X_3^j(\omega) = 0$. Finally, note that Agent j has not held any risk-free assets to maturity by the beginning of period 1, so $z_0^j(\omega) = 0$.

[2]For ease of exposition, we will not consider instances where risk-free assets of different maturities are traded simultaneously. If agents are endowed with shares in a risk-free asset with a one-period maturity, then they will never be endowed with shares in a risk-free asset with a two-period maturity, and vice versa.

In general, agents choose their net sales of the risk-free asset in order to maximize their expected lifetime utility. Agent j's FOC in his/her period-t cell $\{\omega_1 \ldots \omega_k\}$ is

$$M_t^j(\{\omega_1 \ldots \omega_k\})q_t(\omega) \geq M_{t+1}^j(\{\omega_1 \ldots \omega_k\}). \tag{2.19}$$

Notice from (2.19) that the price of the risk-free asset in period $t+1$ is simply one unit of the consumption good, since the risk-free asset matures after one period. Thus, Agent j's period-t WTP for one share of the risk-free asset in his//her cell $\{\omega_1 \ldots \omega_k\}$ is

$$\frac{M_{t+1}^j(\{\omega_1 \ldots \omega_k\})}{M_t^j(\{\omega_1 \ldots \omega_k\})}.$$

If Agent j buys and holds the risk-free asset in period t in state ω, then (2.19) is necessarily an equality, and Agent j's WTP for the risk-free asset determines the risk-free asset's equilibrium price. If (2.19) is a strict inequality, then Agent j is necessarily a seller of the risk-free asset who is short-sale constrained.

Of course, the FOCs (2.7), (2.6), (2.9), (2.10), and (2.11) from section 2.2 continue to apply. However, the marginal utilities used to determine $\overline{p}_1$ and $\overline{p}_2$ are evaluated at the consumption levels in (2.18) instead of those from Section 2.2

2.4.1.1 A Risk-Free Asset in Period 2

A strong bubble equilibrium is possible when agents are endowed in period 2 with a risk-free asset that matures after one period. Ellen's period-2 WTP for the risk-free asset in state H is

$$\frac{M_3^E(H)}{M_2^E(H)},$$

while Frank's period-2 WTP in $\{L, H\}$ is

$$\frac{M_3^F(\{L, H\})}{M_2^F(\{L, H\})}.$$

Condition 3 in Section 2.2 implies

$$\frac{M_3^F(\{L,H\})}{M_2^F(\{L,H\})} > \frac{M_3^E(H)}{M_2^E(H)},$$

which means that Frank's period-2 intertemporal consumption smoothing motive in his cell $\{L, H\}$ is always greater than Ellen's period-2 motive in her cell $\{H\}$. In other words, Frank has a higher WTP for the risk-free asset in a strong bubble equilibrium in period 2, state H. As a result, Frank will purchase all of Ellen's shares in the risk-free asset in period 2 in state H, assuming that Ellen's period-2 WTP for the risk-free asset in L is also lower than Frank's period-2 WTP in his cell $\{L, H\}$. Thus, the equilibrium pattern of trade for the risk-free asset is $Y_2^E(\omega) = n_f^E(2)$ for $\omega = L, H$ and $Y_2^F(\omega) = -n_f^E(2)$. Thus, the period-2 equilibrium risk-free asset price in states L and H, $q_2 = q_2(L) = q_2(H)$, is

$$q_2 = \frac{M_3^F(\{L,H\})}{M_2^F(\{L,H\})}.$$

2.4.1.2 Numerical Example C

To see how a risk-free asset endowed in period 2 can exist within a strong bubble equilibrium, consider a numerical example where Ellen is endowed with $n_f^E(2) = n_f^T(2) = 8$ shares of a risk-free asset in period 2, while Frank never receives any endowment of risk-free asset shares, so $n_f^F(2) = 0$. Furthermore, assume that Ellen and Frank have the endowment structures shown in Table 2.8. Also, assume that $\pi(\omega)$, β, $U(C)$, $n_f^E(2)$, $n_f^F(2)$, and the dividend structure of the risky asset is the same as those in the numerical examples in Liu and White (2019) and Sections 2.2 and 2.3

Table 2.8: Agents' Endowments for Numerical Example C

State	b	L	H
Period 1	17	17	9
Period 2	4	4	4
Period 3	64	64	64

Ellen

State	b	L	H
Period 1	10	9	9
Period 2	36	40	40
Period 3	20	20	12

Frank

In equilibrium, Frank sells his 1 share of the risky asset to Ellen in period 1 in each state of the world, and Ellen sells her 2 shares to Frank in period 2 in states L and H. Since neither Ellen nor Frank is endowed with shares of the risk-free asset in period 1, the equilibrium trades of the risk-free asset is zero in period 1. However, in the period-2 equilibrium, Ellen sells all 8 of her shares of the risk-free asset to Frank in states b, L and H.[3] The equilibrium period-1 risky asset is price $p_1 = 1 \; \forall \; \omega \in \Omega$, and the equilibrium period-2 risky asset price is $p_2 = 2$ for $\omega = \{L, H\}$. Furthermore, the equilibrium period-2 risk-free asset price is $q_2 = q_2(b) = q_2(L) = q_2(H) = 1$. Ellen's and Frank's resulting consumption are shown in Table 2.9.

Table 2.9: Agents' Consumption for Numerical Example C

State	b	L	H
Period 1	16	16	8
Period 2	12	16	16
Period 3	64	64	64

Ellen

State	b	L	H
Period 1	11	10	10
Period 2	28	28	28
Period 3	28	28	28

Frank

As in Sections 2.2 and 2.3, we can use the consumption values in Table 2.9 to make sure the Conditions in (2.5) are not violated, starting with Condition 1 which does indeed hold since

$$\frac{M_2^E(H)}{M_1^E(H)} = \frac{M_2^E(L)}{M_1^E(\{b, L\})} = \frac{1}{2}.$$

Furthermore, we know that Condition 2 holds since

$$\frac{M_2^F(\{L, H\})}{M_1^F(\{L, H\})} = \frac{5}{14} \leq \frac{1}{2}.$$

Finally, Condition 3 holds because

$$\frac{M_3^F(H)}{M_2^F(\{L, H\})} = \frac{1}{2} \geq \frac{M_3^E(H)}{M_2^E(H)} = \frac{1}{4}.$$

[3] Note, a strong bubble equilibrium requires the period-2 risk-free asset prices in states L and H to be equal. However, a strong bubble equilibrium does not require the period-2 risk-free asset price in state b to equal the period-2 price in states L and H, though it does in this example.

2.4.1.3 A Risk-Free Asset in Period 1

We now show that a strong bubble equilibrium cannot exist if a risk-free asset with a one-period maturity is held/traded in period 1. This is shown in the proof of Proposition C1.

Proposition C1: In a three-state model, a strong bubble equilibrium cannot exist if agents can hold a risk-free asset with a one-period maturity in period 1.

Proof. Assume that a strong bubble equilibrium exists, and a risk-free asset with a one-period maturity is available in period 1. Suppose Ellen buys and holds the risk-free asset in period 1, so it is her WTP that determines the risk-free asset' period-1 price in a strong bubble equilibrium. Ellen's WTP for the risk-free asset in H is

$$q_1(H) = \frac{M_2^E(H)}{M_1^E(H)}$$

while her WTP for the risk-free asset in b and L is

$$q_1(\{b, L\}) = \frac{M_2^E(\{b, L\})}{M_1^E(\{b, L\})}.$$

If this is a strong bubble equilibrium, then it must be that $q_1(H) = q_1(\{b, L\})$ otherwise Frank's period-1 cell $\{L, H\}$ will be endogenously refined. This implies

$$\frac{M_2^E(H)}{M_1^E(H)} = \frac{M_2^E(\{b, L\})}{M_1^E(\{b, L\})} > \frac{M_2^E(L)}{M_1^E(\{b, L\})}$$

which violates Condition 1. Thus, it cannot be Ellen's WTP that determines the period-1 price of the risk-free asset in a strong bubble equilibrium.

Now suppose Frank buys and holds the risk-free asset in period 1, so it is his WTP that determines the risk-free asset's period-1 price in a strong bubble equilibrium. This must mean that Frank's WTP for the risk-free asset is greater than Ellen's WTP in any state of the world ω, so

$$\frac{M_2^F(\{L,H\})}{M_1^F(\{L,H\})} \geq \frac{M_2^E(\{b,L\})}{M_1^E(\{b,L\})}.$$

However, this means that Condition 1 and Condition 2 are mutually exclusive, because if Condition 1 holds then it must be that

$$\frac{M_2^F(\{L,H\})}{M_1^F(\{L,H\})} > \frac{M_2^E(L)}{M_1^E(\{b,L\})} = \frac{M_2^E(H)}{M_1^E(H)},$$

which contradicts Condition 2. Thus, it cannot be Frank's WTP that determines the period-1 price of the risk-free asset in a strong bubble equilibrium.

Since neither buys and holds the risk-free asset, neither agent's WTP can determine the period-1 price of the risk-free asset in a strong bubble equilibrium. This implies that a strong bubble equilibrium cannot exist if a risk-free asset with a one-period maturity can be held in period 1. $\square$

2.4.2 A Risk-Free Asset with a Two-Period Maturity

Now consider a risk-free asset that matures after two periods. As in the case of a risk-free asset with a one-period maturity, let $q_t(\omega)$ be the period-t price of one share of the risk-free asset in state ω, and let $Y_t^j(\omega)$ be Agent j's period-t net sales of the risk-free asset in state ω. At the beginning of period 1, Agent j receives an endowment of n_f^j shares of the risk-free asset. Unlike the case of a risk-free asset with one-period maturity, Agent j only receives one endowment of the risk-free asset when the risk-free asset has a two-period maturity. Finally, let $z_t^j(\omega)$ be the quantity of risk-free asset shares that Agent j owns by the end of period t. Thus, if the risk-free asset has a two-period maturity, then Agent j's consumption is

$$
\begin{aligned}
C_1^j(\omega) &= e_1^j(\omega) + q_1(\omega)Y_1^j(\omega) + p_1(\omega)X_1^j(\omega) \\
C_2^j(\omega) &= e_2^j(\omega) + q_2(\omega)Y_2^j(\omega) + p_2(\omega)X_2^j(\omega) \\
C_3^j(\omega) &= e_3^j(\omega) + q_3(\omega)Y_3^j(\omega) + z_3^j(\omega) + p_3(\omega)X_3^j(\omega) + d(\omega)s_3^j(\omega)
\end{aligned}
\tag{2.20}
$$

where $s_t^j(\omega) = s_{t-1}^j(\omega) - X_t^j(\omega)$ and $z_t^j(\omega) = z_{t-1}^j(\omega) - Y_t^j(\omega)$. As in the previous subsection, $q_3(\omega) = 1$ and $p_3(\omega) = d(\omega)$, and $Y_3^j(\omega) = X_3^j(\omega) = 0$.

In general, agents choose their expected net sales of the risk-free asset to maximize their expected lifetime utility. In his/her period-t cell $I_{t,i}^j = \{\omega_1 \ldots \omega_k\}$, Agent j's FOC is

$$
M_t^j(I_{t,i}^j)q_t(\omega) \geq \sum_{\omega \in I_{t,i}^j} M_{t+1}^j(\omega)q_{t+1}(\omega), \tag{2.21}
$$

so his/her WTP for the risk-free asset is

$$
\frac{\sum_{\omega \in I_{t,i}^j} M_{t+1}^j(\omega)q_{t+1}}{M_t^j(I_{t,i}^j)}.
$$

Note that $q_3(\omega) = 1$. Thus, in period 2 (2.21) is simply

$$
M_t^j(I_{t,i}^j)q_2(\omega) \geq M_{t+1}^j(\{\omega_1 \ldots \omega_k\}).
$$

If Agent j is the buyer of the risk-free asset in period t in state ω, then (2.21) is necessarily an equality, and Agent j's WTP for the risk-free asset determines the risk-free asset's equilibrium price. If (2.21) is a strict inequality, then Agent j is necessarily a seller of the risk-free asset who is short-sale constrained.

Of course, the FOCs (2.7), (2.6), (2.9), (2.10), and (2.11) from Appendix A continue to apply. However, the marginal utilities used to determine $\overline{p}_1$ and $\overline{p}_2$ are evaluated at the consumption levels in (2.20) instead of those from Appendix A.

2.4.2.1 A Risk-Free Asset in Period 1

Proposition C2: In a three-state model, a strong bubble equilibrium cannot exist if agents can hold a risk-free asset with a two-period maturity in period 1.

Proof. Assume that a strong bubble equilibrium exists, and a risk-free asset with a two-period maturity is available in period 1. Condition 3 requires that Frank have a larger period-2 WTP for the risk-free asset than Ellen in state H. This must also be true for state L, otherwise Frank's period-2 cell would be endogenously refined. Thus, in a strong bubble equilibrium, Frank's period-2 WTP for the risk-free asset must determine the period-2 price of the risk-free asset in states L and H, so

$$q_2(\{L,H\} = \frac{M_2^F(\{L,H\})}{M_1^F(\{L,H\})},$$

and $q_2(b)$ is the period-2 price of the risk-free asset in state b.

Buying a risk-free asset with two periods left till maturity in period 1 and then selling it in period 2 is equivalent to buying a risk-free asset with a one-period maturity in period 1 that pays an period-2 dividend of $q_2(\{L,H\})$ in states L and H and $q_2(b)$ in state b. Suppose Ellen buys and holds this risk-free asset in period 1, so it is her WTP that determines its period-1 price. Ellen's period-1 WTP for the risk-free asset in her cell $\{b,L\}$ is

$$\frac{M_2^E(b)q_2(b) + M_2^E(L)q_2(\{L,H\})}{M_1^E(\{b,L\})}, \tag{2.22}$$

and her period-1 WTP in her cell $\{H\}$ is

$$\frac{M_2^E(H)q_2(\{L,H\})}{M_1^E(H)}. \tag{2.23}$$

In a strong bubble equilibrium, Ellen's WTPs in (2.22) and (2.23) must be equal, otherwise

Frank could refine his cell $\{L, H\}$. . However, this violates Condition 1 since

$$\frac{M_2^E(H)q_2(\{L,H\})}{M_1^E(H)} = \frac{M_2^E(b)q_2(b) + M_2^E(L)q_2(\{L,H\})}{M_1^E(\{b,L\})} > \frac{M_2^E(L)q_2(\{L,H\})}{M_1^E(\{b,L\})},$$

so Ellen cannot buy and hold the risk-free asset in period 1 in a strong bubble equilibrium. Now suppose Frank buys and holds the risk-free asset in period 1. Frank's WTP for the risk-free asset in his cell $\{L, H\}$ determines the period-1 price of the risk-free asset. This means his period-1 WTP must be greater than or equal to Ellen's period-1 WTP in any state of the world ω, so

$$\frac{M_2^F(\{L,H\})q_2(\{L,H\})}{M_1^F(\{L,H\})} \geq \frac{M_2^E(b)q_2(b) + M_2^E(L)q_2(\{L,H\})}{M_1^F(\{L,H\})}.$$

However, this implies that Conditions 1 and 2 are mutually exclusive because if if Condition 1 is met, then

$$\frac{M_2^F(\{L,H\})q_2(\{L,H\})}{M_1^F(\{L,H\})} > \frac{M_2^E(L)q_2(\{L,H\})}{M_1^F(\{L,H\})} = \frac{M_2^E(H)q_2(\{L,H\})}{M_1^E(H)},$$

which violates Condition 2. Thus, Frank cannot buy and hold the risk-free asset in period 1 in a strong bubble equilibrium. Since, neither agent can buy and hold the risk-free asset in period 1 in a strong bubble equilibrium, then a strong bubble cannot exist if a risk-free asset with a two-period maturity can be held in period 1. □

2.4.3 Summary

In summary, a strong bubble is not possible in a three-state model when a one-period risk-free asset is available for sale in period, though Conlon et al. (2019) show that it is possible in a four-state model. However, a strong bubble is possible in a three-state model if the risk-free asset is introduced in period 2. The former result provides implications for the introduction of new asset markets as an information transmission mechanism. Because intertemporal consumption smoothing incentives vary across states of the world, risk-free asset prices may also differ across states of the world since risk-free asset prices reflect agents' intertemporal consumption smoothing motives. This means that, in the three-state model, the introduction of new asset markets can reveal agents' private information about the true state of the economy, thereby preventing the formation of asset price bubbles. However, this lesson may not be robust in a four-state model.

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