An Explanation and Illustration of the Importance of Infinite Series in Mathematics

Allison Walker

An Explanation and Illustration of the Importance of Infinite Series in Mathematics

by
Allison Walker

A thesis submitted to the faculty of The University of Mississippi in partial fulfillment of the requirements of the McDonnell-Barksdale Honors College.

Oxford
May 2001

Approved by

Eldon Miller
Advisor: Dr. Eldon Miller

Wm. Staton
Reader: Dr. Bill Staton

Frederick E. Laurenzo
Reader: Dr. Fred Laurenzo

It is with great love and gracious heart
that I dedicate this thesis to the memory of
Evelyn White Walker
and
Robert Rex McRaney, Sr.
Each of you saw something special inside of me,
and that thought has never left me in the pursuit
of my dreams.

Acknowledgements

First and foremost, I must thank Dr. Eldon Miller. Without your guidance and support, I would never have had the courage to pursue this. I am sure my occasional lack of enthusiasm and endless questions were a trial, but you never gave up, so I did not either. Because I you, I now know what I am capable of accomplishing.

I also must thank my friends and my "sisters" who took the time to understand what I was going through and sympathize with me. After enduring several computer crashes and sleepless nights, I have made it, but not without you.

And, I would never forgive myself if I did not thank the two people who have helped me the most. Deddy, thank you for reading the rough drafts, and Mama, thank you for teaching me how to use the computer's equation editor. I don't think either of those things God would have listed under the "Duties of a Parent". And thank you both for the encouragement in those "stressful" phone conversations. Without your support I would have quit a long time ago.

ABSTRACT

ALLISON WALKER: An Explanation and Illustration of the Importance of Infinite Series in Mathematics

The purpose of this work is to first, explore exactly what an infinite series is, and second, to explain the importance of infinite series in the everyday calculations of mathematicians, scientists, and engineers. In investigating this topic, I researched the development of infinite series as well as how Taylor and Mclaurin series are derived. I then considered many different problems containing intractable functions. In the consideration of these problems I was able to explore the usefulness of infinite series in solving these intractable problems. The conclusion of this research was the realization that in order to obtain an answer for many of these problems, infinite series have to be employed. Infinite series are not just an alternate way to write an equation, they are useful tools in calculations.

Table of Contents

INTRODUCTION

In a world of Pentium processors, electronic mail, and the information highway, many people turn to computers for answers to some of society's most difficult problems. But even in the twenty-first century, there are some mathematical calculations that the most powerful super computers are unable to complete. This is due to the fact that there are some operations on functions, when expressed in standard terms, that cannot be calculated by a computer. Ironically, several hundred years ago, a scientist discovered an alternate method of writing these functions and performing these operations that stump the machines of the 20^{th} century. Unlike the original notation of the functions, computers can calculate the operations using the alternate notation.

Centuries ago, before the Age of Technology, Sir Isaac Newton introduced the idea of representing functions as the sums of an infinite series (Stewart 655). Although it may seem illogical to represent a function this way, this strategy is useful for integrating functions that are otherwise intractable, for approximating functions by polynomials, and for solving differential equations. The primary goal of this paper is to illustrate usefulness of infinite series in the integration of functions used by mathematicians, scientists, and engineers in everyday calculations.

CHAPTER ONE
Definition of Infinite Series

Before infinite series can be defined, infinite sequences must be explained. A sequence is thought of as a list of infinitely many numbers with a definite order. A sequence is denoted by the expression

$$\{a_1, a_2, a_3, ...\}.$$

The first number in the sequence is referred to as a_1, the second term is a_2, and a_n is the nth term in the sequence. Because this sequence is infinite, it is understood that each a_n has a successor, which is called a_{n+1} (Stewart 598). An infinite series is then defined as the sum of the infinitely many terms of the sequence, and is denoted by

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ... + a_n + ...$$

(Stewart 609).

Section One
Geometric Series:

There are two types of infinite series that this paper will consider; the geometric series is the first series that will be explored. This series is represented by

$$\sum_{n=1}^{\infty} ar^{n-1}\text{, where } a \neq 0.$$

In a geometric series each term is obtained from the term before it; the succeeding term is equal to the proceeding term multiplied by the common ratio r. This fact can be seen in the expansion of the summation

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \ldots.$$

The constant a is the first term, and the second term is determined by multiplying a by r which results in ar. Therefore, the terms that follow in the series increase by the number of r that a is multiplied by. So the expansion of the geometric series is represented by

$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \ldots$$

(Stewart 610-611).

Section Two
Power Series:

The second type of infinite series is referred to as a power series. This type of series is represented by

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \ldots.$$

In this series x is assigned as a variable and the c_n's are referred to as constants (or coefficients) of the series. Generally, a power series centered at a point a is represented by

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$$

(Stewart 643). The main function of power series is to provide an alternate way to represent some of the most important functions that arise in mathematics, physics, and chemistry. Once these important functions are converted into power series, they can be integrated where as this was not possible before (Stewart 648).

CHAPTER 2
Convergence

Before an infinite series can be differentiated or integrated, it is essential to know the convergence or divergence of the series in question. Convergence for a sequence is dependent on the existence of a limit for the sequence; common notation for the limit of a sequence is

$$\lim_{n\to\infty} a_n = L .$$

In general, this notation relates that the value of a_n gets closer and closer to the number L as n increases without bound. The limit for a sequence exists if for every $\varepsilon > 0$ there is a corresponding integer N such that $|a_n - L| < \varepsilon$ or $L-\varepsilon < a_n < L+\varepsilon$ whenever $n > N$. If the limit for a_n exists, then the sequence converges, if not it diverges. The idea of convergence of a sequence can be illustrated by plotting the terms of the sequence on a number line (see Figure One). Any positive number can be chosen for ε. These ε are then put into the inequality $L-\varepsilon < a_n < L+\varepsilon$ with the L that corresponds to the sequence in question. It does not matter how small an interval of $(L-\varepsilon, L+\varepsilon)$ is chosen, there will always be some integer N such that every term of the sequence from $a_{N+1}, a_{N+2}, a_{N+3},\ldots$ must lie in the interval $(L-\varepsilon, L+\varepsilon)$ (Stewart 600).

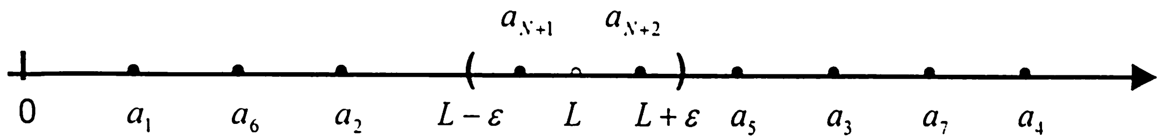

Figure One – Illustration of the Terms of the Sequence on a Number Line. Stewart, James. Calculus, Third Edition. Brooks/Cole Publishing Company, Albany, 1995. (page 600)

To define the convergence of the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots,$$

s_n must first be allowed to be the series' nth partial sum (i.e. s_n is the sum of the first n terms of the series:

$$s_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \dots + a_n .$$

If the sequence $s_1, s_2, s_3, \dots$ is convergent, which means there is some real number s such that

$$\lim_{n\to\infty} s_n = s$$

then the series $\sum_{n=1}^{\infty} a_n$ is convergent and

$$\sum_{n=1}^{\infty} a_n = s .$$

So, the infinite series is equal to s, which is called the sum of the series

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \dots + a_n + \dots = s .$$

This means that by adding sufficiently many terms of the series we can get as close as we like to s. If $\lim_{n\to\infty} s_n$ does not exist, or if $\lim_{n\to\infty} s_n = \pm\infty$, then $\sum_{n=1}^{\infty} a_n$ is said to be divergent (Stewart 610).

Section One
Radius of Convergence:

Now that the convergence of a series of numbers has been explained, it can be said that when considering a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ there is a set of values of x (an interval) for which the series is convergent. For the power series

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

there are only three possibilities for convergence:

(i) The series converges only when $x = a$.

(ii) The series converges for all x.

(iii) There is a positive number R such that the power series converges if $|x-a| < R$. This R is called the Radius of Convergence. In this case the power series diverges for all x such that $|x-a| > R$

(Stewart 645).

It is accepted that the Radius of Convergence for case (i) is zero and for case (ii) it is infinity (Stewart 645). In case (iii) a test of convergence called the Ratio Test, is usually used to find this R. The Ratio Test states that if the

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L \text{, and } 0 \le L < 1,$$

then the series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is absolutely convergent for each x such that

$|x-a| < \frac{1}{L}$. In completing the Ratio test, solving for x gives the Radius of

Convergence, $R = \frac{1}{L}$ (Stewart 636).

Section Two
Interval of Convergence:

There is also an Interval of Convergence for each power series. This interval consists of all the values of x for which the series converges. In case (i) the interval is just the single point a (which is a collapsed interval $[a,a] = \{a\}$). The interval in case (ii) is the infinite interval $(-\infty, \infty)$. The interval becomes more complicated in case (iii); there are four possibilities for the Interval of Convergence. Note that the inequality $|x-a| < R$ can be rewritten as $a - R < x < a + R$. When x is an endpoint of the interval (that is $x = a \pm R$), the series could converge at one or both endpoints or it might diverge at both endpoints. So the four possibilities for the Interval of Convergence are:

$$(a-R, a+R) \qquad (a-R, a+R] \qquad [a-R, a+R) \qquad [a-R, a+R]$$

(A parenthesis refers to divergence at a point, and a bracket refers to convergence). The Ratio Test will always fail when x is an endpoint of the Interval of Convergence, so the endpoints should be checked with another test (Stewart 645).

CHAPTER THREE
Term-by-Term Differentiation

Now that power series and their convergence have been explored, Term-by-Term Differentiation and Integration can be introduced. Converting functions into power series is not useful unless there is a purposeful reason for doing so. The reason in this paper is to integrate or perhaps differentiate functions that are intractable in their original form.

Section One
Derivatives:

First, the derivative and integral of a function requires explanation. The derivative of the function $f(x)$ can be denoted by $f'(x)$ or $\frac{d}{dx}f(x)$. Since we are using power series, the only rules of differentiation needed are:

1) $\frac{d}{dx}(c) = 0$ where c is any constant (Stewart 112)

and for each natural n,

2) $\frac{d}{dx}(x^n) = nx^{n-1}$, and

3) $\frac{d}{dx}(x^{-n}) = -nx^{-n-1}$ (Stewart 112-13).

Several other useful rules in differentiation are:

1) Suppose c is a constant and $f'(x)$ and $g'(x)$ exist, then

    a) If $F(x) = cf(x)$, then $F'(x)$ exists and $F'(x) = cf'(x)$

    b) If $G(x) = f(x) + g(x)$, then $G'(x)$ exists and $G'(x) = f'(x) + g'(x)$

    c) If $H(x) = f(x) - g(x)$, then $H'(x)$ exists and $H'(x) = f'(x) - g'(x)$

(Stewart 114).

2) The Product Rule: If $F(x) = f(x) \times g(x)$ and $f'(x)$ and $g'(x)$ both exist, then $F'(x) = f(x)g'(x) + f'(x)g(x)$ (Stewart 115).

3) The Quotient Rule: If $F(x) = \frac{f(x)}{g(x)}$ and both $f'(x)$ and $g'(x)$ exist, then $F'(x)$ exists and $F'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$ (Stewart 116).

4) The Chain Rule: If $f'(x)$ and $g'(x)$ exist, and $F(x) = f[g(x)]$, then $F'(x)$ exists and $F'(x) = f'[g(x)]g'(x)$ (Stewart 118).

Section Two
Integrals

Now the operation of integration must be considered. For this paper's purposes, we will suppose that $f(x) \geq 0$ over [a,b], so that the integral of f over the interval [a,b] is simply the area between the curve y=f(x) and the x-axis for $a \leq x \leq b$; denoted

$$\int_a^b f(x)dx.$$

For a power series we will need the rule that for n=1,2,3,...,

$$\int_a^b x^n dx = \left.\frac{x^{n+1}}{n+1}\right]_a^b = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

In this case, The Fundamental Theorem of Calculus (part 2), which is denoted as

$$\int_a^b f(x)dx = F(b) - F(a),$$

is used where F is any function such that f'(x)=f(x). Several other useful rules to follow in integration are:

1) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ (c is a constant)

2) $\int_a^b cdx = c(b-a)$ (c is a constant)

3) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$

and of course

4) for n=1,2,3,..., $\int_a^b x^n dx = \frac{b^{n+1} - a^{n+1}}{n+1}$

(Stewart 283-91).

Section Three
Weierstrass M-Test:

Before Term-by-Term Differentiation and Integration can be introduced a concept called the Weierstrass M-Test will be discussed. In Math 556, a famous theorem states that "if $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on the interval I=(c,d) and if c<a<b<d, then $\int_a^b \sum_{n=1}^{\infty} f_n(x)dx = \sum_{n=1}^{\infty}\int_a^b f_n(x)dx$.

The proof of this concept is as follows: [Note: By definition, the series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly on a set of D if and only if the sequence $\{S_n(z)\}_{n=1}^{\infty}$, with $S_n(z) = \sum_{k=1}^{n} f_k(z)$, converges uniformly.] Let $\sum_{n=1}^{\infty} f_n(x)$ be a series which converges uniformly on [a,b]. Then, by definition, the sequence of partial sums $\{S_n(x)\}_{n=1}^{\infty}$, where $S_n(x) = \sum_{k=1}^{n} f_k(x)$, converges uniformly on [a,b]. Hence, we can conclude that

(A) $\int_a^b (\lim_{n\to\infty} S_n(x))dx = \lim_{n\to\infty}\left(\int_a^b S_n(x)dx\right)$ [because $f_n \to f$ implies

$\lim_{n\to\infty}\left(\int_a^b f_n\right) = \int_a^b (\lim_{n\to\infty} f_n)$]

(B) However, $\lim_{n\to\infty} S_n(x)$ is $\sum_{n=1}^{\infty} f_n(x)$, so that the Left Hand Side of (A) is

$\int_a^b \left(\sum_{n=1}^{\infty} f_n(x)\right)dx$, and on the Right Hand Side of (A), we have

$$\int_a^b S_n(x)dx = \int_a^b \left(\sum_{k=1}^{\infty} f_k(x)\right)dx = \int_a^b [f_1(x) + f_2(x) + \ldots + f_k(x)]dx$$

$$= \int_a^b f_1(x)dx + \int_a^b f_2(x)dx + \ldots + \int_q^b f_n(x)dx$$

$$= \sum_{k=1}^{n}\left(\int_a^b f_k(x)dx\right).$$

Hence, the Right Hand Side of (A) becomes

(C) $\lim_{n\to\infty}\left(\int_a^b S_n(x)dx\right) = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\int_a^b f_k(x)dx\right) = \sum_{n=1}^{\infty}\left(\int_a^b f_n(x)dx\right)$.

Equating (B) and (C) gives $\int_a^b\left(\sum_{n=1}^{\infty} f_n(x)\right)dx = \sum_{n=1}^{\infty}\left(\int_a^b f_n(x)dx\right)$.

To use this theorem (which is vital to Term-by-Term differentiation) we must verify that a series converges uniformly on a set S. To do this we generally use the Weierstrass M-test Theorem.

Weierstrass M-Test: If $\{f_n\}_{n=1}^{\infty}$ is a sequence of functions having a common domain 0 such that:

1. for each n∈N there is a $M_n > 0$ such that for all x∈D,

   $|f_n(x)| < M_n$, and

2. $\sum_{n=1}^{\infty} M_n < \infty$,

then the series $\sum_{n=1}^{\infty} f_n(x)$ converges absolutely and uniformly on D.

Proof: Let $\varepsilon > 0$ be given. Let $s_n = \sum_{k=1}^{n} M_k$, so that $\{s_n\}_{n=1}^{\infty}$ is a sequence of numbers. Since $\sum_{k=1}^{n} M_k$ converges, $\{s_n\}_{n=1}^{\infty}$ is a Cauchy sequence, so we can use the given $\varepsilon > 0$ to find an N∈IN such that for all m>n>N we have $|S_m - S_n| = \left|\sum_{k=1}^{m} M_k - \sum_{k=1}^{n} M_k\right| = \sum_{k=n+1}^{m} M_k < \varepsilon$. Now, let $G_n(z) = \sum_{k=1}^{n} f_k(z)$.

We can see that for all m>n>N we have

$$|G_m(z)-G_n(z)| = \left|\sum_{k=1}^{m} f_k(z) - \sum_{k=a}^{n} f_k(z)\right| = \left|\sum_{k=n+1}^{m} f_k(z)\right| \le \sum_{k=n+1}^{m} |f_k(z)| \le \sum_{k=n+1}^{m} M_k < \varepsilon .$$

That is, the sequence $\{G_n(z)\}_{n=1}^{\infty}$ converges absolutely and uniformly on D. This is the definition for $\sum_{n=1}^{\infty} f_n(z)$ to converge absolutely and uniformly on D.

Section Four
Term-by-Term Differentiation and Integration:

Term-by Term Differentiation and Integration is the method that is used to perform these calculations on power series. Term-by-Term Differentiation and Integration states that a power series can be differentiated or integrated term by individual term in the series. The theorem is mathematically defined by stating that if a power series

$$\left(\sum_{n=0}^{\infty} c_n (x-a)^n\right)$$

has a radius of convergence $R > 0$, then the function f defined by

$$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots = \sum_{n=0}^{\infty} c_n (x-a)^n$$

is differentiable, and therefore continuous, on the interval $(a-R, a+R)$ and

(a) $f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \ldots = \sum_{n=1}^{\infty} nc_n (x-a)^{n-1}$

or $\frac{d}{dx}\left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] = \sum_{n=0}^{\infty} \frac{d}{dx}\left[c_n (x-a)^n\right]$

Thus, $f'(x)$ is derived from the individual derivatives of c_0, $c_1(x-a)$, $c_2(x-a)^2, \ldots$, and the terms of the series that follow.

The derivative of the constant c_0 is zero. The derivative of

$$c_1(x-a) = c_1 x - c_1 a$$

is c_1 (This is because $c_1 x$'s derivative is c_1, by Rule 1(a) - see previous page. And since c_1 and a are both constants, the derivative of their product – a constant – is zero). The derivative is taken term by term until a pattern is formed, then a general summation for the derivative is found.

(b) $\int f(x)dx = C + c_0(x-a) + c_1\frac{(x-a)^2}{2} + c_2\frac{(x-a)^3}{3} + \ldots = C + \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1}$

or $\int_a^b \left[\sum_{n=0}^{\infty} c_n (x-a)^n\right] dx = \sum_{n=0}^{\infty} \int_a^b c_n (x-a)^n dx$. (True by the Weierstrass M-Test)

As in the differentiation in (a), the integration in (b) is taken term by term. First, the integral of c_0 is found to be $c_0(x-a)$, this is due to the fact that in this case we are treating the x^n to be $(x-a)^n$. Next the integral of

$$\int c_1(x-a)dx = c_1\frac{(x-a)^2}{2}$$

because of Rule 1. Finally, $c_2(x-a)^2$ is integrated as $c_2\frac{(x-a)^3}{3}$. Now the pattern can be found.

The radius of convergence of the power series is R in (a) and (b), but the Interval of Convergence is not necessarily the same. For finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. The second half of (a) and (b) asserts that the same is true for infinite series as long as the series are power series (Stewart 649-50).

because of Rule 1. Finally, $c_2(x-a)^2$ is integrated as $c_2\frac{(x-a)^3}{3}$. Now the pattern can be found.

The radius of convergence of the power series is R in (a) and (b), but the Interval of Convergence is not necessarily the same. For finite sums, the derivative of a sum is the sum of the derivatives and the integral of a sum is the sum of the integrals. The second half of (a) and (b) asserts that the same is true for infinite series as long as the series are power series (Stewart 649-50).

CHAPTER FOUR
Finding the Coefficients of a Power Series

The question that is now raised is what functions have a power series representation, and how do you find this representation? In order to answer this question, the coefficients, c_n, must be found in terms of the given function f. First, suppose that f is any function that can be represented by a power series and

$f(x) = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$ ($|x-a| < R$). If $x = a$, then

$f(a) = c_0 + c_1(a-a) + c_2(a-a)^2 + \dots = c_0 + (c_1 * 0) + (c_2 * 0) + \dots = c_0$.

Applying Term-by-Term differentiation, which allows $f(x)$ (because it is a power series) to be differentiable, gives

$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \dots$ ($|x-a| < R$). If $x = a$, then

$f'(a) = c_1 + 2c_2(a-a) + 3c_3(a-a) \; + \dots = c_1 + (2c_2 * 0) + (3c_3 * 0) + \dots = c_1$.

Again applying Term-by-Term differentiation gives

$f''(x) = 2c_2 + 2 \cdot 3c_3(x-a) + 3 \cdot 4c_4(x-a)^2 + \dots$ ($|x-a| < R$). If $x = a$, then

$f''(a) = 2c_2 + 6c_3(a-a) + 12c_4(a-a)^2 + \dots = 2c_2 + (6c_3 \cdot 0) + (12c_4 \cdot 0) + \dots = 2c_2$.

Applying Term-by-Term differentiation one last time gives

$f^{(3)}(x) = 2 \cdot 3c_3 + 2 \cdot 3 \cdot 4c_4(x-a) + 3 \cdot 4 \cdot 5c_5(x-a)^2 + \ldots \ (|x-a| < R)$. If $x = a$,

then

$$f^{(3)}(a) = 2 \cdot 3c_3 = 3!c_3 + 4!c_4(a-a) + 5!c_5(a-a) + \ldots = 3!c_3 + (4!c_4 \cdot 0) + (5!c_5 \cdot 0) = 3!c_3$$

[Note: $n! = 1 \cdot \ldots \cdot (n-1) \cdot n$, for example $3! = 1 \cdot 2 \cdot 3$. Also, by definition $0! = 1$.] A pattern can now be seen in regards to the numbers $f^{(n)}(a)$ for the power series. The previous examples found that

$$f^{(0)}(a) = c_0, \ f^{(1)}(a) = 1!c_1, \ f^{(2)}(a) = 1 \cdot 2c_2 = 2!c_2,$$

and

$$f^{(3)}(a) = 1 \cdot 2 \cdot 3c_3 = 3!c_3.$$

From these examples it can be determined that

$$f^{(n)}(a) = n!c_n.$$

The coefficients in terms of f can now be found by solving for the previous equation for c_n. This gives

$$c_n = \frac{f^{(n)}(a)}{n!}$$

(Stewart 653-54).

CHAPTER FIVE
Taylor and Mclaurin Series

Now that the coefficients are defined in terms of f, the new formula for c_n can be substituted into the power series equation

$$\sum_{n=0}^{\infty} c_n (x-a)^n .$$

This substitution gives the equation

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots,$$

and this new infinite series is referred to as the Taylor Series of the function f centered at a. If f has a power series expansion at a, then it must be of the Taylor Series form. Allowing $a = 0$ then gives the infinite series

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots,$$

which is referred to as a Mclaurin series (Stewart 654-55).

Section One
Determining if $f(x)$ is Equal to its Taylor/Mclaurin Series:

Now that two additional types of infinite series, Taylor and Maclaurin series, have been defined, it can now be asked when is $f(x)$ equal to the sum of its Taylor series (as well as its Maclaurin series)? Since the series being dealt with is convergent, it is known that $f(x)$ is equal to the limit of the sequence of partial sums. In the Taylor series, the partial sums are

$$T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n.$$

T_n is a polynomial of degree n called the nth degree Taylor polynomial of f centered at a. In general, $f(x)$ is the sum of its Taylor series if

$$f(x) = \lim_{n\to\infty} T_n(x).$$

Suppose that $R_n(x)$ is the remainder of the series, then

$$R_n(x) = f(x) - T_n(x)$$

and this means that

$$f(x) = T_n(x) + R_n(x).$$

If it can be proved that the

$$\lim_{n\to\infty} R_n(x) = 0,$$

then it can also be seen that the

$$\lim_{n\to\infty} T_n(x) = \lim_{n\to\infty}[f(x) - R_n(x)] = f(x) - \lim_{n\to\infty} R_n(x) = f(x).$$

In order to prove that the $\lim_{n\to\infty} R_n(x) = 0$, Lagrange's form of the remainder term,

$$R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1},$$

is usually used. This formula is used if f has $n+1$ derivatives in an interval I that contains the number a, then for x in I there is a number z strictly between

x and a such that the remainder term in the Taylor series can be expressed as in Lagrange's formula (Stewart 655-57).

CHAPTER SIX
Example of Everyday Use of Infinite Series

Infinite series gained importance in the study of calculus when they were found to aid in the integration of functions that were previously unable to be integrated by standard means. These intractable functions can now be converted into an infinite series, and then each term of the series can be integrated. Without Newton's discovery of representing functions as sums of infinite series, many mathematical calculations would be determined unsolvable, and simple, everyday calculations could not be performed. There are many instances in which an infinite series is used in everyday life; the example this paper explores is finding the normal distribution, which is simply the Bell-Curve, of a high school class's American College Test (ACT) scores. This normal distribution will be beneficial in obtaining the probability that a particular student will achieve a certain ACT score.

The class's ACT scores are called the sample data, and each score is named $x_1, x_2, x_3, \ldots, x_n$ in random order. Using this sample data, the sample's mean is found by dividing the sum of the set of sample data by the number of data in the sample

$(\frac{x_1 + x_2 + x_3 + \ldots + x_n}{n}) = \mu$, with μ representing the mean) (Ostle 35).

Now the sum of squares and the variance can be determined. The sum of squares is equal to

$$SSQ = \sum_{i=1}^{n} (x_i - \mu)^2 \text{ (Ostle 40),}$$

and the variance is referred to as

$$\sigma^2 = \frac{SSQ}{n-1} \text{ (the sum of squares divided by n-1) (Ostle 41).}$$

Finally the standard deviation can be calculated with the equation

$$\sigma = \sqrt{\sigma^2} \text{ (Ostle 41).}$$

Since this is a normal distribution, every x_i must be converted into its standardized value or z-score (z_i). A z-score is obtained by subtracting the mean from x_i and dividing that number by the standard deviation

$$(z_i = \frac{x_i - \sigma}{\sigma}).$$

These calculations create a sample of the standard normal random variables with a mean of zero and a standard deviation of one (Ostle 149).

The probability density function f(z), where

$$f(z) = \frac{1}{\sqrt{2\Pi}} e^{\frac{-z^2}{2}}$$

creates the bell curve in Figure Two. Because this curve has y-axis symmetry, the area under the curve from negative infinity to zero is 0.5

$$(\int_{-\infty}^{0} f(z)dz = 0.5).$$

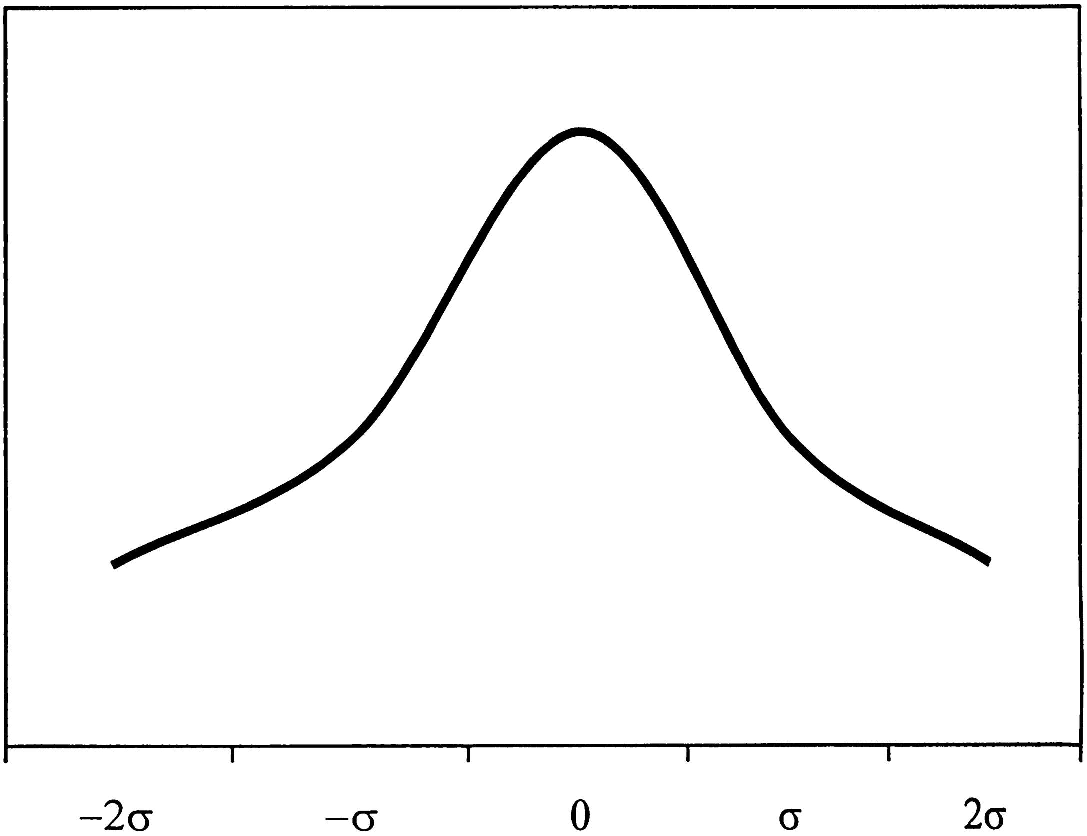

Figure Two – Bell Curve of the Probability Density Fuction.
Adapted from: Engineering Statistics: The Industrial Experience. Ostle, Turner, Hicks, and McElrath. Wadsworth Publishing Company, Belmont, 1996 (page 147)

The probability that z is equal to any α (a certain ACT score that is converted to its standard normal value) is done with the calculation, if $\alpha > 0$

$$P(z = \alpha) = 0.5 + \frac{1}{\sqrt{2\Pi}} \int_0^{\alpha} e^{\frac{-z^2}{2}} dz.$$

This equation requires the integration of $e^{\frac{-x^2}{2}}$ – which is impossible unless infinite series are used. In order to find the infinite series associated with $e^{\frac{-x^2}{2}}$, the function e^x will first be considered (Ostle 146-49).

In this consideration, the Maclaurin series of the function e^x is determined by finding $f^{(n)}(x)$, the nth derivative of f(x), and $f^{(n)}(0)$, which is the nth derivative of f at 0, for all n (see Table One). Substituting this information into the Mclaurin equation, the Mclaurin series for the function e^x is found to be

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots.$$

Once this series for e^x is found, the Mclaurin series for $e^{\frac{-x^2}{2}}$ is derived by substituting $\frac{-x^2}{2}$ for x into the series that was previously derived. This application gives the Mclaurin series for $e^{\frac{-x^2}{2}}$ to be

$$\sum_{n=0}^{\infty} \frac{\left(\frac{-x^2}{2}\right)^n}{n!} = 1 + \frac{-x}{2} + \frac{\left(\frac{-x^2}{2}\right)^2}{2!} + \frac{\left(\frac{-x^2}{2}\right)^3}{3!} + \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n n!}.$$

The Radius of Convergence must be found by applying the Ratio Test.

Table One – The nth Derivative of e^x and e^0 for Several n.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^x	1
1	e^x	1
2	e^x	1
3	e^x	1
4	e^x	1
...	...	...
...	...	...

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{\left(\frac{-x}{2}\right)^{n+1}}{(n+1)!}\frac{n!}{\left(\frac{-x}{2}\right)^n}\right| = \frac{|x|}{n+1} \to 0 = L$$

So, by the Ratio Test, the series converges for all x. The Radius of Convergence is ∞ because the inequality derived through the Ratio Test is 0<1 for all x. The Interval of Convergence is then (-∞, ∞) (Stewart 655-56).

Before integration can take place, the Weierstrass M-Test is applied to assure uniform convergence of the series $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n * n!}$ on the interval I = $[0, \alpha]$.

Let $g_n(x) = \frac{(-1)^n x^{2n}}{2^n * n!}$. Then $|g_n(x)| = \left|\frac{(-1)^n x^{2n}}{2^n * n!}\right| = \frac{|x|^{2n}}{2^n * n!} \leq \frac{\alpha^{2n}}{2^n * n!} = M_n$. So

$\sum M_n = \sum \frac{\alpha^{2n}}{2^n * n!}$. Now the limit from n to infinity must be taken of $\frac{M_{n+1}}{M_n}$. So,

$\lim_{n\to\infty}\frac{M_{n+1}}{M_n} = \lim_{n\to\infty}\frac{\alpha^{2n+2}}{2^{n+1} * (n+1)!} * \frac{2^n * n!}{\alpha^{2n}} = \frac{\alpha^2}{2}\lim_{n\to\infty}\frac{1}{n+1} = 0$. Therefore, by the M-Test, $\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n * n!}$ converges uniformly on I = $[0,\alpha]$. So

$$\int_0^{\alpha} e^{\frac{-x^2}{2}}\,dx = \int_0^{\alpha}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{2^n * n!}\,dx = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n * n!}\int_0^{\alpha} x^{2n}\,dx.$$

Now that an infinite series for $e^{\frac{-x^2}{2}}$ has been derived and its uniform convergence is known, the integral of the z variable (which is z to the 2nth power) must be taken from zero to α

$$\left(\int_0^{\alpha} z^{2n}dz = \left.\frac{z^{2n+1}}{2n+1}\right]_0^{\alpha} = \frac{\alpha^{2n+1}}{2n+1}\right).$$

This answer is then substituted into the infinite series derived for $e^{\frac{-x^2}{2}}$ – this will yield the probability that alpha is equal to Z:

$$0.5 + \frac{1}{\sqrt{2\Pi}}\int_0^{\alpha} e^{\frac{-x^2}{2}}dx = 0.5 + \frac{1}{\sqrt{2\Pi}}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha^{2n+1})}{(2n+1)2^n n!},$$

which is the probability that a student has an ACT score less than or equal to alpha (Ostle 146-149).

CONCLUSION

It almost seems improbable for a calculation that is this simple to become this complicated. In this Age of Technology, many people take for granted things such as infinite series – things that mathematicians of centuries ago discovered. But, as this paper has proved, many everyday calculations would be impossible without these discoveries; even the all-powerful computer would be left without an answer. As humans, we want to make every problem black and white, cut and dry. But sometimes, the straight and narrow path is not what will lead to the answer; sometimes the road less traveled leads to what we are looking for. "Our minds are finite, and yet even in those circumstances of finitude, we are surrounded by possibilities that are infinite, and the purpose of human life is to grasp as much as we can out of that infinitude." – ALFRED NORTH WHITEHEAD

LIST OF REFERENCES

Engineering Statistics: The Industrial Experience. Ostle, Turner, Hicks, and McElrath. Wadsworth Publishing Company: Belmont, 1996.

Stewart, James. Calculus, Third Edition. Brooks/Cole Publishing Company: Albany, 1995.