2010

Five Point Zero Divisor Graphs

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FIVE POINT ZERO DIVISOR GRAPHS

Five Point Zero Divisor Graphs Determined By Equivalence Classes of Zero Divisors

by

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A thesis submitted to the faculty of the University of Mississippi in partial fulfillment of the requirements of the Sally McDonnell Barkesdale Honors College

Oxford

May 2010

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Abstract

We study the zero divisor graphs, determined by equivalence classes of zero divisors of a ring R, with exactly five vertices. In particular, we determine which graphs with exactly five vertices can be realized as the zero divisor graph of a ring. We provide rings for the graphs which are possible, and prove that the rest of graphs can not be realized via any commutative ring. There are thirty-four graphs in total which contain exactly five vertices.

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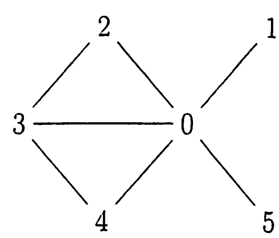
0.1 Introduction

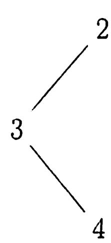
In 1988, I. Beck [?] introduced the idea of the zero divisor graph. By this notion, a ring R is represented by a simple graph, where each vertex corresponds to an element of the ring. Vertices a and b share an edge if and only if $ab = 0$. By a simple graph we mean that there are no loops (i.e., an edge from a to a) or multiple edges between any two vertices. Each of these graphs is connected with diameter two, since every vertex is adjacent to zero.

Example The zero divisor graph for the ring $R = \mathbb{Z}/6\mathbb{Z}$ contains the elements $\{0, 1, 2, 3, 4, 5\}$, and is shown below.



In 1999, Anderson and Livingston [?] elaborated on Beck's idea of zero divisor graphs. In their graph, only zero divisors are represented. Therefore, any element of the ring which is not a zero divisor is not shown. The notation for these graphs is $\Gamma(R)$. These are also simple graphs and are connected.

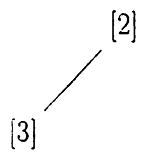
Example (Revisited) The zero divisors of the ring $R = \mathbb{Z}/6\mathbb{Z}$ are $\{2, 3, 4\}$.



$\Gamma(\mathbb{Z}/6\mathbb{Z})$

In 2002, S. Mulay [?] demonstrated how a graph could be constructed from equivalence classes of zero divisors. Once again, these graphs are simple and connected. The notation for a graph constructed from equivalence classes is $\Gamma_E(R)$.

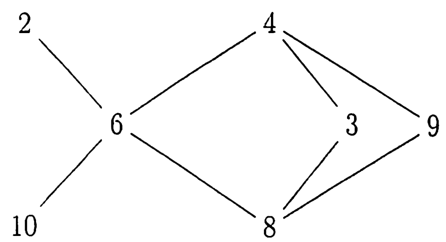
Example (Revisited) The equivalence classes of zero divisors of the ring $R = \mathbb{Z}/6\mathbb{Z}$ are $\{[2], [3]\}$.



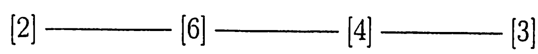
$\Gamma_E(\mathbb{Z}/6\mathbb{Z})$

To illustrate the relationship between $\Gamma(R)$ and $\Gamma_E(R)$, we provide the example below. Note that $\Gamma(R)$ "collapses", in some sense, to $\Gamma_E(R)$, since we are identifying zero divisors.

Example Consider the graphs for $\mathbb{Z}/12\mathbb{Z}$:



$\Gamma(\mathbb{Z}/12\mathbb{Z})$



$\Gamma_E(\mathbb{Z}/12\mathbb{Z})$

The goal of this project is to examine the five point zero divisor graphs constructed from equivalence classes and to determine which five point graphs can be zero divisor graphs of some ring. For each graph which is a zero divisor graph of some ring, we provide a ring associated to it. For those graphs which are not, we prove that no ring exists such that $\Gamma_E(R)$ takes the necessary form. We note that there are thirty-four total graphs with exactly five vertices [?, pp.216-217], but we need only focus on the connected ones in this project since it has been shown [?], [?], [?] that zero divisor graphs are always connected. Therefore we will consider the twenty-one connected five point graphs. See the Appendix for a complete list of the graphs.

Finally, to motivate this project, we provide one more example. The ring $R = (\mathbb{Z}/6\mathbb{Z})[X]$, all polynomials with coefficients from $\mathbb{Z}/6\mathbb{Z}$, is infinite, and therefore the zero divisor graphs of Beck and Anderson & Livingston can not be drawn. However, $\Gamma_E(R)$ takes the same form as $\Gamma_E(\mathbb{Z}/6\mathbb{Z})$ in our earlier example.

0.2 Background

The following is background information regarding rings and related concepts. A standard reference for this material is the book by J. Gallian [?]. The book by H. Matsumura is a standard reference for the material on commutative ring theory [?].

Definition A **ring** R is a set with two binary operations, addition and multiplication, such that for all a, b, c in R:

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. There is an additive identity 0. That is, there is an element 0 in R such that $a + 0 = a$ for all a in R.
4. There is an element $-a$ in R such that $a + (-a) = 0$.
5. $a(bc) = (ab)c$
6. $a(b + c) = ab + ac$ and $(b + c)a = ba + ca$

Note that a simple example of a ring is the set of integers, denoted $\mathbb{Z}$. The "integers modulo n", for example $\mathbb{Z}/6\mathbb{Z}$ or $\mathbb{Z}/12\mathbb{Z}$, are also rings.

Definition A ring R is **commutative** if multiplication is commutative; i.e., if $ab = ba$ for every a, b in R. A ring has **unity** if it has a multiplicative identity; i.e., if there is an element 1 in R such that $1a = a1$ for all a in R.

Definition A **zero divisor** is a nonzero element a of a ring R such that for some nonzero element b in R, $ab = 0$.

Note that $\mathbb{Z}$ is commutative and has unity one, and it contains no zero divisors. Note also that $\mathbb{Z}/12\mathbb{Z}$ contains zero divisors, namely 3 and 4 because (3)(4)=0 in $\mathbb{Z}/12\mathbb{Z}$.

Definition In general, an **annihilator** element b of an element a is one such that $ab = 0$.

Definition An **ideal** A of a ring R is a nonempty subset where if a,b are in A then $a - b$ is in A, and if a is in A and r is in R, then ar and ra are both in A.

Definition The **annihilator ideal** of a is defined as $\text{ann}(a) = \{r \in R \mid ra = 0\}$. It is an ideal of R.

Definition A **unit** in a ring R is a nonzero element u such that there exists another nonzero element u^{-1} such that $uu^{-1} = u^{-1}u = 1$.

Definition A **field** is a commutative ring with unity such that every nonzero element is a unit.

Definition An **equivalence relation**, denoted $\sim$, on a set S is a binary relation that satisfies the following three conditions:

1. $a \sim a$ for all $a \in S$.

2. $a \sim b$ implies $b \sim a$ for all $a, b \in S$.

3. $a \sim b$ and $b \sim c$ imply $a \sim c$ for all $a, b, c \in S$.

The set $[a] = \{x \in S : x \sim a\}$ is called **the equivalence class of** a.

Example For a commutative ring R with unity, denote $a \sim b$ if and only if $\text{ann}(a) = \text{ann}(b)$, for all $a, b \in R$. Note that:

1. $\text{ann}(a) = \text{ann}(a)$ for all $a \in R$.

2. $\text{ann}(a) = \text{ann}(b)$ implies $\text{ann}(b) = \text{ann}(a)$ for all $a, b \in R$.

3. $\mathrm{ann}(a) = \mathrm{ann}(b)$, $\mathrm{ann}(b) = \mathrm{ann}(c)$ imply $\mathrm{ann}(a) = \mathrm{ann}(c)$ for all $a, b, c \in R$.

This is an equivalence relation on R. The set $[a] = \{x \in R : \mathrm{ann}(x) = \mathrm{ann}(a)\}$ is the equivalence class of a.

Definition The **graph of equivalence classes of zero divisors of a ring** R, denoted by $\Gamma_E(R)$, is the simple graph associated to R whose vertices are the classes of zero divisors determined by the above relation, and with each pair of distinct classes $[u], [v]$ joined by an edge if and only if $[u] \cdot [v] = 0$, where $[u] \cdot [v]$ is defined to be $[uv]$. (This is well-defined [?], [?].)

Definition Some graph theory definitions are as follows:

1. A *path of length* n between two vertices u and v is a sequence of distinct vertices w_i of the form $u = w_0 - w_1 - \cdots - w_n = v$ such that $w_{i-1} - w_i$ is an edge for each i. The *distance* between u and v is the length of the shortest path.

2. A *connected* graph is one in which there is a path between any two vertices.

3. The *diameter* of a graph is the greatest distance between any two vertices. For zero divisor graphs determined by equivalence classes, the diameter is three or less.

Finally, a number of strategies will be useful in determining which five point graphs are not zero divisor graphs of any ring.

Strategies

1. If there is an edge between two points on the graph, then we say the two points connected annihilate one another. Specifically, when multiplied together, the products of the two connected points is zero. Note that by

definition, a five point zero divisor graph contains five points, each of which has a distinct group of annihilators.

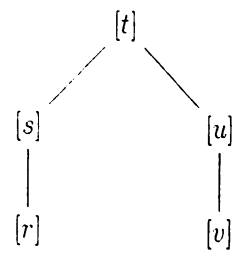
2. If two points on the zero divisor graph appear to have the exact same annihilators (that is they share edges with the exact same other vertices) and they are not adjacent to one another, then at least one is self-annihilating. In other words, at least one of the points must annihilate itself. Otherwise, the two points would represent the exact same class.

3. If two points on the zero divisor graph appear to have the exact same annihilators and they are adjacent to one another, then at least one of those points must not annihilate itself. Otherwise, then two points would represent the exact same class.

4. To show that a five point graph is not the zero divisor graph of some ring, we will assume that it is and then construct a new sixth class from the original five.

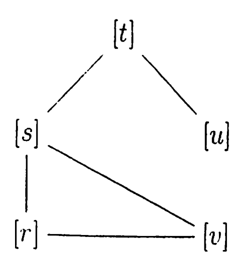
0.3 Negative Results

Claim 0.3.1. *The graph pictured below is not possible.*



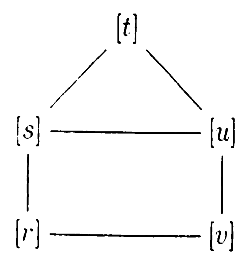
Proof. We will find a new class of zero divisor not represented on the graph. Consider the class $[ru]$. The element is not zero, otherwise there would be an edge directly from $[r]$ to $[u]$. But it is a zero divisor. Note that ann$[ru]$ contains at least s, t, and v. Now examine the annihilators of the individual vertices. Ann$[r]$ does not contain t. Ann$[s]$ does not include v. Ann$[t]$ does not include v. Ann$[u]$ does not include s. Ann$[v]$ does not include s. Thus ann$[ru]$ would represent a new element on this graph. □

Claim 0.3.2. *The graph pictured below is not possible.*



Proof. We will find a new class of zero divisor not represented on the graph. Consider the class $[tv]$. The element is not zero, otherwise there would be an edge directly from t to v. But it is a zero divisor since $r(tv) = 0$. Note that tv is annihilated by at least r, s, and u. Now examine the annihilators of the individual vertices. Ann[r] doesn't contain t. Ann[s] doesn't contain u. Ann[t] doesn't contain r. Ann[u] doesn't contain s. Ann[v] doesn't contain u. Thus Ann[tv] would represent a new element on the graph. □

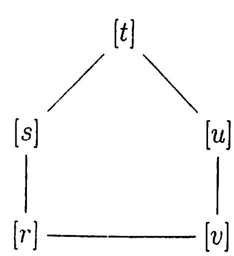
Claim 0.3.3. *The following graph is not possible.*



Proof. We will find a new class of zero divisor not represented on the graph. Consider the class [$r+t$]. It is a zero divisor because the individual elements r and t share an annihilator, but the two do not represent the same class (otherwise they would share a vertex.) The only annihilator of $r+t$ is s since the only element which annihilates both r and t is s. Now examine the annihilators of the individual vertices. Ann[r] includes v. Ann[t] includes u. Ann[s] includes u. Ann[u] includes v. Ann[v] includes u. Thus Ann[$r+t$] would represent a new element on the graph.

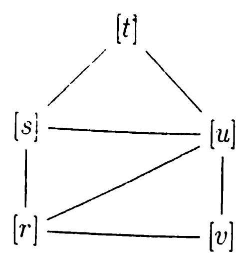
□

Claim 0.3.4. *The following graph is not possible.*



Proof. We will find a new class of zero divisor not represented on the graph. Consider the class [tv]. This element is not zero, otherwise there would be an edge directly from t to v. But it is a zero divisor since $u(tv) = 0$. Note that tv is annihilated by at least s, r, and u. Now examine the annihilators of the individual vertices. Ann$[r]$ doesn't include u. Ann$[s]$ doesn't include u. Ann$[t]$ doesn't include r. Ann$[u]$ doesn't include s. Ann$[v]$ doesn't include s. Thus Ann$[tv]$ would represent a new element on the graph. □

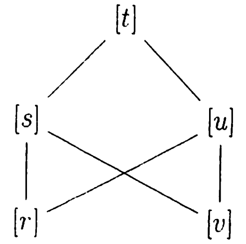
Claim 0.3.5. *The following graph is not possible.*



Proof. We will find a new class of zero divisor not represented on the graph. Consider the class $[t+v]$. It is a zero divisor because the individual elements t and v share an annihilator, but the two do not represent the same class (otherwise they would share a vertex.) The only annihilator of $t+v$ is u. Also, both r and s are not annihilators of $[t+v]$. Now examine the annihilators of the individual vertices. Ann$[r]$ includes s. Ann$[s]$ includes r. Ann$[t]$ includes s. Ann$[u]$ includes s. Ann$[v]$ includes r. Thus Ann$[t+v]$ would represent a new element on the graph.

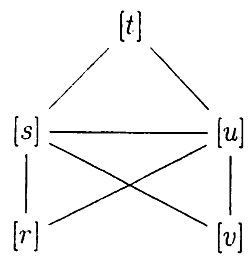
□

Claim 0.3.6. *The following graph is not possible.*



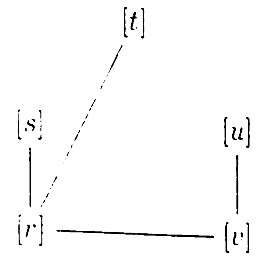
Proof. We will find a new class of zero divisor not represented on the graph. First consider r and v. Since $[r]$ and $[v]$ appear to have the same annihilator, by Strategy (2), we can assume that $r^2 = 0$ because otherwise $[r] = [v]$. Consider s and u. Say $s^2 = 0$ otherwise $[s] = [u]$. We will now find a new class of zero divisor not represented on the graph. Consider $[r + u]$. It is a zero divisor because the individual elements r and u share an annihilator, but the two do not represent the same class (otherwise they would share a vertex.) Ann$[r + u]$ includes r. It does not include s or v. Now examine the annihilators of individual elements. Ann$[r]$ includes s. Ann$[s]$ includes v. Ann$[t]$ includes s. Ann$[u]$ includes v. Ann$[v]$ includes s. Thus Ann$[r + u]$ would represent a new element on the graph. □

Claim 0.3.7. *The following graph is not possible.*



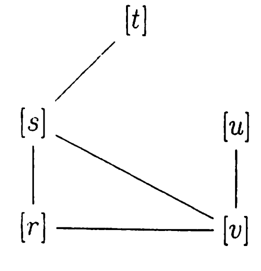
Proof. We will find a new class of zero divisor not represented on the graph. Consider s and u. By Strategy (3) say $u^2 \neq 0$ because otherwise $[s] = [u]$. Consider r and v. By Strategy (2) one must be self-annihilating, otherwise $[r] = [v]$. Thus say $r^2 = 0$. We will now find a new class of zero divisor not represented on the graph. Consider $[r + u]$. It is a zero divisor because the individual elements r and u share an annihilator, but the two do not represent the same class. Ann$[r + u]$ includes s and r. It does not include u, t, or v. Now examine the annihilators of individual elements. Ann$[r]$, Ann $[s]$, Ann$[t]$, and Ann$[v]$ include u. Ann$[u]$ includes v. Thus Ann$[r + u]$ would represent a new element on the graph. □

Claim 0.3.8. *The following graph is not possible.*



Proof. We will find a new class of zero divisor not represented on the graph. Consider $[s]$ and $[t]$. By Strategy (2) say $s^2 = 0$ because otherwise $[s] = [t]$. We will now find a class of zero divisor not included on the graph. Consider the class of $[sv]$. This element is not zero, otherwise there would be an edge directly betwen s and v. But it is a zero divisor because there is some nonzero element which multiplies sv to zero. Not that Ann$[sv]$ includes at least r, u, and s. Now examine the annihilators of the individual elements. Ann$[r]$, Ann$[s]$, and Ann$[t]$ do not include u. Ann$[u]$ does not include r. Ann$[v]$ does not include s. Thus Ann$[sv]$ would represent a new element on the graph. □

Claim 0.3.9. *The following graph is not possible.*



Proof. We will show that this graph cannot possibly be constructed by equivalence classes. Note that $su \neq 0$ but $rsu = tsu = vsu = 0$, so $[su] = [s]$. Moreover, it implies that $su^2 \neq 0$, hence $u^2 \neq 0$. By symmetry, $[tv] = [v]$ and $t^2 \neq 0$. Next, consider $s + v$, which is annihilated by r, but not t or u. The only candidate for

$[s+v]$ is $[r]$. The implications of this are $r^2 = s^2 = v^2 = 0$ since $r(s+v) = s(s+v) = v(s+v) = 0$.

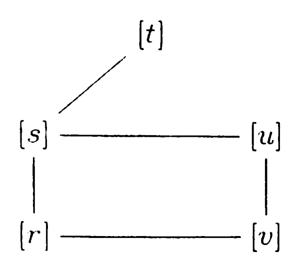
We consider tu, which is annihilated by s and v. The candidates for $[tu]$ are $[r]$, $[s]$, and $[v]$. By symmetry, we need only consider $[r]$ and $[s]$.

Case I: $[tu] = [r]$ This means that $t^2u \neq 0$, $tu^2 \neq 0$, but $t^2u^2 = 0$. It follows that $[t^2u] = [v]$ and $[tu^2] = [s]$. Moreover, $[t^2] \neq [v]$ since t^2 is not annihilated by u; likewise $[u^2] \neq [s]$. On the other hand, t^2 is annihilated by s, hence $[t^2] \neq [u]$; likewise $[u^2] \neq [t]$. Thus, the possibilities for $[t^2]$ are $[r]$, $[s]$, and $[t]$. If $[t^2] = [r]$, then $t^2v = 0$, which contradicts $[tv] = [v]$. For the same reason $[t^2] \neq [s]$. Therefore, the only remaining possibility is the case $[t^2] = [t]$. Then $[u^2]$ has to be $[s]$ since $[t^2u^2] = 0$. But the class of s is self-annihilating, hence $su^2 = 0$, a contradiction.

Case II: $[tu] = [s]$ This means that $t^2u = 0$. Thus $[t^2] = [v]$, and hence $t^2v = 0$ since v is self-annihilating. But this contradicts the fact that $[tv] = [v]$.

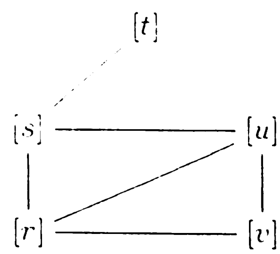
In conclusion, $[tu]$ represents a new class. □

Claim 0.3.10. *The following graph is not possible.*



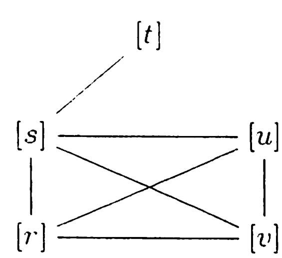
Proof. We will find a new class of zero divisor not represented on the graph.Consider the element $r+v$. By Strategy (2), we can assume that $r^2 = 0$, otherwise $[r] = [u]$. So $r+v$ is annihilated by r but not by s or u. Now we see that $r+v$ represents a new element on the graph because t and u are not annihilated by r, r is annihilated by s, and s and v are annihilated by u. □

Claim 0.3.11. *The following graph is not possible.*



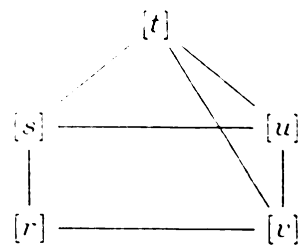
Proof. We will find a new class of zero divisor not represented on the graph. By Strategy (3) we can assume that $r^2 \neq 0$, otherwise $[r] = [u]$. Therefore, $r + v$ is annihilated by u but not by r or s. Because s is in the annihilators of r and t, and r is in the annihilators of s, u, and v, we see that $r + v$ represents a new element on the graph. □

Claim 0.3.12. *The following graph is not possible.*



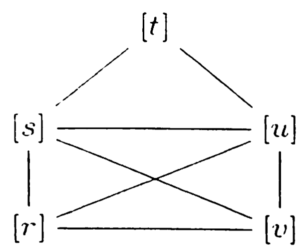
Proof. We will find a new class of zero divisor not represented on the graph. Notice that by Strategy (3), $r^2 \neq 0$ and $v^2 \neq 0$ (without loss of generality) because otherwise $[r] = [u] = [v]$. Now consider the element $r + v$. It is annihilated by s and u but not by r or v. Note that $r(r+v) = r^2 \neq 0$ and $v(r+v) = v^2 \neq 0$. Notice that r, s, u, and v are all annihilated by r or v, and that t is not annihilated by u. Therefore $r + v$ represents a new class. □

Claim 0.3.13. *The following graph is not possible.*



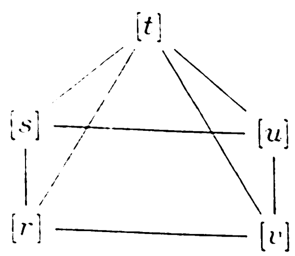
Proof. We will find a new class of zero divisor not represented on the graph. Consider $t + v$. Its annihilators include u and do not include r or s. An examination of the annihilators shows that s is in the annihilators of r, t, and u and r is in the annihilators of s and v. Thus $t + v$ represents a new class. □

Claim 0.3.14. *The following graph is not possible.*



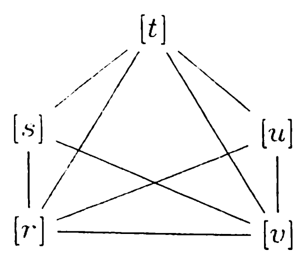
Proof. We will find a new class of zero divisor not represented on the graph. Consider the element $r + s$. By Strategy (3), $r^2 \neq 0$ and $s^2 \neq 0$ otherwise $[r] = [v]$ and $[s] = [u]$. Notice, then that $r + s$ is annihilated by u and v, but not by r, s, or t. An examination of the annihilators shows that r, t, u, and v are all annihilated by s, and that s is annihilated by r. Thus $r + s$ represents a new class. □

Claim 0.3.15. *The following graph is not possible.*



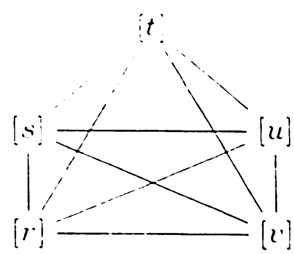
Proof. We will find a new class of zero divisor not represented on the graph. First note that $u^2 = 0$ otherwise $[r] = [u]$. Likewise, $v^2 = 0$ otherwise $[s] = [v]$. Therefore, $u + v$ is annihilated by t, u, and v but not by r or s. However, an examination of the annihilators shows that every vertex is adjacent to either r or s. Thus $\mathrm{Ann}[u+v]$ determines a class distinct from each vertex, and hence $[u+v]$ would represent a new element on the graph. □

Claim 0.3.16. *The following graph is not possible.*



Proof. We will find a new class of zero divisor not represented on the graph. First note that by Strategy (3) $v^2 \neq 0$ otherwise $[r] = [v]$. Also note that $u^2 = 0$ otherwise $[s] = [u]$. Now consider $u + v$. Its annihilators include r, t, and u but not s or v. However, every vertex on the graph is adjacent to one of these two vertices. Thus $[u + v]$ represents a new element on the graph. □

Claim 0.3.17. *The following graph is not possible.*

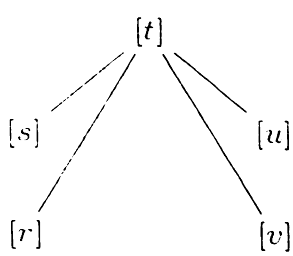


Proof. We will find a new class of zero divisor not represented on the graph. Consider all points. By Strategy (3), $r^2 \neq 0$, $s^2 \neq 0$, $t^2 \neq 0$, and $u^2 \neq 0$, otherwise any combination of these would represent the same class. Now wel will find a class of zero divisor not represented on the graph. Consider $[r + t]$. It is a zero divisor because the individual elements r and t share an annihilator, but the two do not represent the same class. Note that $\text{Ann}[r + t]$ includes s, u, and v. It does not include r or t. Now examine the annihilators of the individual elements. $\text{Ann}[r]$, $\text{Ann}[s]$, $\text{Ann}[t]$, $\text{Ann}[u]$, and $\text{Ann}[v]$ each include r. Thus $\text{Ann}[r + t]$ would represent a new element on the graph. □

0.4 Positive Results

In this section, we show that four of the (connected) five vertex graphs can be realized as $\Gamma_E(R)$ for some ring R. Many of these rings will be factor rings $R = S/I$, where S is a polynomial ring with coefficients from $\mathbb{Z}/p\mathbb{Z}$ for a prime number p and I is some ideal. Note that if $I = (a, b)$, then a general element of I looks like $s_1 a + s_2 b$, where $s_1, s_2 \in S$. Also, $I^2 = (a^2, ab, b^2)$ and $I^3 = (a^3, a^2b, ab^2, b^3)$. A general element of $R = S/I$ looks like $r = s + I$, where $r = 0$ if and only if $s \in I$.

Proposition 0.4.1. *Set* $R = \dfrac{(\mathbb{Z}/2\mathbb{Z})[[X, Y, Z]]}{(X^3, X^2Y, X^2Z, Y^2, Z^2, XYZ)}$ *and let* $\mathfrak{m}$ *be the ideal* (x, y, z), *where* x *(resp.,* y, z*) represents the coset of* X *(resp.,* Y, Z*) in* R. *We claim that* $\Gamma_E(R)$ *has the graph shown below.*



First of all, note that the nonzero generators of $\mathfrak{m}^2$ are (x^2, xy, xz, yz) and that $\mathfrak{m}^3 = 0$ in R; i.e., *each of* x, y, *and* z *annihilates every element in* $\mathfrak{m}^2$. Since every polynomial of degree three is equal to zero in R, a general element of R looks like $a + bx + cy + dz + ex^2 + fxy + gyz + hxz$, where the coefficients a, b, c, d, e, f, g, and h are all either 0 or 1. However, whenever $a \neq 0$, this polynomial is a unit since the other terms all lie in $\mathfrak{m}$ [?, p. 4]. Therefore, *the only possible zero divisors live in* $\mathfrak{m}$ *and have the form* $bx + cy + dz + ex^2 + fxy + gyz + hxz$.

- $\operatorname{ann}(x^2) = \mathfrak{m}$. This (first) class will be represented by $[x^2]$.

Note that all the generators of $\mathfrak{m}$ annihilate x^2. Therefore $\mathfrak{m} \subseteq \operatorname{ann}(x^2)$. On the other hand, we already have $\operatorname{ann}(x^2) \subseteq \mathfrak{m}$. Thus, $\operatorname{ann}(x^2) = \mathfrak{m}$.

Remark 0.4.2. The same argument applies to the elements xy, yz, xz, and any combination of these terms. Therefore, all of these terms, and any combination of them, determine the class $[x^2]$.

Remark 0.4.3. We claim that for any zero divisor α, $\text{ann}(\alpha + x^2) = \text{ann}(\alpha)$; i.e., $[\alpha + x^2] = [\alpha]$. To see this, note that each zero divisor β lives in $\mathfrak{m} = \text{ann}(x^2)$, hence $\beta(\alpha + x^2) = \beta\alpha$ equals zero if and only if $\beta \in \text{ann}(\alpha)$. Moreover, this same argument applies to xy, yz, xz, and any combination of these terms. Therefore, *only linear combinations of the zero divisors of degree one will provide new equivalence classes.*

- $\text{ann}(x) = (\mathfrak{m}^2)$. This (second) class will be represented by $[x]$.

Note that x is annihilated by every element in $\mathfrak{m}^2$, hence $\mathfrak{m}^2 \subseteq \text{ann}(x)$. We need to establish the reverse containment. Let $\alpha \in \text{ann}(x)$, where $\alpha = bx+cy+dz$. Then $0 = x(bx + cy + dz) = bx^2 + cxy + dxz$. Since $b, c, d \in \mathbb{Z}_2$ and x^2, xy and xz are linearly independent over $\mathbb{Z}_2$, we must have $b = c = d = 0$. Therefore, no degree one term annihilates x; i.e., $\text{ann}(x) \subseteq (\mathfrak{m}^2)$, and hence, $\text{ann}(x) = (\mathfrak{m}^2)$.

- $\text{ann}(y) = (y, \mathfrak{m}^2)$. This (third) class will be represented by $[y]$.

Note that y is annihilated by every element in $\mathfrak{m}^2$ and y itself, hence $(y, \mathfrak{m}^2) \subseteq \text{ann}(y)$. We need to establish the reverse containment. Let $\alpha \in \text{ann}(y)$, where $\alpha = bx + cy + dz$. Then $0 = y(bx + cy + dz) = bxy + dyz$. Since $b, d \in \mathbb{Z}_2$ and xy and xz are linearly independent over $\mathbb{Z}_2$, we must have $b = d = 0$. Therefore, the only degree one term that annihilates y is y itself; i.e., $\text{ann}(y) \subseteq (y, \mathfrak{m}^2)$, and hence, $\text{ann}(y) = (y, \mathfrak{m}^2)$.

- An analogous argument, with the roles of y and z switched, shows $\text{ann}(z) = (z, \mathfrak{m}^2)$. This (fourth) class will be represented by $[z]$.

- $\text{ann}(y + z) = (y + z, \mathfrak{m}^2)$. This (fifth) class will be represented by $[z]$.

Note that $y+z$ is annihilated by every element in $\mathfrak{m}^2$ and $y+z$ itself, hence $(y+z, \mathfrak{m}^2) \subseteq \text{ann}(y+z)$. We need to establish the reverse containment. Let $\alpha \in \text{ann}(y+z)$, where $\alpha = bx + cy + dz$. Then $0 = (y+z)(bx+cy+dz) = bxy + bxz + (c+d)yz$. Note that if $b = 0$ and $c = d = 1$, then the right hand side is zero. Therefore, there is some dependence among these terms. We consider all the cases carefully in the chart below. (Note that any triple containing only one nonzero term has already been considered in the classes above.)

(b, c, d)	$bxy + bxz + (c+d)yz$	
(1, 1, 0)	$xy + xz + yz$	$\neq 0$ in R
(1, 0, 1)	$xy + xz + yz$	$\neq 0$ in R
(0, 1, 1)	$2yz$	$= 0$ in R
(1, 1, 1)	$xy + xz$	$\neq 0$ in R

Therefore, the only degree one term that annihilates $y+z$ is $y+z$ itself; i.e., $\text{ann}(y+z) \subseteq (y+z, \mathfrak{m}^2)$, and hence, $\text{ann}(y+z) = (y+z, \mathfrak{m}^2)$.

Finally, in the remainder of our argument, we show that the remaining degree one terms do not result in any new classes.

∘ $\text{ann}(x+y) = \mathfrak{m}^2$; i.e., $[x+y] = [x]$.

Note that $x+y$ is annihilated by every element in $\mathfrak{m}^2$, hence $\mathfrak{m}^2 \subseteq \text{ann}(x+y)$. We need to establish the reverse containment. Let $\alpha \in \text{ann}(x+y)$, where $\alpha = bx + cy + dz$. Then $0 = (x+y)(bx+cy+dz) = bx^2 + (b+c)xy + dxz + dyz$. We consider all the cases carefully in the chart below. (Note that any triple containing only one nonzero term has already been considered in the classes above.)

(b, c, d)	$bx^2 + (b+c)xy + dxz + dyz$	
(1, 1, 0)	$x^2 + 2xy = x^2$	$\neq 0$ in R
(1, 0, 1)	$x^2 + xy + xz + yz$	$\neq 0$ in R
(0, 1, 1)	$xy + xz + yz$	$\neq 0$ in R
(1, 1, 1)	$x^2 + 2xy + xz + yz = x^2 + (x+y)z$	$\neq 0$ in R

Therefore, no degree one term annihilates $x + y$; i.e., $\mathrm{ann}(x + y) \subseteq (\mathfrak{m}^2)$, and hence, $\mathrm{ann}(x + y) = (\mathfrak{m}^2)$; i.e., $[x + y] = [x]$.

Remark 0.4.4. By a symmetric argument, $[x + z] = [x]$.

∘ $\mathrm{ann}(x + y + z) = \mathrm{ann}(x)$.

Note that $x+y+z$ is annihilated by every element in $\mathfrak{m}^2$, hence $\mathfrak{m}^2 \subseteq \mathrm{ann}(x+y+z)$. We need to establish the reverse containment. Let $\alpha \in \mathrm{ann}(x+y+z)$, where $\alpha = bx+cy+dz$. Then $0 = (x+y+z)(bx+cy+dz) = bx^2+(b+c)xy+(b+d)xz+(c+d)yz$. We consider all the cases carefully in the chart below. (Note that any triple containing only one nonzero term has already been considered in the classes above.)

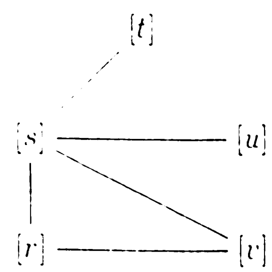
(b, c, d)	$bx^2 + (b+c)xy + (b+d)xz + (c+d)yz$	
(1, 1, 0)	$x^2 + 2xy + xz + yz = x^2 + xz + yz$	$\neq 0$ in R
(1, 0, 1)	$x^2 + xy + 2xz + yz$	$\neq 0$ in R
(0, 1, 1)	$xy + xz + 2yz = xy + xz$	$\neq 0$ in R
(1, 1, 1)	$x^2 + 2xy + 2xz + 2yz = x^2$	$\neq 0$ in R

Therefore, no degree one term annihilates $x+y+z$; i.e., $\mathrm{ann}(x+y+z) \subseteq (\mathfrak{m}^2)$, and hence, $\mathrm{ann}(x + y + z) = (\mathfrak{m}^2)$; i.e., $[x + y + z] = [x]$.

In conclusion, to see that the graph above is $\Gamma_E(R)$, take $t = x^2, s = x, r = y, v = z$, and $u = y + z$.

Proposition 0.4.5. *Set* $R = \dfrac{(\mathbb{Z}/2\mathbb{Z})[[X,Y,Z]]}{(X^3, Y^3, Z^3, XY, X^2Z, XZ^2, Y^2Z, YZ^2, X^2+Y^2)}$ *and let* $\mathfrak{m}$ *be the ideal* (x, y, z), *where* x *(resp.,* y, z*) represents the coset of* X *(resp.,*

$Y, Z)$ *in* R. *We claim that* $\Gamma_E(R)$ *has the graph shown below.*



First of all, note that the nonzero generators of $\mathfrak{m}^2$ are (x^2, xz, y^2, yz, z^2) and that $\mathfrak{m}^3 = 0$ in R; i.e., *each of* x, y, *and* z *annihilates every element in* $\mathfrak{m}^2$. Therefore, a general element of R looks like $a + bx + cy + dz + ex^2 + fy^2 + gz^2 + hxz + iyz$, where the coefficients a, b, c, d, e, f, g, h, and i are all either 0 or 1. However, whenever $a \neq 0$, this polynomial is a unit since the other terms all lie in $\mathfrak{m}$ [?, p. 4]. Therefore, *the only possible zero divisors live in* $\mathfrak{m}$ *and have the form* $bx + cy + dz + ex^2 + fy^2 + gz^2 + hxz + iyz$.

- $\text{ann}(x^2) = \mathfrak{m}$. This (first) class will be represented by $[x^2]$.

Note that all the generators of $\mathfrak{m}$ annihilate x^2. Therefore $\mathfrak{m} \subseteq \text{ann}(x^2)$. On the other hand, we already have $\text{ann}(x^2) \subseteq \mathfrak{m}$. Thus, $\text{ann}(x^2) = \mathfrak{m}$.

Remark 0.4.6. The same argument applies to the elements xz, y^2, yz, z^2, and any combination of these terms. Therefore, all of these terms, and any combination of them, determine the class $[x^2]$.

Remark 0.4.7. We claim that for any zero divisor α, $\text{ann}(\alpha + x^2) = \text{ann}(\alpha)$; i.e., $[\alpha + x^2] = [\alpha]$. To see this, note that each zero divisor β lives in $\mathfrak{m} = \text{ann}(x^2)$, hence $\beta(\alpha + x^2) = \beta\alpha$ equals zero if and only if $\beta \in \text{ann}(\alpha)$. Moreover, this same argument applies to xz, y^2, yz, z^2, and any combination of these terms. Therefore, *only linear combinations of the zero divisors of degree one will provide new equivalence classes.*

- $\mathrm{ann}(x) = (y, \mathfrak{m}^2)$. This (second) class will be represented by $[x]$.

Note that x is annihilated by every element in $\mathfrak{m}^2$ and y, hence $(y, \mathfrak{m}^2) \subseteq \mathrm{ann}(x)$. We need to establish the reverse containment. Let $\alpha \in \mathrm{ann}(x)$, where $\alpha = bx + cy + dz$. Then $0 = x(bx + cy + dz) = bx^2 + dxz$. Since $b, d \in \mathbb{Z}_2$ and x^2 and xz are linearly independent over $\mathbb{Z}_2$, we must have $b = d = 0$. Therefore, the only degree one term that annihilates x is y; i.e., $\mathrm{ann}(x) \subseteq (y, \mathfrak{m}^2)$, and hence, $\mathrm{ann}(x) = (y, \mathfrak{m}^2)$.

- An analogous argument, with the roles of x and y switched, shows $\mathrm{ann}(y) = (x, \mathfrak{m}^2)$. This (third) class will be represented by $[y]$.

- $\mathrm{ann}(z) = (\mathfrak{m}^2)$. This (fourth) class will be represented by $[z]$.

Note that z is annihilated by every element in $\mathfrak{m}^2$, hence $(\mathfrak{m}^2) \subseteq \mathrm{ann}(z)$. We need to establish the reverse containment. Let $\alpha \in \mathrm{ann}(z)$, where $\alpha = bx + cy + dz$. Then $0 = z(bx + cy + dz) = bxz + cyz + dz^2$. Since $b, c, d \in \mathbb{Z}_2$ and xz, yz and z^2 are linearly independent over $\mathbb{Z}_2$, we must have $b = c = d = 0$. Therefore, no degree one term annihilates z; i.e., $\mathrm{ann}(z) \subseteq (\mathfrak{m}^2)$, and hence, $\mathrm{ann}(z) = (\mathfrak{m}^2)$.

- $\mathrm{ann}(x + y) = (x + y, \mathfrak{m}^2)$. This (fifth) class will be represented by $[x + y]$.

Note that $x + y$ is annihilated by every element in $\mathfrak{m}^2$ and $x + y$, hence $(x + y, \mathfrak{m}^2) \subseteq \mathrm{ann}(x + y)$. We need to establish the reverse containment. Let $\alpha \in \mathrm{ann}(x + y)$, where $\alpha = bx + cy + dz$. Then $0 = (x + y)(bx + cy + dz) = bx^2 + cy^2 + dxz + dyz$. Note that if $b = c = 1$ and $d = 0$, then the RHS is zero. Therefore, there is some dependence among these terms. We consider all the cases carefully in the chart below. (Note that any triple containing only one nonzero term has already been considered in the classes above.)

(b, c, d)	$bx^2 + cy^2 + dxz + dyz$	
(1, 1, 0)	$x^2 + y^2$	$= 0$ in R
(1, 0, 1)	$x^2 + xz + yz$	$\neq 0$ in R
(0, 1, 1)	$y^2 + xz + yz$	$\neq 0$ in R
(1, 1, 1)	$x^2 + y^2 + z^2 = z^2$	$\neq 0$ in R

Therefore, the only degree one term that annihilates $x+y$ is $x+y$ itself; i.e., $\text{ann}(x+y) \subseteq (x+y, \mathfrak{m}^2)$, and hence, $\text{ann}(x+y) = (x+y, \mathfrak{m}^2)$.

Finally, in the remainder of our argument, we show that the remaining degree one terms to do not result in any new classes.

∘ $\text{ann}(x+z) = \mathfrak{m}^2$; i.e., $[x+z] = [z]$.

Note that $x+z$ is annihilated by every element in $\mathfrak{m}^2$, hence $(\mathfrak{m}^2) \subseteq \text{ann}(x+z)$. We need to establish the reverse containment. Let $\alpha \in \text{ann}(x+z)$, where $\alpha = bx + cy + dz$. Then $0 = (x+z)(bx+cy+dz) = bx^2 + (b+d)xz + cyz + dz^2$. We consider all the cases carefully in the chart below. (Note that any triple containing only one nonzero term has already been considered in the classes above.)

(b, c, d)	$bx^2 + (b+d)xz + cyz + dz^2$	
(1, 1, 0)	$x^2 + xz + yz$	$\neq 0$ in R
(1, 0, 1)	$x^2 + z^2$	$\neq 0$ in R
(0, 1, 1)	$xz + yz + z^2$	$\neq 0$ in R
(1, 1, 1)	$x^2 + yz + z^2$	$\neq 0$ in R

Therefore, no degree one term annihilates $x+z$; i.e., $\text{ann}(x+z) \subseteq (\mathfrak{m}^2)$, and hence, $\text{ann}(x+z) = (\mathfrak{m}^2)$; i.e., $[x+z] = [x]$.

Remark 0.4.8. By a symmetric argument, $[y+z] = [z]$.

∘ $\text{ann}(x+y+z) = \mathfrak{m}^2$; i.e., $[x+z] = [z]$.

Note that $x+y+z$ is annihilated by every element in $\mathfrak{m}^2$, hence $(\mathfrak{m}^2) \subseteq$

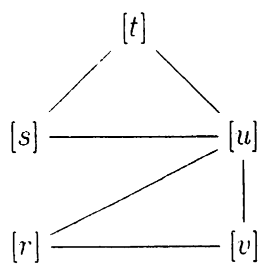
$\text{ann}(x+y+z)$. We need to establish the reverse containment. Let $\alpha \in \text{ann}(x+y+z)$, where $\alpha = bx + cy + dz$. Then $0 = (x+y+z)(bx+cy+dz) = bx^2 + cy^2 + dz^2 + (b+d)xz + (c+d)yz$. We consider all the cases carefully in the chart below. (Note that any triple containing only one nonzero term has already been considered in the classes above.)

(b, c, d)	$bx^2 + cy^2 + dz^2 + (d+b)xz + (d+c)yz$	
(1, 0, 0)	$x^2 + xz$	$\neq 0$ in R
(0, 1, 0)	$y^2 + yz$	$\neq 0$ in R
(0, 0, 1)	$z^2 + xz + yz$	$\neq 0$ in R
(1, 1, 0)	$x^2 + y^2 + xz + yz$	$\neq 0$ in R
(1, 0, 1)	$x^2 + z^2 + yz$	$\neq 0$ in R
(0, 1, 1)	$y^2 + z^2 + xz$	$\neq 0$ in R
(1, 1, 1)	z^2	$\neq 0$ in R

Therefore, no degree one term annihilates $x+y+z$; i.e., $\text{ann}(x+y+z) \subseteq (\mathfrak{m}^2)$, and hence, $\text{ann}(x+y+z) = (\mathfrak{m}^2)$; i.e., $[x+y+z] = [z]$.

In conclusion, to see that the graph above is $\Gamma_E(R)$, take $s = x^2, r = x, v = y, u = z$, and $t = x + y$.

Proposition 0.4.9. *Set* $R = \dfrac{(\mathbb{Z}/3\mathbb{Z})[[X,Y]]}{(X^3, Y^3, XY, (X+Y)(X+2Y))}$ *and let* $\mathfrak{m}$ *be the ideal* (x, y), *where* x *(resp.,* y*) represents the coset of* X *(resp.,* Y*) in* R. *We claim that* $\Gamma_E(R)$ *has the graph shown below.*



First of all, note that the nonzero generators of $\mathfrak{m}^2$ are (x^2, y^2) and that $\mathfrak{m}^3 = 0$ in R; i.e., *both x and y, annihilate every element in* $\mathfrak{m}^2$. Therefore, a general element of R looks like $a + bx + cy + dx^2 + ey^2$, where the coefficients a, b, c, d, e, and m are all either 0, 1 or 2. However, whenever $a \neq 0$, this polynomial is a unit since the other terms all lie in $\mathfrak{m}$ [?, p. 4]. Therefore, *the only possible zero divisors live in* $\mathfrak{m}$ *and have the form* $bx + cy + dx^2 + ey^2$.

- $\operatorname{ann}(x^2) = \mathfrak{m}$. This (first) class will be represented by $[x^2]$.

Note that all the generators of $\mathfrak{m}$ annihilate x^2. Therefore $\mathfrak{m} \subseteq \operatorname{ann}(x^2)$. On the other hand, we already have $\operatorname{ann}(x^2) \subseteq \mathfrak{m}$. Thus, $\operatorname{ann}(x^2) = \mathfrak{m}$.

Remark 0.4.10. The same argument applies to $2x^2$, cy^2 and $bx^2 + cy^2$, for any $b, c \in \mathbb{Z}_3$. Therefore, all of these terms, and any combination of them, determine the class $[x^2]$.

Remark 0.4.11. We claim that for any zero divisor α, $\operatorname{ann}(\alpha + x^2) = \operatorname{ann}(\alpha)$; i.e., $[\alpha + x^2] = [\alpha]$. To see this, note that each zero divisor β lives in $\mathfrak{m} = \operatorname{ann}(x^2)$, hence $\beta(\alpha + x^2) = \beta\alpha$ equals zero if and only if $\beta \in \operatorname{ann}(\alpha)$. Moreover, this same argument applies to $2x^2$, cy^2, and $bx^2 + cy^2$. Therefore, *only linear combinations of the zero divisors of degree one will provide new equivalence classes.*

- $\operatorname{ann}(x) = (y, \mathfrak{m}^2)$. This (second) class will be represented by $[x]$.

Note that x is annihilated by every element in $\mathfrak{m}^2$ and y, hence $(y, \mathfrak{m}^2) \subseteq \operatorname{ann}(x)$. We need to establish the reverse containment. Let $\alpha \in \operatorname{ann}(x)$, where $\alpha = bx + cy$. Then $0 = x(bx + cy) = bx^2$, which is zero only when $b = 0$. Therefore, the only degree one term that annihilates x is y; i.e., $\operatorname{ann}(x) \subseteq (y, \mathfrak{m}^2)$, and hence, $\operatorname{ann}(x) = (y, \mathfrak{m}^2)$.

- An analogous argument, with the roles of x and y switched, shows that

$\operatorname{ann}(y) = (x, \mathfrak{m}^2)$. This (third) class will be represented by $[y]$.

• $\operatorname{ann}(x + y) = (x + 2y, \mathfrak{m}^2)$. This (fourth) class will be represented by $[x + y]$.

Note that $x + y$ is annihilated by every element in $\mathfrak{m}^2$ and $x + 2y$, hence $(x + 2y, \mathfrak{m}^2) \subseteq \operatorname{ann}(x + y)$. We need to establish the reverse containment. Let $\alpha \in \operatorname{ann}(x + y)$, where $\alpha = bx + cy$. Then $0 = (x + y)(bx + cy) = bx^2 + cy^2$. Note that if $b = 1$ and $c = 2$, then the right hand side is zero. Therefore, there is some dependence among these terms. We consider all the cases carefully in the chart below.

(b, c)	$bx^2 + cy^2$	
(1, 0)	x^2	$\neq 0$ in R
(0, 1)	y^2	$\neq 0$ in R
(1, 1)	$x^2 + y^2$	$\neq 0$ in R
(2, 0)	$2x^2$	$\neq 0$ in R
(0, 2)	$2y^2$	$\neq 0$ in R
(1, 2)	$x^2 + 2y^2$	$= 0$ in R
(2, 1)	$2x^2 + y^2 = 2(x + 2y^2)$	$= 0$ in R
(2, 2)	$2x^2 + 2y^2 = 2(x^2 + y^2)$	$\neq 0$ in R

Therefore, the only degree one terms that annihilate $x+y$ are $x+2y$ and $2x+y$; i.e., $\operatorname{ann}(x + y) \subseteq (x + 2y, \mathfrak{m}^2)$, and hence, $\operatorname{ann}(x + y) = (x + 2y, \mathfrak{m}^2)$.

• $\operatorname{ann}(x + 2y) = (x + y, \mathfrak{m}^2)$. This (fifth) class will be represented by $[x + 2y]$.

Note that $x + 2y$ is annihilated by every element in $\mathfrak{m}^2$ and $x + y$, hence $(x + y, \mathfrak{m}^2) \subseteq \operatorname{ann}(x + 2y)$. We need to establish the reverse containment. Let $\alpha \in \operatorname{ann}(x + 2y)$, where $\alpha = bx + cy$. Then $0 = (x + 2y)(bx + cy) = bx^2 + 2cy^2$. Note that if $b = c = 1$, then the right hand side is zero. Therefore, there is some dependence among these terms. We consider all the cases carefully in the chart below.

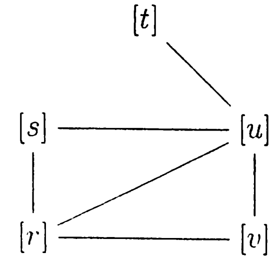
(b, c)	$bx^2 + 2cy^2$	
(1, 0)	x^2	$\neq 0$ in R
(0, 1)	$2y^2$	$\neq 0$ in R
(1, 1)	$x^2 + 2y^2$	$= 0$ in R
(2, 0)	$2x^2$	$\neq 0$ in R
(0, 2)	y^2	$\neq 0$ in R
(1, 2)	$x^2 + y^2$	$\neq 0$ in R
(2, 1)	$2x^2 + 2y^2$	$\neq 0$ in R
(2, 2)	$2x^2 + y^2 = 2(x^2 + 2y^2)$	$= 0$ in R

Therefore, the only degree one terms that annihilate $x + 2y$ are $x + y$ and $2x + 2y$; i.e., $\operatorname{ann}(x + 2y) \subseteq (x + y, \mathfrak{m}^2)$, and hence, $\operatorname{ann}(x + 2y) = (x + y, \mathfrak{m}^2)$.

Finally, in the remainder of our argument, we show that the remaining degree one terms do not result in any new classes. Note that $\operatorname{ann}(2x + 2y) = \operatorname{ann}(x + y)$ since $2x + 2y = 2(x + y)$ and 2 is a unit in R; i.e., $\alpha \in \operatorname{ann}(2x + 2y)$ if and only if $\alpha(x + y) = 0$. Likewise, $\operatorname{ann}(2x + y) = \operatorname{ann}(x + 2y)$ since $2x + y = 2(x + 2y)$ and 2 is a unit in R. Thus, the five classes above are the only distinct classes.

In conclusion, to see that the graph above is $\Gamma_E(R)$, take $u = x^2, t = x, s = y, r = x + y$, and $v = x + 2y$.

Proposition 0.4.12. *Set $R = \mathbb{Z}/p^6\mathbb{Z}$. We claim that $\Gamma_E(R)$ has the graph shown below.*

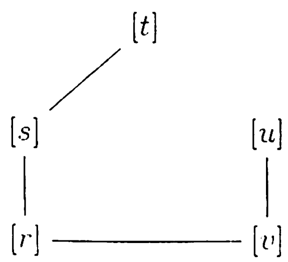


Note that every nonzero element in R is either a unit or a zero divisor. In particular, $\overline{r}$ in R is a unit if and only if $\gcd(r, p) = 1$ and $\overline{r}$ is a zero divisor in R if and only if $\gcd(r, p) = p$. Since p is a prime, $\overline{r}$ is a zero divisor in R if and only if $\gcd(r, p) = p$. The classes $[p]$, $[p^2]$, $[p^3]$, $[p^4]$, $[p^5]$ are distinct since $p \in \text{ann}(p^5)$ but p is not in any of the other annihilators. Likewise, $p^2 \in \text{ann}(p^4)$ but p^2 is not in any of the other (smaller) annihilators. Continue in this way to see that all these classes are distinct. Finally, consider a zero divisor $\overline{n}$ of R. We can write $\overline{n}$ as $\overline{up^k}$, where $\gcd(u, p) = 1$; i.e., $\overline{u}$ is a unit in R and has an inverse $\overline{u}^{-1}$. Clearly $\text{ann}(\overline{p^k}) \subseteq \text{ann}(\overline{up^k})$ since any element that annihilates $\overline{p^k}$ will annihilate a product of $\overline{p^k}$. Let $t \in \text{ann}(\overline{up^k})$. Then $\overline{0} = \overline{tup^k} \Rightarrow \overline{u^{-1}} \cdot \overline{0} = \overline{u^{-1}} \cdot \overline{tup^k} \Rightarrow \overline{0} = \overline{tp^k}$; i.e., $\text{ann}(\overline{up^k}) = \text{ann}(\overline{p^k})$. In conclusion, the five classes already identified are the only distinct classes.

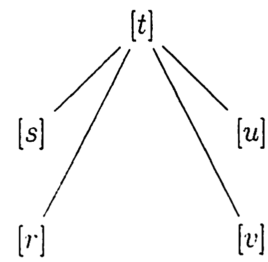
In conclusion, to see that the graph above is $\Gamma_E(R)$, take $t = p, s = p^2, v = p^3, r = p^4$, and $u = p^5$.

In summary, graphs (2), (6), (9), and (11) in the Appendix can each be realized as the zero divisor graph of some ring R. The remaining graphs in the Appendix can not.

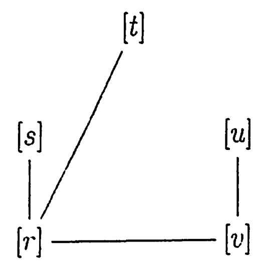
0.5 Appendix



(1)



(2)



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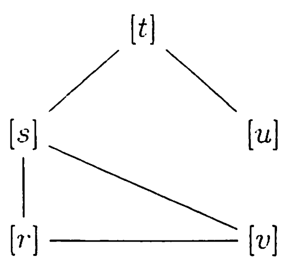

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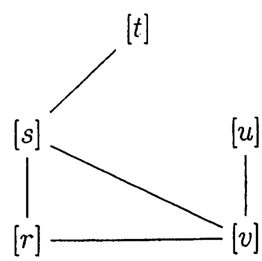

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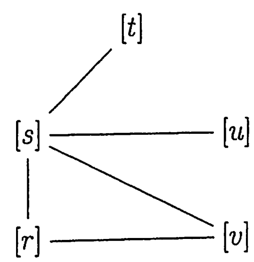

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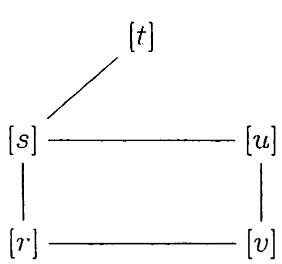

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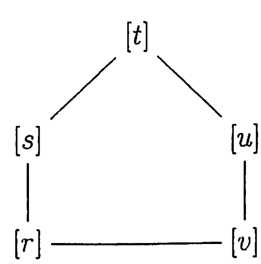

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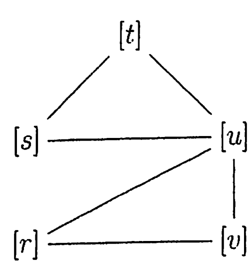

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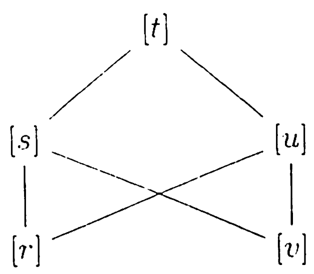

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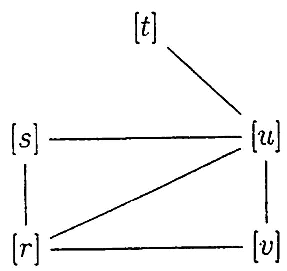
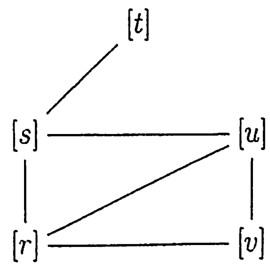


(11)



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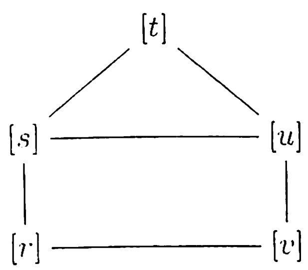
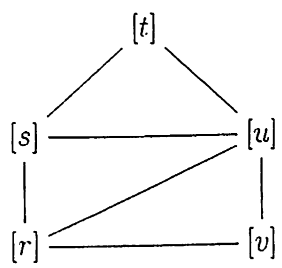


(13)



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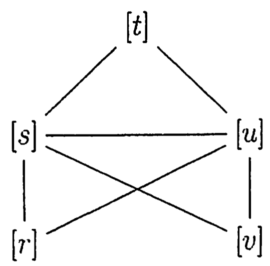
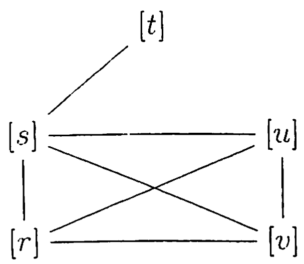


(15)



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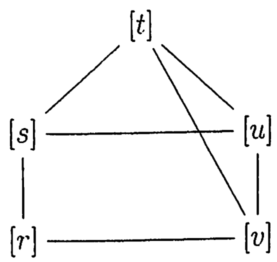
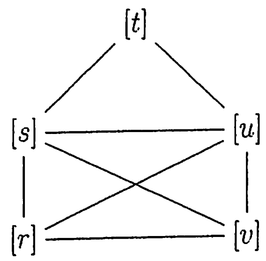
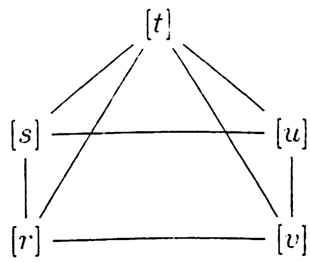


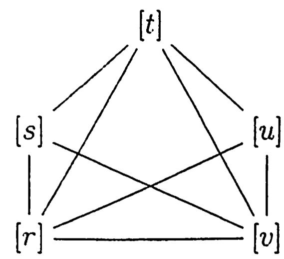
(17)



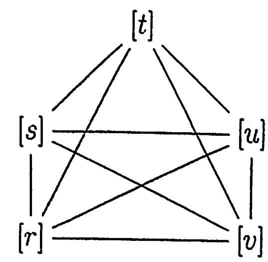
(18)



(19)



(20)



(21)

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