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Alpha-Representability of Vector Lattices

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α-REPRESENTABILITY OF VECTOR LATTICES

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ABSTRACT

MATTHEW J. STEPHENSON: α-Representability of Vector Lattices
(Under the direction of Gerard Buskes)

The primary topic of this thesis is representation of vector lattices. The related theory of Boolean algebras is used as a tool to this end. In [1], Brown and Nakano present a theorem that establishes what we will call σ-representability for vector lattices in this thesis. We cast their proof in the light of the Boolean algebra of bands. Consequently, we show that it is the Loomis-Sikorski Theorem which makes their proof work. We then exploit this insight to study α-representability for greater cardinal numbers α.

A primary goal of this thesis is to be self-contained. As a result of this, the bulk of the text introduces definitions and theorems, as presented by other authors. We have liberally used existing theorems and proofs and give credit when ideas and/or exact proofs are borrowed. The novelty in our approach comes in connecting the work of Brown and Nakano with the Loomis-Sikorski Theorem and isolating our definition of α-representability.

Contents

1. Preliminaries

By $\overline{\mathbb{R}}$ we denote the extended real numbers (i.e. $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}$).

The complement of a subset U of X is denoted by U^c or $X\backslash U$. If $f : X_1 \to X_2$ is a map between sets X_1 and X_2 and $U \subset X_2$, then $f^{-1}(U) = \{x \in X_1 : f(x) \in U\}$.

If $U \subset X$, then the characteristic function of U is denoted by

$$1_U(x) = \begin{cases} 1 & : x \in U \\ 0 & : x \notin U \end{cases}$$

for $x \in X$

A map $f : X_1 \to X_2$ is said to be *surjective* or *onto* if for every $y \in X_2$, there exists an element $x \in X_1$ such that $f(x) = y$. A map $f : X_1 \to X_2$ is said to be *injective* or *one-to-one* if $f(x_1) = f(x_2)$ implies $x_1 = x_2$ for all $x_1, x_2 \in X_1$. If f is both injective and surjective, then we say that f is *bijective*.

A set is called *directed* if it is equipped with a binary operation that is transitive and has the property that any two elements have an upper bound. That is, if Y is a nonempty set, then $(Y, \leq)$ a directed set if

(1) For $x, y, z \in Y$, if $x \leq y$ and $y \leq z$, it follows that $x \leq z$,

(2) For $x, y \in Y$, there exists a z in Y such that $z \geq x$ and $z \geq y$.

A sequence is a function on the natural numbers. A *net* is a function on a directed set. Every sequence is therefore a net. Given a set X and a directed set Γ, the map $x : \Gamma \to X$ is denoted by $(x_\gamma)_{\gamma\in\Gamma}$ or shortly (x_γ).

Let X be a set. A collection τ of subsets of X that contains X and $\emptyset$ and is closed under unions and finite intersections is called a *topology* on X. The sets contained in τ are called *open sets*, and their complements are called *closed sets*. The pair (X, τ) is called a *topological space*. Often, if the context is clear, X itself will be called a topological space.

In the rest of this section, let X be a topological space. For a point $x \in X$, a *neighborhood* of x is any set that contains an open set containing x. A subset U of X is open if and only if for every point x of U there exists a neighborhood that contains x and is completely contained in U. A function $f : X_1 \to X_2$ between two topological spaces is said to be *continuous* if for every open set U in X_2, we have $f^{-1}(U)$ is open in X_1.

A function $f : X_1 \to X_2$ is a *homeomorphism* if it is continuous, bijective, and its inverse is continuous. Two topological spaces X_1 and X_2 are said to be homeomorphic if there exists a homeomorphism $X_1 \to X_2$.

The *closure* of U is the smallest closed set that contains U and is denoted by U^-. A subset of X is closed if and only if it is equal to its closure. A point $x \in X$ is in the closure of U if and only if every neighborhood of x has nonempty intersection with U.

The *interior* of V is the largest open set that is contained in V and is denoted by V°. A subset of X is open if and only if it is equal to its interior. A point $x \in X$ is in the interior of V if and only if there exists a neighborhood of x that is completely contained in V. We have $V^\circ = V^{c-c}$.

$U \subset X$ is *nowhere dense* if $U^{-\circ} = \emptyset$. Consequently, a nowhere dense subset of X contains no nonempty open set. A set is *meagre* if it is the union of countably many nowhere dense sets. If a statement about elements of X is true except on a meagre subset of X, then we say the statement is "almost everywhere" true. X is a *Hausdorff space* if for any two distinct points x and y, there exist disjoint open sets U and V such that $x \in U$ and $y \in V$.

$U \subset X$ is said to be *regular open* if it is equal to the interior of its closure, i.e. $U = U^{-\circ}$. Every regular open set is open. A space X is called *extremally disconnected* if every regular open subset of X is additionally closed. Thus a set is regular open if and only if it has clopen closure. Moreover, X is called *zerodimensional* if the clopen subsets of X form a base for the topology of X, that is, if every open subset of X can be written as the union of the clopen subsets of X. Next we show that extremally disconnected spaces are zerodimensional (see [**6**], page 85).

THEOREM 1.1. *If X is an extremally disconnected compact Hausdorff space, then X is zerodimensional.*

PROOF. Take $x \in X$ and a neighborhood V of x. There exists an open set U such that $x \in U \subset \overline{U} \subset V$. Hence, $x \in U^{-\circ} \subset V$, which implies that V can be written as the union of regular open sets. Hence, the regular open subsets of X form a base for the topology of X, and since X is extremally disconnected, these sets are all clopen. $\square$

2. Boolean Algebras, Preliminaries

To introduce Boolean algebras, we need some definitions. Basic definitions are found in [**11**] unless otherwise noted.

DEFINITION 2.1. For a nonempty set X, the binary relation $\leq$ is a *partial ordering* if the following hold.

(1) $x \leq x$ for every $x \in X$,

(2) $x \leq y$ and $y \leq z$ implies $x \leq z$, and

(3) $x \leq y$ and $y \leq x$ implies $x = y$.

The pair $(X, \leq)$ is called a *partially ordered set.*

If Y is a nonempty subset of a partially ordered set $(X, \leq)$ and $x_0 \in X$ is such that $y \leq x_0$ for every $y \in Y$, then x_0 is an *upper bound* of Y. Moreover, if x_0 is an upper bound of Y and $x_0 \leq x'$ for all upper bounds x' of Y, then we call x_0 the *supremum* or *least upper bound* of Y. *Lower bounds* and *infima* are defined similarly.

DEFINITION 2.2. A partially ordered set $(X, \leq)$ is called a *lattice* if every subset with two elements has a supremum and an infimum.

Let $(X, \leq)$ be a lattice, and let x, y be elements of X. We denote the supremum of $\{x, y\}$ by $x \vee y$. The infimum of $\{x, y\}$ will be denoted by $x \wedge y$. If X has a smallest element, we denote it by $\mathbf{0}$ and call it the zero element. If X has a largest element, we denote it by $\mathbf{1}$ and call it the unit element.

In a lattice $(X, \leq)$ with a zero and unit element, x' is the *complement* of x if $x \vee x' = \mathbf{1}$ and $x \wedge x' = \mathbf{0}$. If every element of a lattice has a complement, then we say that the lattice is *complemented.* We claim that every element of a lattice has at most one complement (see [**11**], page 6). Indeed, suppose y and z are both complements of x, i.e. $x \vee y = x \vee z = \mathbf{1}$ and $x \wedge y = x \wedge z = \mathbf{0}$. It follows that

$$y = y \vee \mathbf{0} = y \vee (x \wedge z) = (y \vee x) \wedge (y \vee z) = \mathbf{1} \wedge (y \vee z) = y \vee z.$$

Hence, $y \geq z$. Similarly, $z \geq y$. Therefore, $y = z$, and the complement of x is uniquely determined. We denote the complement of an element x in a lattice by x'.

In a lattice $(X, \leq)$, if $x \leq z$ and $y \leq z$, then $x \vee y \leq z$. It follows that

$$(x \wedge y) \vee (x \wedge z) \leq x \wedge (y \vee z) \tag{2.1}$$

because $x \wedge y \leq x \wedge (y \vee z)$ and $x \wedge z \leq x \wedge (y \vee z)$. A lattice $(X, \leq)$ is called *distributive* if there is equality in (2.1), that is, if

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$$

for all x, y, z in X. We are now equipped to present the definition of a Boolean algebra.

DEFINITION 2.3. A *Boolean algebra* is a complemented, distributive lattice with a unit element and a zero element.

Next we provide the prime example of a Boolean algebra: the collection of all subsets of a given set X under inclusion.

EXAMPLE 2.4. **$\mathcal{P}(X)$ is a Boolean algebra**

Let $\mathcal{P}(X)$ be the collection of all subsets of a nonempty set X, and take $U, V, W \in \mathcal{P}(X)$. We show that the operation of inclusion forms a partial ordering. U is trivially a subset of itself. If $U \subset V$ and $V \subset W$, then $U \subset W$. Finally, if $U \subset V$ and $V \subset U$, it follows that every element of U is in V and vice versa. Therefore, $U = V$.

Suppose that $U \subset W$ and $V \subset W$. It follows that $U \cup V \subset W$, and therefore, $U \cup V$ is the least element of $\mathcal{P}(X)$ that contains both U and V. We conclude that $U \vee V = U \cup V \subset X$. By a similar argument, $U \wedge V = U \cap V \subset X$. Hence, $\mathcal{P}(X)$ is a lattice. For any $U \in \mathcal{P}(X)$, we have $\emptyset \subset U \subset X$. Then $\emptyset$ is the zero element of $\mathcal{P}(X)$, and X is the unit element of $\mathcal{P}(X)$. We claim that $X \backslash U$ is the complement of U, for

$U \vee X \backslash U = U \cup X \backslash U = X$ and $U \wedge X \backslash U = U \cap X \backslash U = \emptyset$. The final requirement, the distributive law, is an elementary exercise in set theory. Consider

$$\begin{aligned} x \in U \wedge (V \vee W) &\Leftrightarrow x \in U \cap (V \cup W) \\ &\Leftrightarrow x \in U \text{ and } x \in V \cup W \\ &\Leftrightarrow (x \in U \text{ and } x \in V) \text{ or } (x \in U \text{ and } x \in V) \\ &\Leftrightarrow x \in U \cap V \text{ or } x \in U \cap W \\ &\Leftrightarrow x \in (U \cap V) \cup (U \cap W) \\ &\Leftrightarrow x \in (U \wedge V) \vee (U \wedge W). \end{aligned}$$

Therefore, we conclude that $\mathcal{P}(X)$ under inclusion is a Boolean algebra. This concrete example of a Boolean algebra will be important in the third chapter when representation is discussed.

We define a *field of sets* to be any subset of $\mathcal{P}(X)$ that forms a Boolean algebra under the set-theoretical properties of union, intersection, and complementation.

EXAMPLE 2.5. **$\mathcal{RO}(X)$ is a Boolean algebra**

Let $\mathcal{RO}(X)$ be the collection of regular open subsets of a topological space X (see [5], page 66). The partial ordering on $\mathcal{RO}(X)$ is inclusion, as in Example 2.4. Recall that an open set is said to be regular open if it is equal to the interior of its closure. Note that $U^{\circ} = U^{c-c}$ for any $U \subset X$. Certainly, X and $\emptyset$ are both regular open subsets of X, i.e. they are the unit element and zero element in $\mathcal{RO}(X)$, respectively. For $U, V \in \mathcal{RO}(X)$, we prove that

$$\begin{gathered} U \wedge V = U \cap V \\ (U \vee V) = (U \cup V)^{-\circ}, \text{ and} \\ U' = U^{-c}. \end{gathered}$$

First, we establish inclusions to be used later. Note that

$$Y \subset Z \subset X \Rightarrow Z^{-c} \subset Y^{-c} \tag{2.2}$$

because the closure operation preserves inclusions and complementation reverses them. As an immediate consequence, $Y^{-\circ} \subset Z^{-\circ}$ if $Y \subset Z$. Moreover,

$$Y \text{ is open} \Rightarrow Y \subset Y^{-\circ} \tag{2.3}$$

because $Y^{-\circ}$ is the largest among all open subsets of Y^{-}, and Y is an open subset of Y^{-}.

Step 1: If U is open, then $U^{-c} \in \mathcal{RO}(X)$

If U is an open subset of X, then $(U^{-c})^{-\circ} \subset U^{-c}$ because $U \subset U^{-\circ}$. For the reverse inclusion, we consider the open set U^{-c} and apply (2.3), which gives $U^{-c} \subset (U^{-c})^{-\circ}$. Therefore, $U^{-c} = (U^{-c})^{-\circ}$, and $U^{-\circ}$ is a regular open subset of X.

Step 2: $\mathcal{RO}(X)$ is a lattice

Since $(U \cup V)^{-c}$ is open, it follows from Step 1 that $(U \cup V)^{-c-c} = (U \cup V)^{-\circ}$ is regular open. Moreover, $U = U^{-\circ} \subset (U \cup V)^{-\circ}$ because $U \subset U \cup V$. Similarly, $V = V^{-\circ} \subset (U \cup V)^{-\circ}$. Suppose $W \in \mathcal{RO}(X)$ such that $U \subset W$ and $V \subset W$. It follows that $U \cup V \subset W$. Hence, $(U \cup V)^{-\circ} \subset W^{-\circ} = W$. Therefore, $U \vee V = (U \cup V)^{-\circ}$.

Next we show that $U \cap V$ is regular open. In one direction, $U \cap V$ is a subset of both U and V, and therefore $(U \cap V)^{-\circ} \subset U^{-\circ} \cap V^{-\circ}$. We will use the following series of inclusions to establish the converse:

$$U^{-\circ} \cap V^{-\circ} \subset (U^{-\circ} \cap V)^{-\circ} \subset (U \cap V)^{-\circ-\circ} = (U \cap V)^{-\circ}.$$

For the verification of the inclusions, we first claim that $U \cap V^{-} \subset (U \cap V)^{-}$ if U is open. Indeed, take a point x in $U \cap V^{-}$ and let P be an arbitrary neighborhood of x. Since $P \cap U$ is also a neighborhood of x and $x \in V^{-}$, it follows that $P \cap U$ intersects V. Thus there is some point in $P \cap (U \cap V)$, and we infer that $x \in (U \cap V)^{-}$. It follows that

$$U \cap V^- \subset (U \cap V)^- \Rightarrow (U \cap V)^{-c} \subset U^c \cup V^{-c} \Rightarrow U \cap V^{-\circ} \subset (U \cap V)^{-\circ}$$

for subsets U, V of X where U is open. We twice use the final inclusion to obtain the desired result, first by replacing U with the open set $U^{-\circ}$ and then by interchanging the roles of U and V. Thus

$$U^{-\circ} \cap V^{-\circ} \subset (U^{-\circ} \cap V)^{-\circ} \subset (U \cap V)^{-\circ-\circ} = (U \cap V)^{-\circ},$$

where the final equality follows because any subset of the form $Y^{-\circ}$ is regular open. Therefore, $(U \cap V)^{-\circ} = U^{-\circ} \cap V^{-\circ} = U \cap V$, which implies that $U \cap V \in \mathcal{RO}(X)$.

Certainly, $U \cap V \subset U$ and $U \cap V \subset V$. Furthermore, if $W \in \mathcal{RO}(X)$ such that $W \subset U$ and $W \subset V$, then $W \subset U \cap V$. Hence, $U \wedge V = U \cap V$.

We conclude that $\mathcal{RO}(X)$ is a lattice.

Step 4: $\mathcal{RO}(X)$ is distributive.

Indeed, for $U, V, W \in \mathcal{RO}(X)$,

$$\begin{aligned} U \wedge (V \vee W) = U \cap (V \cup W)^{-\circ} &= U^{-\circ} \cap (V \cup W)^{-\circ} \\ &= (U \cap (V \cup W))^{-\circ} \\ &= ((U \cap V) \cup (U \cap W))^{-\circ} \\ &= (U \wedge V) \vee (U \wedge W). \end{aligned}$$

Step 5: $U' = U^{-c}$

We show that $U \wedge U^{-c} = \emptyset$ and $U \vee U^{-c} = X$ for $U \in \mathcal{RO}(X)$. The former follows immediately from $U^{-c} \subset U^c$. To establish the latter, we use that the boundary $U^- \cap U^c$ of any open set U is a nowhere dense closed set and therefore that it contains no nonempty open set. Indeed, if the boundary did contain a nonempty open set, then that set would intersect U^-, but it would not intersect U. Thus the complement of the boundary $U \cup U^{-c}$ is a dense open set, and its closure is the whole space X. Therefore, $(U \cup U^{-c})^{-c} = \emptyset$, and $U \vee U^{-c} = (U \cup U^{-c})^{-c-c} = X$.

Hence, $\mathcal{RO}(X)$ is a Boolean algebra.

Throughout the remainder of this section, let B be a Boolean algebra. In the remainder of this section, definitions come from [5] unless noted otherwise. Having provided two examples of Boolean algebras, we proceed to define ideals.

DEFINITION 2.6. A nonempty subset I of B is called an *ideal* if

(1) $x \vee y \in I$ whenever $x, y \in I$ and

(2) $x \in I$ whenever $y \in I$ and $x \leq y$

A *principal ideal* of B is defined by $(x) = \{y \in B : y \leq x\}$ for $x \in B$. An ideal I of B is called a *proper ideal* if $I \neq B$. A proper ideal I of B is called a *maximal ideal* if no other proper ideal of B contains I. Maximal ideals contain either x or x', but not both, for every $x \in B$. Suppose that for some $x \in B$, neither x nor x' is in a maximal ideal I. The smallest ideal containing $I \cup \{x\}$, which is the set $\{y \vee z : y \in I \text{ and } z \leq x\}$, is a proper ideal (since it does not contain x') that properly contains I (since $x \notin I$). This contradicts the maximality of I, and therefore I contains either x or x'. Furthermore, if $x, x' \in I$, then $x \vee x' = \mathbf{1} \in I$, which contradicts the condition that I be a proper ideal of B.

For a Boolean algebra B and an ideal I of B, we say that x is equivalent to y, or $x \sim y$, if $x - y \in I$ and $y - x \in I$, where $x - y = x \wedge y'$. We show that this relation is indeed an equivalence relation, i.e. that it is reflexive, symmetric, and transitive.

Certainly, $x \sim x$ since $x - x = 0 \in I$ for any ideal I. Moreover, if $x \sim y$, then $x - y \in I$ and $y - x \in I$, and therefore $y \sim x$. Hence, the relation is symmetric. For transitivity, suppose $x \sim y$ and $y \sim z$. Note that

$$x \wedge z' \leq (x \wedge y') \vee (y \wedge z').$$

The right-hand side, which is the supremum of two elements of I, is in I, and therefore $x \wedge z'$ is also in I. Similarly, $z \wedge x' \in I$ because $z \wedge y'$ and $y \wedge x'$ are in I and $z \wedge x' \leq (z \wedge y') \vee (y \wedge x')$. Hence, $x \sim z$, and $\sim$ is an equivalence relation.

The *equivalence class* containing an element x consists of all z such that $z \sim x$ and is denoted by $[x]$. The following statements are equivalent:

(1) $x \sim y$

(2) $x \in [y]$

(3) $[x] = [y]$.

By definition of equivalence classes, $x \in [y]$ if and only if $x \sim y$. Furthermore, suppose $[x] = [y]$, and let z be in both $[x]$ and $[y]$. It follows that $z \sim x$ and $z \sim y$ and that $x \sim y$ by transitivity. Conversely, suppose that $x \sim y$, and take $z \in [x]$. Then $z \sim x \sim y$, and $z \in [y]$. Thus $[x] \subset [y]$. Similarly, we have the reverse inclusion $[y] \subset [x]$, and therefore $[x] = [y]$.

The *quotient algebra* B/I is the set of all equivalence classes. B/I is a Boolean algebra with the operations defined by

$$[x] \vee [y] = [x \vee y], \quad [x] \wedge [y] = [x \wedge y], \quad [x]' = [x']$$

for $[x], [y] \in B/I$.

If I is a maximal ideal of B, then $B/I = \{[\mathbf{0}], [\mathbf{1}]\}$. Certainly, $I = [\mathbf{0}]$. Take $x \in B \backslash I$. It follows from the maximality of I that $x' \in I$. Since $x' \wedge \mathbf{1} = x' \in I$ and $x \wedge \mathbf{1}' = x \wedge \mathbf{0} = \mathbf{0} \in I$, we have $[x] = [\mathbf{1}]$.

Definition 2.7. Let B_1 and B_2 be Boolean algebras. A map $\phi : B_1 \to B_2$ is a *Boolean homomorphism* if it satisfies

(1) $\phi(x \vee y) = \phi(x) \vee \phi(y)$ and

(2) $\phi(x') = \phi(x)'$.

If a Boolean homomorphism $\phi : B_1 \to B_2$ is bijective, then ϕ is an *isomorphism*, and B_1 and B_2 are said to be *isomorphic*.

Theorem 2.8. *Let B_1 and B_2 be Boolean algebras. A bijective map $\phi : B_1 \to B_2$ is an isomorphism if and only if both ϕ and ϕ^{-1} preserve order.*

PROOF. Suppose that ϕ and ϕ^{-1} preserve order. Then $\phi(\mathbf{0}) \leq \phi(x) \leq \phi(\mathbf{1})$ for all $x \in B_1$, and it follows that $\phi(\mathbf{0}) = \mathbf{0}$ and $\phi(\mathbf{1}) = \mathbf{1}$. Moreover, $\phi(x) \leq \phi(x \vee y)$ and $\phi(y) \leq \phi(x \vee y)$ for all $x, y \in B_1$. Thus $\phi(x) \vee \phi(y) \leq \phi(x \vee y)$. If $\phi(z) \geq \phi(x) \vee \phi(y)$ for some $z \in B$, then $z \geq x \vee y$ because ϕ^{-1} preserves order. Then $\phi(z) \geq \phi(x \vee y)$. Hence, $\phi(x \vee y)$ is the least element such that $\phi(x \vee y) \geq \phi(x)$ and $\phi(x \vee y) \geq \phi(y)$, i.e. $\phi(x) \vee \phi(y) = \phi(x \vee y)$.

Next we show that $\phi(x)' = \phi(x')$ for all $x \in B_1$. Certainly, $\mathbf{0} \leq \phi(x), \phi(x') \leq \mathbf{1}$. If y is another element of B_1 such that $\phi(y) \geq \phi(x) \vee \phi(x')$, then $y \geq x \vee x' = \mathbf{1}$. Therefore, $\phi(y) \geq \mathbf{1}$, and it follows that $\mathbf{1} = \phi(x) \vee \phi(x')$. Similarly, if z is an element of B_1 such that $\phi(z) \leq \phi(x) \vee \phi(x')$, then $z \leq x \vee x'$. Thus $z \leq x \wedge x'$, and $\phi(z) \leq \phi(x \wedge x') = \phi(\mathbf{0}) = \mathbf{0}$. Hence, $\phi(x) \wedge \phi(x') = \mathbf{0}$, and we conclude that ϕ is an isomorphism.

Suppose ϕ is an isomorphism. It follows that

$$x \leq y \Leftrightarrow x \vee y = y \Leftrightarrow \phi(x) \vee \phi(y) = \phi(y) \Leftrightarrow \phi(x) \leq \phi(y).$$

Therefore, ϕ and ϕ^{-1} preserve order (see [9], page 16). $\square$

The map $\phi : B \to B/I$ defined by $\phi(x) = [x]$ for $x \in B$ is called the *natural homomorphism* (see [9], page 30). It is clear that ϕ is a homomorphism because $\phi(x \vee y) = [x \vee y] = [x] \vee [y] = \phi(x) \vee \phi(y)$.

3. Vector Lattices, Preliminaries

In this section, we introduce vector lattices. Basic definitions in this section are found in [**11**] unless otherwise stated.

Given a vector space V over $\mathbb{R}$ equipped with a partial ordering $\leq$, we call $(V, \leq)$ an *ordered vector space* if

(1) $f \leq g \Rightarrow f + h \leq g + h$ for $f, g, h \in V$ and
(2) $f \geq 0 \Rightarrow af \geq 0$ for all $a \in \mathbb{R}_{\geq 0}$ and $f \in V$.

We now define vector lattices.

DEFINITION 3.1. We call $(E, \leq)$ a *vector lattice* if

(1) $(E, \leq)$ is an ordered vector space and
(2) $(E, \leq)$ is a lattice with respect to the partial ordering.

We will denote a vector lattice $(E, \leq)$ shortly by E. Throughout this section, E will be a vector lattice. Next we present several examples of vector lattices.

EXAMPLE 3.2. **$\mathbb{R}^n$ is a vector lattice**

The most basic examples of vector lattices are the n-dimensional Euclidean spaces $\mathbb{R}^n$ with regular addition and multiplication and coordinatewise ordering (that is, for $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, $x \leq y$ when $x_k \leq y_k$ for $k = 1, 2, \ldots, n$). The supremum of two elements of $\mathbb{R}^n$ is the coordinatewise supremum: If $x = (x_1, x_2, \ldots, x_n)$ and $y = (y_1, y_2, \ldots, y_n)$, then $(x_1 \vee y_1, x_2 \vee y_2, \ldots, x_n \vee y_n)$ is the smallest element of $\mathbb{R}^n$ that is greater than both x and y. Similarly, the infimum of x and y is $(x_1 \wedge y_1, x_2 \wedge y_2, \ldots, x_n \wedge y_n)$. Thus $\mathbb{R}^n$ is a lattice. That the lattice structure of $\mathbb{R}^n$ is compatible with the partial ordering follows from basic properties of real numbers. Let $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n)$, and $z = (z_1, z_2, \ldots, z_n)$ be elements of $\mathbb{R}^n$. Indeed, if $x \leq y$, then $x_i \leq y_i$ for $i = 1, 2, \ldots, n$. Therefore,

$x_i + z_i \leq y_i + z_i$ for $i = 1, 2, \ldots, n$, and $x + z \leq y + z$. Moreover, suppose that $x \geq 0$, and let a be a nonnegative real number. Since $x_i \geq 0$ for each i, it follows that $ax_i \geq 0$ for every i, which implies that $ax \geq 0$.

EXAMPLE 3.3. **$\mathbb{R}^X$ is a vector lattice**

Let X be a nonempty set or a topological space. By $\mathbb{R}^X$ we denote all maps $X \to \mathbb{R}$. For elements f and g of $\mathbb{R}^X$, we say that $f \leq g$ if $f(x) \leq g(x)$ for all $x \in X$. Then $\mathbb{R}^X$ is a lattice because the map h defined by $h(x) = f(x) \vee g(x)$ is the least element of $\mathbb{R}^X$ that is greater than f and g. Similarly, $f \wedge g \in \mathbb{R}^X$. If $f \leq g$, then for any $h \in \mathbb{R}^X$, it is clear that $f + h \leq g + h$ because $(f+h)(x) = f(x) + h(x) \leq g(x) + h(x) = (g+h)(x)$ for all $x \in X$. Moreover, if $f \geq 0$, then $af(x) \geq 0$ for all x and for any nonnegative real number a.

We denote the positive part of an element f of a vector lattice by $f^+ = f \vee 0$. Furthermore, we write $f^- = (-f) \vee 0$ and $|f| = (-f) \vee f$. The *positive cone* of a vector lattice E is defined by $E^+ = \{f \in E : f \geq 0\}$. An immediate consequence of these definitions is that if $f \in E$, then $f^+, f^-, |f| \in E^+$.

Next we provide several equalities for elements of E (see [**11**], page 17).

THEOREM 3.4. *Let f and g be elements of E.*

(1) $f = f^+ - f^-$,

(2) $f^+ \wedge f^- = 0$,

(3) $|f| = f^+ + f^-$

(4) $f \vee g + f \wedge g = f + g$ *and* $f \vee g - f \wedge g = |f - g|$.

(5) $f \vee g = \frac{1}{2}(f+g) + \frac{1}{2}|f-g|$ *and* $f \wedge g = \frac{1}{2}(f+g) - \frac{1}{2}|f-g|$,

PROOF. (1) $f^+ - f = (f \vee 0) - f = 0 \vee (-f) = f^-$.

(2) $0 = -f^- + f^- = (f \wedge 0) + f^- = (f + f^-) \wedge f^- = f^+ \wedge f^-$.

(3) $|f| = f \vee (-f) = \{(2f) \vee 0\} - f = 2f^+ - (f^+ - f^-) = f^+ - f^-$.

(4) Note that

$$f \vee g = \{(f-g) \vee 0\} + g = (f-g)^+ + g = (g-f)^+ + f \text{ and}$$

$$f \wedge g = f + \{0 \wedge (g - f)\} = f - \{0 \vee (f - g)\} = f - (f - g)^+.$$

It follows that $f \vee g + f \wedge g = f + g$, and

$$f \vee g - f \wedge g = (f - g)^+ + (g - f)^+ = (f - g) \vee (g - f) = |f - g|.$$

(5) Follows from adding and subtracting the equalities of (4).

□

If $\{f_\gamma\}$ is a collection of elements in E such that $f_0 = \sup\{f_\gamma\}$, then $f_0 \wedge h = \sup\{f_\gamma \wedge h\}$ for all $h \in E$. Similarly if $f_1 = \inf\{f_\gamma\}$, then $f_1 \vee h = \inf\{f_\gamma \vee h\}$ for all h (see [**11**], page 21). Indeed, let h be an element of E. It is clear that $f_0 \wedge h$ is an upper bound of $\{f_\gamma \wedge h\}$. Suppose g is an upper bound of this set. Then $g \geq f_\gamma \wedge h = f_\gamma + h - (f_\gamma \vee h) \geq f_\gamma + h - (f_0 \vee h)$. Thus $g - h + (f_0 \vee h) \geq f_\gamma$ for all γ, and therefore $g - h + (f_0 \vee h) \geq f_0$. It follows that $g \geq f_0 + h - (f_0 \vee h) = f_0 \wedge h$. Hence $f_0 \wedge h = \sup\{f_\gamma \wedge h\}$. The proof for $f_1 \vee h$ is similar.

We now present a theorem known as the Riesz decomposition property (see [**11**], page 22).

THEOREM 3.5. *Let f, g_1, and g_2 be elements in E^+ such that $f \leq g_1 + g_2$. Then there exist elements $f_1, f_2 \in E^+$ such that $f_1 \leq g_1$, $f_2 \leq g_2$, and $f = f_1 + f_2$.*

PROOF. Let $f_1 = f \wedge g_1$. Define f_2 by $f_2 = f - f_1$ so that $f = f_1 + f_2$. Then $f_1 \in E^+$, and $f_1 \leq g_1$. Since $f_1 \leq f$, it follows that $f_2 \in E^+$. Moreover, we have

$$f_2 = f - f_1 = f(f \wedge g_1) = f + \{(-f) \vee (-g_1)\} = 0 \vee (f - g_1) \leq 0 \vee g_2 = g_2.$$

This completes the proof. □

EXAMPLE 3.6. $C(X)$ **is a vector lattice**

Another example of a vector lattice is the set of all real-valued continuous functions on a topological space X, denoted by $C(X)$. The space $C(X)$ is partially ordered pointwise in the same way as $\mathbb{R}^X$. We show that the pointwise supremum and infimum of two continuous functions are indeed continuous. Let $(f \vee g)_P$ denote the pointwise supremum of $\{f, g\}$ as in Example 3.3. Note that continuity is preserved under addition, subtraction, and taking of absolute values. Therefore, $(f \vee g)_P = \frac{1}{2}(f + g) + \frac{1}{2}|f - g|$ is a continuous function that is necessarily below all other bounds of f and g in $C(X)$. We infer that $f \vee g = (f \vee g)_P$, where the left-hand side refers to the supremum in $C(X)$. By a similar argument, we use (5) of Theorem 3.4 to show that $\{f, g\}$ has an infimum in $C(X)$. Hence, $C(X)$ is a lattice.

The proof that the partial ordering is compatible with the vector space holds precisely as in Example 3.3

The positive cone E^+ is an example of a Riesz subspace of E. We call a subset F of E a *Riesz subspace* if F is closed under infima and suprema. Note that if $f, g \in E^+$, then $f \vee g \geq 0$ and $f \wedge g \geq 0$, and therefore $f \vee g, f \wedge g \in E^+$.

A *function lattice* is a Riesz subspace of $\mathbb{R}^X$ for a nonempty set X. Examples 3.3 and 3.6 are both function lattices.

An *ideal* I is a Riesz subspace such that if $f \in I$ and $|g| \leq |f|$, then $g \in I$. A *band* D is an ideal in E such that for every subset of D that has a supremum, the supremum is in D. For an ideal I in E, we define $[I]$ to be the smallest band in E that contains I. It follows that $[I] = \{f \in E : |f| = \sup F\,,\ F \subset I^+\}$ (see [**11**], page 32).

DEFINITION 3.7. An vector lattice E is called *Archimedean* if for every f in E^+, the infimum of the set $\{n^{-1}f : n = 1, 2, \ldots\}$ is zero.

Every Riesz subspace of an Archimedean vector lattice is Archimedean.

A subspace G of a vector lattice E is said to be *order dense* if for every $f > 0$ in E, there exists an element g in G such that $0 < g \leq f$.

A sequence $\{f_n\}$ in E is said to be *uniformly convergent* to f if for some $0 < g \in E^+$ there exists a sequence ϵ_n in $\mathbb{R}$ decreasing to 0 such that $|f - f_n| \leq \epsilon_n g$ for all n.

Next we define Riesz homomorphisms. Recall that a map ϕ between vector spaces V_1 and V_2 is linear if $\phi(af + bg) = a\phi(f) + b\phi(g)$ for all $f, g \in V_1$ and all scalars a

and b. Let F be a vector lattice. A map $\phi : E \to F$ is called a *Riesz homomorphism* if $\phi(f \vee g) = \phi(f) \vee \phi(g)$ for every $f, g \in E$.

A bijective Riesz homomorphism is a *Riesz isomorphism.* If $\phi : E \to F$ is a Riesz isomorphism, then E and F are said to be isomorphic.

THEOREM 3.8. *Let $\phi : E \to F$ be a Riesz homomorphism. Then $\phi(f) \geq \phi(g)$ whenever $f \geq g$.*

PROOF. Note that $\phi(0) = 0$ because for any scalar a, namely $a = 0$, we have $\phi(af) = a\phi(f)$. Let $h \in E$, and suppose $h \geq 0$. Then $h \vee 0 = h$, and

$$\phi(h) = \phi(h \vee 0) = \phi(h) \vee \phi(0) = \phi(h) \vee 0 \geq 0.$$

Now suppose that $f \geq g$. It follows that $f - g \geq 0$ and that $\phi(f - g) = \phi(f) - \phi(g) \geq \phi(0)$. Therefore, $\phi(f) \geq \phi(g)$. □

Let I an ideal in E. The *quotient space* E/I consists of the sets $[f] = \{f_1 \in E : f_1 - f \in I\}$, for each f in E. Quotient spaces are endowed with a partial ordering. Given $[f], [g] \in E/I$, we say that $[f] \leq [g]$ whenever there exist elements $f_1 \in [f]$ and $g_1 \in [g]$ such that $f_1 \leq g_1$. Under this ordering, we claim that the quotient space is actually a vector lattice (see **[11]**, page 116).

THEOREM 3.9. *The quotient space E/I, where I is an ideal of E, is a vector lattice.*

PROOF. We first show that $\leq$, as defined above, is a partial ordering. Clearly $[f] \leq [f]$ because $f \leq f$. If $[f] \leq [g]$ and $[g] \leq [h]$, then there exist elements $f_1 \in [f]$, $g_1, g_2 \in [g]$, and $h_1 \in [h]$ such that $f_1 \leq g_1$ and $g_2 \leq h_1$. Therefore

$$f_1 \leq g_1 = g_2 + (g_1 - g_2) \leq h_1 + (g_1 - g_2).$$

Since $g_1 - g_2 \in I$, it follows that $h_2 = h_1 + (g_1 - g_2) \in [h]$. Hence, $f_1 \leq h_2$, and $[f] \leq [h]$.

Furthermore, if $[f] \leq [g]$ and $[g] \leq [f]$, then there exist elements $f_1, f_2 \in [f]$ and $g_1, g_2 \in [g]$ such that $f_1 \leq g_1$ and $g_2 \leq f_2$. Note that

$$0 \leq g_1 - f_1 \leq (g_1 - f_1) + (f_2 - g_2) = (f_2 - f_1) + (g_1 - g_2) \in I.$$

Since I is an ideal, $f_1 - g_1 \in I$ and $[f] = [f_1] = [g_1] = [g]$. This establishes that $\leq$ is a partial ordering.

We next prove that the partial ordering is compatible with the structure of the vector space. If $[f] \leq [g]$, then there exist elements $f_1 \in [f]$ and $g_1 \in [g]$ such that $f_1 \leq g_1$. The result follows because E is a vector lattice. That is, $f_1 \leq g_1 \Rightarrow af_1 \leq ag_1$ for all $a \geq 0$, and therefore $a[f] \leq a[g]$. To show that $[f] \leq [g] \Rightarrow [f]+[h] \leq [g]+[h]$, take $f_1 \in [f]$ and $g_1 \in [g]$ such that $f_1 \leq g_1$, and let h_1 be in $[h]$. Again since E is a lattice with respect to its partial ordering, $f_1 + h_1 \leq g_1 + h_1$ with $f_1 + h_1 \in [f+h] = [f] + [h]$ and $g_1 + h_1 \in [g] + [h]$. Thus $[f] + [h] \leq [g] + [h]$.

It now remains to be shown that suprema and infima exist in E/I. Take $[f], [g] \in E/I$. It follows from $f \vee g \geq f$ and $f \vee g \geq g$ that $[f \vee g] \geq [f]$ and $[f \vee g] \geq [g]$. Hence, we only need to show that for every upper bound $[h]$ of $[f]$ and $[g]$, we have $[h] \geq [f \vee g]$. If $[h]$ is an upper bound of $[f]$ and $[g]$, then for any $f' \in [f], g' \in [g]$, and $h' \in [h]$, there exist elements $j_1, j_2 \in I$ such that $h' \geq f' + j_1$ and $h' \geq g' + j_2$. Therefore, if $j = j_1 \wedge j_2$, it follows that $h' \geq f' + j$ and $h' \geq g' + j$. Thus $h' \geq (f'+j) \vee (g'+j) = (f' \vee g') + j$, which implies that $[h] \geq [f \vee g]$. Hence, $[f] \vee [g] = [f \vee g]$.

To show that E/I contains infima, note that $f + g = f \vee g + f \wedge g$. Thus $[f \wedge g] = [f] + [g] - [f] \vee [g] = [f] \wedge [g]$. Therefore E/I is a vector lattice. $\square$

The *sum of two subsets* F *and* G of E is defined by

$$F + G = \{f_1 + f_2 : f_1 \in F, f_2 \in G\}.$$

LEMMA 3.10. *If F and G are ideals in E, then $F + G$ is also an ideal in E.*

PROOF. To show this, let $f \in F + G$, and suppose $|g| \leq |f|$. Since $f = f_1 + f_2$ where $f_1 \in F$ and $f_2 \in G$, we have $g^+ \leq |g| \leq |f| \leq |f_1| + |f_2|$. Moreover, g^+ can be decomposed into $g^+ = g_1 + g_2$ where $0 \le g_1 \le |f_1|$ and $0 \le g_2 \le |f_2|$ by Theorem 3.5. Since F and G are ideals, it follows that $g_1 \in F$ and $g_2 \in G$. Therefore, $g^+ \in F + G$. Similarly $g^- \in F + G$, and $g = g^+ - g^- \in F + G$. $\square$

It, however, is not true that the sum of two bands is a band. For a counterexample, consider the vector lattice $E = C([-1,1])$, the continuous functions on $[-1,1]$, and let the bands D_1 and D_2 be defined by

$$D_1 = \{f \in E : f = 0 \text{ on } [0,1]\}, \text{ and}$$

$$D_2 = \{f \in E : f = 0 \text{ on } [-1,0]\}.$$

The sum of D_1 and D_2 consists of all $f \in E$ for which $f(0) = 0$. The sum $D_1 + D_2$ is an ideal, but not a band, in E. Indeed, the band generated by $D_1 + D_2$ is the entire space E (see [**11**], page 31).

The *disjoint complement* of a subset F of E is defined by

$$F^d = \{f \in E : f \perp g \text{ for all } g \in F\},$$

where $f \perp g$ if and only if $|f| \wedge |g| = 0$

LEMMA 3.11. *If F is a subset of E, then F^d is a band in E.*

PROOF. Let H be a subset of F^d with $\sup H = f_0$, and let $g \in E$ be such that $|g| \wedge |f| = 0$ for every element f of H. It follows that

$$f_0^+ \wedge |g| = \sup(f^+ \wedge |g| : f \in H) = 0.$$

Moreover, $f_0^- \wedge |g| \le f^- \wedge |g| = 0$ for every $f \in H$. Therefore $f_0^- \wedge |g| = 0$, and

$$|f_0| \wedge |g| = (f_0^+ + f_0^-) \wedge |g| = (f_0^+ \wedge |g|) + (f_0^- \wedge |g|) = 0.$$

Hence, if $f \perp g$ for all $f \in H$, then $f_0 = \sup H$ is also disjoint with g. Since every f in H is disjoint with F, the supremum f_0 is in F^d. Therefore F^d is a band. $\square$

LEMMA 3.12. *If I is an ideal in E, then $I^d = [I]^d$.*

PROOF. Note that if $F \subset G \subset E$, then $G^d \subset F^d$. Therefore $[I]^d \subset I^d$. For the reverse inclusion, let $f \in I^d$ and $g \in [I]$. Since $|g| = \sup I_0$ for a subset I_0 of I^+, it follows that $|g| \wedge |f| = 0$. Therefore $f \perp g$, and $I^d \subset [I]^d$. □

LEMMA 3.13. *If F and G are Riesz subspaces of E, then $(F+G)^d = F^d \cap G^d$.*

PROOF. Since $F \subset F + G$ and $G \subset F + G$, it follows that $(F+G)^d \subset F^d \cap G^d$. Conversely, suppose $f \in F^d \cap G^d$, and take $g = g_1 + g_2 \in F + G$ such that $g_1 \in F$ and $g_2 \in G$. Then

$$|f| \wedge |g| \leq |f| \wedge (|g_1| + |g_2|) \leq |f| \wedge |g_1| + |f| \wedge |g_2| = 0.$$

Therefore $f \in (F+G)^d$. □

LEMMA 3.14. *If E is Archimedean, then $D = D^{dd}$ for all bands D in E.*

PROOF. Certainly $D \subset D^{dd}$. Suppose there exists an element g' in D^{dd} but not in D. If $g = |g'|$, then $0 < g \in D^{dd} \backslash D$. Let $M_g = \{h \in D : 0 < h < g\}$. It is clear the g is an upper bound of M_g, but it cannot be the supremum because $M_g \subset D$ and $g \notin D$. Let f' be an upper bound of M_g such that $g \leq f'$ does not hold. The element $f = g \wedge f'$ is then an upper bound of M_g such that $0 < f < g$. Thus $f \in D^{dd}$, and $0 < g - f \in D^{dd}$.

We claim that there exists an element $j \in D$ such that $0 < j < g - f$, for if not, then $|g - f| \wedge |h| = 0$ for all $h \in D$, and $g - f \in D^d \cap D^{dd} = \{0\}$. This is impossible because $g - f > 0$. Hence for every $h \in M_g$, we have $j + h \in D$ and $0 < j + h \leq j + f < g$, and therefore $j + h \in M_g$. In particular if $h = j$, then it follows that $2j \in M_g$. By induction, $nj \in M_g$ for $n = 1, 2, \ldots$, that is, $0 < nj < g$ for $n = 1, 2, \ldots$. This cannot occur in an Archimedean vector lattice. Therefore there exists no element in $D^{dd} \backslash D$. We conclude that $D = D^{dd}$ for every band of E (see **[11]**, page 41). □

The structure of Boolean algebras is closely related to the structure of vector lattices, and in fact, their relation forms the basis for this thesis. We now exhibit a natural way in which to associate with each Archimedean vector lattice a Boolean algebra (see [**11**], page 46).

THEOREM 3.15. *If E is an Archimedean vector lattice, then $\mathcal{B}(E) = \{D \subset E : D \text{ is a band}\}$ is a Boolean algebra.*

PROOF. We claim that $\mathcal{B}(E)$ is a Boolean algebra with respect to partial ordering under inclusion and operations determined by

$$D_1 \wedge D_2 = D_1 \cap D_2, \quad D_1 \vee D_2 = [D_1 + D_2], \quad D_1' = D^d.$$

The proof comes in three parts.

(i) $\mathcal{B}$ **is a lattice.** Under the partial ordering of inclusion, the largest subset of two sets is the intersection. Let D_1 and D_2 be in $\mathcal{B}$. For a nonempty subset $D \subset D_1 \cap D_2$ with $\sup D = f$, it follows that f is in D_1 and D_2 since they are both bands. Therefore $f \in D_1 \cap D_2$, and $D_1 \cap D_2 \in \mathcal{B}$. Hence, $D_1 \wedge D_2 = D_1 \cap D_2$.

Let $D_1, D_2 \in \mathcal{B}$. Suppose that $D_1 \subset D$ and $D_2 \subset D$ for some $D \in \mathcal{B}$. We show that $[D_1 + D_2] \subset D$ and therefore that $D_1 \vee D_2 = [D_1 + D_2]$. Note that

$$[D_1 + D_2] =^{(1)} [D_1 + D_2]^{dd} =^{(2)} (D_1 + D_2)^{dd} =^{(3)} (D_1^d \cap D_2^d)^d. \tag{3.1}$$

Equality (1) follows from Lemma 3.14. Moreover, (2) follows from Lemma 3.12 and (3) from Lemma 3.13. Then $D^d \subset D_1^d \cap D_2^d$ and thus

$$[D_1 + D_2] = (D_1^d \cap D_2^d)^d \subset D^{dd} = D.$$

Hence, $\mathcal{B}(E)$ is a lattice.

(ii) $\mathcal{B}$ **contains complements.** We claim that for an element D of $\mathcal{B}$, its complement is the disjoint complement D^d. Note that $F \cap F^d = \{0\}$ for any subspace F of E. It follows that

$$D \wedge D^d = D \cap D^d = \{0\}$$

and

$$D \vee D^d = (D^d \cap D^{dd})^d = \{0\}^d = E$$

The latter equation follows from (3.1) with $D_1 = D$ and $D_2 = D^d$. Hence, every element in $\mathcal{B}$ has a complement, indeed D^d.

(iii) $\mathcal{B}$ **is distributive.** Given $D, D_1, D_2 \in \mathcal{B}$, that $\mathcal{B}$ is a lattice implies that

$$(D \wedge D_1) \vee (D \wedge D_2) \leq D \wedge (D_1 \vee D_2).$$

For the reverse inequality let $f \in (D \cap (D_1 + D_2))^+$. Then f can be written as

$$f = f_1 + f_2, \quad f_1 \in D^+, f_2 \in D^+, f_1 \in D_1^+, f_2 \in D_2^+.$$

Thus $f_1 \in D \cap D_1$ and $f_2 \in D \cap D_2$, and $f \in (D \cap D_1) + (D \cap D_2)$. Hence, $D \cap (D_1 + D_2) \subset D \cap D_1 + D \cap D_2$. Then

$$\begin{aligned} D \wedge (D_1 \vee D_2) &= D \cap [D_1 + D_2] \\ &= [D \cap (D_1 + D_2)] \\ &\subset [(D \cap D_1) + (D \cap D_2)] \\ &= (D \wedge D_1) \vee (D \wedge D_2). \end{aligned}$$

Therefore, $\mathcal{B}$ is a distributive lattice and hence a Boolean algebra. □

Next we present a theorem that relates the Boolean algebras of bands of two specific vector lattices (see **[6]**).

THEOREM 3.16. *If F is an order dense Riesz subspace of an Archimedean vector lattice E, then the Boolean algebras $\mathcal{B}(E)$ and $\mathcal{B}(F)$ are isomorphic.*

PROOF. Define $\phi : \mathcal{B}(E) \to \mathcal{B}(F)$ by $\phi(D) = D \cap F$. We first prove that for a band D of $\mathcal{B}(E)$, the intersection $D \cap F$ is a band of F. Take a subset $\{f_\gamma\}$ of $D \cap F$ whose supremum over elements of F exists in F. We show that the supremum is in D as well. By $\sup_F\{f_\gamma\}$ we denote the supremum over F of $\{f_\gamma\}$, and similarly,

$\sup_E\{f_\gamma\}$ is the supremum over the whole space E. Note that $\sup_E\{f_\gamma\} \leq \sup_F\{f_\gamma\}$ because $\sup_F\{f_\gamma\}$ is an upper bound of $\{f_\gamma\}$ in E. Suppose $\sup_E\{f_\gamma\} \neq \sup_F\{f_\gamma\}$. Then $\sup_F\{f_\gamma\} - \sup_E\{f_\gamma\} > 0$, and by the order denseness of F, there exists an element $f \in F$ such that $0 < f \leq \sup_F\{f_\gamma\} - \sup_E\{f_\gamma\}$. It follows that for every element of $\{f_\gamma\}$,

$$f_\gamma \leq \sup_E\{f_\gamma\} \leq \sup_F\{f_\gamma\} - f < \sup_F\{f_\gamma\}.$$

Thus $\sup_F\{f_\gamma\} - f \in F$ is an upper bound of the set $\{f_\gamma\}$ in F that is strictly less than the supremum, which contradicts the definition of supremum. Therefore, $\sup_E\{f_\gamma\} = \sup_F\{f_\gamma\}$. Since D is a band in E, it follows that $\sup_E\{f_\gamma\} \in D$. Hence, $\sup_F\{f_\gamma\} \in D \cap F$, and $D \cap F$ is a band in F.

We show that ϕ is surjective. Let D be an element of $\mathcal{B}(F)$. We claim that $\phi(D^{dd}) = D^{dd} \cap F = D$. Certainly, D^{dd} is a band in E, and thus $D^{dd} \cap F$ is a band in F. Since $D \subset F$ and $D \subset D^{dd}$, it follows that $D \subset D^{dd} \cap F$. To prove the reverse inclusion, we use the property of Archimedean spaces that $D = D^{dd}$ for all bands D. Since D is a band in F but not necessarily E, this property implies the equality $D = D^{dd}_F = \{f \in F : |f| \wedge |g| = 0 \text{ for all } g \in D\}$. Moreover, $D^{dd} \cap F \subset D^{dd}_F = D$. Therefore, $\phi(D^{dd}) = D$, and ϕ is a surjective map.

We prove the injectivity of ϕ. Suppose $\phi(D_1) = \phi(D_2)$. Then $D_1 \cap F = D_2 \cap F$. Suppose $D_1 \neq D_2$. Without loss of generality, we may assume there exists an f in $D_1 \backslash D_2$. Since D_1 is an ideal, $f = f^+ - f^-$, where $f^+, f^- \in D_1^+$. It follows that f^+ or f^- is not in D_2, for otherwise f would be in D_2. We may, therefore, assume that $f \in D_1^+ \backslash D_2$. Note that $f \neq 0$ because $0 \in D_2$. Moreover, $f \notin F$ because $D_1 \cap F = D_2 \cap F$. Therefore, there exists an element $g \in F$ such that $0 < g \leq f$. Since B_1 is an ideal, $g \in D_1 \cap F = D_2 \cap F$. Consider the subset G of all such elements of F: $G = \{g \in F : 0 \leq g \leq f\}$. Certainly, G is a subset of $D_1 \cap F$, which is a band of F. Therefore, the supremum of G, if it exists, is in $D_1 \cap F$. We claim that $\sup G = f$, but $f \notin F$, which provides a contradiction.

To establish that $\sup G = f$, suppose that there exists an upper bound h of G for which it is not true that $h \leq f$. Then $h' = f \wedge h$ is an upper bound of G such that

$h' < f$. Thus there exists an element $g \in F$ such that $0 < g \le f - h' < f$. Therefore, $g \in G$, and $g \le h'$. It follows that $2g = g + g \le h' + (f - h') = f$, which implies that $2g \in G$. By induction, $ng \in G$ for $n = 1, 2, \ldots$, that is, $0 < g \le fn^{-1}$ for all n. However, F is Archimedean, and this contradicts the definition of an Archimedean vector lattice. Hence, $f = \sup G$, and ϕ is injective.

Having established that ϕ is bijective, we show that ϕ is bipositive (that $\phi(D_1) \subset \phi(D_2)$ if and only if $D_1 \subset D_2$). If $D_1 \subset D_2$, then $\phi(D_1) = D_1 \cap F \subset D_2 \cap F = \phi(D_2)$. Suppose that $D_1 \cap F \subset D_2 \cap F$. As established in the proof that ϕ is surjective, $\phi^{-1}(D_1 \cap F) = (D_1 \cap F)^{dd}$ and $\phi^{-1}(D_2 \cap F) = (D_2 \cap F)^{dd}$. It follows that $(D_2 \cap F)^d \subset (D_1 \cap F)^d$ and that $\phi(D_1 \cap F) = (D_1 \cap F)^{dd} \subset (D_2 \cap F)^{dd} = \phi(D_2 \cap F)$.

Therefore, ϕ is an isomorphism between $\mathcal{B}(E)$ and $\mathcal{B}(F)$.

$\square$

4. Representation of Boolean Algebras

The fundamental representation theorem of Boolean algebras was developed by Stone and states that every Boolean algebra is isomorphic to a field of sets (see [**10**]). In Theorem 4.2, the field of sets consists of all the two-valued homomorphisms $x : B \to \{0, 1\}$. We first give a lemma to show that such maps exist (see [**5**], page 188).

LEMMA 4.1. *If $x \neq 0$ is an element of a Boolean algebra B, then there exists a two-valued homomorphism ϕ on B such that $\phi(x) = 1$.*

PROOF. Let x be a nonzero element of B, and let C be the ideal generated by x'. Since $C = (x') = \{y \in B : y \leq x'\}$ and $x' \neq \mathbf{1}$, we infer that $\mathbf{1} \notin C$ and that C is a proper ideal of B. Let C_1 be a maximal ideal containing C. We claim the quotient algebra B/C_1 is isomorphic to the set $\{0, 1\}$. Let $\psi : B \to B/C_1$ be the natural homomorphism that sends y to $[y]$, and let $\theta : B/C_1 \to \{0, 1\}$ be the map defined by $\theta([\mathbf{0}]) = 0$ and $\theta([\mathbf{1}]) = 1$. The composition $\phi = \theta \circ \psi$ is the desired 2-valued homomorphism. Consider

$$\phi(y) = \theta(\psi(y)) = \theta([y]) = \begin{cases} 0 & : y \in C_1 \\ 1 & : y \notin C_1. \end{cases}$$

Since C_1 is maximal and $x' \in C_1$, it follows that $x \notin C_1$ and that $\phi(x) = 1$. $\square$

THEOREM 4.2. *Let B be a Boolean algebra, and let X be the set of 2-valued homomorphisms on B. Then B is isomorphic to a subset of $\mathcal{P}(X)$ via the map*

$$f(x) = \{\phi \in X : \phi(x) = 1\}$$

for all $x \in B$.

PROOF. We first show that f is a homomorphism. If x, y are elements of B, then

$$\begin{aligned} f(x \vee y) &= \{\phi \in X : \phi(x \vee y) = 1\} \\ &= \{\phi \in X : \phi(x) \vee \phi(y) = 1\} \\ &= \{\phi \in X : \phi(x) = 1 \text{ or } \phi(y) = 1\} \\ &= \{\phi \in X : \phi(x) = 1\} \cup \{\phi \in X : \phi(y) = 1\} \\ &= f(x) \vee f(y). \end{aligned}$$

The first equality is the definition of f. The second uses the homomorphic properties of ϕ. The third holds because ϕ can only take the values 0 and 1, while the fourth uses the definition of union. Moreover,

$$\begin{aligned} f(x') &= \{\phi \in X : \phi(x') = 1\} \\ &= \{\phi \in X : \phi(x)' = 1\} \\ &= \{\phi \in X : \phi(x) = 0\} \\ &= \{\phi \in X : \phi(x) = 1\}' \\ &= f(x)'. \end{aligned}$$

To establish that f is one-to-one, we show that the kernel of f contains only $\mathbf{0}$, where the kernel of f is all elements of B that are mapped by f to the empty set. If $x \neq \mathbf{0}$, then there exists a 2-valued homomorphism ϕ such that $\phi(x) = 1$ by the previous lemma. Thus f maps every nonzero element of B onto a nonempty set, and therefore the kernel of f contains only $\mathbf{0}$. Hence, if $f(x) = f(y)$, then $f(x) \wedge f(y)' = f(x \wedge y') = \emptyset$, and $x \wedge y' = \mathbf{0}$. Similarly, $x' \wedge y = \mathbf{0}$, and we conclude that $x = y$.

We verify that $f(B) \subset \mathcal{P}(X)$ is a field of sets, i.e. $f(B)$ contains the empty set and is closed under union and complements. Since there exists no two-valued homomorphism that maps $\mathbf{0}$ to 1, it follows that $f(\mathbf{0}) = \emptyset \in f(B)$. Let $Y, Z \in f(B)$.

Then there exist $y, z \in B$ such that $f(y) = Y$ and $f(z) = Z$. It follows that

$$\begin{aligned} Y \cup Z &= \{\phi \in X : \phi(y) = 1\} \cup \{\phi \in X : \phi(z) = 1\} \\ &= \{\phi \in X : \phi(y) \vee \phi(z) = 1\} \\ &= \{\phi \in X : \phi(y \vee z) = 1\} \\ &= f(y \vee z) \in f(B) \end{aligned}$$

and

$$\begin{aligned} Y' &= \{\phi \in X : \phi(y) = 1\}' \\ &= \{\phi \in X : \phi(y) = 0\} \\ &= \{\phi \in X : \phi(y)' = 0\} \\ &= \{\phi \in X : \phi(y') = 1\} \\ &= f(y') \in f(B). \end{aligned}$$

We conclude that every Boolean algebra B is isomorphic to a field of sets. □

The representation presented in the previous theorem has several variations. Instead of using the subsets of 2-valued homomorphisms on B, one could also use maximal ideals or maximal filters, as in Theorem 4.3.

In addition, every Boolean algebra B is representable as a space of functions rather than sets. For a set X, the set of all functions from X to $\{0, 1\}$, denoted by 2^X, is isomorphic to $\mathcal{P}(X)$ via the map that sends a subset of X to its characteristic function. Recall from our conventions that for a subset Y of X, the characteristic function of Y is defined as

$$1_Y(x) = \begin{cases} 1 & : x \in Y \\ 0 & : x \notin Y \end{cases}$$

for $x \in X$. If $g : \mathcal{P}(X) \to 2^X$ is defined by $g(Y) = 1_Y$ for $Y \in \mathcal{P}(X)$, then g is an isomorphism. To verify this, take $Y, Z \subset X$, and note that

$$g(Y \vee Z) = 1_{Y \cup Z}(x) = \begin{cases} 1 : x \in Y \text{ or } x \in Z \\ 0 : x \notin Y \text{ and } x \notin Z \end{cases} = 1_Y(x) \vee 1_Z(x) = g(Y) \vee g(Z).$$

Moreover, if $g(Y) = g(Z)$, then $1_Y(x) = 1_Z(x)$ for all $x \in X$, which implies that $Y = Z$ and g is one-to-one. A Boolean algebra can be embedded into 2^X through the composition $g \circ f$ of g with the function f from Theorem 4.2.

Another field of sets that may be used in place of the 2-valued homomorphisms is the collection of all closed and open (clopen) subsets of a topological space X, which we denote by $b(X)$. The significant contribution of Stone was the determination of precisely what sets may be used to establish this correspondence. If X is a zerodimensional compact Hausdorff space, then $b(X)$ is a Boolean algebra under the operation of inclusion. If there exists a zerodimensional compact Hausdorff space X such that $b(X)$ and a Boolean algebra B are isomorphic, then we call X the *Stone space* of B. The proof of the following theorem is taken from [6], page 117.

THEOREM 4.3. *Every Boolean algebra has a Stone space, which is unique up to homeomorphism.*

PROOF. Let B be a Boolean algebra. A subset A of B is called a *filter* if

(1) A contains $\mathbf{1}$ but not $\mathbf{0}$ and
(2) $x \wedge y \in A$ if and only if $x \in A$ and $y \in A$.

Define S to be the set of all maximal filters, that is, every filter that is not contained properly in another filter of B.

Suppose $A \in S$ and x is an element of B that is not contained in A. Then, the set $A' = \{y \in B : y \vee x' \in A\}$ contains $\mathbf{1}$ because $\mathbf{1} \vee x' = \mathbf{1} \in A$. Moreover, A' satisfies property (2) of filters because

$$\begin{aligned} y, z \in A' \Leftrightarrow y \vee x' \in A \text{ and } z \vee x' \in A &\Leftrightarrow (y \vee x') \wedge (z \vee x') \in A \\ &\Leftrightarrow (y \wedge z) \vee x' \in A \\ &\Leftrightarrow y \wedge z \in A', \end{aligned}$$

but A' contains x and hence properly contains A. By the maximality of A, it is true that $\mathbf{0} \in A'$ so that A' is not a filter. Thus $\mathbf{0} \vee x' = x' \in A$.

Conversely, if $A \in S$ and $x \in A$, then $x \wedge x' = \mathbf{0} \notin A$. Therefore, if A is a maximal filter, then $x \in A$ if and only if $x' \notin A$. For $x \in B$, define a subset S_x of S by $S_x = \{A \in S : x \in A\}$.

From the second property of filters, we have

$$S_{x \wedge y} = S_x \cap S_y. \tag{4.1}$$

Since $x \in A$ if and only if $x' \notin A$, it follows that

$$S \backslash S_x = S_{x'}, \tag{4.2}$$

and therefore that

$$S_{x \vee y} = S_x \cup S_y. \tag{4.3}$$

Certainly,

$$S_{\mathbf{0}} = \emptyset, \ \ S_{\mathbf{1}} = S. \tag{4.4}$$

From 4.1 and 4.4, it follows that $\{S_x : x \in B\}$ is a base for a topology on S. By 4.2, every S_x is clopen, and therefore S is zerodimensional. If A_1 and A_2 are distinct elements of S, then there exists an element $x \in A_1 \backslash A_2$, which implies that $A_1 \in S_x$ and $A_2 \in S_{x'}$ but $S_x \cap S_{x'} = \emptyset$. Therefore, S is a Hausdorff space.

To prove that S is compact, we show that if $B_1 \subset B$ for which $S = \bigcup\{S_x : x \in B_1\}$, then there exists a finite set $B_2 \subset B_1$ such that $S = \bigcup\{S_x : x \in B_2\}$. We prove the contrapositive. Let $B_1 \subset B$ be such that $S \neq \bigcup\{S_x : x \in B_2\}$ for all finite subsets B_2 of B_1. Define

$$I = \{z \in B : z \geq (\sup B_2)' \text{ for some finite } B_2 \subset B_1\}.$$

It follows that $\mathbf{1} \in I$ and that I satisfies the second property of filters. By 4.3 and the assumption that every B_2 is finite, $S_{\sup B_2} = \bigcup\{S_x : x \in B_2\} \neq S$, which establishes that $\sup B_2 \neq \mathbf{1}$ and $(\sup B_2)' \neq \mathbf{0}$. Therefore, $\mathbf{0} \notin B_2$ for all B_2, and I is a filter. I is contained in a maximal filter $I' \in S$. If $x \in B_1$, then $x \leq \sup B_1$ and $x' \geq (\sup B_1)'$. It follows that $x' \in I \subset I'$ and $I' \notin S_x$. Hence, $S \neq \bigcup\{S_x : x \in B_1\}$, and S is compact.

Having shown that S is a zerodimensional compact Hausdorff space, we prove that $b(S)$ is isomorphic to B. By 4.1 and 4.2, the map $x \mapsto S_x$ is a Boolean homomorphism from B to $b(S)$. To establish bijectivity, first let $U \in b(S)$. Since U is open, there exists a $B_1 \subset B$ such that $U = \bigcup\{S_x : x \in B_1\}$, and we may assume that B_1 is finite because S is compact. Therefore, $\bigcup\{S_x : x \in B_1\} = S_{\sup B_1} = U$, and the map is surjective.

Take $x, y \in B$ such that $x \neq y$. Suppose without loss of generality that $y > x$. Let J be the filter defined by $J = \{z \in B : z \vee y \geq x\}$. Certainly, J is a filter because $\mathbf{1} \vee y = \mathbf{1} \geq x$ while $x < y = \mathbf{0} \vee y$. Moreover,

$$\begin{aligned} z_1, z_2 \in J &\Leftrightarrow z_1 \vee y \geq x \text{ and } z_2 \vee y \geq x \\ &\Leftrightarrow (z_1 \vee y) \wedge (z_2 \vee y) \geq x \wedge x \\ &\Leftrightarrow (z_1 \wedge z_2) \vee y \geq x \\ &\Leftrightarrow z_1 \wedge z_2 \in J. \end{aligned}$$

Then J is contained in a maximal filter J', and it follows that $y' \in J'$ and $x \in J'$. Therefore, $J' \in S_x \cap S_{y'} = S_x \backslash S_y$, which implies that $S_x \neq S_y$. Thus the map is injective.

Hence, S is a Stone space of B. Next we show that any other Stone space of B is homeomorphic to S.

Let T be a Stone space of B, and let $x \mapsto T_x$ be a Boolean isomorphism of B onto $b(T)$. There exists a map $\phi : S \to T$ such that

$$\phi(s) \in T_x \text{ if and only if } s \in S_x \; (s \in S, x \in B).$$

Indeed, ϕ is a homeomorphism of S onto T.

□

DEFINITION 4.4. If the supremum (infimum) of an arbitrary collection of α many elements of a Boolean algebra B exists, then B is said to be *α-complete.*
If the supremum (infimum) of any collection of elements of B exists, then we say B is *complete.*

Recall the Boolean algebra $\mathcal{B}(E)$ of Theorem 3.15. Consider the arbitrary intersection $\bigcap D$ of bands of $\mathcal{B} = \mathcal{B}$ and a subset F of this intersection that has a supremum. Certainly, F is contained in every band D, and we infer that $\sup F \in D$ for each D and therefore that $\sup F \in \bigcap D$. Hence, $\bigcap D \in \mathcal{B}$, and $\mathcal{B}$ is complete.

DEFINITION 4.5. An ideal I of an α-complete Boolean algebra is an α-ideal if the supremum of any collection of at most α elements in I is in I.

DEFINITION 4.6. A homomorphism $\phi : B_1 \to B_2$ is said to be an α-homomorphism if $\bigvee \phi(x_\gamma) = \phi(x)$ for a collection of at most α elements $x_\gamma \in B_1$ with $\bigvee x_\gamma = x$, assuming all such suprema exist.

The following theorem connects the Boolean algebra $b(X)$ with the Boolean algebra of bands $\mathcal{B}(E)$ for an Archimedean vector lattice E.

THEOREM 4.7. *For an Archimedean vector lattice E, $\mathcal{B}(E)$ is Boolean isomorphic to $b(X)$, where X is an extremally disconnected compact Hausdorff space.*

PROOF. The existence of a unique space X was proven in Theorem 4.3. Given that the algebra $\mathcal{B}(E)$ has a Stone space, we will show the space to be extremally disconnected. Since $\mathcal{B}(E)$ is a complete Boolean algebra and $\mathfrak{b}(X)$ is isomorphic to $\mathcal{B}(E)$, it follows that $\mathfrak{b}(X)$ is also complete. Therefore the arbitrary union of clopen subsets of X is clopen. Let U be a regular open set in X. Then U can be written as the union of clopen subsets of the zerodimensional space X, and therefore U is clopen. Hence, all regular open subsets of X are clopen, and X is extremally disconnected. $\square$

The natural question to ask following Stone's representation theorem is, "Is every σ-complete Boolean algebra isomorphic to a σ-complete field of sets?" The answer turns out to be negative. To provide a counterexample, we introduce several new terms.

DEFINITION 4.8. A Boolean algebra B is said to be α-distributive if the sets I and J have cardinality at most α and

$$\bigwedge_{i\in I}\bigvee_{j\in J} x_{i,j} = \bigvee_{\phi\in J^I}\bigwedge_{i\in I} x_{i,\phi(i)}$$

where J^I denotes the set of maps from I into J and each $x_{i,j} \in B$, given that all the infima and suprema exist.

Every σ-complete field of sets is σ-distributive. This statement holds because the set-theoretical union (intersection) of elements x_γ coincides with the Boolean supremum (infimum) of x_γ whenever the union (intersection) of all x_γ belongs to the field of sets (see [**9**], page 68).

A *Borel set* of the real numbers is any set that can be formed from open subsets of $\mathbb{R}$ by using the operations of countable union, countable intersection, and complementation. By definition, the field of Borel sets forms a σ-complete algebra. Let B be the field of all Borel sets of the real numbers, and let I be the ideal consisting of all subsets of B that are at most countable. I is an ideal because any subset of an at most countable set is indeed at most countable. Moreover, I is a σ-ideal since the countable union of at most countable sets is at most countable.

We claim the Boolean algebra B/I is σ-complete but not isomorphic to a σ-complete field of sets. First we show that B/I is σ-complete because both B and I

are σ-complete. Note that if $x, y \in B$, then

$$x - y \in I \Leftrightarrow x \wedge y' \in I \Leftrightarrow [x \wedge y'] = [0] \Leftrightarrow [x] \wedge [y]' = [0] \Leftrightarrow [x] \leq [y].$$

Let $\{x_n\}$ be a collection of at most countably many elements of B, and let $x = \sup\{x_n\}$. For all n, we have $x_n - x = x_n \wedge (\sup\{x_n\})' = 0 \in I$, and therefore $[x_n] \leq [x]$. Take $x_0 \in B$ such that $[x_n] \leq [x_0]$ for all n. It follows that $x_n - x_0 \in I$ and that $x - x_0 = \sup\{x_n - x_0\} \in I$ since I is a σ-ideal. Thus $[x] \leq [x_0]$, and we infer that $[x]$ is the supremum of the collection $\{[x_n]\}$. More generally, if $\{x_n\}$ is a countable collection of elements in a σ-complete Boolean algebra and quotients are taken by a σ-ideal, then

$$\sup_n [x_n] = [\sup_n x_n]. \tag{4.5}$$

Hence, B/I is σ-complete.

The quotient algebra B/I is not σ-distributive (see [9], page 61). Let $B_{n,i}$ be the set of all numbers of the form

$$\sum_{j=1}^{\infty} \frac{a_j + 1}{2^{j+1}}$$

where $a_j = -1$ or 1 for $j \neq n$, and $a_n = i$ $(i = \pm 1, n = 1, 2, \ldots)$. The sets $B_{n,i}$ belong to B because they are unions of finite numbers of closed subintervals of the closed unit interval U. Define $A_{n,i} = [B_{n,i}]$. The following statement establishes a counterexample to the σ-distributivity of B/I:

$$\cap_{n=1}^{\infty} (A_{n,-1} \cup A_{n,1}) = [U] \neq [0] = \cup_{\phi \in \Phi} \cap_{n=1}^{\infty} A_{n,\phi(n)} \tag{4.6}$$

where Φ is the set of maps from the natural numbers to the set $\{-1, 1\}$.

It follows from (4.5) and the equality $\cap_{n=1}^{\infty} B_{n,\phi(n)} = 0$ that

$$\cap_{n=1}^{\infty} A_{n,\phi(n)} = \cap_{n=1}^{\infty} [B_{n,\phi(n)}] = [0].$$

Since the action of taking arbitrary unions does not affect $[0]$, the right-hand side of (4.6) holds.

We conclude that B/I is a σ-complete Boolean algebra that is not σ-distributive and therefore not isomorphic to a σ-complete field of sets.

Since not every σ-complete Boolean algebra can be represented as a σ-complete field of sets, we turn to quotients to create a representation. The following fundamental theorem independently discovered by Loomis and Sikorski in 1947 answers the question of representability for σ-complete Boolean algebras (see [7]).

THEOREM 4.9. *(Loomis-Sikorski Theorem) Every σ-complete Boolean algebra is representable as a σ-complete field of sets modulo a σ-ideal.*

For all higher cardinals, the existence of Boolean algebras which cannot be represented in such a way has been established. In [3], Chang sets out necessary and sufficient conditions for Boolean algebras to be α-representable. In [9], Sikorski expands on this to form an extensive list of necessary and sufficient conditions for Boolean algebras to be α-representable.

DEFINITION 4.10. A Boolean algebra is *α-representable* if it is isomorphic to an α-complete field of sets modulo an α-ideal.

5. Representation of Vector Lattices

An important theorem in the representation of vector lattices, discovered by Maeda and Ogasawara, states that any Archimedean vector lattice can be represented as an order dense subset of a vector lattice of functions with values in the extended real numbers.

Let X be an extremally disconnected space. $C^{\infty}(X)$ denotes the set of all continuous functions $f : X \to \overline{\mathbb{R}}$, such that $f^{-1}(\mathbb{R})^C$ is meagre, that is, where the function takes finite values almost everywhere. If $f, g \in C^{\infty}(X)$, then the subset of X on which f or g takes finite values is also meagre. Outside of this meagre subset, $f + g$ and fg are well-defined, continuous functions. Under pointwise partial ordering, $C^{\infty}(X)$ is a vector lattice. Indeed, $C(X) \subset C^{\infty}(X)$.

THEOREM 5.1. *(Maeda-Ogasawara) If X is the Stone space of $\mathcal{B}(E)$, then there exists a Riesz isomorphism from E to an order dense subspace of $C^{\infty}(X)$.*

Let (x_γ) be a net in a vector lattice E. If $x_{\gamma 1} \geq x_{\gamma 2}$ whenever $\gamma 1 \geq \gamma 2$, then we say (x_γ) is *increasing*, or $x_\gamma \uparrow$. We write $x_\gamma \uparrow x$ if (x_γ) is increasing and the supremum of (x_γ) is equal to $x \in E$

DEFINITION 5.2. Let G be an ideal of a vector lattice E. Then G is an α*-ideal* if for any collection of at most α elements in G whose supremum exists in E, the supremum exists in G.

An ideal is a band if it is an α-ideal for every cardinal α.

DEFINITION 5.3. A vector lattice is called α*-complete* if all of its nonempty subsets of cardinality at most α that are bounded above (below) have a supremum (infimum).

DEFINITION 5.4. Let E and F be vector lattices. A homomorphism $\phi : E \to F$ is called an *α-homomorphism* if

$$f_\gamma \uparrow f \Rightarrow \phi(f_\gamma) \uparrow \phi(f)$$

for a net $(f_\gamma)_{\gamma \in \Gamma}$, where the directed set Γ contains at most α elements.

We present a lemma before the proof of the following theorem (see **[11]**, page 129).

LEMMA 5.5. *Given the following statements, (1)$\Rightarrow$(2)$\Rightarrow$(3).*

(1) If $\{f_n\}$ is an increasing sequence in G^+ that uniformly converges to some f, then $f \in G$.

(2) If $f, g \in E^+$ and $(nf - g)^+ \in G$ for all n, then $f \in G$.

(3) E/G is Archimedean

PROOF. (1)$\Rightarrow$(2) Take $f, g \in E^+$ such that $(nf - g)^+ \in G$ for all n, and assume that (1) holds. Since

$$0 \leq f - (f - n^{-1}g)^+ = |f^+ - (f - n^{-1}g)^+| \leq |f - (f - n^{-1}g)| = n^{-1}g,$$

we infer that the increasing sequence $h_n = (f - n^{-1}g)^+$ converges uniformly to f. Moreover, $h_n \in G$ for all n by assumption. It follows from (1) that $f \in G$.

(2)$\Rightarrow$(3) Assume (2) holds and take $f, g \in E^+$ such that $n[f] \leq [g]$ for all n. Then $[0] = (n[f] - [g])^+ = [(nf - g)^+]$ for all n, and $(nf - g)^+ \in G$. By (2), $f \in G$ so that $[0] = [f]$. Therefore, E/G is Archimedean. □

In fact, the converses are true as well, but the proof is not necessary for the following theorem.

THEOREM 5.6. *Let E be an Archimedean vector lattice and I a σ-ideal in E. Then, E/I is Archimedean.*

PROOF. Suppose I is a σ-ideal in an Archimedean vector lattice E, and take an increasing sequence $\{f_n\}$ in I^+ that converges uniformly to some f (i.e. $g \in E^+$ such that there exists a sequence $\epsilon_n \downarrow 0$ for which $|f - f_n| \leq \epsilon_n g$ for all n). Since E is

Archimedean, $\epsilon_n g \downarrow 0$, and therefore $0 \leq f_n \uparrow f$. Furthermore, $f \in I$ because I is a σ-ideal. By the previous lemma, E/I is Archimedean. $\square$

Since every α-ideal is necessarily a σ-ideal, it follows immediately that quotients by α-ideals are Archimedean vector lattices. The following theorem states that the *natural homomorphism*, which takes f to $[f]$ for all $f \in E$, is actually an α-homomorphism when the ideal used to take quotients is an α-ideal.

THEOREM 5.7. *Let E be a vector lattice, and let I be an α-ideal in E. If $\phi : E \to E/I$ is defined as $\phi(f) = [f]$ for $f \in E$, then ϕ is an α-homomorphism.*

PROOF. To show that ϕ is an α-homomorphism, we prove that

$$f_\gamma \uparrow f \Rightarrow \phi(f_\gamma) \uparrow \phi(f)$$

for a net of cardinality at most α. Take a net (f_γ) such that $f_\gamma \in E$ and $\sup(f_\gamma) = f$.

Since ϕ is a homomorphism, $f \geq f_\gamma$ implies that $\phi(f) \geq \phi(f_\gamma)$, and therefore the net $(\phi(f_\gamma))$ is bounded above by $\phi(f)$. Next, we show that $\phi(g) \geq \phi(f)$ for any upper bound $\phi(g)$ of $(\phi(f_\gamma))$. Equivalently, we may show that 0 is the least upper bound of the difference $(\phi(f) - \phi(f_\gamma))$.

Suppose $\phi(g) \leq \phi(f) - \phi(f_\gamma)$ for all γ. Then $\phi(g - f + f_\gamma) \leq 0$, and $\phi(g - f + f_\gamma)^+ = 0$. It follows that $\phi((g - f + f_\gamma)^+) = 0$ because ϕ is a homomorphism. Therefore, $(g - f + f_\gamma)^+$ is in the kernel of ϕ, which is the α-ideal I. The net $((g - f + f_\gamma)^+)$ increases with supremum equal to g^+. Therefore, $g^+ \in I$, which implies that $0 = \phi(g^+) = \phi(g)^+$ and that $\phi(g) \leq 0$. It follows that 0 is the infimum of $(\phi(f) - \phi(f_\gamma))$ and that $\phi(f_\gamma) \uparrow \phi(f)$. Hence, ϕ is an α-homomorphism. $\square$

We now arrive at a crucial definition of this thesis.

DEFINITION 5.8. A vector lattice is α-representable if it is isomorphic to a function lattice modulo an α-ideal.

As an immediate consequence of this definition, $C(X)$ is α-representable for every X and every cardinal α.

6. α-Representability of E via $\mathcal{B}(E)$

The following definitions generalize the common notions of nowhere dense sets and sets of the first category (see [9], page 85).

DEFINITION 6.1. A set is *α-closed* if it is the intersection of at most α clopen sets. A set is *α-nowhere dense* if it is a subset of a nowhere dense α-closed set. A set is of the *α-category* if it is the union of at most α sets that are α-nowhere dense.

The following lemma of Sikorski (see [9], page 120) plays a subtle but significant role in Theorem 6.4.

LEMMA 6.2. *A Boolean algebra is α-representable if and only if, in its Stone space, no nonempty open set is of the α-category.*

By "α-almost everywhere", we mean "at all points except on a set of α-category."

COROLLARY 6.3. *Let X be the Stone space of an α-representable Boolean algebra, and let $f, g \in C^{\infty}(X)$. If $f = g$ α-almost everywhere, then $f = g$ on X.*

PROOF. We claim that $\{x \in X : f(x) \neq g(x)\} = \emptyset$. Since $f, g \in C^{\infty}(X)$, the difference $f - g$ is a continuous function $X \to \overline{\mathbb{R}}$. By the definition of continuity, the set $\{x \in X : f(x) \neq g(x)\} = \{(f-g)^{-1}(y) : y \in \overline{\mathbb{R}} \backslash \{0\}\}$ is open in X because $\overline{\mathbb{R}} \backslash \{0\}$ is open in $\overline{\mathbb{R}}$. Therefore $\{x \in X : f(x) \neq g(x)\}$ is an open set of α-category. By Lemma 6.2 $\{x \in X : f(x) \neq g(x)\} = \emptyset$, and $f(x) = g(x)$ for all $x \in X$. $\square$

We now present the primary theorem of this thesis.

THEOREM 6.4. *If E is an Archimedean vector lattice such that $\mathcal{B}(E)$ is α-representable, then E is α-representable*

PROOF. Let L be the order dense subset of $C^\infty(X)$ that is isomorphic to E by Theorem 5.1. Let $\mathfrak{L}$ be a set of real-valued functions on X defined by: $f \in \mathfrak{L}$ if there is a $g \in L$ such that the set $\{x \in X : f(x) \neq g(x)\}$ is of α-category.

Step 1: $\mathfrak{L}$ is a vector lattice under pointwise ordering

Take $f_1, f_2 \in \mathfrak{L}$, and let g_1, g_2 be the corresponding functions in L to which f_1 and f_2, respectively, are α-almost everywhere equal. At every point x for which $f_1(x) = g_1(x)$ AND $f_2(x) = g_2(x)$, we have $f_1(x) \vee f_2(x) = g_1(x) \vee g_2(x)$. The remaining points lie within the union of two sets of α-category, which is itself of α-category. Thus the pointwise supremum of f_1 and f_2 equals $g_1 \vee g_2$ except on a set of α-category. Hence, $f_1 \vee f_2 \in \mathfrak{L}$. Similarly, $f_1 \wedge f_2 \in \mathfrak{L}$.

With this in mind, we define a mapping $\phi : \mathfrak{L} \to L$ by $\phi(f) = g_f$, where $f = g_f$ except on a set of α-category.

Step 2: ϕ is well-defined

To show that ϕ is well-defined, suppose $f \in \mathfrak{L}$ such that $\phi(f) = g_1$ and $\phi(f) = g_2$. We show that $g_1 = g_2$. By definition, $f \neq g_1$ on a subset $U \subset X$ of α-category, and $f \neq g_2$ on a similar set V. It is then clear that $g_1 = f = g_2$ everywhere in the complement of $U \cup V$ and possibly at points in $U \cup V$ as well. Equivalently, the set on which $g_1 \neq g_2$ is a subset of $U \cup V$, which as the union of two sets of α-category, once again is of α-category. From Corollary 6.3 it follows that $g_1(x) = g_2(x)$ for all $x \in X$.

Step 3: ϕ is surjective

An element g of $L \subset C^\infty(X)$ only takes the values of $+\infty$ and $-\infty$ on a meagre subset U of X. Since a nowhere dense set is α-nowhere dense, it follows that a meagre set is of the α-category. Therefore g takes real values except on a set of α-category. Hence, if $h(x) = 0$ for $x \in U$ and $h(x) = g(x)$ otherwise, then h is a real-valued function that equals g except on a set of α-category (i.e. $\phi(h) = g$).

Step 4: ϕ is a homomorphism

$\phi(f_1 \vee f_2) = f_1 \vee f_2$ α-almost everywhere. Moreover, $\phi(f_1)$ and $\phi(f_2)$ are α-almost everywhere equal to f_1 and f_2, respectively. Therefore $\phi(f_1) \vee \phi(f_2) = f_1 \vee f_2$ everywhere except on a set of α-category. It follows that $\phi(f_1 \vee f_2) = \phi(f_1) \vee \phi(f_2)$ α-almost everywhere and thus everywhere on X by Corollary 6.3 because $\phi(f_1 \vee f_2)$ and $\phi(f_1) \vee \phi(f_2)$ are both continuous functions on X.

Step 5: $Ker(\phi)$ is an ideal

Since ϕ is a Riesz homomorphism, the kernel of ϕ is an ideal. Indeed, take an element f of $N = Ker(\phi)$. Then f has the same zero set as $|f|$ so that $|f| \in N$. If $|g| \leq |f|$ for some $g \in \mathfrak{L}$, then $|g| = 0$ whenever $|f| = 0$. Therefore $|g| \in N$, which in turn implies that $g \in N$. Since N is an ideal, by Theorem 3.9 the quotient space $\mathfrak{L}/N$ is a vector lattice.

Step 6: L and $\mathfrak{L}/N$ are isomorphic

We define a map $\psi : \mathfrak{L}/N \to L$ by $\psi([f]) = \phi(f)$. The map ψ is well-defined because ϕ is well-defined, for if $\psi([f]) = \phi(f) = g_1$ and $\psi([f]) = \phi(f) = g_2$, then $g_1 = g_2$. We claim that ψ is a homomorphism as well because

$$\psi([f] \vee [g]) = \psi([f \vee g]) = \phi(f \vee g) = \phi(f) \vee \phi(g) = \psi([f]) \vee \psi([g]).$$

The surjectivity of ψ follows from that of ϕ. If $g \in L$, then $g = \phi(f) = \psi([f])$ for some $[f] \in \mathfrak{L}/N$. Moreover, suppose $\psi([f]) = \psi([g])$. Then $\phi(f) = \phi(g)$, and $\phi(f - g) = 0$. Hence, $f - g \in Ker(\phi) = N$, and $[f] = [g]$. We conclude that the map ψ is bijective and thus an isomorphism.

Step 7: N is an α-ideal

Take a net of at most α many positive elements in N that increase with supremum f, i.e. $f_\gamma \uparrow f$ with $f_\gamma \in N$. We need to show that $f \in N$, that is, that f is equal to 0 except on a set of α category. This is equivalent to showing that $\{x : \phi(f(x)) \neq 0\} \neq \emptyset$. Let $\phi(f) = g$, and suppose $[g > 0] \neq \emptyset$, where the notation $[g > 0]$ is shorthand for $\{x : g(x) > 0\}$. There exists an $x \in X$ such that $g(x) > 0$, and there exists an open (and closed, since X is extremally disconnected) set U containing x such that $g(y) > \varepsilon$ for all $y \in U$ and some $\varepsilon > 0$.

Let A be defined as the set of points on which at least one f_γ takes a positive value: $A = \bigcup_\gamma [f_\gamma > 0]$. For each f_γ, the set $[f_\gamma > 0]$ is of α-category because $\phi(f_\gamma) = 0$. Hence, the union of α many such sets is also of the α-category. In addition, let the set B be defined as all the points in X where f and g are not equal, i.e. $B = [f \neq g]$. Since $\phi(f) = g$, it follows that B is also of the α-category. We define a map h by

$$h(x) = \begin{cases} \varepsilon/2 & : x \in A^c \cap B^c \cap U \\ 0 & : x \notin A^c \cap B^c \cap U. \end{cases}$$

Consider the difference $f - h$. As the map h is defined, it is continuous outside of U and α-almost everywhere continuous in U. Thus h is continuous except on a set of α-category. It follows that $h \in \mathfrak{L}$ and that the difference $f - h$ is also in $\mathfrak{L}$. At all points outside of the set $A^c \cap B^c \cap U$, certainly $f_\gamma \leq f - h = f$. If $x \in A^c \cap B^c \cap U$, then $f(x) = g(x)$ because $x \notin B$ and $g(x) > \varepsilon$ because $x \in U$. Moreover, $f_\gamma(x) = 0$ for all $x \in A^c \cap B^c \cap U$ and all γ. Therefore, $f - h = g - h > \varepsilon - \varepsilon/2 = \varepsilon/2 > 0$, and $f_\gamma \leq f - h$ on $A^c \cap B^c \cap U$. Furthermore, $f - h < f$, which contradicts the minimality of f as the supremum of (f_γ). We conclude that $\phi(f) = g = 0$, which implies that $f \in N$ and that N is an α-ideal.

Hence, E is isomorphic to $\mathfrak{L}/N$, which is a function lattice modulo an α-ideal. □

The Loomis-Sikorski Theorem and Theorem 6.4 together give the result of Theorem 3 of [**1**].

COROLLARY 6.5. *If E is an Archimedean vector lattice, then E is σ-representable.*

PROOF. Every Boolean algebra, particularly $\mathcal{B}(E)$, is σ-representable (see [**9**], page 123). Note that a set is of the σ-category precisely when it is meagre. It follows from Lemma 6.2 that a Boolean algebra is σ-representable if and only if no nonempty open set in its Stone space is meagre. Moreover, Stone spaces are compact Hausdorff spaces, and no nonempty open subset of a compact Hausdorff space is meagre (see

[4], Theorem 7.2). We conclude that $\mathcal{B}(E)$ is σ-representable and therefore that E is σ-representable by Theorem 6.4. $\square$

As another corollary, if F is an order dense subspace of an Archimedean vector lattice E and the Boolean algebra $\mathcal{B}(E)$ is α-representable, it follows that $\mathcal{B}(F)$ is α-representable by Theorem 3.16 and therefore that both E and F are α-representable as vector lattices.

7. Questions and a Conjecture

Question 1: Can Theorem 6.4 be proven constructively, that is, without the use of the order dense subspace of $C^\infty(X)$ guaranteed by Maeda and Ogasawara?

Question 2: Does the converse of Theorem 6.4 hold? Is it true that if a vector lattice is α-representable, then the Boolean algebra generated by its bands is α-representable?

CONJECTURE 7.1. *The natural embedding $\mathfrak{L}/N \hookrightarrow \mathbb{R}^X/N$ (where $\mathfrak{L}, N,$ and X are as in Theorem 6.4) is an α-homomorphism.*

If this conjecture holds for α equal to the cardinality of the natural numbers, then we obtain Theorem 3.2 of [**2**].

8. Index of terms

The following table outlines how items are denoted throughout this thesis.

- Sets - X, Y, Z
- Topological spaces - X
- Subsets of topological spaces - U, V, W, Y, Z
- Points of a topological space - x, y, z
- Real numbers - a, b, c
- Boolean algebras - A, B, C
- Elements of a Boolean algebra - x, y, z
- Vector lattices - $E, F, G, \ldots$
- Elements of a vector lattice - $f, g, h, \ldots$
- Maps - ϕ, π, ψ, f, g, h
- Cardinal numbers - α

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