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A Random Walk, Absorbtion, and Enumeration of Returns

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RANDOM WALKS, ABSORPTION, AND ENUMERATION OF RETURNS:
The Myth of Eternal Return

by
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This paper is dedicated to my high school math teacher, Mrs. Knox Hardin.

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ABSTRACT

JOANNA C. ROCHESTER: Random Walks, Absorption, and Enumeration of Returns
(Under the direction of William Staton)

The general notion of a Markov Chain is introduced in Chapter 1, and a theorem is proven characterizing the two-state Markov Chain. The concept central to the thesis, the Random Walk, is introduced in Chapter 2 and a thorough analysis is presented of a Random Walk in one dimension with a single absorbing state. A theorem is proven which provides probabilities of absorption for arbitrary starting points. The theoretical results are then tested by computer simulation, yielding a very satisfying match with the predictions of the theorem. Finally, in Chapter 3 a modification of the Random Walk without absorbing states is presented and analyzed. The expected number of returns to the origin is derived, and again is tested by computer simulation. In this case as well, the simulated results provided a nice empirical verification of the theoretical result. Techniques employed in the work vary across the undergraduate curriculum. Linear algebra appears in the powers of matrices and in the use of eigenvalues in Chapter 1. Probability concepts such as independence play a small role. First order linear differential equation techniques are used in Chapter 2. Infinite series are evaluated in Chapters 2 and 3, and linear homogeneous recursion equations are solved in Chapter 3. Some use is made of Catalan Numbers and Binomial Coefficients

as well. The computing necessary for the simulations in Chapter 2 and Chapter 3 was done in the MATLAB language on the University of Mississippi's supercomputer accessed through Sweetgum. This thesis was typeset using LATEX.

Contents

1 Preliminaries

Many situations studied in the natural and social sciences involve systems which can be in one of an array of states, and which, from time to time, evolve from one state into another. In some such systems, there is what might be called a systemic memory, whereby the system takes into consideration its past, speaking figuratively, when deciding how to move forward. In other situations, there is no such memory, so that the system, when in a certain state one, has the same probability of evolving into another state two regardless of how the system arrived in state one. It is such systems, called Markov chains, which are studied in this thesis. The main focus will be a particular type of Markov chain called a Random Walk, defined more formally later, in which states can be pictured as integer points of a line and transitions are made only to nearby points. Once again, using figurative language, the questions considered in this thesis will involve "escaping" and "returning". The techniques employed here will range across many courses I have taken as an undergraduate. There will be matrices, eigenvalues, infinite series, recursion equations, linear differential equations, binomial coefficients, and Catalan numbers. In this thesis, I will proceed from the general to the specific. At the outset, very general definitions will be given, especially the definition of a Markov Chain. After illustrating with examples, I will prove a theorem completely characterizing the 2-state Markov Chains. I will then proceed to a particularly interesting and useful instance of the Markov Chain concept, the Random Walk. Most of the research in this thesis is devoted to instances of this concept. I begin with a number of definitions.

Definition 1 *[2] An mxn matrix, where m and n are positive integers, is an array* $\{a_{ij} | 1 \leq i \leq m, 1 \leq j \leq n\}$ *of numbers. An n-vector is an nx1 matrix.*

Definition 2 *[2] If* $A = \{a_{ij}\}$ *and* $B = \{b_{ij}\}$ *are mxn matrices, then the sum* $A + B$ *is the mxn matrix C whose entry* $c_{ij} = a_{ij} + b_{ij}$.

Definition 3 *[2] If A is an mxn matrix and B is and nxp matrix, then the product AB is the mxp matrix C whose entry* $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

Definition 4 *[2] If* $A = \{a_{ij}\}$ *is an nxn matrix and* $1 \leq k, l \leq n$, *then the minor* M_{kl} *is the* $(n-1)x(n-1)$ *matrix obtained by removing each entry whose first index is k or whose second index is l.*

Definition 5 *[2] The determinant of an nxn matrix* $A = \{a_{ij}\}$ *is defined inductively as follows:*

$$det(A) = \begin{cases} a_{11} & \text{if } n=1 \\ \sum_{i=1}^{n} (-1)^{1+i} det(M_{1i}) & \text{if } n > 1 \end{cases}$$

Definition 6 *[2] A scalar* λ *is said to be an eigenvalue of* A *if there exists a nonzero vector* x *such that* $Ax = \lambda x$.

Definition 7 *[2] A matrix is said to be stochastic if its entries are nonnegative and the entries in each column add up to 1.*

Definition 8 *[4] A Markov chain is a finite or countable infinite set of states* $\{S_i\}$ *along with a set* $\{p_{ij}\}$ *of transition probabilities with the stipulation that for fixed i,* $\sum_j p_{ij} = 1$.

One should conceptualize a Markov chain as a system of states evolving at discrete time intervals. The transition probability p_{ij} represents the probability that at any of the discrete times, the system evolves from state S_i into state S_j.

Definition 9 *[4] A transition matrix* A *of a Markov chain is a stochastic matrix where each entry* a_{ij} *represents the probability of moving from state* i *to state* j.

It should be noted that the appearance of the matrix will vary with the numbering of the states. This is of no consequence.

Definition 10 *[4] A stationary matrix of a Markov chain is* $\lim_{n\to\infty}(A^n)$, *if the limit exists, where* A *is a transition matrix of that Markov chain.*

Now that the idea of a Markov Chain has been presented, I would like to stop and give a few practical examples of how Markov Chains can be used. The first example that I would like to introduce is a well-known problem called the "Gambler's ruin," see [4].

Imagine that you enter a casino with a fortune of $k and gamble, $1 at a time, with probability p of doubling your stake and probability q of losing it. The resources of the casino are regarded as infinite, so there is no upper limit to you fortune. But, what is the probability that you leave broke?

Let $h_i = probability(\text{hit } 0)$, for $i = 1, 2, \ldots$ then h is the minimal nonnegative solution to

$$h_0 = 1,$$

$$h_i = ph_{i+1} + qh_{i-1}, \text{ for i=1,2, } \ldots$$

If $p \neq q$ this recurrence relation has a general solution

$$h_i = A + B(\tfrac{q}{p})^i.$$

If $p < q$, which is the case in most successful casinos, then the restriction $0 \leq h_i \leq 1$ forces $B = 0$, so $h_i = 1$ for all i. If $p > q$, then since $h_0 = 1$ we get a family of solutions

$$h_i = (\tfrac{q}{p})^i + A(1 - (\tfrac{q}{p})^i);$$

for a nonnegative solution we must have $A \geq 0$, so the minimal nonnegative solution is $h_i = (\frac{q}{p})^i$. Finally, if $p = q$ the recurrence relation has a general solution $h_i = A + Bi$ and again the restriction $0 \leq h_i \leq 1$ forces $B = 0$, so $h_i = 1$ for all i. Thus, even if you find a fair casino, you are certain to end up broke. This is why the paradox is called gamblers' ruin.

To further explain the idea of the Markov chain, I would like to illustrate the simple two-state Markov chain. In this instance, there are only two possible states that you can be in. So, at all times, you are either in state 1 or state 2. Then, for each unit of time, if you are already in state 1, you have a certain probability, say α, of moving from state 1 to state 2. Therefore, you have a $1 - \alpha$ probability of staying in state 1. Similarly, if you are already in state 2, you have a probability, say β of moving from state 2 to state 1. Thus you have a $1 - \beta$ probability of staying in state 2. This type of Markov chain can be expressed with a very simple transition matrix:

$$A = \begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix}$$

At this point, I will prove a very satisfactory little theorem characterizing the two-state Markov chain defined above. This is original work, presented to me as an exercise, although the theorem was of course, already known.

Theorem 1 *For a two-state Markov chain with transition matrix*

$$\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix},$$

the stationary matrix is

$$\begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}$$

unless $\alpha = \beta = 1$.

Proof of Theorem 1. To solve for the stationary matrix, we must first find the eigenvalues of the transition matrix. Therefore, we find

$$\begin{vmatrix} (1-\alpha)-\lambda & \alpha \\ \beta & (1-\beta)-\lambda \end{vmatrix} = 1-\alpha-\lambda-\beta+\alpha\beta+\beta\lambda-\lambda+\alpha\lambda+\lambda^2-\alpha\beta$$

Canceling then leaves $\lambda^2 + \lambda(\alpha+\beta-2) + 1 - \alpha - \beta$ This can be factored into $(\lambda-1)(\lambda-1+\alpha+\beta)$ leaving our eigenvalues to be 1 and $1-\alpha-\beta$.

Now that we have our eigenvalues, we can make a diagonal matrix so that we can write our transition matrix in the form $A = xDx^{-1}$. Using the eigenvalues, this diagonal matrix becomes $D = \begin{pmatrix} 1 & 0 \\ 0 & 1-\alpha-\beta \end{pmatrix}$.

To find the stationary matrix, we must first consider the transition matrix A raised to the nth power. Hence we have $A^n = x \begin{pmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{pmatrix} x^{-1}$.

To continue, we want to diagonalize by multiplying the diagonal matrix by x on the left and x^{-1} on the right to find the particular diagonal matrix for this problem. So we have

$$A^n = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12}(1-\alpha-\beta)^n \\ a_{21} & a_{22}(1-\alpha-\beta)^n \end{pmatrix}.$$

Continuing, we have

$$A^n = \begin{pmatrix} a_{11} & a_{12}(1-\alpha-\beta)^n \\ a_{21} & a_{22}(1-\alpha-\beta)^n \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$$

$$= \begin{pmatrix} a+b(1-\alpha-\beta)^n & c+d(1-\alpha-\beta)^n \\ e+f(1-\alpha-\beta)^n & g+h(1-\alpha-\beta)^n \end{pmatrix}$$

where a,b,c,d,e,f,g,h are expressions involving the entries of x.

At this point, we will call the ij entry of A^n by the name P_{ij}^n where i and j take on the values 1 and 2. We will start by looking at P_{11}^n. First notice that $P_{11}^0 = 1 = a + b$ and $P_{11}^1 = 1 - \alpha = a + b(1-\alpha-\beta)$.

Now we have a system of two equations and two unknowns. The solution to this system is $a = \frac{\beta}{\alpha+\beta}$ and $b = \frac{\alpha}{\alpha+\beta}$. Hence, we have

$$P_{11}^n = \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n.$$

Moving on, we will next look at P_{12}^n. Again, we will first notice that $P_{12}^n = c + d(1-\alpha-\beta)$. Then we have $P_{12}^0 = 1 = c + d$ and $P_{12}^1 = \alpha = c + d(1-\alpha-\beta)$. Now we have another system of equations. Here the solution is $c = \frac{\alpha}{\alpha+\beta}$ and $d = \frac{-\alpha}{\alpha+\beta}$. Thus, $P_{12}^n = \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n$.

Next we will focus on P_{21}^n. First we note that $P_{21}^n = e + f(1-\alpha-\beta)^n$. Then, $P_{21}^0 = 1 = e + f$ and $P_{21}^1 = \beta = e + f(1-\alpha-\beta)$. Once again we have a system of equations with solution $e = \frac{\beta}{\alpha+\beta}$ and $f = \frac{-\beta}{\alpha+\beta}$. Hence,

$$P_{21}^n = \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n.$$

Finally, we want to look at P_{22}^n. Note that $P_{22}^n = g + h(1-\alpha-\beta)^n$. So, $P_{22}^0 = 1 = g + h$ and $P_{22}^1 = 1-\beta = g + h(1-\alpha-\beta)$. This system leaves us with a solution $g = \frac{\alpha}{\alpha+\beta}$ and $h = \frac{\beta}{\alpha+\beta}$. Thus we have

$$P_{22}^n = \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n.$$

We now will look at the transition matrix raised to the nth power as a whole. Thus we have

$$A^n = \begin{pmatrix} \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n & \frac{\alpha}{\alpha+\beta} - \frac{\alpha}{\alpha+\beta}(1-\alpha-\beta)^n \\ \frac{\beta}{\alpha+\beta} - \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n & \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}(1-\alpha-\beta)^n \end{pmatrix}.$$

Now we note that $-1 < (1-\alpha-\beta) < 1$ so as $n \to \infty$, $(1-\alpha-\beta)^n \to 0$. Thus, as $n \to \infty$, the second part of each P_{ij} becomes 0. So now we see that we have the stationary matrix

$$\lim_{n\to\infty} A^n = \begin{pmatrix} \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \\ \frac{\beta}{\alpha+\beta} & \frac{\alpha}{\alpha+\beta} \end{pmatrix}. \qquad \square$$

This theorem provides us with a beautiful outcome that I would like to take note of as a wonderful conclusion to this first chapter. Note that in the final, stationary matrix, both entries in the first column and both entries in the second column are exactly the same. What we can gather from this is that it makes no difference where one starts. The fraction of time spent in a given state is independent of the starting state.

2 Doom

Moving on, we will spend this chapter looking at a specific type of Markov Chain, a random walk. First I will add a few more definitions.

Definition 11 *[4] A random walk in one dimension is a Markov chain in which the set of states is a set of integers and in which the transition probabilities are all zeros except possibly for $p_{n,n}$, $p_{n,n+1}$, and $p_{n,n-1}$. In other words, at a given time, transitions are only made to adjacent states.*

Definition 12 *[4] A state i is called an absorbing state if there is no escape from this state. In other words, $p_{i,j}$ is 0 unless $i = j$.*

A random walk on the positive integers is considered, with one as an absorbing state and constant transition probabilities left and right from every positive state. Using a variety of mathematical tools, the probability, a_n, of escaping to infinity from state n is calculated. We consider a conceptually simple random walk on the positive integers as follows: Let $p, q \geq 0$ and $p + q = 1$. If the system is in state one at time t, the system stays in state one at time $t + 1$. If the system is in state $n > 1$ at time t, it moves to state $n - 1$ with probability p and to state $n + 1$ with probability q.

In this situation, state one is called the absorbing state. Once there, the system is doomed to remain there ever after. We wish to investigate the probability that the system ends in state one given that it starts in state n. For this purpose, we define:

a_n = the probability that the system starting in state n ends in state one as $t \to \infty$.

Informally, we can think of a_n as the probability of doom for a system in state n, and $1-a_n$ as the probability of escape. Clearly, we begin with $a_1 = 1$. The probability a_n is related to a_{n-1} and a_{n+1} in the obvious way: for $n \geq 1$, $a_{n+1} = pa_n + qa_{n+2}$, using the multiplication of probabilities and the fact that moving left and moving right at time t are disjoint events. Rearranging yields

$$a_{n+2} = \frac{1}{q}a_{n+1} - \frac{p}{q}a_n.$$

Here we are assuming $q \neq 0$, which is valid except in the clearly doomed situation where every possible move is to the left. It will be helpful to use this recursion to express all a_n, $n \geq 2$ in terms of a_2. A bit of computation shows

for $n \geq 2$ $a_n = (\frac{qa_2-p}{q-p}) + \frac{p^{n-1}(1-a_2)}{q^{n-2}(q-p)}$

which will be extraordinarily useful below, and which can be verified with a fairly routine induction argument.

It is now clear that if we can determine a_2, the probability of eventual doom for a system starting in state 2, we will have an explicit formula for every a_n. To this end, we note that every path from state 2 to state 1 involves an odd number of steps, say $2k+1$ steps. First, a sequence of $2k$ steps starting to the right and ending at state 2, followed by a single step to the left. Each path from state 2 back to state 2 must involve an equal number of steps right and steps left and the number of steps left must never exceed the number of steps right in any initial segment of the path. Hence, when $k = 1$, there is only one such path RL. When $k = 2$, there are 2 such paths RRLL and RLRL. For $k = 3$, we have 5 paths, RRRLLL, RRLRLL, RLRRLL, RLRLRL, and RRLLRL. Let C_k denote the number of paths of $2k$ steps. Each path has probability $p^k q^k$, and since each path is followed by a single step left, we have

$$a_2 = \sum_{k=0}^{\infty} C_k p^{k+1} q^k.$$

Now, the quantity we have called C_k is a well-known combinatorial sequence called the Catalan number, defined precisely as we have defined our C_k. There is an elegant closed formula for these Catalan numbers, in particular

$$C_k = \frac{1}{k+1}\binom{2k}{k}. \text{ [1]}$$

Our task of determining a_2 now reduces to finding

$$\begin{aligned} a_2 &= \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} p^{k+1} q^k \\ &= p \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} (pq)^k. \end{aligned}$$

For this purpose, we define

$$f(x) = \sum_{k=0}^{\infty} \frac{1}{k+1}\binom{2k}{k} x^{k+1}$$

and we note $a_2 = \frac{f(pq)}{q}$.

We pause to note the following recursion formula for the binomial coefficients in our series:

$\binom{2k}{k} = (4 - \frac{2}{k})\binom{2k-2}{k-1}$. [1][3]

which is simple cancelation of fractions.

Now, $f'(x) = \sum_{k=0}^{\infty} \binom{2k}{k} x^k$

and $\begin{cases} f(0) = 0 \\ f'(0) = 1 \end{cases}$.

Using the recursion

$f'(x) = 1 + x \sum_{k=1}^{\infty} (4 - \frac{2}{k})\binom{2k-2}{k-1} x^{k-1}$

$= 1 + 4xf'(x) - 2 \sum_{k=1}^{\infty} \frac{1}{k}\binom{2k-2}{k-1} x^k$

$= 1 + 4xf'(x) - 2f(x),$

or

$f'(x) + \frac{2}{1-4x} f(x) = \frac{1}{1-4x}.$

This, happily, is a routine first order linear differential equation encountered in the first part of an introductory differential equations text [5]. To solve, one finds an integrating factor, here $\frac{1}{\sqrt{1-4x}}$ and solves:

$(\frac{f(x)}{\sqrt{1-4x}})' = \frac{1}{(1-4x)^{\frac{3}{2}}}.$

This translates with the aid of the initial conditions into

$f(x) = \frac{1-\sqrt{1-4x}}{2}.$

Hence,

$$a_2 = \frac{f(pq)}{q} = \frac{1-\sqrt{1-4pq}}{2q}$$

$$a_2 = \frac{1-\sqrt{1-4(q)(1-q)}}{2q}$$

$$a_2 = \frac{1-\sqrt{4q^2-4q+1}}{2q}$$

$$a_2 = \begin{cases} \frac{1-(1-2q)}{2q} & \text{if } p \geq q \\ \\ \frac{1-(2q-1)}{2q} & \text{if } p < q \end{cases}$$

So,

$$a_2 = \begin{cases} 1 & \text{if } p \geq q \\ \\ \frac{p}{q} & \text{if } p < q. \end{cases}$$

It must be noted that our derivation is not valid for $p = q$. The fact that $a_2 = 1$ when $p = q$ follows from continuity.

Now that we have determined a_2, we have an easy computation to derive a formula for every a_n:

i) If $p \geq q$, $a_2 = 1$ and so

$$a_n = \frac{qa_2 - p}{q-p} + \frac{p^{n-1}}{q^{n-2}}\left(\frac{1-a_2}{q-p}\right)$$

$$= \frac{q-p}{q-p} + 0$$

$$= 1,$$

and doom is statistically certain;

ii) If $p < q$, $a_2 = \frac{p}{q}$ and so

$$a_n = \frac{qa_2 - p}{q-p} + \frac{p^{n-1}}{q^{n-2}}\left(\frac{1-a_2}{q-p}\right)$$

$$= 0 + \frac{p^{n-1}}{q^{n-2}}\left(\frac{1-\frac{p}{q}}{q-p}\right)$$

$$= \frac{p^{n-1}}{q^{n-2}}\frac{q}{q}\left(\frac{1-\frac{p}{q}}{q-p}\right)$$

$$= \left(\frac{p}{q}\right)^{n-1}.$$

and for reasonable values of p, escape is conceivable from any state other than state one.

This is a complete and satisfactory solution to the original question we asked. For concreteness, we illustrate with $p = \frac{1}{3}$:

$a_2 = \frac{1}{2}$

$a_{10} = \frac{1}{512}$

$a_{100} = \frac{1}{2^{99}} \approx 0$

For $p = .49$:

$a_2 = \frac{49}{51} \approx .96$

$a_{10} \approx .698$

$a_{100} \approx .019$

and, happily, we can report high likelihood of escaping doom for those not starting too close!

In order to test this result empirically , I have written, with the assistance of Professor Tristan Denley, a program in the MATLAB language to simulate the random walks which are evaluated theoretically above. Of course, in the theory above, an infinite number of steps is assumed in each walk. In practice, this is impossible. The following program runs simulations of a size N steps where N is given as input. Each run begins at state n and moves left with probability p and right with probability $1-p$, where n and p are also given as inputs. 100 repetitions are run and the average number of absorbtions is recorded. The simulation program, called "hillavmod", is as follows:

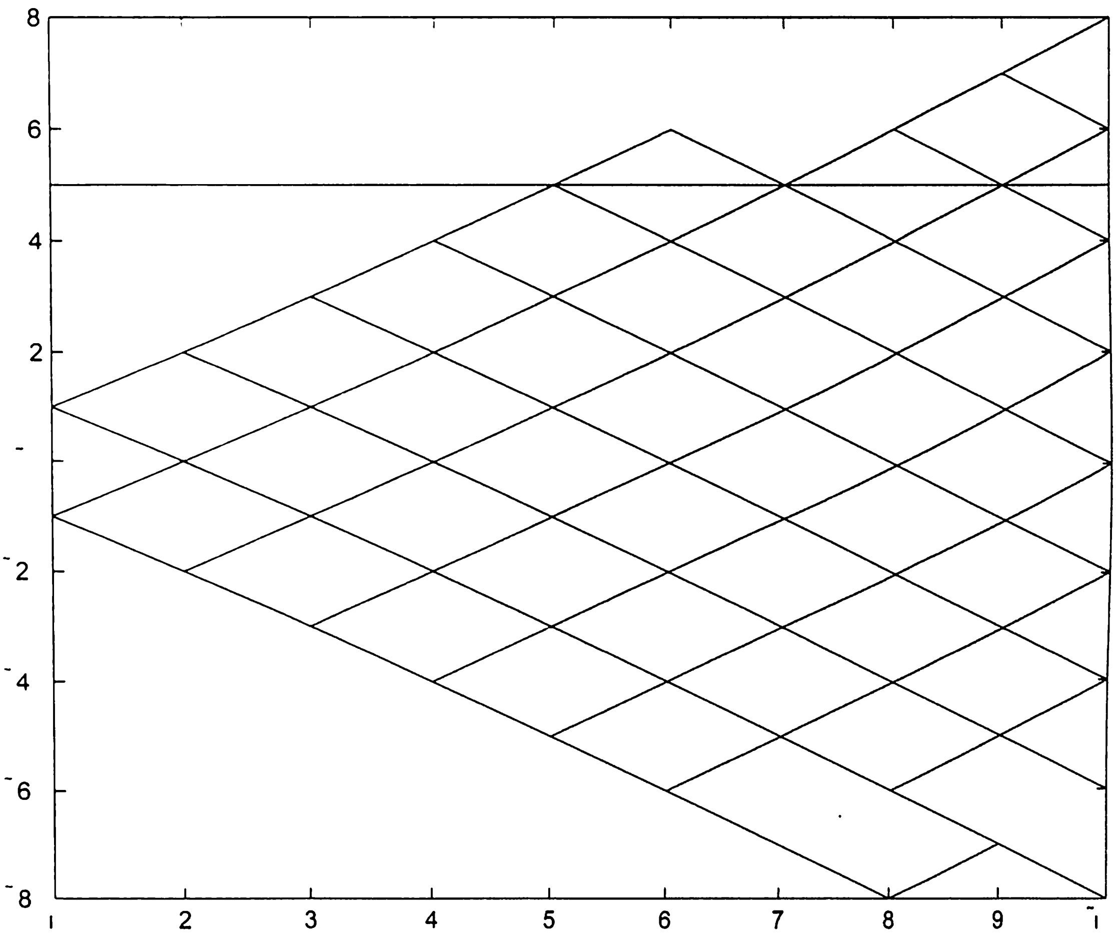
```
function [average, add] = mountain(N, p, n)
add = zeros(1, 100);
for i = 1 : 100
c=rand(1,N);
d = (c¡p);
f=2*d-ones(1,N);
for j=1:N ps(j)=sum(f(1,1:j)); end
cross=sum((ps==n));
add(i)=cross;
x=1:N;
plot(x,ps,'r')
hold on
```

```
plot(x,n*ones(1,N))

    end

average = sum(add > 0)/100.0;
```

The following is the graphical output of "hillavmod."



Now we will look at a comparison of theoretical and simulated results with different probabilities p, and starting at different states n, on the number line.

n	p	Theoretical	Simulation 1	Simulation 2	Simulation 3	Simulation 4
2	.47	.887	.87	.91	.86	.80
2	.40	.667	.55	.62	.73	.75
2	.55	1	.97	1	.99	.98
5	.48	.723	.68	.73	.68	.71
8	.49	.756	.74	.68	.69	.71

Without any formal statistical analysis, I conclude the chapter by noting that the simulated results seem to cluster beautifully around the theoretical prediction.

3 Counting Returns

In this chapter, a modified random walk is considered, a walk with no absorbing state expanded into two dimensions. The problem considered is enumerating the number of returns to the diagonal line $y = x$. If the problem were to be visualized in one dimension, the formulation of the problem would be counting returns to the origin.

In particular, we consider a random walk in which states are pairs (a, b) of non-negative integers with transition probabilities as follows: From (a, b) there is probability p of moving to $(a, b+1)$ and probability $q = 1 - p$ of moving to $(a+1, b)$. This can be pictured as lattice points in the first quadrant with probability p of moving north one step and probability q of moving east one step. If one starts at the origin, then any state lying on the line $y = x$ represents a moment when the number of steps north and the number of step east are equal. The question may be also asked starting from any point, and in this chapter, starting states $(n, 0)$ on the x-axis will be considered. In such cases, hitting the line $y = x$ means that the number of steps north is n more than the number of steps east.

To derive an expression for the expected number of hits starting from $(n, 0)$, we note that the line $y = x$ may be hit at any point $(n+k, n+k)$ where $k \geq 0$. Such a hit requires exactly k steps east and $n+k$ steps north. This can happen in $\binom{n+2k}{k}$

ways, each of which has probability $p^{n+k}q^k$. Summing over $k \geq 0$ yields the following series expression for the expectation:

$$\sum_{k=0}^{\infty} \binom{n+2k}{k} p^{n+k} q^k$$

or

$$p^n \sum_{k=0}^{\infty} \binom{n+2k}{k} (pq)^k.$$

In order to sum this series, we define the function $f_n(x) = \sum_{k=0}^{\infty} \binom{n+2k}{k} x^k$ and note that the desired expectation is $p^n f_n(pq)$. We are in the happy situation of having available from chapter two an expression for $f_0(x)$, in particular, $f_0(x) = \sum_{k=0}^{\infty} \binom{2k}{k} x^k = \frac{1}{\sqrt{1-4x}}$.

Now, to begin searching for a recursive relationship among the f_n's, note that $\binom{2k+2}{k+1} = \binom{2k+1}{k+1} + \binom{2k+1}{k} = 2\binom{2k+1}{k}$. Hence,

$$f_1(x) = \sum_{k=0}^{\infty} \binom{2k+1}{k} x^k$$

$$= \frac{1}{2} \sum_{k=0}^{\infty} \binom{2k+2}{k+1} x^k$$

$$= \frac{1}{2x} \sum_{k=0}^{\infty} \binom{2k+2}{k+1} x^{k+1}$$

$$= \frac{1}{2x} \sum_{r=1}^{\infty} \binom{2r}{r} x^r \text{ (shifting indices by } r = k+1)$$

$$= \frac{1}{2x}(f_0(x) - 1)$$

$$= \frac{1}{2x}\left(\frac{1}{\sqrt{1-4x}} - 1\right).$$

With closed expressions for $f_0(x)$ and $f_1(x)$ as a basis, we now seek a 2nd degree recursion relation among the f_n's. In particular, we show that for $n \geq 2$, $f_n(x) =$

$\frac{1}{x}f_{n-1}(x) - \frac{1}{x}f_{n-2}(x)$. To show this, having $f_0(x)$ and $f_1(x)$ in hand, we consider for $n \geq 2$,

$$
\begin{aligned}
f_n(x) &= \sum_{k=0}^{\infty} \binom{n-2k}{k} x^k \\
&= \sum_{k=0}^{\infty} [\binom{n+1-2k}{k-1} - \binom{n-2k}{k-1}] x^k \\
&= \sum_{k=0}^{\infty} \binom{n-1-2k}{k-1} x^k - \sum_{k=0}^{\infty} \binom{n-2k}{k-1} x^k \\
&= \sum_{r=1}^{\infty} \binom{n-2r-1}{r} x^{r-1} - \sum_{r=1}^{\infty} \binom{n-2r-2}{r} x^{r-1} \text{ (shifting indices by } r = k+1) \\
&= \frac{1}{x} \sum_{r=1}^{\infty} \binom{n-1-2r}{r} x^r - \frac{1}{x} \sum_{r=1}^{\infty} \binom{n-2-2r}{r} x^r \\
&= \frac{1}{x}(f_{n-1}(x) - 1) - \frac{1}{x}(f_{n-2}(x) - 1) \\
&= \frac{1}{x} f_{n-1}(x) - \frac{1}{x} f_{n-2}(x).
\end{aligned}
$$

We have a 2nd order recursion relation satisfied by the various functions $f_n(x)$. Indeed the recursion relation is linear and homogeneous, so standard techniques may be applied [1][3]. Rewrite to obtain

$$f_n(x) - \frac{1}{x}f_{n-1}(x) + \frac{1}{x}f_{n-2}(x) = 0.$$

The auxiliary equation is

$$r^2 - \frac{1}{x}r + \frac{1}{x} = 0.$$

The critical values of r are easily obtained using the quadratic formula with $a = 1$, $b = \frac{-1}{x}$ and $c = \frac{1}{x}$, to yield

$$r_1 = \frac{1+\sqrt{1-4x}}{2x}$$

$$r_2 = \frac{1-\sqrt{1-4x}}{2x}.$$

It follows that for all n,

$$f_n(x) = c_1(\frac{1+\sqrt{1-4x}}{2x})^n + c_2(\frac{1-\sqrt{1-4x}}{2x})^n$$

with constant functions c_1 and c_2 to be determined by the known values of $f_0(x)$ and $f_1(x)$. Hence, when $n = 0$, we have

$$\frac{1}{\sqrt{1-4x}} = c_1 + c_2,$$

and when $n = 1$ we have

$$\frac{1}{2x}(\frac{1}{\sqrt{1-4x}} - 1) = c_1(\frac{1+\sqrt{1-4x}}{2x}) + c_2(\frac{1-\sqrt{1-4x}}{2x}).$$

It is routine to show that

$c_1 = 0$ and

$$c_2 = \frac{1}{\sqrt{1-4x}}.$$

Putting all pieces together, we now have

$$f_n(x) = \frac{1}{\sqrt{1-4x}}(\frac{1-\sqrt{1-4x}}{2x})^n.$$

The expected number of hits of the diagonal given the starting point $(n, 0)$ is

$$p^n f_n(pq) = p^n(\frac{1}{\sqrt{1-4pq}})(\frac{1-\sqrt{1-4pq}}{2pq})^n.$$

This concludes the proof of the following theorem:

Theorem 2 *In the random walk of this chapter, the expected number of returns, starting from* $(n, 0)$, *is*

$$p^n(\frac{1}{\sqrt{1-4pq}})(\frac{1-\sqrt{1-4pq}}{2pq})^n,\ p \neq \frac{1}{2}.$$

In case $p < q$, this formula depends heavily on n, as one would expect. Since the tendency $p < \frac{1}{2}$ would lead to drifting "downward", the expected number of returns would clearly be largest if one started at the origin. In the case $p > \frac{1}{2}$, the following little theorem might be unexpected.

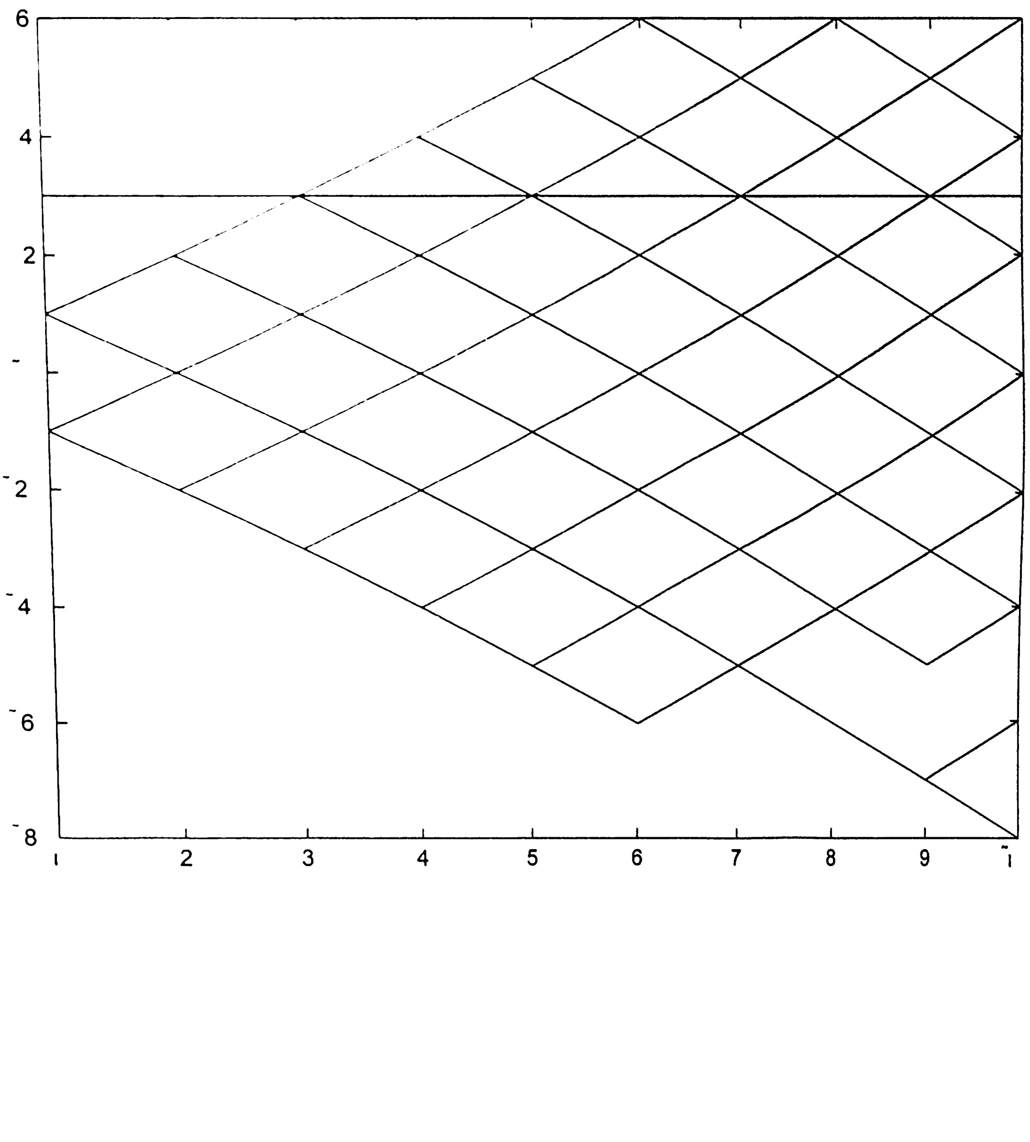
Theorem 3 *If $p > \frac{1}{2}$, the expected number of returns is exactly $\frac{1}{2p-1}$, independent of n.*

Proof of Theorem 3. If $p > \frac{1}{2}$, then $\sqrt{1-4pq} = \sqrt{1-4p(1-p)} = \sqrt{4p^2-4p+1} = 2p-1$. Now, $(\frac{1-\sqrt{1-4pq}}{2pq})^n = (\frac{1-(2p-1)}{2pq})^n = (\frac{2-2p}{2pq})^n = (\frac{2q}{2pq})^n = (\frac{1}{p})^n$ and we have, from Theorem 2, that the expected number of crossings is $p^n(\frac{1}{2p-1})(\frac{1}{p})^n = \frac{1}{2p-1}$. □

While I did not anticipate that this expectation would be independent of n, once noted, I quickly saw the intuitive explanation. If the expectation from $(0,0)$ is E, then, starting from $(n,0)$, one is certain to cross the diagonal because $p > \frac{1}{2}$. At the moment of first hitting the diagonal, the situation is exactly as if one had started from $(0,0)$, so the expectation should be E.

To run a simulation of this new result, I modified the simulation program that was presented in Chapter 2 so that it counts and averages crossings instead of absorbtions. Now we will compare the theoretical result with the simulated result for various choices of N, p, and n and show the graphs of the simulated results.

The following is the graphical output of "hillav."



n	p	Theoretical	Simulation 1	Simulation 2	Simulation 3	Simulation 4
2	.75	2	1.64	2.12	2.12	2.03
2	.45	6.69	6.49	6.15	7.16	6.12
0	.47	16.67	17.3	14.38	15.01	17.05
3	.4	1.48	1.65	1.55	1.42	1.21
10	.6	5	4.4	4.89	4.37	5.54

Again, as in Chapter 2, I note the close agreement between the simulated and the theoretical.

4 Future Questions

There are numerous questions that can be thought of as extensions or modifications of the work I have done. One could put in more absorbing states. In the one-dimensional situation, this would be essentially the Gambler's Ruin Problem. One could look into higher dimensions, but in that case, it is problematic about what to count. Returns to any point or line would be extraordinarily rare. One could count returns to a given plane in 3 dimensions, say the x-y plane, but that would be essentially the same as a 2-dimensional problem. Perhaps it would be of interest to look at unbounded solids of rotation, such as an hyperboloid, pick a starting point, and compute probabilities of remaining inside, or outside, the solid. Such questions could be asked in any dimension.

I am interested in the techniques I have used to try to find sums for strange-looking infinite series. It seems interesting to me that I was able to use differential equations to sum infinite series, which reminded me of the use of infinite series to solve differential equations in Mathematics 353. Perhaps infinite series whose coefficients involve well-known numbers such as binomial coefficients or Catalan numbers or Fibonacci numbers can often be summed using techniques like those in this thesis. There must surely be an interesting relationship between the linear homogeneous recursion relation I studied in Mathematics 301 and the linear homogeneous differential equations

I studied in Mathematics 353. In both these situations, the solution involved roots of a characteristic polynomial. If the opportunity arises, I will welcome the chance to think more about these questions.

References

[1] Daniel I.A. Cohen. *Basic Techniques of Combinatorial Theory.* John Wiley and Sons, New York, 1978.

[2] Stephen J. Leon. *Linear Algebra with Applications, 5th edition.* Prentice-Hall, Upper Saddle River, New Jersey, 1999.

[3] Daniel A. Marcus. *Combinatorics.* The Mathematical Association of America, Washington, D.C., 1998.

[4] J.R. Norris, *Markov Chains.* Cambridge University Press, New York, 1998.

[5] Gary Peterson and James Sochacki, *Linear Algebra and Differential Equations.* Addison Wesley, Boston, 2002.