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On the Variance in Selberg’s Central Limit Theorem Dissertation

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ON THE VARIANCE IN SELBERG'S CENTRAL LIMIT THEOREM

DISSERTATION

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ABSTRACT

Number theorists have made great progress in understanding the distribution of the prime numbers by studying properties of the Riemann zeta-function. A celebrated and classical result of Selberg is that the real and imaginary parts of the logarithm of Riemann zeta-function are normally distributed on the critical line. Selberg proved this using the method of moments. It is known that any model of the logarithm of the Riemann zeta-function near the critical line requires input from the primes and the zeros of the zeta function. We refine Selberg's calculation for the variance of the real part of the logarithm of the Riemann zeta-function (the second moment) assuming the Riemann Hypothesis (RH) and carefully studying the pair correlation of the zeros. This uses ideas of Montgomery and Goldston and tools from Fourier analysis. Then we consider the distribution of real and imaginary parts of the logarithm of the Riemann zeta-function in short intervals, proving an asymptotic for the mean-square of the differences of shifted values uniformly in bounded intervals. Our results generalize previous work of Fujii and establish a conjecture of Berry for the number variance of the zeros, assuming RH, and a conjecture for the pair correlation of zeta zeros in long ranges.

DEDICATION

I dedicate my life and this work to my God and Savior, Jesus Christ. Thank you for being with me throughout this process. Thank you for your strength to work on this project, your help to see it through to the end, and for sustaining me through it all. “Fear not, for I am with you; be not dismayed, for I am your God; I will strengthen you, I will help you, I will uphold you with my righteous right hand.” ∼ Isaiah 41:10

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1 INTRODUCTION

In this chapter, I introduce some of the results that I have proved in the theory of the Riemann zeta-function.

1.1 Riemann zeta-function

In keeping with standard notation, we will denote a complex number as $s = \sigma + it$ with the *real part* of s denoted as $\mathrm{Re}(s) = \sigma$ and the *imaginary part* of s as $\mathrm{Im}(s) = t$. The prevailing methodology in analytic number theory is to extract statistical information concerning a sequence $\{a_n\}$ using analytic tools such as a power series or a Dirichlet series. Perhaps the most famous of all Dirichlet series is the *Riemann zeta-function*, which is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s},$$

for $\sigma > 1$. By the Fundamental Theorem of Arithmetic, we also write $\zeta(s)$ as the *Euler product*

$$\zeta(s) = \prod_p (1 - p^{-s})^{-1},$$

for $\sigma > 1$. Here and throughout this thesis, we use p to denote a prime number. In his only paper on number theory in 1859, Bernhard Riemann [27] made the pivotal connection between the study of the distribution of the prime numbers and the study of $\zeta(s)$. He proved that $\zeta(s)$ is analytic on $\mathbb{C} \setminus \{1\}$ with a simple pole at $s = 1$, and that it satisfies a *functional equation*, written as

$$\pi^{-\frac{s}{2}}\Gamma(\tfrac{s}{2})\zeta(s) = \pi^{-\frac{(1-s)}{2}}\Gamma(\tfrac{1-s}{2})\zeta(1-s).$$

From the functional equation, Riemann found that the zeros of $\zeta(s)$ for $\sigma < 0$ were simple and coincided with the poles of $\Gamma\left(\frac{s}{2}\right)$, which are located at $s = -2, -4, -6, \ldots$. These are called the *trivial zeros* of $\zeta(s)$ because it is easy to determine their location and multiplicity. He conjectured that $\zeta(s)$ has infinitely many zeros in the strip of the complex plane where $0 \le \sigma \le 1$, now called the *critical strip* [8, pp. 59]. Such zeros are now known as the *non-trivial zeros* of $\zeta(s)$. The commonly used notation for such zeros is $\rho = \beta + i\gamma$. In regard to these zeros, Riemann postulated the following:

Riemann Hypothesis. *All zeros of $\zeta(s)$ in the critical strip are located on the line $\sigma = \frac{1}{2}$.*

Still unsolved, the Riemann Hypothesis (RH) is commonly regarded as one of the most significant open problems in pure mathematics. Riemann [27] was also interested in counting the number of non-trivial zeros of $\zeta(s)$ with $0 < \gamma \le T$, now denoted $N(T)$. We define $N(T)$ formally as

$$N(T) = \#\{\rho = \beta + i\gamma : 0 \le \beta \le 1,\ 0 < \gamma \le T\}.$$

The asymptotic expression that Riemann conjectured for the size of $N(T)$ was later proved by von Mangoldt [8]: if $T \ge 2$ is not the ordinate of a zero, then

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} + S(T) + O(T^{-1}), \tag{1.1}$$

where

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\tfrac{1}{2} + iT\right) = \frac{1}{\pi}\operatorname{Im}\log\zeta(\tfrac{1}{2} + iT). \tag{1.2}$$

Here, the argument is calculated by continuous variation along the line segments joining 2, $2+iT$, and $\frac{1}{2}+iT$ starting with the value 0. This result illustrates the deep connection between the distribution of the zeros of the zeta function and analytic properties of $\log\zeta(s)$. It can be shown that $S(T) \ll \log T$ (see [16, pp.14]). Consequently, for $T \geq 2$, we have

$$N(T) = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T). \tag{1.3}$$

1.2 Selberg's central limit theorem

We now turn our attention to the study of the distribution of $\log\zeta(\frac{1}{2}+it)$. Since this is a complex logarithm, we can write it as

$$\begin{aligned}\log\zeta(\tfrac{1}{2}+it) &= \operatorname{Re}\log\zeta(\tfrac{1}{2}+it) + i\operatorname{Im}\log\zeta(\tfrac{1}{2}+it)\\ &= \log|\zeta(\tfrac{1}{2}+it)| + i\arg\zeta(\tfrac{1}{2}+it)\\ &= \log|\zeta(\tfrac{1}{2}+it)| + i\pi S(t),\end{aligned}$$

where $S(t)$ is defined in (1.2). We know detailed information about the vertical distribution of the zeros of $\zeta(s)$ since $S(T) \ll \log T$. In 1924, Littlewood [18] proved that

$$\int_0^T S(t)\,\mathrm{d}t \ll \log T \quad \text{and} \quad S(T) \ll \frac{\log T}{\log\log T},$$

assuming RH. The order of magnitude of these estimates have never been improved, but the inequalities have been sharpened [4]. In later work [19], assuming RH, Littlewood showed that

$$\int_0^T |S(t)|\,\mathrm{d}t \ll T\log\log T.$$

In a remarkable set of papers, Selberg was able to asymptotically estimate all even moments of $S(t)$ first assuming RH [28] and then later without any conditions [29].

Theorem (Selberg). *Assume RH. If $k \in \mathbb{N}$ and $T \geq 3$, then*

$$\int_0^T S(t)^{2k}\,\mathrm{d}t = \frac{(2k)!}{k!(2\pi)^{2k}} T(\log\log T)^k \left[1 + O\left(\frac{1}{\log\log T}\right)\right]. \tag{1.4}$$

Unconditionally Selberg proved the same main term but a slightly weaker error term. Though he does not explicitly state it, on RH his method gives

$$\int_0^T S(t)^{2k+1}\,\mathrm{d}t \ll T\,(\log\log T)^k$$

for the odd moments of $S(t)$. In other words, Selberg proved the moments of $S(t)$ are Gaussian. In this way, since Gaussian distributions are determined by their moments, Selberg [30] deduces a *central limit theorem* for $S(t)$:

$$\frac{1}{T}\ \mathrm{meas}\left\{T \leq t \leq 2T : \frac{\pi S(t)}{\sqrt{\frac{1}{2}\log\log T}} \in [a,b]\right\} = \frac{1}{\sqrt{\pi}}\int_a^b e^{-x^2/2}\,\mathrm{d}\,x + O\left(\frac{T\log\log\log T}{\sqrt{\log\log T}}\right).$$

This tells us $\pi S(t)$ is normally distributed for $t \in [T, 2T]$ with mean 0 and variance $\frac{1}{2}\log\log T$, when T is large.

Selberg also considered the moments of $\mathrm{Re}\log\zeta(\frac{1}{2}+it) = \log|\zeta(\frac{1}{2}+it)|$. His work was never published, but the details were worked out by Tsang [34] who proved the following result using Selberg's methods:

Theorem (Selberg/Tsang). *Assume RH. If $k \in \mathbb{N}$ and $T \geq 3$, then*

$$\int_0^T \log^{2k}|\zeta(\tfrac{1}{2}+it)|\,\mathrm{d}t = \frac{(2k)!}{k!\,2^{2k}} T(\log\log T)^k \left[1 + O\left(\frac{1}{\log\log T}\right)\right]. \tag{1.5}$$

Similarly, for the odd moments, the Selberg/Tsang method gives

$$\int_0^T \big(\log|\zeta(\tfrac{1}{2}+it)|\big)^{2k+1}\,\mathrm{d}t \ll T\,(\log\log T)^k.$$

A corresponding central limit theorem for $\log|\zeta(\frac{1}{2}+it)|$ follows from the work of Selberg [30] and Tsang:

$$\frac{1}{T}\,\mathrm{meas}\left\{T\le t\le 2T: \frac{\log|\zeta(\frac{1}{2}+it)|}{\sqrt{\frac{1}{2}\log\log T}}\in[a,b]\right\}=\frac{1}{\sqrt{\pi}}\int\limits_a^b e^{-x^2/2}\,\mathrm{d}\,x+O\left(\frac{T(\log\log\log T)^2}{\sqrt{\log\log T}}\right).$$

As with the normal distribution for $\pi S(t)$, this means that $\log|\zeta(\frac{1}{2}+it)|$ is normally distributed for $t\in[T,2T]$ with mean 0 and variance $\frac{1}{2}\log\log T$, when T is large.

1.3 The variance in Selberg's central limit theorem

Any model of $\log\zeta(s)$ near the critical line relies on information from the primes and the zeros of $\zeta(s)$. Selberg arrives at the main term of his formula in (1.4) using information from the primes. The information about the zeros is cleverly contained in his error term. Recall that the *variance* of a distribution is given by its second moment. On RH, when $k=1$ in (1.4), we see that Selberg's result gives

$$\int\limits_0^T S(t)^2\,\mathrm{d}t=\frac{T}{2\pi^2}\log\log T+O(T),$$

as $T\to\infty$ for the variance of $S(t)$. Goldston [12] gave a refined estimate for the variance of $S(t)$ in Selberg's Central Limit Theorem utilizing both the primes and the zeros of $\zeta(s)$ in his representation formula for $\log\zeta(s)$. He does so through methods relying, in part, on Montgomery's work [22] on the pair correlation of the zeros of $\zeta(s)$. In particular, Goldston uses the following function, which was originally introduced by Montgomery in his work on the pair correlation of zeros of $\zeta(s)$:

$$F(\alpha) = F(\alpha, T) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma'), \tag{1.6}$$

for real $\alpha \geq 0$ and $T \geq 2$ with $w(u) = \frac{4}{4+u^2}$. Note that $w(0) = 1$, $w \geq 0$, and w decays to zero rapidly. The weight function $w(u)$ localizes the sum to pairs of zeros that are close to one another. We discuss this function in greater detail in Chapter 3. Using (1.6), Goldston obtains the following result:

Theorem (Goldston, 1987). *Assume RH and let $F(\alpha)$ be defined by* (1.6)*. Then, as $T \to \infty$,*

$$\int_0^T |S(t)|^2 \, \mathrm{d}t = \frac{T}{2\pi^2}\log\log T + \frac{aT}{\pi^2} + o(T),$$

where the constant a is given by

$$a = \frac{1}{2}\left(\gamma_0 + \sum_{m=2}^{\infty}\sum_{p}\left(\frac{1}{m^2} - \frac{1}{m}\right)\frac{1}{p^m} + \int_1^{\infty}\frac{F(\alpha)}{\alpha^2}\mathrm{d}\alpha\right) \tag{1.7}$$

and γ_0 is Euler's constant and the sum over p runs over the primes.

The term with $F(\alpha)$ captures the information from the zeros using Montgomery's pair correlation method. In this way, we see that Goldston's result contains information from both the primes and the zeros in the definition of the constant a.

Analogously, the case $k = 1$ of the Selberg/Tsang result in (1.5) gives

$$\int_0^T \log^2 |\zeta(\tfrac{1}{2} + it)| \, \mathrm{d}t = \frac{T}{2}\log\log T + O(T).$$

Our first theorem is an analogue of Goldston's more precise result for the second moment of the real part of $\log\zeta(\frac{1}{2} + it)$.

Theorem 1.3.1. *Assume RH and let $F(\alpha)$ be defined by* (1.6)*. Then, as $T \to \infty$,*

$$\int_0^T \log^2 |\zeta(\tfrac{1}{2}+it)|\,\mathrm{d}t = \frac{T}{2}\log\log T + aT + o(T), \tag{1.8}$$

where the constant a is defined in (1.7).

It is important to note that the constant a can be estimated using the strong form of Montgomery's pair correlation conjecture [22, pg.183]:

Conjecture (Montgomery). *Assume RH. Then for any fixed M,*

$$F(\alpha) = 1 + o(1)$$

uniformly for $1 \le \alpha \le M$.

Assuming this conjecture, the constant a in Theorem 1.3.1 satisfies

$$a = \frac{1}{2}\left(\gamma_0 + \sum_{m=2}^{\infty}\sum_p \left(\frac{1}{m^2} - \frac{1}{m}\right)\frac{1}{p^m} + 1\right) \approx 0.7005\ldots.$$

Assuming RH, Goldston [12, Thm. 2] made progress towards Montgomery's conjecture by showing that

$$\frac{2}{3} - \epsilon < \int_1^{\infty} \frac{F(\alpha)}{\alpha^2}\mathrm{d}\alpha < 2$$

for all $\epsilon > 0$ and for T sufficiently large. The values of $\frac{2}{3}$ and 2 can be improved slightly using the recent work of [3]. However, the cost of this slight improvement is an exorbitant amount of calculations.

Though the statement of our first main result is very similar to Goldston's theorem, the proofs are considerably different. From the formula for $N(t)$ in (1.1), we see that the function $S(t)$ is bounded near the zeros of $\zeta(\frac{1}{2}+it)$ with a jump discontinuity equal to the multiplicity of the zero if $\zeta(\frac{1}{2}+it) = 0$. On the other hand, $\log|\zeta(\frac{1}{2}+it)|$ is not bounded near the zeros of $\zeta(\frac{1}{2}+it)$, and can be arbitrarily large in the negative direction. These logarithmic singularities do not substantially change the end result, but they do cause considerable difficulty within the proof and lead our proof to differ from Goldson's in a number of ways. Another major difference between our work and Goldston's is that our proof relies on a delicate cancellation of main terms, which we accomplish through the introduction of the function $g(x)$ defined in Chapter 2. Though Goldston's proof does not rely on an analogous cancellation of main terms, remarkably the constants in the second-order terms of the two results end up having the same shape!

1.4 Distribution of $\log|\zeta(\frac{1}{2}+it)|$ in short intervals

Following the work of Selberg, Fujii [10, 11] considered the $2k^{th}$ moments of the difference $S\left(t+\frac{2\pi\delta}{\log T}\right)-S(t)$, with $0<\delta\ll\log T$. For T sufficiently large, he showed that

Theorem (Fujii, 1981). *Let $0<\delta\ll\log T$ and $k\in\mathbb{N}$. Then as $T\to\infty$*

$$\begin{aligned}\int_0^T &\left[S\Big(t+\tfrac{2\pi\delta}{\log T}\Big)-S(t)\right]^{2k}\mathrm{d}t\\ &=\frac{(2k)!}{2\pi^{2k}k!}T(2\log(2+2\pi\delta))^k+O\Big(T(\log(2+2\pi\delta))^{k-\frac{1}{2}}\Big).\end{aligned} \tag{1.9}$$

Fujii's method also gives an upper bound for odd moments [10, pg. 140], which also leads to a Gaussian distribution. These unconditional results only gives an asymptotic formula when $\delta=\delta(T)\to\infty$ as a function of T tending to infinity with $\delta(T)=O(\log T)$. If $\delta\asymp 1$, then the main term is the same size as the error term, and this result does not give an asymptotic formula as both terms on the right-hand side of (1.9) are of size T. In this range of δ, information from the zeros is necessary to understand the main term. However, Selberg and Fujii are employing only information from the primes to produce their main terms, and information from the zeros is included in their error terms. In order for the main term to come only from the primes, we need $\theta = \frac{2\pi\delta}{\log T}$ to be big enough so that p^{it} and $p^{i(t+\theta)}$ act like independent random variables as t varies. This happens only if $\delta = \delta(T) \to \infty$.

In order to prove a main term in (1.9) when $\delta \ll 1$, Fujii [9] applies Goldston's methods [12] to his own work, and, assuming RH, he demonstrates the following:

Theorem (Fujii, 1990). *Assume RH. For* $0 < \delta = o(\log T)$*, as* $T \to \infty$*, we have*

$$\pi^2 \int_0^T \left[S\left(t + \frac{2\pi\delta}{\log T}\right) - S(t)\right]^2 \mathrm{d}t = T\left\{ \int_0^1 \frac{1-\cos(2\pi\delta\alpha)}{\alpha}\,\mathrm{d}\alpha + \int_1^\infty \frac{F(\alpha)[1-\cos(2\pi\delta\alpha)]}{\alpha^2}\mathrm{d}\alpha \right\} + o(T). \tag{1.10}$$

Notice that this uses information from the zeros, in the form of $F(\alpha)$, to give an asymptotic formula in the range $\delta \ll 1$. Also note that Fujii is assuming $\delta = o(\log T)$ in (1.10). Our second result refines Fujii's work by proving an asymptotic formula for any δ with $0 < \delta \ll \log T$, showing that new main terms arise for larger values of δ. To achieve this, we must overcome significant technical challenges as more careful consideration of the error terms is needed. Our result relies on finer information from both the primes and the zeros of $\zeta(s)$. We state our results in terms of the *von Mangoldt* function defined as

$$\Lambda(n) = \begin{cases} \log p, & n = p^k,\ k \geq 1, \\ 0, & \text{otherwise}, \end{cases} \tag{1.11}$$

and a variation of Montgomery's function $F(\alpha)$, which was introduced by Chan [5] in his study of the pair correlation of zeta zeros in longer ranges:

$$F_\delta(\alpha) = F_\delta(\alpha, T) := \frac{2\pi}{T\log T} \sum_{0<\gamma,\gamma'\le T} T^{i\alpha\left(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\right)} w\left(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\right). \qquad (1.12)$$

This is a renormalization of Chan's original definition. With these definitions, we prove the following theorem.

Theorem 1.4.1. *Assume RH. Let* $0 < \delta \ll \log T$. *For* $y \ge 1$, *define*

$$E(y) = \sum_{m\le y} \Lambda(m)^2 - y\log y + y. \qquad (1.13)$$

Then, as $T \to \infty$,

$$\begin{aligned}
&\pi^2 \int_0^T \left[S\left(t+\frac{2\pi\delta}{\log T}\right) - S(t)\right]^2 \mathrm{d}t \\
&\quad = T\left\{\int_0^1 \frac{1-\cos(2\pi\delta\alpha)}{\alpha}\,\mathrm{d}\alpha + \frac{1}{2}\int_1^\infty \frac{2F(\alpha)-F_\delta(\alpha)-F_{-\delta}(\alpha)}{\alpha^2}\,\mathrm{d}\alpha\right\} \\
&\quad\quad + T\,c\Big(\frac{2\pi\delta}{\log T}\Big) + o(T),
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^T \left[\log\left|\zeta\left(\frac{1}{2}+it+i\frac{2\pi\delta}{\log T}\right)\right| - \log\left|\zeta\left(\frac{1}{2}+it\right)\right|\right]^2 \mathrm{d}t \\
&\quad = T\left\{\int_0^1 \frac{1-\cos(2\pi\delta\alpha)}{\alpha}\,\mathrm{d}\alpha + \frac{1}{2}\int_1^\infty \frac{2F(\alpha)-F_\delta(\alpha)-F_{-\delta}(\alpha)}{\alpha^2}\,\mathrm{d}\alpha\right\} \\
&\quad\quad + T\,c\Big(\frac{2\pi\delta}{\log T}\Big) + o(T);
\end{aligned}$$

where

$$c(v) := \int_1^\infty \frac{E(y)}{y^2\log^3 y}\left[-v\log y\,\sin(v\log y) + \sin^2\left(\frac{v\log y}{2}\right)(\log y+2)\right]\mathrm{d}y - \frac{v^2}{2}. \qquad (1.14)$$

From the definition of $E(y)$, we see $E(y) = -y \log y + y$ for $1 \le y < 2$. Also by the prime number theorem, we know $E(y) = O(y/\log^2 y)$ as $y \to \infty$ unconditionally. A stronger estimate for $E(y)$ can be proved assuming RH. These facts, together with the inequality $|\sin x| \le |x|$, imply that $c(v)$ is well-defined, $c(v) \ll v^2$ for small v, and $c(v) \ll 1$ large v.

In comparison to Fujii's work in (1.10), it is important to note that there is new input from the zeros contained in the function F_δ, and new input from the primes contained in the functions $E(y)$ and $c(v)$. Such input is not present within Fujii's work due to his restrictions on δ. In particular, when $\delta = o(\log T)$ as in (1.10), the term $T\, c\left(\frac{2\pi\delta}{\log T}\right)$ is absorbed into the error term and F_δ written in terms of F. When $\delta \asymp \log T$, these represent new main terms. Furthermore when $\delta = o(\log T)$, we will show $F_\delta(\alpha)$ reduces to the analogous term in (1.10) involving $F(\alpha)$. This reduction, while founded on simple principles, is quite subtle and requires another technical but straightforward modification of Montgomery's theorem for $F(\alpha)$ to control some of the error terms (see Section 3.3). When $\delta = O(\log T)$, the first variance in Theorem 1.4.1 should match Fujii's work in (1.9). To see this, we note that the second and third terms on the right-hand side of our result are absorbed into the error bound since the functions c, $F(\alpha)$, and $F_\delta(\alpha)$ are bounded on average. Then it can be shown that the interval involving $\cos(2\pi\delta\alpha)$ matches the main term in (1.9).

We will handle the proofs of Theorems 1.3.1 and 1.4.1 simultaneously by proving a variety of preliminary results for the sums over primes (Chapter 6) and the sums over the zeros of $\zeta(s)$ (Chapter 7). These results involve methods from Fourier analysis, Montgomery's work on the pair correlation of zeta zeros, and classical prime number sum estimates. We give an overview of these concepts in Chapters 2, 3, and 4. As we noted for Theorem 1.3.1, the proofs of the imaginary and real parts of $\log \zeta(1/2 + it)$ in short intervals are analogous in many ways, but the proof for the real part is significantly more difficult. For this reason, we give the details only for the latter. It is important to note that, although we present the main steps of the proofs of Theorem 1.3.1 and Theorem 1.4.1 in parallel, the proof of Theorem 1.3.1 is independent of the proof of Theorem 1.4.1. Additionally, we will use Theorem 1.3.1 to control some of the error terms in some steps for Theorem 1.4.1 (see Lemma 6.2.2). The proofs of Theorems 1.3.1 and 1.4.1 will be submitted in paper that is joint work with Micah B. Milinovich and Oscar E. Quesada-Herrera.

1.5 A conjecture of Berry

The Hilbert-Pólya conjecture states that the imaginary parts of the zeros of $\zeta(s)$ are the eigenvalues of some self-adjoint operator, and this would imply RH. In 1973, with his pair correlation approach, Montgomery [22] conjectured that the zeros of $\zeta(s)$ are distributed as the eigenvalues of a random matrix from the Gaussian unitary ensemble (GUE). Numerical evidence by Odlyzko [26] suggests that this holds for short-range statistics between zeros, such as the distribution of the gap between consecutive zeros $\gamma_{n+1} - \gamma_n$. However, Odlyzko's evidence shows that the GUE model fails for long-range statistics, such as the correlation between zeros that are very far apart. In this case, these long-range statistics are better described in terms of primes, instead of GUE statistics.

Berry (see [1]) proposed a conjectural model for the zeros of $\zeta(s)$, where they are the eigenvalues of a quantum Hamiltonian operator. His model is expected to conform to the behavior of both short-range and long-range statistics, as described above. In 1988, Berry used his model to conjecture an asymptotic formula for

$$\pi^2 \int_0^T \left[S\left(t + \frac{2\pi\delta}{\log T}\right) - S(t) \right]^2 \mathrm{d}t. \tag{1.15}$$

In the *universal regime* of his model, when $\delta = o(\log T)$, his conjectured asymptotic formula for (1.15) matches exactly the variance of the GUE of random matrices.

The *non-universal regime* of his model, when $\delta \asymp \log T$, is no longer described by the predictions of GUE, and incorporates additional input from the primes. See [1, Equations (19) and (21)].

Conjecture 1 (Berry, 1988)**.** *Let $0 < \delta \ll \log T$. Then, as $T \to \infty$, we have:*
(a): *If $\delta = o(\log T)$, then*

$$\pi^2 \int_0^T \Big[S\Big(t + \frac{2\pi\delta}{\log T}\Big) - S(t)\Big]^2 \mathrm{d}t$$
$$= T\left[\log(2\pi\delta) - \mathrm{Ci}(2\pi\delta) - 2\pi\delta\mathrm{Si}(2\pi\delta) + \pi^2\delta - \cos(2\pi\delta) + 1 + \gamma_0\right] + o(T).$$

(b): *If $\delta \asymp \log T$, then*

$$\pi^2 \int_0^T \Big[S\Big(t + \frac{2\pi\delta}{\log T}\Big) - S(t)\Big]^2 \mathrm{d}t$$
$$= T\left[\sum_{n\le T} \frac{\Lambda^2(n)}{n\log^2 n}\left(1 - \cos\left(\frac{2\pi\delta\,\log n}{\log T}\right)\right) + 1\right] + o(T).$$

Here γ_0 is Euler's constant,

$$\mathrm{Si}(x) := \int_0^x \frac{\sin u}{u}\,\mathrm{d}u, \quad \textit{and} \quad \mathrm{Ci}(x) := -\int_x^\infty \frac{\cos u}{u}\,\mathrm{d}u. \tag{1.16}$$

In 1990, Fujii [9] proved an asymptotic formula for (1.15), assuming RH, in the universal regime where $\delta = o(\log T)$. In particular, assuming RH and the Strong Pair Correlation Conjecture, he proves Berry's conjecture in the universal regime (part (a) above). However, Fujii's proof relies on the fact that $\frac{\delta}{\log T} \to 0$ as $T \to \infty$ in numerous places, and it is not obvious that his proof can be modified to establish part (b). Our proof of part (b) involves a more delicate analysis of both primes and zeta zeros not present in Fujii's original work.

We show that our formula in Theorem 1.4.1 reduces to Fujii's in the universal regime, and, assuming RH and a strong version of the Pair Correlation Conjecture due to Chan, our formula implies Berry's conjecture, in both the universal and the non-universal regimes. Although Berry never conjectures the range of δ for which part (b) of Conjecture 1 holds, we verify his conjecture holds in the range $\delta \ll \log T$. Conceivably part (b) continues to hold for δ in a much longer range. We require the following generalization of Montgomery's Pair Correlation Conjecture due to Chan [5, Conjecture 1.1]:

Conjecture 2 (Chan)**.** *For* $|\alpha| \geq 1$ *and* $\delta = o(\log^{\frac{4}{3}} T)$, *we have*

$$F_\delta(\alpha) = e^{-2\pi i \alpha \delta} w\left(\frac{2\pi\delta}{\log T}\right)(1 + o(1)),$$

uniformly for α *in compact intervals.*

Assuming Chan's conjecture, we use Theorem 1.4.1 to prove a new case of Berry's conjecture in Chapter 7.

Theorem 1.5.1. *Assume RH and Conjecture 2. Then, Conjecture 1 holds.*

2 FOURIER ANALYSIS

2.1 Basic properties

Within our results we handle concepts from Fourier analysis (e.g. [21,31]). We briefly review some standard facts. For $j \in L^1$, we define the *Fourier transform* of j to be

$$\widehat{j}(\xi) := \int_{-\infty}^{\infty} j(x) e^{-2\pi i \xi x} \, \mathrm{d}\, x$$

for all $\xi \in \mathbb{R}$. In effect the Fourier transform of a real-valued function j is the continuous analog of a Fourier series. If it turns out that $\widehat{j} \in L^1$, then the original function can be recovered from its Fourier transform using the *Fourier inversion formula*,

$$j(x) = \int_{-\infty}^{\infty} \widehat{j}(\xi) e^{2\pi i x \xi} \, \mathrm{d}\, \xi$$

for all $x \in \mathbb{R}$. We call the two functions $j(x)$ and $\widehat{j}(\xi)$ a *Fourier pair.* Now if $j, k \in L^2$, the *convolution* $j * k$ is defined by

$$(j * k)(x) = \int_{-\infty}^{\infty} j(x - u) k(u) \, \mathrm{d}\, u \tag{2.1}$$

for all $x \in \mathbb{R}$. Recall that within Fourier transforms the convolution $j * k$ behaves like a piecewise product. That is, for all $\xi \in \mathbb{R}$,

$$\widehat{j * k}(\xi) = \widehat{j}(\xi) \widehat{k}(\xi). \tag{2.2}$$

2.2 Functions arising in our work

We now give a brief list, with proofs of relevant properties, of the particular functions and Fourier transforms required in our proofs of Theorem 1.3.1 and Theorem 1.4.1. The function $f(v)$ serves as the test function in the sums over primes, and the function $h(v)$ serves as the test function in the sums over zeros. The function $g(v)$ is used within our proofs in order to elucidate cancellation between terms involving $h(v)$, $\widehat{h}(u)$, and $f(v)$. First, we define $f : [0, 2) \to \infty$ such that

$$f(v) := v \int_0^\infty \frac{\sinh(u(1-v))}{\cosh u} \, \mathrm{d}\, u. \tag{2.3}$$

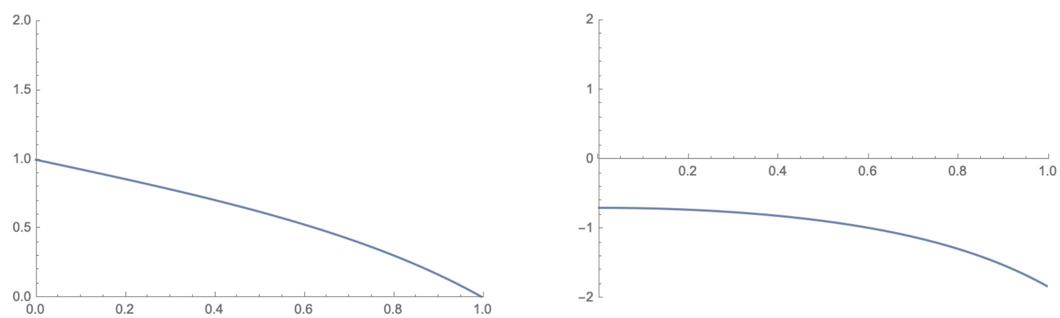

Figure 2.1: Graph of $f(v)$ for $v \in [0, 1]$ on the left, and graph of $f'(v)$ for $v \in [0, 1]$ on the right.

The function $f(v)$ is a smooth, decreasing function on $[0, 1]$. It straightforward to show that $f(0) = 1$, $f(1) = 0$, and $f(v)$ and $f'(v)$ are uniformly bounded for all $v \in [0, 1]$ (see Figure 2.1). This implies $f^2(v)$ is uniformly bounded for $v \in [0, 1]$.

Next define $g : (-2, 2) \to \mathbb{R}$ by

$$g(x) := \int_0^\infty \frac{e^{-y} \cosh(xy)}{\cosh(y)} \mathrm{d}y. \tag{2.4}$$

Observe, by the Dominated Convergence Theorem, that $g \in C^\infty(-2, 2)$. By the

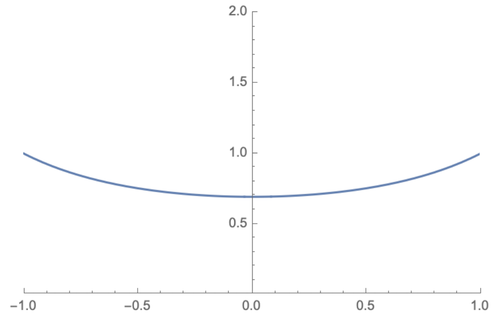

Figure 2.2: Graph of $g(x)$ for $x \in [-1, 1]$.

definition of g, we see that g is even, $g(\pm 1) = 1$, and $0 \le g(x) \le 1$ for all $x \in [-1, 1]$ (see Figure 2.2). This implies $|g(x)| \ll 1$ for all $x \in [0, 1]$. Also by [15, Eq. 3.552-3] we have $g(0) = \log 2$. We now investigate the relationship between $g(x)$ and $f(x)$.

Lemma 2.2.1. *For* $v \in (-2, 2)$*, we have* $g(0^+) = -f'(0^+)$ *and*

$$g(v) = \frac{1 - f(v)}{v}.$$

Proof. Observe that for $v > 0$, we know

$$\frac{1}{v} = \int_0^\infty e^{-vu} \, \mathrm{d}\, u.$$

Then

$$\begin{aligned}
\frac{1 - f(v)}{v} &= \int_0^\infty e^{-vu} \, \mathrm{d}\, u - \frac{f(v)}{v} \\
&= \int_0^\infty \left(e^{-vu} - \frac{\sinh(u(1 - v))}{\cosh u} \right) \mathrm{d}\, u \\
&= \int_0^\infty \left(\frac{e^{-u} \left(e^{vu} + e^{-vu} \right)}{e^u + e^{-u}} \right) \mathrm{d}\, u \\
&= \int_0^\infty \left(\frac{e^{-u} \cosh(vu)}{\cosh u} \right) \mathrm{d}\, u = g(v),
\end{aligned}$$

as claimed. The calculation of $g(0^+)$ follows from the definition of the derivative. $\square$

Now we define $h : \{\mathbb{R}\}\backslash 0 \to \mathbb{R}$ by

$$h(v) := \cos v \int_0^\infty \frac{u}{u^2+v^2} \frac{du}{\cosh u}. \tag{2.5}$$

Note that h is even and $h(v)$ is unbounded in a neighborhood of $v = 0$. However, as the next lemma shows, $h \in L^1(\mathbb{R})$ and the Fourier transform of $h(v)$ can be written in terms of $g(v)$.

Lemma 2.2.2. *Let $h(v)$ be defined as in* (2.5)*. Then $h \in L^1(\mathbb{R})$ and*

$$\widehat{h}(a) = \pi \begin{cases} g(2\pi a), & 0 \le 2\pi a \le 1, \\ \dfrac{1}{2\pi a}, & 2\pi a > 1. \end{cases}$$

Proof. We will first show that $h \in L^1(\mathbb{R})$. By the definition of h, observe that

$$\int_{-\infty}^\infty |h(v)| \,\mathrm{d}\, v \le 2 \int_0^\infty \frac{1}{\cosh u} \int_0^\infty \frac{u}{u^2+v^2} \,\mathrm{d}\, v \,\mathrm{d}\, u = \pi \int_0^\infty \frac{du}{\cosh u} = \pi^2,$$

which implies $h \in L^1(\mathbb{R})$ as claimed. Next we calculate the Fourier transform of $h(v)$ using the well known Fourier pair

$$\varphi(x) = e^{-2\pi|x|} \quad \text{and} \quad \widehat{\varphi}(y) = \frac{1}{\pi} \frac{1}{1+y^2}. \tag{2.6}$$

Let $a \in \mathbb{R}$. Since h is even, we may assume $a \ge 0$. Thus, using the variable change $w = \frac{v}{u}$ and (2.6), it follows that

$$
\begin{aligned}
\widehat{h}(a) &= \int_{-\infty}^{\infty} h(v)e(-av)\,\mathrm{d}\,v \\
&= \int_{0}^{\infty} \frac{u}{\cosh u} \int_{-\infty}^{\infty} \frac{\cos v}{u^2+v^2}\ e^{-2\pi i a v}\,\mathrm{d}\,v\,\mathrm{d}\,u \\
&= \frac{1}{2}\int_{0}^{\infty} \frac{1}{\cosh u} \int_{-\infty}^{\infty} \frac{(e^{u(\frac{1}{2\pi}-a)2\pi i w} + e^{u(-\frac{1}{2\pi}-a)2\pi i w})}{1+w^2}\mathrm{d}w\,\mathrm{d}\,u \\
&= \frac{\pi}{2}\int_{0}^{\infty} \frac{1}{\cosh u}(e^{-u|1-2\pi a|} + e^{-u|1+2\pi a|})\,\mathrm{d}\,u \\
&= \begin{cases} \pi \int_0^\infty \frac{e^{-u}}{\cosh u}\cosh(2\pi a u)\,\mathrm{d}\,u, & 0 \le 2\pi a \le 1, \\ \pi \int_0^\infty \frac{e^{-2\pi a u}}{\cosh u}\cosh(u)\,\mathrm{d}\,u, & 2\pi a > 1, \end{cases} \\
&= \begin{cases} \pi\ g(2\pi a), & 0 \le 2\pi a \le 1, \\ \dfrac{1}{2a}, & 2\pi a > 1, \end{cases}
\end{aligned}
$$

as seen in Figure 2.3. Hence, the proof is complete. □

Utilizing (2.2) and Lemma 2.2.2, we define $k : \mathbb{R} \to \mathbb{R}$ as

$$
k(u) = \frac{\widehat{h * h}(u)}{\pi^2} = \frac{\widehat{h}^2(u)}{\pi^2} = \begin{cases} g^2(2\pi u), & |2\pi u| \le 1, \\ \dfrac{1}{4\pi^2 u^2}, & |2\pi u| > 1. \end{cases} \tag{2.7}
$$

By construction, we note that $k(u)$ is nonnegative and even. Also by (2.4) we see $k(u)$ is increasing for $u \le \frac{1}{2\pi}$, decreasing for $u > \frac{1}{2\pi}$, and

$$
k(0) = g^2(0) = \log^2 2 \tag{2.8}
$$

(see Figure 2.3). Moreover, the order of magnitude of $\widehat{k}(y)$ is straightforward to bound.

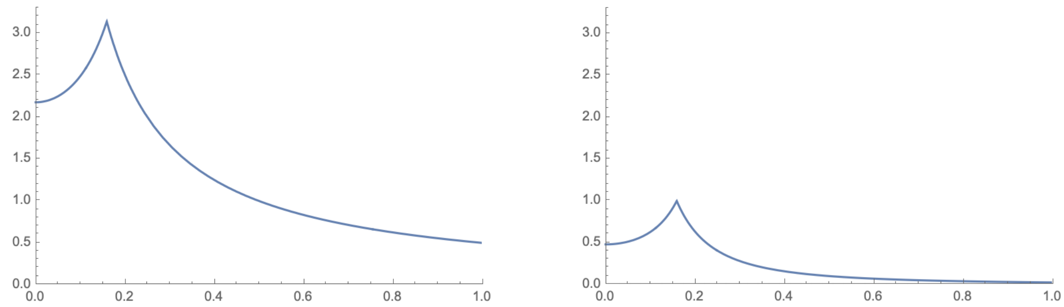

Figure 2.3: Graph of $\widehat{h}(a)$ for $a \in [0,1]$ on the left, and graph of $k(u)$ for $u \in [0,1]$ on the right.

Lemma 2.2.3. *If $k(v)$ is defined as in (2.7), then*

$$\widehat{k}(y) \ll \min\left(1, \frac{1}{y^2}\right).$$

Proof. Note since k is nonnegative and even, we have by the triangle inequality that $|\widehat{k}(y)| \leq \widehat{k}(0)$. This implies

$$|\widehat{k}(y)| \leq \widehat{k}(0) = 2\int_0^\infty k(u)\,\mathrm{d}\,u \leq 2\int_0^{\frac{1}{2\pi}} k(u)\,\mathrm{d}\,u + \frac{1}{2}\int_{\frac{1}{2\pi}}^\infty \frac{1}{u^2}\,\mathrm{d}\,u \leq 2\pi.$$

Also, by definition of Fourier transform, since $k(u)$ is nonnegative and $k'(u)$ is uniformly bounded, we have

$$\begin{aligned} |\widehat{k}(y)| &= \left| \frac{k(u)\sin(2\pi u y)}{\pi y}\Bigg|_0^\infty - \frac{1}{\pi y}\int_0^\infty k'(u)\sin(2\pi u y)\,\mathrm{d}\,u \right| \\ &= \left| \frac{1}{\pi y}\int_0^\infty k'(u)\sin(2\pi u y)\,\mathrm{d}\,u \right| \\ &\ll \frac{1}{y}\left[\frac{|k'\left(\frac{1}{2\pi}+0\right) - k'\left(\frac{1}{2\pi}-0\right)|}{y} + \frac{1}{y}\int_0^\infty |k''(u)|\,\mathrm{d}\,u\right] \\ &\ll \frac{1}{y^2}. \end{aligned}$$

Hence, $\widehat{k}(y) \ll \min\left(1, \frac{1}{y^2}\right)$, which completes the proof. ☐

3 PAIR CORRELATION OF ZETA ZEROS

3.1 Montgomery's work on pair correlation of zeta zeros

In 1973, Montgomery [22] conjectured a result for the *pair correlation* function of the zeros of $\zeta(s)$. That is, he investigated the distribution function of the differences of ordinates, $\gamma - \gamma'$, between non-trivial zeros of $\zeta(s)$. To do this, he introduced the function

$$F(\alpha) = \left(\frac{T}{2\pi}\log T\right)^{-1} \sum_{0<\gamma,\gamma'\leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma - \gamma'),$$

as defined in (1.6), with $w(u) = \frac{4}{4+u^2}$. The function w has the affect of focusing the sum defining F on pairs of nearby zeros. Some of his conclusions on the function $F(\alpha)$ are contained in the following theorem.

Theorem 3.1.1 (Montgomery)**.** *For $\alpha \geq 0$ and $T \geq 2$, let $F(\alpha)$ be defined as in (1.6). Then $F(\alpha)$ is real, even, and nonnegative. Moreover, uniformly for fixed $\alpha \in [0,1]$, we have*

$$F(\alpha) = \alpha + o(1) + T^{-2\alpha}\log T(1 + o(1)),$$

as $T \to \infty$.

Here, the error term is of size $O\left(\sqrt{\frac{\log\log T}{\log T}}\right)$. Montgomery [22] initially proved this result for $\alpha \in (0,1)$ and it was later refined to the above form in Goldston and Montgomery's work [13]. The fact that $F(\alpha) \geq 0$ for all α was proved by Mueller [25].

Montgomery's primary focus was the sums over differences $\gamma - \gamma'$ of the form

$$\sum_{0<\gamma,\gamma' \leq T} \widehat{r}\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) w(\gamma-\gamma').$$

It follows from (1.3) that the average spacing between successive zeros with $\gamma \in (0, T]$ is

$$\frac{(T-0)}{N(T)} \sim \frac{T}{\frac{T\log T}{2\pi}} = \frac{2\pi}{\log T}$$

as $T \to \infty$. We see that Montgomery was interested in studying the "normalized" spacings of zeros, since he divided by the size of the average gap between zeros inside the function $\widehat{r}$. These are exactly the types of sums that arise in the proof of our main theorems.

In our work, we use Montgomery's methods and estimate such sums over zeros using information about $F(\alpha)$ from Theorem 3.1.1. In our calculations, we will write our function $\widehat{r}(\alpha)$ in terms of the convolution of $F(\alpha)$ and $r(\alpha)$ to yield

$$\begin{aligned}
\sum_{0\leq\gamma,\gamma' \leq T} \widehat{r}\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) w(\gamma-\gamma') &= \int_{-\infty}^{\infty} r(\alpha)\left(\sum_{0\leq\gamma,\gamma'\leq T} T^{i\alpha(\gamma-\gamma')} w(\gamma-\gamma')\right) \mathrm{d}\alpha \\
&= \frac{T\log T}{2\pi} \int_{-\infty}^{\infty} r(\alpha)F(\alpha)d\alpha \\
&= \frac{T\log T}{2\pi}\,(F * r)(0). \qquad (3.1)
\end{aligned}$$

Although Montgomery confined his focus to functions $\widehat{r}$ with the condition that $\mathrm{supp}(r) \subseteq [-1, 1]$, we consider $\widehat{r}$ with unbounded support. The utility of (3.1) comes from the fact that we can find asymptotics for the distribution function over the differences $\gamma - \gamma'$ as long as the function has a "well-behaved" kernel. Then for even $r \in L^1$ such that $\widehat{r} \in L^1$, we will use Montgomery's results for $F(\alpha)$ and show that

$$\sum_{0<\gamma,\gamma'\le T} \widehat{r}\left((\gamma-\gamma')\frac{\log T}{2\pi}\right) w(\gamma-\gamma')$$
$$= \frac{T\log T}{2\pi}\left(r(0) + 2\int_0^1 \alpha\, r(\alpha)\mathrm{d}\alpha + 2\int_1^\infty r(\alpha)F(\alpha)\mathrm{d}\alpha + o(1)\right).$$

For the specific choice of $r(u) = k(u)$ given in (2.7), the second integral on the right-hand side appears in the constant a in Theorem 1.3.1.

3.2 A variation of Montgomery's theorem

In 2004, Chan [5] generalized Montgomery's function $F(\alpha)$ by constructing the function $F_\delta(\alpha)$

$$F_\delta(\alpha) = \frac{2\pi}{T\log T}\sum_{0<\gamma,\,\gamma'\le T} T^{i\alpha\left(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\right)} w\left(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\right),$$

as defined in (1.12), with $w(u) = \frac{4}{4+u^2}$. He turned his attention to the study of F_δ in order to better describe the distribution of the gaps between the zeros of $\zeta(s)$ in longer ranges. The properties of the function F_δ are contained in the following theorem.

Theorem 3.2.1 (Chan)**.** *For $\alpha \ge 0$ and $T \ge 2$, let $F(\alpha)$ be defined as in* (1.12)*. Then*

$$\overline{F_\delta(\alpha)} = F_\delta(-\alpha) = F_{-\delta}(\alpha). \tag{3.2}$$

Moreover, uniformly for fixed $\alpha \in [0,1]$, we have

$$F_\delta(\alpha) = \left(T^{-2\alpha}\log T + \alpha\, w\left(\frac{2\pi\delta}{\log T}\right)e^{-2\pi i\alpha\delta}\right)(1+o(1)),$$

as $T \to \infty$.

Notice that the error term above is of size $\sqrt{\frac{\log\log T}{\log T}}$. Chan [5, Theorem 1.1] originally showed this result for the real part of F_δ and for $|\alpha| < 1$, and this can be extended to $|\alpha| \le 1$ using the argument of Goldston and Montgomery [13].

3.3 Pair correlation in longer ranges

In this section, we show how Theorem 1.4.1 simplifies to Fujii's results in (1.10) when $\delta = o(\log T)$. To achieve an extension of Fujii's work for a larger range of δ, we used Chan's function F_δ, as defined in (1.12), in our calculations of the sums over zeta zeros. However, in order to show that our work equates to that of Fujii for $\delta = o(\log T)$, it suffices to show that the term involving $F_\delta(\alpha)$ reduces to the corresponding term in (1.10) involving $F(\alpha)$ in the following lemma. To accomplish this, we apply Montgomery's work on the function $F(\alpha)$ to an analogous function, which we define below.

Lemma 3.3.1. *Let $0 < \delta \ll \log T$. Then as $T \to \infty$,*

$$\frac{1}{2}\int_1^\infty \frac{2F(\alpha) - F_\delta(\alpha) - F_{-\delta}(\alpha)}{\alpha^2}\,\mathrm{d}\alpha = \int_1^\infty \frac{F(\alpha)\,[1-\cos(2\pi\delta\alpha)]}{\alpha^2}\,\mathrm{d}\alpha + O\bigg(\frac{\delta}{\log T}\bigg).$$

Note that the error term is only smaller than the main term if $\delta = o(\log T)$. In order to prove this lemma, we will use Montgomery's methods [22] to prove results analagous to those involving $F(\alpha)$ for the function $\widetilde{F}_{\sigma_0}(\alpha)$, which we define as

$$\widetilde{F}_{\sigma_0}(\alpha) := \frac{2\pi}{T\log T}\sum_{0<\gamma,\gamma'\le T} T^{i\alpha(\gamma-\gamma')} w_{\sigma_0}(\gamma-\gamma'), \tag{3.3}$$

for $\frac{1}{2} < \sigma_0 < \frac{3}{2}$ with $w_{\sigma_0}(u) := \frac{4\sigma_0^2}{4\sigma_0^2+u^2}$. We recover Montgomery's function $F(\alpha)$ by taking $\sigma_0 = 1$. Note that $\widetilde{F}_{\sigma_0}(\alpha)$ is even. Moreover, since

$$\widehat{w_{\sigma_0}}(y) = 2\pi\sigma_0 e^{-4\pi\sigma_0|y|}, \tag{3.4}$$

we have the identity

$$\widetilde{F}_{\sigma_0}(\alpha) = \frac{4\pi^2\sigma_0}{T\log T}\int_{-\infty}^{\infty} e^{-4\pi\sigma_0|y|}\left|\sum_{0<\gamma\leq T} T^{i\alpha\gamma}e^{2\pi y\gamma}\right|^2 \,\mathrm{d}y.$$

In particular, $\widetilde{F}_{\sigma_0}(\alpha) \geq 0$. Following Montgomery, we will prove the following asymptotic formula for $\widetilde{F}_{\sigma_0}(\alpha)$:

Remark 1. *For any small $\varepsilon > 0$, we have*

$$\widetilde{F}_{\sigma_0}(\alpha) = \sigma_0 T^{-2|\alpha|\sigma_0}\log T(1+o(1)) + |\alpha| + o(1),$$

uniformly for $0 \leq |\alpha| \leq 1-\varepsilon$, as $T \to \infty$.

Proof. We essentially follow Montgomery's argument. In Montgomery's explicit formula [22, pg. 185], we take $\sigma = \frac{1}{2} + \sigma_0$ such that $\frac{1}{2} < \sigma_0 < \frac{3}{2}$. Thus, for $x \geq 1$, we have

$$\begin{aligned}2\sigma_0\sum_{\gamma}\frac{x^{i\gamma}}{\sigma_0^2+(t-\gamma)^2} &= -\,x^{-\sigma_0}\sum_{n\leq x}\frac{\Lambda(n)n^{\sigma_0-1/2}}{n^{it}} - x^{\sigma_0}\sum_{n>x}\frac{\Lambda(n)}{n^{1/2+\sigma_0+it}} \\ &\quad + x^{-\sigma_0+it}(\log\tau + O(1)) + O(x^{1/2}\tau^{-1}), \qquad (3.5)\end{aligned}$$

where $\tau = |t|+2$, and the implied constants depend only on σ_0 (which we henceforth assume to be fixed, e.g. we may take $\sigma_0 = \frac{1}{\sqrt{2}}$). We can abbreviate (3.5) by writing $L(x,t) = R(x,t)$. Now we consider $\int_0^T |L(x,t)|^2\,\mathrm{d}t$ and $\int_0^T |R(x,t)|^2\,\mathrm{d}t$. For the left-hand side of (3.5), we have

$$\begin{aligned}\int_0^T |L(x,t)|^2\,\mathrm{d}t &= \int_0^T \left|2\sigma_0\sum_{\gamma}\frac{x^{i\gamma}}{\sigma_0^2+(t-\gamma)^2}\right|^2 \mathrm{d}t \\ &= 4\sigma_0^2\sum_{\gamma,\gamma'} x^{i(\gamma-\gamma')}\int_0^T \frac{\mathrm{d}t}{(\sigma_0^2+(t-\gamma)^2)\,(\sigma_0^2+(t-\gamma')^2)}.\end{aligned}$$

Next we need to the truncate the sum over zeros and then extend the integral to the entire real-axis. An argument of Montgomery [22, pg. 187] can be used mutatis mutandis to show that

$$\int_0^T |L(x,t)|^2 \, \mathrm{d}t = 4\sigma_0^2 \sum_{0<\gamma,\gamma'\le T} x^{i(\gamma-\gamma')} \int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(\sigma_0^2+(t-\gamma)^2)\,(\sigma_0^2+(t-\gamma')^2)} + O(\log^3 T). \tag{3.6}$$

This integral can be evaluated using the calculus of residues, giving

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}t}{(\sigma_0^2+(t-\gamma)^2)\,(\sigma_0^2+(t-\gamma')^2)} = \frac{2\pi}{\sigma_0}\cdot\frac{1}{(\gamma-\gamma')^2+4\sigma_0^2} = \frac{2\pi}{4\sigma_0^3}\, w_{\sigma_0}(\gamma-\gamma').$$

Hence for $x = T^\alpha$, we have

$$\begin{aligned}\int_0^T |L(x,t)|^2 \, \mathrm{d}t &= \frac{2\pi}{\sigma_0} \sum_{0<\gamma,\,\gamma'\le T} x^{i(\gamma-\gamma')} w_{\sigma_0}(\gamma-\gamma') + O(\log^3 T)\\ &= \frac{1}{\sigma_0}\widetilde{F}_{\sigma_0}(\alpha)\, T\log T + O(\log^3 T).\end{aligned} \tag{3.7}$$

Now we evaluate the right-hand side of (3.5). For $x \ge 1$, we have

$$\begin{aligned}\int_0^T |R(x,t)|^2 \, \mathrm{d}t = \int_0^T \Bigg| &-x^{-\sigma_0}\sum_{n\le x}\frac{\Lambda(n)n^{\sigma_0-1/2}}{n^{it}} - x^{\sigma_0}\sum_{n>x}\frac{\Lambda(n)}{n^{1/2+\sigma_0+it}}\\ &+x^{-\sigma_0+it}(\log\tau+O(1)) + O\left(\frac{x^{1/2}}{\tau}\right)\Bigg|^2 \mathrm{d}t.\end{aligned} \tag{3.8}$$

For the mean-square of the third term on the right-hand side above, we have

$$\begin{aligned}\int_0^T \left|x^{-\sigma_0+it}(\log\tau+O(1))\right|^2 \, \mathrm{d}t &= x^{-2\sigma_0}\int_0^T \log^2\tau |x^{it}|^2 \, \mathrm{d}t + O\left(Tx^{-\sqrt{2}}\right)\\ &= Tx^{-2\sigma_0}\left(\log^2 T + O(\log T)\right)\end{aligned}$$

for all $x \geq 1, T \geq 2$. Using Montgomery and Vaughan's Mean Value Theorem in Lemma 6.1.1, we compute the mean square of the Dirichlet series. That is, using

$$\int_0^T \left| \sum_n a_n n^{-it} \right|^2 \mathrm{d}t = \sum_n |a_n|^2 (T + O(n)),$$

we have

$$\begin{aligned} &\int_0^T \left| -x^{-\sigma_0} \sum_{n \leq x} \frac{\Lambda(n) n^{\sigma_0 - 1/2}}{n^{it}} - x^{\sigma_0} \sum_{n > x} \frac{\Lambda(n)}{n^{1/2+\sigma_0+it}} \right|^2 \mathrm{d}t \\ &\qquad = x^{-2\sigma_0} \sum_{n \leq x} \frac{\Lambda(n)^2}{n^{1-2\sigma_0}} (T + O(n)) + x^{2\sigma_0} \sum_{n > x} \frac{\Lambda(n)^2}{n^{1+2\sigma_0}} (T + O(n)). \end{aligned} \tag{3.9}$$

By Lemma 4.1.9, we have

$$\sum_{n \leq x} \frac{\Lambda(n)^2}{n^{1-2\sigma_0}} = \frac{x^{2\sigma_0}(2\sigma_0 \log x - 1) + 1}{4\sigma_0^2} + O\left(x^{2\sigma_0 - 1/2} \log^3 x\right).$$

Using similar logic to that found in Lemma 4.1.9, the second sum on the right-hand side of (3.9) yields

$$\sum_{n > x} \frac{\Lambda(n)^2}{n^{1+2\sigma_0}} = \frac{x^{-2\sigma_0}(2\sigma_0 \log x + 1)}{4\sigma_0^2} + O\left(x^{2\sigma_0 - 1/2} \log^3 x\right).$$

Therefore

$$\begin{aligned} T \left(x^{-2\sigma_0} \sum_{n \leq x} \frac{\Lambda(n)^2}{n^{1-2\sigma_0}} + x^{2\sigma_0} \sum_{n > x} \frac{\Lambda(n)^2}{n^{1+2\sigma_0}} \right) &= T \left(\tfrac{4\sigma_0(\log x + 1)}{4\sigma_0^2} + O(1) \right) \\ &= \frac{T \log x}{\sigma_0} + O(T). \end{aligned}$$

It can be shown using a similar argument that

$$x^{-2\sigma_0}\sum_{n\le x}\frac{\Lambda(n)^2}{n^{-2\sigma_0}}+x^{2\sigma_0}\sum_{n>x}\frac{\Lambda(n)^2}{n^{2\sigma_0}} \ll x\log x.$$

Consequently for $x = T^\alpha$ we have

$$\begin{aligned}\int_0^T |R(T^\alpha,t)|^2\,\mathrm{d}t &= \frac{T\log x}{\sigma_0}+Tx^{-2\sigma_0}\log^2 T\\ &\quad +O\left(Tx^{-2\sigma_0}\log T\right)+O(T)+O(x\log x)\\ &= \frac{\alpha T}{\sigma_0}\log T+T^{1-2\alpha\sigma_0}\log^2 T\\ &\quad +O\left(T^{1-2\alpha\sigma_0}\log T\right)+O(T)+O(T\log T)\\ &= T\log T\left(T^{-2\alpha\sigma_0}\log T(1+o(1))+\frac{\alpha}{\sigma_0}+o(1)\right).\end{aligned}$$

Combining the above result with (3.7) completes the proof. □

Using Remark 3.3.1 and the fact that $\widetilde{F}_{\sigma_0}(\alpha)\ge 0$, an argument of Goldston [12, Lemma A] shows that

$$\int_1^\beta \widetilde{F}_{\sigma_0}(\alpha)\,\mathrm{d}\alpha \ll \beta. \tag{3.10}$$

Finally, we can use this to recover Fujii's result from our estimates.

Proof of Lemma 3.3.1. We know the identity

$$\begin{aligned}&2F(\alpha)-F_\delta(\alpha)-F_{-\delta}(\alpha)\\ &\quad=\frac{8\pi^2}{T\log T}\int_{-\infty}^{\infty} e^{-4\pi|u|}\left[1-\cos\Big(2\pi\delta\alpha+\tfrac{(2\pi)^2}{\log T}u\Big)\right]\left|\sum_{0<\gamma\le T}T^{i\alpha\gamma}e^{2\pi iu\gamma}\right|^2\mathrm{d}u.\end{aligned}$$

By the mean value theorem, $\cos\Big(2\pi\delta\alpha + \frac{(2\pi)^2}{\log T}u\Big) = \cos(2\pi\delta\alpha) + O\big(\frac{\delta\,|u|}{\log T}\big)$. Using the identity for $F(\alpha)$, we obtain

$$2F(\alpha) - F_\delta(\alpha) - F_{-\delta}(\alpha) = F(\alpha)\,[1 - \cos(2\pi\delta\alpha)]$$
$$+ O\Bigg(\frac{\delta}{\log T}\int\limits_{-\infty}^{\infty} e^{-4\pi|u|}|u|\,\Bigg|\sum_{0<\gamma\le T} T^{i\alpha\gamma}e^{2\pi i u\gamma}\Bigg|^2\,\mathrm{d}u\Bigg).$$

Using the estimate $|u| \ll e^{4\pi|u|\varepsilon}$ and (3.4), we find

$$2F(\alpha) - F_\delta(\alpha) - F_{-\delta}(\alpha) = F(\alpha)\,[1 - \cos(2\pi\delta\alpha)] + O\big(\tfrac{\delta}{\log T}\widetilde{F}_{\sigma_0}(\alpha)\big),$$

where $\sigma_0 = 1 - \varepsilon$ (we may take any $0 < \varepsilon < \frac{1}{2}$). Now, (3.10) implies that

$$\int_1^\infty \frac{\widetilde{F}_{\sigma_0}(\alpha)}{\alpha^2}\,\mathrm{d}\alpha \ll 1.$$

Hence, the proof is complete. □

4 PRIME NUMBER SUMS

4.1 Classical results

Let $\pi(x)$ denote the number of primes p less than or equal to x. That is,

$$\pi(x) = \sum_{p \le x} 1. \tag{4.1}$$

The classical Chebyshev prime counting functions, $\theta(x)$ and $\psi(x)$, are defined by

$$\theta(x) = \sum_{p \le x} \log p \quad \text{and} \quad \psi(x) = \sum_{n \le x} \Lambda(n), \tag{4.2}$$

where $\Lambda(n)$ is defined in (1.11). The Prime Number Theorem (PNT), a central result in number theory, can be written as any of the following three equivalent statements as $x \to \infty$:

$$\pi(x) \sim \frac{x}{\log x}, \quad \theta(x) \sim x, \quad \text{and} \quad \psi(x) \sim x.$$

The PNT was first conjectured independently by Legendre and Gauss in the late 1700's and proved independently by de la Vallée Poussin and Hadamard in 1896. For an overview of these topics, see [8, Ch. 7]. Almost all of the prime number sum estimates needed in this thesis follow from the prime number theorem, but some of the results in this chapter follow from weaker results of Chebyshev and Mertens. We have made an effort to indicate when the PNT is necessary for our results.

Important partial progress towards a proof of the PNT was made by Chebyshev and Mertens. For instance, Chebyshev proved the following estimates:

Theorem. (Chebyshev) *For $x \geq 2$, we have*

$$\pi(x) \asymp \frac{x}{\log x}, \quad \theta(x) \asymp x, \text{ and } \psi(x) \asymp x.$$

Other useful bounds for prime number sums were supplied by Mertens. Some of Mertens' prime number sum estimates, commonly referred to as *Mertens' Theorems*, are given below.

Theorem. (Mertens) *For $x \geq 2$, we have*

(a) $\displaystyle\sum_{n \leq x} \frac{\Lambda(n)}{n} = \log x + O(1),$

(b) $\displaystyle\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1),$

(c) $\displaystyle\sum_{p \leq x} \frac{1}{p} = \log\log x + b + O\left(\frac{1}{\log x}\right),$

where γ_0 is Euler's constant and $b = \gamma_0 - \displaystyle\sum_{k=2}^{\infty} \sum_{p} \frac{1}{kp^k}$.

Note that the constant b in Mertens' Theorem appears in the constant in Theorem 1.3.1.

4.1.1 Sums in our results

Using the classical results above, we now introduce the prime number sum estimates that occur in our work. The proofs are included for completeness.

Lemma 4.1.1. *For $x \geq 2$, we have*

$$\sum_{n \leq x} \frac{\Lambda^2(n)}{n} = \frac{\log^2 x}{2} + O(1).$$

Proof. Recall the von Mangoldt function $\Lambda(n)$ is defined as in (1.11). Also, by the PNT with error term, we know $\theta(x) = \sum_{p \leq x} \log p = x + O(\frac{x}{\log^3 x})$. Splitting the sum into primes and primes powers yields

$$\sum_{n\le x}\frac{\Lambda^2(n)}{n} = \sum_{p\le x}\frac{\log^2 p}{p}+\sum_{m=2}^{\infty}\sum_{p^m\le x}\frac{m^2\log^2 p}{p^m}.$$

The second sum on the right-hand side is uniformly bounded for $x \geq 2$. For the first sum on the right-hand side, summing by parts gives

$$\begin{aligned}
\sum_{p\le x}\frac{\log^2(p)}{p} &= \int_{2^-}^{x}\frac{\log u}{u}\mathrm{d}(\theta(u)) \\
&= \theta(u)\frac{\log u}{u}\Bigg|_{2^-}^{x} - \int_2^x \frac{\theta(u)(1-\log u)}{u^2}\,\mathrm{d}\,u \\
&= \log x + O(1) - \int_2^x \frac{(1-\log u)}{u}\,\mathrm{d}u + O\left(\int_2^x \frac{1-\log u}{u\log^3 u}\,\mathrm{d}u\right) \\
&= \frac{\log^2 x}{2} + O(1).
\end{aligned}$$

This proves the lemma. □

Lemma 4.1.2. *For $x \geq 2$, we have*

(a) $$\sum_{2\le n\le x}\frac{\Lambda(n)}{\log^2 n} \ll \frac{x}{\log^2 x},$$

(b) $$\sum_{2\le n\le x}\frac{\Lambda^2(n)}{\log^2 n} \ll \frac{x}{\log x}.$$

Proof. To prove part (a), we use partial summation, integration by parts, and Chebyshev estimates for $\psi(x)$ to find that

$$
\begin{aligned}
\sum_{2\le n\le x}\frac{\Lambda(n)}{\log^2 n} &= \int_{2^-}^{x}\frac{d\psi(u)}{\log^2 u}\\
&\ll \frac{u}{\log^2 u}\Bigg|_{2^-}^{x} + 2\int_{2}^{x}\frac{1}{\log^3 u}\,\mathrm{d}\,u\\
&\ll \frac{x}{\log^2 x}.
\end{aligned}
$$

We deduce part (b) from part (a). Since $\Lambda(n)\le \log n$ for all $n\in\mathbb{N}$, we have

$$
\sum_{2\le n\le x}\frac{\Lambda^2(n)}{\log^2 n}\le \log x\sum_{2\le n\le x}\frac{\Lambda(n)}{\log^2 n}\ll \frac{x}{\log x}.
$$

This completes the proof of the lemma. □

Lemma 4.1.3. *For $x\ge 2$, we have*

$$
\sum_{2\le n\le x}\frac{\Lambda^2(n)}{n\log^2 n}=\log\log x+\gamma_0+\sum_{m=2}^{\infty}\sum_{p}\left(\frac{1}{m^2}-\frac{1}{m}\right)\frac{1}{p^m}+O\left(\frac{1}{\log x}\right).
$$

Proof. By splitting the sum into the sums over primes and sums over primes powers, we have

$$
\sum_{2\le n\le x}\frac{\Lambda^2(n)}{n\log^2 n} = \sum_{p\le x}\frac{\Lambda^2(p)}{p\log^2 p}+\sum_{m=2}^{\infty}\sum_{p^m\le x}\frac{1}{m^2p^m}.
$$

Estimating the tail of the second sum on the right-hand side gives

$$
\sum_{m=2}^{\infty}\sum_{p^m>x}\frac{1}{m^2p^m}\le\sum_{p\le\sqrt{x}}\frac{1}{p^2}\sum_{m=2}^{\infty}\frac{1}{m^2}\ll\sum_{p\le\sqrt{x}}\frac{1}{p^2}\le\sum_{n\le\sqrt{x}}\frac{1}{n^2}\ll\frac{1}{\sqrt{x}}.
$$

Thus by part (c) of Mertens' Theorem, we have

$$\begin{aligned}
\sum_{2\le n\le x} \frac{\Lambda^2(n)}{n\log^2 n} &= \sum_{p\le x}\frac{1}{p} + \sum_{m=2}^{\infty}\sum_{p^m}\frac{1}{m^2p^m} + O\Big(\frac{1}{\sqrt{x}}\Big) \\
&= \log\log x + \gamma_0 - \sum_{m=2}^{\infty}\sum_{p}\frac{1}{mp^m} + \sum_{m=2}^{\infty}\sum_{p^m}\frac{1}{m^2p^m} + O\left(\frac{1}{\log x}\right) \\
&= \log\log x + \gamma_0 + \sum_{m=2}^{\infty}\sum_{p}\left(\frac{1}{m^2} - \frac{1}{m}\right)\frac{1}{p^m} + O\left(\frac{1}{\log x}\right).
\end{aligned}$$

This proves the lemma. □

Lemma 4.1.4. *For $x \ge 2$, we have*

$$\left(\sum_{2\le n\le x}\frac{\Lambda(n)}{n^{1/2}\log n}\right)^2 \ll \frac{x}{\log^2 x}.$$

Proof. Using partial summation, integration by parts, and Chebyshev estimates for $\psi(x)$ gives

$$\begin{aligned}
\left(\sum_{2\le n\le x}\frac{\Lambda(n)}{n^{1/2}\log n}\right)^2 &= \left(\frac{\psi(u)}{\sqrt{u}\log u}\Bigg|_{2^-}^{x} + \int_2^x \frac{\psi(u)}{u^{3/2}\log^2 u}\,\mathrm{d}\,u\right)^2 \\
&\ll \left(\frac{\sqrt{u}}{\log u}\Bigg|_{2^-}^{x} + \int_2^x \frac{1}{\sqrt{u}\,\log^2 u}\,\mathrm{d}\,u\right)^2 \\
&\ll \frac{x}{\log^2 x},
\end{aligned}$$

as claimed. □

Lemma 4.1.5. *For $x \ge 2$, we have*

$$\sum_{2\le n\le x}\frac{\Lambda(n)}{n\log n} \ll \log\log x.$$

Proof. Using a similar argument to that in the proof of Lemma 4.1.5, it follows that

$$\sum_{2\le n\le x}\frac{\Lambda(n)}{n\log n}\ll\int_{2^-}^{x}\frac{d\psi(u)}{u\log u}\ll\log\log x,$$

which completes the proof. □

Lemma 4.1.6. *For $x\ge 2$, we have*

$$\sum_{2\le n\le x}\frac{\Lambda(n)\log\log 3n}{\log n}\ll\frac{x\log\log 3x}{\log x}.$$

Proof. By using partial summation, integration by parts, and Chebyshev estimates for $\psi(x)$, we have

$$\begin{aligned}\sum_{2\le n\le x}\frac{\Lambda(n)\log\log 3n}{\log n} &\le \log\log 3x\int_{2^-}^{x}\frac{d\psi(u)}{\log u}\\ &\ll \log\log 3x\left[\frac{u}{\log u}\bigg|_{2^-}^{x}+\int_{2^-}^{x}\frac{1}{\log^2 u}\,\mathrm{d}\,u\right]\\ &\ll \frac{x\log\log 3x}{\log x},\end{aligned}$$

as claimed. □

Lemma 4.1.7. *For $n,m\ge 2$, we have*

$$\sum_{\substack{m\\ 1\le|m-n|\le n/2}}\frac{\Lambda(m)}{\left|\log\frac{m}{n}\right|}\ll n\log n\log\log 3n.$$

Proof. We follow an argument of Gonek [14, Lemma 1]. First, we observe that

$$\sum_{\substack{m\\ 1\le|m-n|\le n/2}}\frac{\Lambda(m)}{\left|\log\frac{m}{n}\right|}=\sum_{n/2\le m\le n}\frac{\Lambda(m)}{\left|\log\frac{m}{n}\right|}+\sum_{n\le m\le 3n/2}\frac{\Lambda(m)}{\left|\log\frac{m}{n}\right|}. \tag{4.3}$$

Then by [14, Lemma 1], for $n \geq 2$ we have

$$\sum_{k \leq n/2} \frac{\Lambda(N-k)}{k} \ll \log n \log\log 3n,$$

and

$$\sum_{k \leq n/2} \frac{\Lambda(N+k)}{k} \ll \log n \log\log 3n.$$

The proof of such bounds relies on the Brun-Titchmarch inequality [24, Theorem 3.9]. Observe for the first sum in (4.3), because $\log\left(1-\frac{k}{n}\right) \gg \frac{k}{n}$ for small $\frac{k}{n}$, the substitution $m = N - k$ gives

$$\begin{aligned}
\sum_{n/2 \leq m \leq n} \frac{\Lambda(m)}{\left|\log \frac{m}{n}\right|} &= \sum_{n/2 \leq n-k \leq n} \frac{\Lambda(n-k)}{|\log \frac{n-k}{n}|} \\
&\ll n \sum_{k \leq n/2} \frac{\Lambda(n-k)}{k} \\
&\ll n \log n \log\log 3n
\end{aligned}$$

for $n \geq 2$. Similarly, for the second sum in (4.3), the substitution $m = N + k$ yields

$$\begin{aligned}
\sum_{n \leq m \leq 3n/2} \frac{\Lambda(m)}{\left|\log \frac{m}{n}\right|} &= \sum_{n \leq n+k \leq 3n/2} \frac{\Lambda(n+k)}{|\log \frac{n+k}{n}|} \\
&\ll n \sum_{k \leq n/2} \frac{\Lambda(n+k)}{k} \\
&\ll n \log n \log\log 3n,
\end{aligned}$$

for $n \geq 2$. Therefore,

$$\sum_{\substack{m \\ 1 \leq |m-n| \leq n/2}} \frac{\Lambda(m)}{\left|\log \frac{m}{n}\right|} \ll n \log n \log\log 3n,$$

which completes the proof. $\square$

Lemma 4.1.8. *Assuming RH, we have*

$$\sum_{n\le x}\Lambda^2(n) = x\log x - x + O(x^{1/2}\log^3 x).$$

Proof. First we observe that

$$\sum_{n\le x}\Lambda^2(n) = \sum_{n\le x}\Lambda(n)\log n + O\left(x^{1/2}\log x\right),$$

since, by Chebyshev estimates,

$$\sum_{n\le x}\Lambda^2(n) - \sum_{n\le x}\Lambda(n)\log n \ll \sum_{p\le\sqrt{x}}\log^2 p \ll \pi\left(\sqrt{x}\right)\log^2 x \ll x^{1/2}\log x.$$

Then, using integration by parts, we have

$$\begin{aligned}\sum_{n\le x}\Lambda(n)^2 &= \sum_{n\le x}\Lambda(n)\log n + O\left(x^{1/2}\log x\right)\\ &= \int_{2^-}^{x}\log u\,\mathrm{d}\,\psi(u) + O\left(x^{1/2}\log x\right)\\ &= x\log x - x + O\left(x^{1/2}\log^3 x\right),\end{aligned}$$

since, on RH, $\psi(x) = x + O\left(x^{1/2}\log^2 x\right)$. □

Lemma 4.1.9. *For $\sigma_0\in\mathbb{R}$ such that $\frac{1}{2} < \sigma_0 < \frac{3}{2}$ and $x\ge 1$, it follows that*

$$x^{-2\sigma_0}\sum_{n\le x}\frac{\Lambda^2(n)}{n^{1-2\sigma_0}} = \frac{x^{2\sigma_0}(2\sigma_0\log x - 1) + 1}{4\sigma_0^2} + O\left(x^{2\sigma_0-1/2}\log^3 x\right).$$

Proof. Let $\sum_{n\le x}\Lambda(n)^2 = P(x)$. By Lemma 4.1.8 and by integration by parts, we have

$$\begin{aligned}
\sum_{n\le x}\frac{\Lambda(n)^2}{n^{1-2\sigma_0}} &= P(u)u^{2\sigma_0-1}\Big|_{2^-}^{x} + (1-2\sigma_0)\int_2^x \frac{P(u)}{u^{2-2\sigma_0}}\,\mathrm{d}\,u \\
&= \left(u^{2\sigma_0}\log u - u^{2\sigma_0}\right)\Big|_2^x + (1-2\sigma_0)\int_2^x \left(u^{2\sigma_0-1}\log u - u^{2\sigma_0-1}\right)\mathrm{d}\,u \\
&\quad + O\left(x^{2\sigma_0-1/2}\log^3 x\right) \\
&= \frac{x^{2\sigma_0}(2\sigma_0\log x - 1)+1}{4\sigma_0^2} + O\left(x^{2\sigma_0-1/2}\log^3 x\right).
\end{aligned}$$

Thus, the proof is complete. □

Lemma 4.1.10. *For $x \ge 2$, we have*

$$\sum_{m>3x/2}\frac{\Lambda(m)}{m^{1+\frac{1}{\log x}}\log m} \ll \frac{1}{\log x}.$$

Proof. Using partial summation and Chebyshev estimates for $\psi(x)$ gives

$$\begin{aligned}
\sum_{m>3x/2}\frac{\Lambda(m)}{m^{1+\frac{1}{\log x}}\log m} &= \int_{\frac{3x}{2}}^{\infty}\frac{\mathrm{d}\,\psi(u)}{u^{1+\frac{1}{\log x}}\log u} \\
&= \frac{\psi(u)}{u^{1+\frac{1}{\log x}}\log u}\Bigg|_{\frac{3x}{2}}^{\infty} + \int_{\frac{3x}{2}}^{\infty}\frac{\psi(u)(1+\frac{1}{\log x})}{u^{2+\frac{1}{\log x}}\log u}\,\mathrm{d}\,u \\
&\ll \frac{1}{\log u}\Bigg|_{\frac{3x}{2}}^{\infty} + \frac{1}{\log x}\int_{\frac{3x}{2}}^{\infty}\frac{1}{u^{1+\frac{1}{\log x}}}\,\mathrm{d}\,u \ll \frac{1}{\log x},
\end{aligned}$$

as claimed. □

5 A REPRESENTATION FORMULA FOR $\log|\zeta(1/2+it)|$

5.1 Preliminaries

In order to prove Theorems 1.3.1 and 1.4.1, following the ideas and notation developed by Goldston [12], we need to obtain a representation formula for $\log|\zeta(1/2+it)|$. This formula is written in terms of a Dirichlet polynomial supported over prime powers and a sum over the zeros of $\zeta(s)$. The proof is based on Montgomery's explicit formula [22, Lemma] and, in the case of $\log|\zeta(1/2+it)|$, the formula requires the use of two of the auxiliary functions introduced in Section 2.2.

Proposition 5.1.1. *Assuming RH, for $x \geq 4$, $t \geq 1$, $t \neq \gamma$, we have*

$$\log|\zeta(\tfrac{1}{2}+it)| = \sum_{n\leq x} \frac{\Lambda(n)}{\log n}\frac{\cos(t\log n)}{n^{1/2}} f\left(\frac{\log n}{\log x}\right) - \sum_{\gamma} h[(t-\gamma)\log x] + \frac{\log 2 \log\frac{t}{2\pi}}{2\log x} + O\left(\frac{x^{1/2}}{t\log^2 x}\right).$$

We use Proposition 5.1.1 to obtain expressions for the quantities we want to compute in Theorems 1.3.1 and 1.4.1. Before we consider the proofs of these theorems, we adopt some notation for these expressions. Throughout our work, we let $T \geq 4$, and we let $\delta = \delta(T)$ be a function of T such that $0 < \delta \ll \log T$. For $t \geq 1$, denote

$$A(t) := \sum_{n\leq x} \frac{\Lambda(n)\cos(t\log n)}{n^{1/2}\log n} f\left(\frac{\log n}{\log x}\right) \quad \text{and} \quad B(t) := \sum_{\gamma} h[(t-\gamma)\log x], \tag{5.1}$$

so that $A(t)$ contains the information from the primes and $B(t)$ contains the information from the zeros in Proposition 5.1.1. Additionally, we denote

$$
\begin{aligned}
G_1 &:= \int_1^T |A(t)|^2 \,\mathrm{d}t, \qquad G_2 := \int_1^T \left|A\left(t+\tfrac{2\pi\delta}{\log T}\right) - A(t)\right|^2 \,\mathrm{d}t,\\
H_1 &:= 2\int_1^T A(t)\,\log|\zeta(\tfrac12+it)|\,\mathrm{d}t,\\
H_2 &:= 2\int_1^T \left[A\left(t+\tfrac{2\pi\delta}{\log T}\right) - A(t)\right]\left[\log\left|\zeta\left(\tfrac12+it+it\tfrac{2\pi\delta}{\log T}\right)\right| - \log\left|\zeta(\tfrac12+it)\right|\right]\,\mathrm{d}t,\\
R_1 &:= \int_1^T |B(t)|^2 \,\mathrm{d}t, \qquad R_2 := \int_1^T \left|B\left(t+\tfrac{2\pi\delta}{\log T}\right) - B(t)\right|^2 \,\mathrm{d}t.
\end{aligned}
\tag{5.2}
$$

In the next step, we use Proposition 5.1.1 to write the objects in Theorems 1.3.1 and 1.4.1 in terms of the above expressions $G_i,\ H_i,$ and R_i.

Proposition 5.1.2. *Assume RH, let $0<\delta\ll\log T$, and let $4\le x\le T$. Then, as $T\to\infty$, we have:*

$$
\text{(a)} \int_1^T \log^2|\zeta(\tfrac12+it)|\,\mathrm{d}t = R_1 + H_1 - G_1 + \frac{\log^2 2T\log^2 T}{4\log^2 x} + O\left(\frac{T}{\log x}\right) + O\left(\frac{\sqrt{xR_1}}{\log x}\right)
$$

and

$$
\text{(b)} \int_1^T \left[\log\left|\zeta\left(\tfrac12+it+i\tfrac{2\pi\delta}{\log T}\right)\right| - \log\left|\zeta(\tfrac12+it)\right|\right]^2 \mathrm{d}t = R_2 + H_2 - G_2 + O\left(\frac{x}{\log^4 x}\right) + O\left(\frac{\sqrt{xR_2}}{\log^2 x}\right).
$$

The remainder of this chapter is devoted to proving Propositions 5.1.1 and 5.1.2. In the following chapters, we will consider the contributions from the primes, G_i and H_i, and the contributions from the zeta zeros, R_i, separately. We will then estimate these quantities to conclude Theorems 1.3.1 and 1.4.1.

5.2 Proof of Proposition 5.1.1

Proof. Assuming RH, we begin with an explicit formula Montgomery [22, pg.185] used to detail the association between the primes and the zeros of $\zeta(s)$. For $\rho = \frac{1}{2}+i\gamma$, $x \geq 1$, and $s = \sigma + it$ such that $1 < \sigma < 2$, $s \neq 1$, $s \neq \rho$, and $s \neq -2n$, we have

$$\begin{aligned}(2\sigma - 1)\sum_{\gamma} &\frac{x^{i(\gamma-t)}}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2} \\ &= x^{\sigma-\frac{1}{2}}\frac{\zeta'}{\zeta}(\sigma+it) - x^{\frac{1}{2}-\sigma}\frac{\zeta'}{\zeta}(1-\sigma+it) \\ &+ \sum_{n\leq x}\frac{\Lambda(n)}{n^{it}}\left(\frac{x^{\sigma-\frac{1}{2}}}{n^{\sigma}} - \frac{x^{\frac{1}{2}-\sigma}}{n^{1-\sigma}}\right) + x^{\frac{1}{2}-it}\left(\frac{1-2\sigma}{(\sigma-it)(1-\sigma-it)}\right) \\ &+ \sum_{n=1}^{\infty}\frac{x^{-\frac{1}{2}-it}x^{-2n}(2\sigma-1)}{(2n+1-\sigma+it)(2n+\sigma+it)}.\end{aligned} \tag{5.3}$$

This formula stems from an aggregation of the logarithmic derivative of the functional equation [17] of $\zeta(s)$ when $s = \sigma+it$ and $s = 1-\sigma+it$, as appropriate. We now turn our attention to the relationship between $\operatorname{Re}\zeta(1-\sigma+it)$ and $\operatorname{Re}\zeta(\sigma+it)$. We use the fact that the functional equation for $\zeta(s)$ can be written as $\zeta(s) = \chi(s)\zeta(1-s)$, where $\chi(s) = \frac{\pi^{s-\frac{1}{2}}\Gamma((1-s)/2)}{\Gamma(s/2)}$. Taking the real part of the logarithmic derivative of the aformentioned expression of $\zeta(s)$ gives

$$\operatorname{Re}\frac{\zeta'}{\zeta}(s) = \operatorname{Re}\frac{\chi'}{\chi}(s) - \operatorname{Re}\frac{\zeta'}{\zeta}(1-s),$$

and by the reflection principle, $\operatorname{Re}\frac{\zeta'}{\zeta}(1-s) = \operatorname{Re}\frac{\zeta'}{\zeta}(1-\overline{s})$. Therefore

$$\operatorname{Re}\frac{\zeta'}{\zeta}(1-\sigma+it) = -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it) + \operatorname{Re}\frac{\chi'}{\chi}(s).$$

Using Stirling's formula for the logarithmic derivative of $\Gamma(s)$, we know $\frac{\chi'}{\chi}(s) = -\log\left(\frac{|t|}{2\pi}\right) + O\left(\frac{\sigma^2}{t}\right)$ for $t \geq 1$. Thus, for $t \geq 1$,

$$\operatorname{Re}\frac{\zeta'}{\zeta}(1-\sigma+it) = -\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it) - \log\left(\frac{t}{2\pi}\right) + O\left(\frac{\sigma^2}{t}\right). \tag{5.4}$$

Then taking real parts of (5.3), using (5.4), and rearranging gives

$$\begin{aligned}(x^{\sigma-\frac{1}{2}}+&x^{\frac{1}{2}-\sigma})\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it)\\ &= \sum_{\gamma}\cos((t-\gamma)\log x)\frac{2(\sigma-\frac{1}{2})}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}\\ &\quad - \sum_{n\leq x}\Lambda(n)\cos(t\log n)\left(\frac{x^{\sigma-\frac{1}{2}}}{n^{\sigma}} - \frac{x^{\frac{1}{2}-\sigma}}{n^{1-\sigma}}\right)\\ &\quad - x^{\frac{1}{2}}\operatorname{Re}\left(\frac{x^{-it}(1-2\sigma)}{(\sigma-it)(1-\sigma-it)}\right) - x^{\frac{1}{2}-\sigma}\log\left(\frac{t}{2\pi}\right)\\ &\quad - \operatorname{Re}x^{-\frac{1}{2}-it}\sum_{n=1}^{\infty}\frac{x^{-2n}(2\sigma-1)}{(2n+1-\sigma+it)(2n+\sigma+it)} + O\left(\frac{x^{\frac{1}{2}-\sigma}\sigma^2}{t}\right).\end{aligned} \tag{5.5}$$

Observe that because the third sum on the right-hand side of (5.5) remains the same whether σ or $1-\sigma$ is used, we assume $\sigma \geq \frac{1}{2}$ and deduce that

$$\begin{aligned}\left|x^{-\frac{1}{2}-it}\sum_{n=1}^{\infty}\frac{x^{-2n}(2\sigma-1)}{(2n+1-\sigma+it)(2n+\sigma+it)}\right| &\ll \left|x^{-5/2}(\sigma-\tfrac{1}{2})\right|\sum_{n=1}^{\infty}\frac{1}{(2n+1-\sigma+it)^2}\\ &\ll \frac{x^{-5/2}(\sigma-\frac{1}{2})}{t},\end{aligned}$$

for $t \geq 1$. Thus, for $t \neq \gamma$, we know $\log|\zeta(\frac{1}{2}+it)| = -\int_{1/2}^{\infty}\operatorname{Re}\frac{\zeta'}{\zeta}(\sigma+it)\mathrm{d}\sigma$. Then we divide by $(x^{\sigma-\frac{1}{2}}+x^{\frac{1}{2}-\sigma}) = 2\cosh\big((\sigma-\frac{1}{2})\log x\big)$ and integrate (5.5) from $\frac{1}{2}$ to ∞ to find

$$
\begin{aligned}
&\log|\zeta(\tfrac{1}{2}+it)| \\
&=\sum_{n\le x}\Lambda(n)\cos(t\log n)\int_{1/2}^{\infty}\left(\frac{x^{\sigma-\frac{1}{2}}}{n^{\sigma}}-\frac{x^{\frac{1}{2}-\sigma}}{n^{1-\sigma}}\right)\frac{d\sigma}{2\cosh\big((\sigma-\frac{1}{2})\log x\big)} \\
&\quad + x^{1/2}\,\mathrm{Re}\left[x^{-it}\int_{1/2}^{\infty}\frac{(\frac{1}{2}-\sigma)}{(\sigma-it)(1-\sigma-it)}\frac{d\sigma}{\cosh\big((\sigma-\frac{1}{2})\log x\big)}\right] \\
&\quad -\sum_{\gamma}\cos((t-\gamma)\log x)\int_{1/2}^{\infty}\frac{(\sigma-\frac{1}{2})}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}\frac{d\sigma}{\cosh\big((\sigma-\frac{1}{2})\log x\big)} \qquad (5.6)\\
&\quad +\frac{\log\frac{t}{2\pi}}{2}\int_{1/2}^{\infty}\frac{x^{\frac{1}{2}-\sigma}}{\cosh\big((\sigma-\frac{1}{2})\log x\big)}d\sigma + O\left(\frac{x^{-5/2}}{t}\int_{1/2}^{\infty}\frac{|\sigma-\frac{1}{2}|}{\cosh\big((\sigma-\frac{1}{2})\log x\big)}d\sigma\right) \\
&\quad + O\left(\frac{1}{t^{3/2}}\int_{1/2}^{\infty}\frac{x^{\frac{1}{2}-\sigma}t^{\sigma}}{\cosh\big((\sigma-\frac{1}{2})\log x\big)}d\sigma\right),
\end{aligned}
$$

for $x\ge 4$, $t\ge 1$, and $t\ne\gamma$. By using the substitution, $u=(\sigma-\frac{1}{2})\log x$, the first error term on the right-hand side of (5.6) is

$$
\frac{x^{-5/2}}{t}\int_{1/2}^{\infty}\frac{\sigma-\frac{1}{2}}{\cosh\big((\sigma-\frac{1}{2})\log x\big)}d\sigma=\frac{x^{-5/2}}{t}\int_{0}^{\infty}\frac{u/\log x}{\cosh u}\frac{du}{\log x}\ll\frac{1}{t\log^2 x}.
$$

Utilizing the same substitution value for the second error term on the right-hand side of (5.6) gives

$$
\frac{1}{t}\int_{1/2}^{\infty}\frac{x^{\frac{1}{2}-\sigma}\sigma^2}{\cosh\big((\sigma-\frac{1}{2})\log x\big)}d\sigma\ll\frac{1}{t\log^3 x}\int_{0}^{\infty}\frac{e^{-u}u^2}{\cosh u}du\ll\frac{1}{t\log^3 x}.
$$

Again, by the same substitution, the integral in the first main term on the right-hand side of (5.6) yields

$$\int_{1/2}^{\infty}\left(\frac{x^{\sigma-\frac{1}{2}}}{n^{\sigma}}-\frac{x^{\frac{1}{2}-\sigma}}{n^{1-\sigma}}\right)\frac{d\sigma}{2\cosh\left((\sigma-\frac{1}{2})\log x\right)}=\frac{1}{n^{1/2}\log x}\int_{0}^{\infty}\left(\frac{e^{u}}{n^{u/\log x}}-\frac{e^{-u}}{n^{-u/\log x}}\right)\frac{du}{2\cosh u}$$

$$=\frac{1}{n^{1/2}\log x}\int_{0}^{\infty}\frac{\sinh\left(u\left(1-\frac{\log n}{\log x}\right)\right)}{\cosh u}\,\mathrm{d}\,u$$

$$=\frac{1}{n^{1/2}\log n}f\left(\frac{\log n}{\log x}\right).$$

The function f is the test function contained in the sum over primes (see Section 2.2), and it is the first of three auxiliary functions that will arise in our work. For the second main term on the right-hand side of (5.6), we have

$$x^{1/2}\,\mathrm{Re}\left(x^{-it}\int_{1/2}^{\infty}\frac{(\frac{1}{2}-\sigma)}{(\sigma-it)(1-\sigma-it)}\frac{d\sigma}{\cosh\left((\sigma-\frac{1}{2})\log x\right)}\right)$$

$$=-x^{1/2}\,\mathrm{Re}\left(\int_{0}^{\infty}\frac{x^{-it}}{(((\frac{1}{2}-it)\log x)^2-u^2)}\frac{u}{\cosh u}\,\mathrm{d}\,u\right)$$

$$\ll x^{1/2}\,\mathrm{Re}\left(\int_{0}^{\infty}\frac{u}{|\cosh u|}\frac{1}{t\log^2 x}\,\mathrm{d}\,u\right)$$

$$\ll\frac{x^{1/2}}{t\log^2 x}.$$

Similarly, for the third main term on the right-hand side of (5.6), we have

$$-\sum_{\gamma}\cos((t-\gamma)\log x)\int_{\frac{1}{2}}^{\infty}\frac{(\sigma-\frac{1}{2})}{(\sigma-\frac{1}{2})^2+(t-\gamma)^2}\frac{d\sigma}{\cosh\left((\sigma-\frac{1}{2})\log x\right)}$$

$$=-\sum_{\gamma}\cos((t-\gamma)\log x)\int_{0}^{\infty}\frac{u}{(u^2+(t-\gamma)^2\log^2 x)}\frac{du}{\cosh u}$$

$$=-\sum_{\gamma}h[(t-\gamma)\log x],$$

where h is the weight function in the sum over the zeros of $\zeta(s)$ (see Section 2.2). Finally, the fourth main term on the right-hand side of (5.6) reduces to

$$\frac{\log\frac{t}{2\pi}}{2}\int\limits_{1/2}^{\infty}\frac{x^{\frac{1}{2}-\sigma}}{\cosh\big((\sigma-\frac{1}{2})\log x\big)}d\sigma=\frac{\log\frac{t}{2\pi}}{\log x}\int\limits_{0}^{\infty}\frac{1}{e^{2u}+1}du=\frac{\log 2\log\frac{t}{2\pi}}{2\log x}.$$

Combining all the terms of (5.6) completes the proof. □

5.3 Proof of Proposition 5.1.2

Proof. Let $0<\delta\ll\log T$ and $4\leq x\leq T$. By rearranging the terms in Proposition 5.1.1 and using the notation in (5.1), as $T\to\infty$, we have

$$-B(t)+O\left(\frac{x^{1/2}}{t\log^2 x}\right)=\log|\zeta(\tfrac{1}{2}+it)|-A(t)-\frac{\log 2\log\frac{t}{2\pi}}{2\log x}. \tag{5.7}$$

Note that the representation formula in Proposition 5.1.1 requires $t\geq 1$, and the error terms are unbounded if $t\to 0$. Thus, integrating the above expression from 1 to T yields

$$\begin{aligned}
&R_1+O\left(\int\limits_1^T\frac{x^{1/2}}{t\log^2 x}B(t)\,\mathrm{d}t\right)+O\left(\int\limits_1^T\frac{x}{t^2\log^4 x}\,\mathrm{d}t\right)\\
&=\int\limits_1^T\log^2|\zeta(\tfrac{1}{2}+it)|\,\mathrm{d}t-H_1+G_1+\int\limits_1^T\frac{\log^2 2\log^2\frac{t}{2\pi}}{4\log^2 x}\,\mathrm{d}t\\
&\quad+O\left(\frac{1}{\log x}\int\limits_1^T A(t)\log t\,\mathrm{d}t\right)-\frac{\log 2}{2\log x}\int\limits_1^T\log|\zeta(\tfrac{1}{2}+it)|\log\tfrac{t}{2\pi}\,\mathrm{d}t.
\end{aligned} \tag{5.8}$$

Using Cauchy-Schwarz, the first error term on the left-hand side of (5.8) gives

$$\int\limits_1^T\frac{x^{1/2}}{t\log^2 x}B(t)\,\mathrm{d}t\ll\left(\int\limits_1^T\left|\frac{x^{1/2}}{t\log^2 x}\right|^2\mathrm{d}t\cdot R_1\right)^{1/2}\ll\frac{\sqrt{xR_1}}{\log^2 x}.$$

The second error term on the left-hand side of (5.8) reduces to

$$\int_1^T \frac{x}{t^2 \log^4 x}\,\mathrm{d}t \ll \frac{x}{\log^4 x}\left(\frac{1}{T}-1\right) \ll \frac{x}{\log^4 x}.$$

On the right-hand side of (5.8), the fourth main term is

$$\int_1^T \frac{\log^2 2}{4}\frac{\log^2 \frac{t}{2\pi}}{\log^2 x}\,\mathrm{d}t \quad = \quad \frac{\log^2 2}{4\log^2 x}\int_1^T \log^2 \tfrac{t}{2\pi}\,\mathrm{d}t = T\log^2 T\frac{\log^2 2}{4\log^2 x} + O\left(\frac{T}{\log x}\right).$$

For the first error term on the right-hand side of (5.8), since $f(v)$ is uniformly bounded for all $v \in [0,1]$, $|\cos v| \leq 1$ for all $v \in \mathbb{R}$, and $\int_1^T n^{it}\log t\,\mathrm{d}t \ll \log T$, by Lemma 4.1.4 we observe that

$$\int_1^T A(t)\frac{\log t}{\log x}\,\mathrm{d}t \ll \frac{\log T}{\log x}\sum_{n\leq x}\frac{\Lambda(n)}{n^{1/2}\log n} \ll \frac{\sqrt{x}}{\log x}.$$

Lastly, we consider the fifth main term on the right-hand side of (5.8). By [20, Lemma 2.2], we know $\int_1^T \log|\zeta(\frac{1}{2}+it)|\,\mathrm{d}t \ll \frac{\log T}{(\log\log T)^2}$. Then, integrating by parts

$$\begin{aligned}
-\frac{\log 2}{2\log x}&\int_1^T \log|\zeta(\tfrac{1}{2}+it)|\log\tfrac{t}{2\pi}\,\mathrm{d}t \\
&= -\frac{\log 2}{2\log x}\left[\log\tfrac{t}{2\pi}\int_1^t \log|\zeta(\tfrac{1}{2}+iu)|\,\mathrm{d}u\Big|_1^T - \int_1^T\frac{1}{t}\int_1^t \log|\zeta(\tfrac{1}{2}+iu)|\,\mathrm{d}u\,\mathrm{d}t\right] \\
&\ll \frac{1}{\log x}\left[\frac{\log^2 t}{(\log\log t)^2}\Big|_1^T - \int_1^T \frac{\log t}{t(\log\log t)^2}\,\mathrm{d}t\right] \\
&\ll \frac{\log T}{(\log\log T)^2}.
\end{aligned}$$

By combining and rearranging all the estimates for the terms of (5.8), we have

$$\int_1^T \log^2 |\zeta(\tfrac{1}{2}+it)| \,\mathrm{d}t = H_1 + R_1 - G_1 - \frac{\log^2(2) T \log^2 T}{4\log^2 x} + O\left(\frac{T}{\log x}\right) + O\left(\frac{\sqrt{x R_1}}{\log^2 x}\right),$$

which completes part (a).

For part (b), we shift the variable t by a factor of $\frac{2\pi\delta}{\log T}$ in (5.7) to yield

$$\log\left|\zeta\left(\tfrac{1}{2}+it+i\tfrac{2\pi\delta}{\log T}\right)\right| = A\left(t+\tfrac{2\pi\delta}{\log T}\right) - B\left(t+\tfrac{2\pi\delta}{\log T}\right) + \frac{\log 2 \log\left(\frac{t}{2\pi}+\frac{\delta}{\log T}\right)}{2\log x} + O\left(\frac{x^{1/2}}{\left(t+\frac{2\pi\delta}{\log T}\right)\log^2 x}\right), \tag{5.9}$$

at $T\to\infty$. We want to consider the mean-square of the difference of $\log|\zeta(\frac{1}{2}+it)|$ and $\log\left|\zeta\left(\frac{1}{2}+it+i\frac{2\pi\delta}{\log T}\right)\right|$. Thus taking the difference between (5.7) and (5.9) and rearranging terms yields

$$\begin{aligned}\log\left|\zeta\left(\tfrac{1}{2}+it+i\tfrac{2\pi\delta}{\log T}\right)\right| &- \log|\zeta(\tfrac{1}{2}+it)| \\ &= \left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right] - \left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right] \\ &\quad + \frac{\log 2\left[\log\left(\frac{t}{2\pi}+\frac{\delta}{\log T}\right)-\log\frac{t}{2\pi}\right]}{2\log x} \\ &\quad + O\left(\frac{x^{1/2}}{\left(t+\frac{2\pi\delta}{\log T}\right)\log^2 x}\right) + O\left(\frac{x^{1/2}}{t\log^2 x}\right),\end{aligned} \tag{5.10}$$

Observe that for $t > 1$ and $x \geq 4$, by the power series expansion of $\log(1+z)$ for small z, the third term on the right-hand side of (5.10) reduces to

$$\begin{aligned}
\frac{\log 2\left[\log\left(\frac{t}{2\pi}+\frac{\delta}{\log T}\right)-\log\frac{t}{2\pi}\right]}{2\log x} &\ll \frac{1}{\log x}\left[\log\left(\frac{t}{2\pi}+\frac{\delta}{\log T}\right)-\log\frac{t}{2\pi}\right] \\
&= \frac{1}{\log x}\log\left(1+\frac{2\pi\delta}{t\log T}\right) \\
&\ll \frac{\delta}{t\log^2 x} \ll \frac{\log T}{t\log^2 x},
\end{aligned}$$

since $\delta \ll \log T$. Similarly, because $t > 1$ our two error terms on the right-hand side of (5.10) give

$$\frac{x^{1/2}}{\left(t+\frac{2\pi\delta}{\log T}\right)\log^2 x}+\frac{x^{1/2}}{t\log^2 x}, \ll \frac{x^{1/2}}{t\log^2 x}.$$

Thus our difference in (5.10) simplifies to

$$\begin{aligned}
\log\left|\zeta\left(\tfrac{1}{2}+it+i\tfrac{2\pi\delta}{\log T}\right)\right| - \log|\zeta(\tfrac{1}{2}+it)| \\
&= \left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]-\left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right] \\
&\quad + O\left(\frac{x^{1/2}}{t\log^2 x}\right).
\end{aligned}$$

Squaring both sides of the above equation yields

$$
\begin{aligned}
&\log\left|\zeta\left(\tfrac{1}{2}+it+i\tfrac{2\pi\delta}{\log T}\right)\right|-\log|\zeta(\tfrac{1}{2}+it)| \\
&\qquad = \left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]^2+\left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right]^2 \\
&\qquad\quad -2\left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]\left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right] \\
&\qquad\quad +O\left(\frac{x}{t^2\log^4 x}\right)+O\left(\left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]\frac{x^{1/2}}{t\log^2 x}\right) \\
&\qquad\quad +O\left(\left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right]\frac{x^{1/2}}{t\log^2 x}\right) \\
&\qquad = -\left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]^2+\left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right]^2 \\
&\qquad\quad +2\left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]\left[\log\left|\zeta\left(\tfrac{1}{2}+it+i\tfrac{2\pi\delta}{\log T}\right)\right|-\log|\zeta(\tfrac{1}{2}+it)|\right] \\
&\qquad\quad +O\left(\frac{x}{t^2\log^4 x}\right)+O\left(\left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]\frac{x^{1/2}}{t\log^2 x}\right) \\
&\qquad\quad +O\left(\left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right]\frac{x^{1/2}}{t\log^2 x}\right).
\end{aligned}
$$

As in part (a), we must integrate the above expression from 1 to T. This gives

$$
\begin{aligned}
&\int_1^T\left[\log\left|\zeta\left(\tfrac{1}{2}+it+i\tfrac{2\pi\delta}{\log T}\right)\right|-\log|\zeta(\tfrac{1}{2}+it)|\right]^2\mathrm{d}t \\
&\quad = -\int_1^T\left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]^2\mathrm{d}t+\int_1^T\left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right]^2\mathrm{d}t \\
&\qquad +2\int_1^T\left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]\left[\log\left|\zeta\left(\tfrac{1}{2}+it+i\tfrac{2\pi\delta}{\log T}\right)\right|-\log|\zeta(\tfrac{1}{2}+it)|\right]\mathrm{d}t \qquad (5.11) \\
&\qquad +O\left(\int_1^T\frac{x}{t^2\log^4 x}\,\mathrm{d}t\right)+O\left(\int_1^T\left[A\left(t+\tfrac{2\pi\delta}{\log T}\right)-A(t)\right]\frac{x^{1/2}}{t\log^2 x}\,\mathrm{d}t\right) \\
&\qquad +O\left(\int_1^T\left[B\left(t+\tfrac{2\pi\delta}{\log T}\right)-B(t)\right]\frac{x^{1/2}}{t\log^2 x}\,\mathrm{d}t\right)
\end{aligned}
$$

When we integrate, certain cross terms will arise that can be included within the error bounds. We first consider the cross term containing $A(t)$, the sum over the primes. Since $f(v)$ is uniformly bounded for all $v\in[0,1]$ and $|\cos v|\le 1$ for all

$v \in \mathbb{R}$, by Lemma 4.1.4 observe that

$$\int_1^T \left[A\left(t + \tfrac{2\pi\delta}{\log T}\right) - A(t)\right] \frac{x^{1/2}}{t \log^2 x} \, \mathrm{d}t \ll \int_1^T \frac{x^{1/2}}{t \log^2 x} \sum_{n \le x} \frac{\Lambda(n)}{n^{1/2} \log n} \, \mathrm{d}t \ll \frac{x}{\log^2 x}. \quad (5.12)$$

Next we consider the error term containing $B(t)$, the sum over the zeros of $\zeta(s)$. Using Cauchy-Schwarz, we find that

$$\int_1^T \left[B\left(t + \tfrac{2\pi\delta}{\log T}\right) - B(t)\right] \frac{x^{1/2}}{t \log^2 x} \, \mathrm{d}t \ll \left(\int_1^T \left| \frac{x^{1/2}}{t \log^2 x} \right|^2 \mathrm{d}t \cdot R_2 \right)^{1/2} \ll \frac{\sqrt{R_2}}{\log^2 x}.$$

Inputting the estimates for the cross terms in (5.11) gives

$$\int_1^T \left[\log \left|\zeta\left(\tfrac{1}{2} + it + i\tfrac{2\pi\delta}{\log T}\right)\right| - \log |\zeta(\tfrac{1}{2} + it)| \right]^2 \mathrm{d}t = R_2 + H_2 - G_2 + O\left(\frac{x}{\log^2 x}\right) + O\left(\frac{\sqrt{x R_2}}{\log^2 x}\right).$$

Consequently, the proof is complete. $\square$

6 CONTRIBUTION FROM THE PRIMES

In this chapter, we estimate the expression $H_i - G_i$ for $i = 1, 2$, where G_i and H_i are defined in (5.2). We need to obtain intermediate expressions for G_i and H_i separately. In order to obtain and estimate these expressions, we state two lemmas and combine these results with some of our estimates for the sums over primes in Chapter 4.

6.1 Preliminary estimates for G_i and H_i

The first lemma is Montgomery & Vaughan's well-known mean-value estimate for Dirichlet polynomials.

Lemma 6.1.1 (Montgomery & Vaughan). *If a_n is a sequence of complex numbers such that the sum $\sum_{n\le x} n|a_n|^2$ converges, then for all $n \in \mathbb{N}$,*

$$\int_0^T \left|\sum_{n\le x} a_n n^{it}\right|^2 = \sum_{n\le x} |a_n|^2 \Big(T + O(n)\Big).$$

The proof of this result is found in [23, Cor. 3]. The second lemma will be used for the expressions H_1 and H_2. We use some estimates of Goldston and Titchmarsh, together with some trigonometric identities, to obtain expressions for the real and imaginary parts of integrals of $\log \zeta(\frac{1}{2} + it)$ times trigonometric functions. Some of these results appear explicitly in [12] (part (b)) and implicitly in [9] (part (d)). We collect them all in the following lemma, for the reader's convenience.

Lemma 6.1.2. *Assume RH, let $T \geq 2$, let $h \in \mathbb{R}$, and let $n \geq 2$ be an integer. Denote*

$$\mathcal{E} = \mathcal{E}(n, T) := n^{1/2} \log\log 3n + \frac{n^{1/2} \log T}{\log n}.$$

Then, the following estimates hold:

(a) $$\int_1^T \log|\zeta(\tfrac{1}{2} + it)| \cos(t \log n)\, \mathrm{d}t = \frac{T}{2} \frac{\Lambda(n)}{n^{1/2} \log n} + O(\mathcal{E})$$

(b) $$\int_1^T \pi S(t) \sin(t \log n)\, \mathrm{d}t = -\frac{T}{2} \frac{\Lambda(n)}{n^{1/2} \log n} + O(\mathcal{E})$$

(c) $$\int_1^T \log|\zeta(\tfrac{1}{2} + it)| \left[\cos((t+h)\log n) + \cos((t-h)\log n) - 2\cos(t \log n)\right] \mathrm{d}t = -T\, \frac{\Lambda(n)[1 - \cos(h \log n)]}{n^{1/2} \log n} + O(\mathcal{E})$$

(d) $$\int_1^T \pi S(t) \left[\sin((t+h)\log n) + \sin((t-h)\log n) - 2\sin(t \log n)\right] \mathrm{d}t = T\, \frac{\Lambda(n)[1 - \cos(h \log n)]}{n^{1/2} \log n} + O(\mathcal{E})$$

Proof. For part (a), we use a variation of Goldston's modification [12, p. 167] to an argument of Titchmarsh [33] combined with Lemmas 4.1.5 and 4.1.7. We begin with $\int_{\mathscr{C}} \log\zeta(s)\, n^s \mathrm{d}s$, where $\mathscr{C}$ is the rectangular path intersecting the coordinates

$$\{(\tfrac{1}{2}, 0),\ (1 + \tfrac{1}{\log n}, 0),\ (1 + \tfrac{1}{\log n}, iT),\ (\tfrac{1}{2}, iT)\}$$

with semi-circular indentations to avoid the singularities of $\log\zeta(s)$. Using Cauchy's Theorem and allowing the radius of the indentations to tend to zero yields

$$
\begin{aligned}
i \int_0^T \log\zeta\left(\tfrac{1}{2}+it\right) n^{1/2+it}\,\mathrm{d}t
&= \int_{1/2}^{1+\frac{1}{\log n}} \log|\zeta(\sigma)| n^{\sigma} \mathrm{d}\sigma + i \int_0^T \log\zeta\big(1+\tfrac{1}{\log n}+it\big) n^{1+\frac{1}{\log n}+it}\,\mathrm{d}t \\
&\quad - \int_{1/2}^{1+\frac{1}{\log n}} \log\zeta(\sigma+iT) n^{\sigma+iT} \mathrm{d}\sigma \\
&= I_1 + iI_2 - I_3.
\end{aligned}
$$

Recall the Laurent series for $\zeta(s)$ about $s=1$ is $\zeta(s) = \frac{1}{s-1} + O(1)$. This implies

$$
\zeta(\sigma+it) = \frac{1}{\sigma-1+it} + O(1),
$$

for $|\sigma-1| \leq 1$. Hence, $|\zeta(\sigma+it)| \ll \frac{1}{\sigma-1}$ in this range. Therefore

$$
I_1 \ll n^{1+\frac{1}{\log n}} \int_{1/2}^{1+\frac{1}{\log n}} |\log|\sigma-1||\mathrm{d}\sigma \ll n.
$$

Furthermore, we know from [32] that

$$
\log\zeta(s) = \sum_{|t-\gamma|\leq 1} \log(s-\rho) + O(\log|\tau|),
$$

for $|\tau| = |t|+2$ and $-1 \leq \sigma \leq 2$, where we choose the branch of the logarithm so that $|\operatorname{Im}\log(s-\rho)| < \pi$ with $\rho = \frac{1}{2}+i\gamma$. Then by RH, since there are $O(\log T)$

terms in the sum, we have

$$\log|\zeta(\sigma+iT)| \ll \log T \log\left(\frac{1}{\sigma-\frac{1}{2}}\right),$$

with $\frac{1}{2}<\sigma\le 5/4$ and $T\ge 2$. Therefore,

$$I_3 \ll \int_{\frac{1}{2}}^{1+\frac{1}{\log n}} |\log\zeta(\sigma+iT)| n^{\sigma+iT}\mathrm{d}\sigma \ll n^{1+\frac{1}{\log n}} \log T \int_{\frac{1}{2}}^{1+\frac{1}{\log n}} \log\left(\frac{1}{\sigma-\frac{1}{2}}\right) n^{\sigma}\mathrm{d}\sigma \ll \frac{n\log T}{\log n}.$$

Finally, by the Dirichlet series of $\log\zeta(s)=\sum_{m=2}^{\infty}\frac{\Lambda(m)}{\log m}m^{-s}$, we have

$$I_2 = i\sum_{m=2}^{\infty}\frac{\Lambda(m)\, n^{1+\frac{1}{\log n}}}{m^{1+\frac{1}{\log n}}\log m}\int_0^T\left(\frac{n}{m}\right)^{it}\mathrm{d}t = \frac{iT\Lambda(n)}{\log n} + O\left(\sum_{\substack{m=2\\ m\neq n}}^{\infty}\frac{n\Lambda(m)}{m^{1+\frac{1}{\log n}}\log m}\frac{1}{|\log\frac{m}{n}|}\right).$$

By Lemmas 4.1.5 and 4.1.7, the error term on the right-hand side of (6.1) for $n\ge 2$ reduces to

$$\begin{aligned}
&\sum_{\substack{m=2\\ m\neq n}}^{\infty}\frac{n\Lambda(m)}{m^{1+\frac{1}{\log n}}\log m}\frac{1}{|\log\frac{m}{n}|}\\
&\quad\ll n\sum_{m<n/2}\frac{\Lambda(m)}{m\log m} + \frac{1}{\log n}\sum_{n/2\le m\le n}\frac{\Lambda(m)}{|\log\frac{m}{n}|}\left(\frac{2}{n}\right)^{\frac{1}{\log n}}\\
&\qquad + \frac{1}{\log n}\sum_{n\le m\le 3n/2}\frac{\Lambda(m)}{|\log\frac{m}{n}|}\left(\frac{2}{3n}\right)^{\frac{1}{\log n}} + n\sum_{m>3n/2}\frac{\Lambda(m)}{m^{1+\frac{1}{\log n}}\log m}\\
&\quad\ll n\sum_{m<n/2}\frac{\Lambda(m)}{m\log m} + \frac{1}{\log n}\sum_{\substack{m\\ 1\le|m-n|\le n/2}}\frac{\Lambda(m)}{|\log\frac{m}{n}|} + n\sum_{m>3n/2}\frac{\Lambda(m)}{m^{1+\frac{1}{\log n}}\log m}.\\
&\quad\ll n\log\log 3n.
\end{aligned}$$

Therefore, we have

$$I_2 = iT\frac{\Lambda(n)}{\log n} + O\left(n\log\log 3n\right).$$

Combining the results for I_1, I_2, and I_3 yields

$$\int_0^T \log\zeta(\tfrac{1}{2}+it)n^{1/2+it}\,\mathrm{d}t = T\frac{\Lambda(n)}{\log n} + O\left(n\log\log 3n\right) + O\left(n\frac{\log T}{\log n}\right). \quad (6.1)$$

Similarly, considering $\int_{\mathscr{C}} \log(\zeta(s))n^{-s}\mathrm{d}s$ gives

$$\int_0^T \log\zeta(\tfrac{1}{2}+it)n^{-1/2-it}\,\mathrm{d}t = O\left(n^{-1/2}\log T\right). \quad (6.2)$$

Consequently, by combining (6.1) and (6.2) and taking real parts, we have

$$\begin{aligned}
&\int_0^T \log|\zeta(\tfrac{1}{2}+it)|\cos(t\log n)\,\mathrm{d}t \\
&\quad= \frac{1}{2}\left(\mathrm{Re}\left(\int_0^T \log\zeta(\tfrac{1}{2}+it)n^{it}\,\mathrm{d}t\right) + \mathrm{Re}\left(\int_0^T \log\zeta(\tfrac{1}{2}+it)n^{-it}\,\mathrm{d}t\right)\right) \\
&\quad= \mathrm{Re}\left(\frac{1}{2n^{1/2}}\int_0^T \log\zeta(\tfrac{1}{2}+it)n^{\frac{1}{2}+it}\,\mathrm{d}t + \frac{n^{1/2}}{2}\int_0^T \log\zeta(\tfrac{1}{2}+it)n^{-\frac{1}{2}-it}\,\mathrm{d}t\right) \\
&\quad= \frac{T}{2}\frac{\Lambda(n)}{n^{1/2}\log n} + O(\mathcal{E}),
\end{aligned}$$

which proves part (a). Part (b) is [12, Equation (6.3)]. Parts (c) and (d) follow from parts (a) and (b), respectively, after considering the trigonometic identities

$$\cos((t+h)\log n) + \cos((t-h)\log n) - 2\cos(t\log n) = -2\cos(t\log n)[1-\cos(h\log n)],$$

and

$$\sin((t+h)\log n)+\sin((t-h)\log n)-2\sin(t\log n)=-2\sin(t\log n)[1-\cos(h\log n)].$$

Hence the proof is complete. $\square$

6.2 Expressions for G_i and H_i

Using the lemmas in Section 6.1, we are now ready to obtain our intermediate expressions for G_i and H_i. We begin with the following lemma.

Lemma 6.2.1 (G_i)**.** *Let $0<\delta\ll\log T$, and let $4\le x\le T$. Let G_1 and G_2 be defined in* (5.2)*. Then,*

$$\text{(a) } G_1=\frac{T}{2}\sum_{n\le x}\frac{\Lambda(n)^2}{n\log^2 n}f^2\left(\frac{\log n}{\log x}\right)+O\left(\frac{x}{\log x}\right)$$

$$\text{(b) } G_2=T\sum_{n\le x}\frac{\Lambda(n)^2}{n\log^2 n}f^2\left(\frac{\log n}{\log x}\right)\left[1-\cos\left(\frac{2\pi\delta\,\log n}{\log T}\right)\right]+O\left(\frac{x}{\log^2 x}\right).$$

Proof. Let $0<\delta\ll\log T$, and $4\le x\le T$. For part (a), recall $\cos(t\log n)=\mathrm{Re}(n^{-it})$. Then by Lemma 6.1.1 and the identity $(\mathrm{Re}\,z)^2=\frac{1}{2}|z|^2+\frac{1}{2}\mathrm{Re}(z^2)$, we have

$$\begin{aligned}
G_1 &= \int_1^T\left|\sum_{n\le x}\frac{\Lambda(n)}{\log n}\frac{\cos(t\log n)}{n^{1/2}}f\left(\frac{\log n}{\log x}\right)\right|^2 \mathrm{d}t\\
&= \frac{T}{2}\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n}f^2\left(\frac{\log n}{\log x}\right)+O\left(\sum_{n\le x}\frac{\Lambda^2(n)}{\log^2 n}f^2\left(\frac{\log n}{\log x}\right)\right)\\
&\quad+\frac{1}{2}\,\mathrm{Re}\int_1^T\left(\sum_{n\le x}\frac{\Lambda(n)}{n^{1/2+it}\log n}f\left(\frac{\log n}{\log x}\right)\right)^2\mathrm{d}t. \qquad (6.3)
\end{aligned}$$

For the second term on the right-hand side of (6.3), since $f^2(v)$ is uniformly bounded for $v \in [0,1]$, Lemma 4.1.2 part (b) implies that

$$O\left(\sum_{n\le x}\frac{\Lambda^2(n)}{\log^2 n}f^2\left(\frac{\log n}{\log x}\right)\right) = O\left(\frac{x}{\log x}\right).$$

We estimate the third term on the right-hand side of (6.3) by interchanging the sums and the integral, integrating term-by-term, observing $f(v)$ is unformly bounded on $v \in [0,1]$, and then by applying Lemma 4.1.4:

$$\begin{aligned}
&\frac{1}{2}\operatorname{Re}\int_1^T\left(\sum_{n\le x}\frac{\Lambda(n)}{n^{1/2+it}\log n}f\left(\frac{\log n}{\log x}\right)\right)^2 \mathrm{d}t\\
&\quad=\frac{1}{2}\sum_{n\le x}\sum_{m\le x}\frac{\Lambda(n)\Lambda(m)f\left(\frac{\log n}{\log x}\right)f\left(\frac{\log m}{\log x}\right)}{(nm)^{1/2}\log n\log m}\operatorname{Re}\left(\int_1^T (nm)^{-2it}\,\mathrm{d}t\right)\\
&\quad=\frac{1}{2}\sum_{n\le x}\sum_{m\le x}\frac{\Lambda(n)\Lambda(m)f\left(\frac{\log n}{\log x}\right)f\left(\frac{\log m}{\log x}\right)}{(nm)^{1/2}\log n\log m}\operatorname{Re}\left(\frac{e^{-2iT\log(nm)}}{-2i\log(nm)}\right)\\
&\quad=O\left(\left(\sum_{n\le x}\frac{\Lambda(n)}{n^{1/2}\log n}\right)^2\right)=\frac{x}{\log^2 x},
\end{aligned}$$

since $\log(mn) \ge \log 2 \gg 1$ since m and n run over prime powers. Therefore,

$$G_1 = \frac{T}{2}\sum_{m\le x}\frac{\Lambda(m)^2}{m\log^2 m}f^2\left(\frac{\log m}{\log x}\right) + O\left(\frac{x}{\log x}\right),$$

which completes part (a). Now we turn our attention to G_2. Observe that

$$
\begin{aligned}
G_2 &= \int_1^T \left[A\left(t+\tfrac{2\pi\delta}{\log T}\right) - A(t)\right]^2 \,\mathrm{d}t \\
&= \int_1^T \sum_{n\le x} \frac{\Lambda(n)\Lambda(m) f\left(\frac{\log n}{\log x}\right) f\left(\frac{\log m}{\log x}\right)}{n^{1/2}m^{1/2}\log n\log m} \Big(\cos\Big((t+\tfrac{2\pi\delta}{\log T})\log n\Big)\cos\Big((t+\tfrac{2\pi\delta}{\log T})\log m\Big) \\
&\qquad -2\cos\Big((t+\tfrac{2\pi\delta}{\log T})\log n\Big)\cos(t\log m) + \cos(t\log n)\cos(t\log m)\Big) \\
&= \sum_{n\le x} \frac{\Lambda^2(n) f^2\left(\frac{\log n}{\log x}\right)}{n\log^2 n} \int_1^T \Big(\cos^2((t+\tfrac{2\pi\delta}{\log T})\log n) - 2\cos\Big((t+\tfrac{2\pi\delta}{\log T})\log n\Big)\cos(t\log n) \\
&\qquad + \cos^2(t\log n)\Big)\,\mathrm{d}t + O\left(\sum\sum_{\substack{n,m\le x\\ n\ne m}} \frac{\Lambda(n)\Lambda(m)}{n^{1/2}m^{1/2}\log n\log m}\right).
\end{aligned}
$$

Using various trigonometric identities, observe that

$$
\begin{aligned}
&\cos^2((t+\tfrac{2\pi\delta}{\log T})\log n) - 2\cos\Big((t+\tfrac{2\pi\delta}{\log T})\log n\Big)\cos(t\log n) + \cos^2(t\log n) \\
&\quad = 1 + \frac{\cos\Big(2(t+\tfrac{2\pi\delta}{\log T})\log n\Big) + \cos(2t\log n)}{2} - \cos\Big(\frac{2\pi\delta\,\log n}{\log T}\Big) \\
&\qquad - \cos\Big((2t+\tfrac{2\pi\delta}{\log T})\log n\Big) \\
&\quad = \left[1-\cos\left(\frac{2\pi\delta\,\log n}{\log T}\right)\right]\left[1-\cos\Big((2t+\tfrac{2\pi\delta}{\log T})\log n\Big)\right].
\end{aligned}
$$

Furthermore, because

$$
\int_1^T \left[1-\cos\left(\frac{2\pi\delta\,\log n}{\log T}\right)\right]\cos\Big((2t+\tfrac{2\pi\delta}{\log T})\log n\Big)\,\mathrm{d}t \ll 1,
$$

our estimate for G_2 reduces to

$$G_2 = T\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n}f^2\left(\frac{\log n}{\log x}\right)\left[1-\cos\left(\frac{2\pi\delta\,\log n}{\log T}\right)\right]$$
$$+\,O\left(\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n}f^2\left(\frac{\log n}{\log x}\right)\right)+O\left(\sum_{\substack{n,m\le x\\ n\ne m}}\sum\frac{\Lambda(n)\Lambda(m)}{n^{1/2}m^{1/2}\log n\log m}\right). \tag{6.4}$$

For the first error term on the right-hand side of (6.4), we have by Lemma 4.1.3 that

$$\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n}f^2\left(\frac{\log n}{\log x}\right)\ll \log\log x.$$

Similarly, by Lemma 4.1.4, the second error term on the right-hand side of (6.4) yields

$$\sum_{\substack{n,m\le x\\ n\ne m}}\sum\frac{\Lambda(n)\Lambda(m)}{n^{1/2}m^{1/2}\log n\log m} \ll \left(\sum_{n\le x}\frac{\Lambda(n)}{n^{1/2}\log n}\right)^2 \ll \frac{x}{\log^2 x}.$$

Hence

$$G_2 = T\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n}f^2\left(\frac{\log n}{\log x}\right)\left(1-\cos\left(\tfrac{2\pi\delta\log n}{\log T}\right)\right)+O\left(\frac{x}{\log^2 x}\right),$$

as claimed for part (b). □

Next, we obtain expressions for H_i using Lemma 6.1.2. As mentioned in the Introduction, in the next lemma we will use Theorem 1.3.1 to control some of the error terms in part (b), which is the part relevant to Theorem 1.4.1.

Lemma 6.2.2 (H_i)**.** *Assuming RH, let* $0 < \delta \ll \log T$*, and let* $4 \le x \le T$*. Let* H_1 *and* H_2 *be defined in* (5.2)*. Then,*

(a) $H_1 = T\sum_{n\le x}\frac{\Lambda(n)^2}{n\log^2 n} f\left(\frac{\log n}{\log x}\right) + O\left(\frac{x\log\log x\log T}{\log^2 x}\right)$

(b) $H_2 = 2T\sum_{n\le x}\frac{\Lambda(n)^2}{n\log^2 n} f\left(\frac{\log n}{\log x}\right)\left[1-\cos\left(\frac{2\pi\delta\log n}{\log T}\right)\right] + O\left(\frac{x\log\log x\log T}{\log^2 x}\right)$

Proof. Assuming RH, let $0 < \delta \ll \log T$, and let $4 \le x \le T$. Then by the definition of H_1 in (5.2), observe that

$$\begin{aligned} H_1 &= 2\int_1^T \log|\zeta(\tfrac{1}{2}+it)| \sum_{n\le x}\frac{\Lambda(n)}{\log n}\frac{\cos(t\log n)}{n^{1/2}} f\left(\frac{\log n}{\log x}\right) \mathrm{d}t \\ &= 2\sum_{n\le x}\frac{\Lambda(n)}{n^{1/2}\log n} f\left(\frac{\log n}{\log x}\right)\int_1^T \log|\zeta(\tfrac{1}{2}+it)|\cos(t\log n)\,\mathrm{d}t. \end{aligned}$$

Using part (a) of Lemma 6.1.2, we know

$$\begin{aligned} H_1 &= T\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n} f\left(\frac{\log n}{\log x}\right) \\ &+ O\left(\sum_{n\le x}\frac{\Lambda(n)\log\log 3n}{\log n} f\left(\frac{\log n}{\log x}\right)\right) + O\left(\sum_{n\le x}\frac{\Lambda(n)\log T}{\log^2 n}\right). \end{aligned} \tag{6.5}$$

For the first error term on the right-hand side of (6.5), since $f(v)$ is uniformly bounded for $v \in (0,1]$, by Lemma 4.1.6 we have

$$\sum_{n\le x}\frac{\Lambda(n)\log\log 3n}{\log n} f\left(\frac{\log n}{\log x}\right) \ll \frac{x\log\log 3x}{\log x} \ll \frac{x\log\log x\log T}{\log^2 x}.$$

Similarly, by part (a) of Lemma 4.1.2, the second error term of (6.5) is

$$\sum_{n\le x}\frac{\Lambda(n)\log T}{\log^2 n} \ll \frac{x\log T}{\log^2 x}.$$

Combining the estimates of the error terms for (6.5) yields

$$H_1 \quad = \quad T\sum_{n\le x}\frac{\Lambda^2(n)}{n\log^2 n}f\left(\frac{\log n}{\log x}\right)+O\left(\frac{x\log\log x\log T}{\log^2 x}\right),$$

which completes part (a) of the proof. For part (b), we first let $L(t) := \log\left|\zeta(\frac{1}{2}+it)\right|$. Then we rearrange the terms of H_2, which is defined in (5.2), and use a change of variables to find that

$$\begin{aligned} H_2 &= 2\int_1^T \left[A(t+\tfrac{2\pi\delta}{\log T})-A(t)\right]\left[L(1+\tfrac{2\pi\delta}{\log T})-L(t)\right]\mathrm{d}t \\ &= -2\int_1^T L(t)\left[A(t+\tfrac{2\pi\delta}{\log T})+A(t-\tfrac{2\pi\delta}{\log T})-2A(t)\right]\mathrm{d}t \\ &\quad +O\left(\int_1^{1+\frac{2\pi\delta}{\log T}}|L(t)|\left|A(t)-A(t-\tfrac{2\pi\delta}{\log T})\right|\mathrm{d}t\right)+O\left(\int_T^{T+\frac{2\pi\delta}{\log T}}|L(t)|\left|A(t)-A(t-\tfrac{2\pi\delta}{\log T})\right|\mathrm{d}t\right). \end{aligned} \tag{6.6}$$

Notice the first error term in the previous expression is $O(1)$ since $\delta \ll \log T$. Now, observe by Lemma 4.1.4 that $A(t)$ is bounded pointwise by $\frac{\sqrt{x}}{\log x}$ for all $x \ge 2$. Thus, using Theorem 1.3.1 to bound the integral involving $|L(t)|$, by the Cauchy-Schwarz inequality, the second error term on the right-hand side of (6.6) gives

$$\int_T^{T+\frac{2\pi\delta}{\log T}} |L(t)|\,|A(t)-A(t-\tfrac{2\pi\delta}{\log T})|\,\mathrm{d}t \ll \frac{\sqrt{x}}{\log^{3/2}x}\left(\int_T^{T+\frac{2\pi\delta}{\log T}}|L(t)|^2\,\mathrm{d}t\right)^{1/2} \ll \frac{\sqrt{xT}}{\log^2 x}\ll\frac{T}{\log x},$$

since $4 \le x \le T$ and $\delta \ll \log T$. Using part (c) of Lemma 6.1.2, H_2 simplifies to

$$\begin{aligned}
H_2 &= -2\sum_{n\le x}\frac{\Lambda(n)f\left(\frac{\log n}{\log x}\right)}{n^{1/2}\log n}\int_0^T L(t)\,\Big[\cos\Big((t+\tfrac{2\pi\delta}{\log T})\log n\Big) \\
&\qquad + \cos\Big((t-\tfrac{2\pi\delta}{\log T})\log n\Big) - 2\cos(t\log n)\Big]\,\mathrm{d}t + O\left(\frac{T}{\log x}\right) \\
&= 2T\sum_{n\le x}\frac{\Lambda^2(n)f\left(\frac{\log n}{\log x}\right)}{n\log^2 n}\Big[1-\cos\Big(\tfrac{2\pi\delta}{\log T}\log n\Big)\Big] \\
&\quad + O\left(\sum_{n\le x}\frac{\Lambda(n)\log\log 3n}{\log n}\right) + O\left(\sum_{n\le x}\frac{\Lambda(n)\log T}{\log^2 n}\right) + O\left(\frac{T}{\log x}\right).
\end{aligned} \tag{6.7}$$

For the first error term on the right-hand side of (6.7), by Lemma 4.1.6, we know

$$\sum_{n\le x}\frac{\Lambda(n)\log\log 3n}{\log n} \ll \frac{x\log\log 3x}{\log x} \ll \frac{x\log\log x\log T}{\log^2 x}.$$

Again by part (a) of Lemma 4.1.2, the second error term on the right-hand side of (6.7) yields

$$\sum_{n\le x}\frac{\Lambda(n)\log T}{\log^2 n} \ll \frac{x\log T}{\log^2 x}.$$

Consequently, since $4 < x \le T$

$$H_2 = 2T\sum_{n\le x}\frac{\Lambda^2(n)f\left(\frac{\log n}{\log x}\right)}{n\log^2 n}\Big[1-\cos\Big(\tfrac{2\pi\delta\log n}{\log T}\Big)\Big] + O\left(\frac{x\log\log x\log T}{\log^2 x}\right),$$

as claimed. □

6.3 Estimating $H_i - G_i$

We now estimate $G_i + H_i$ asymptotically. We will obtain some cancellation in the sums due to the function g, which is defined in (2.4). In this section, we will deviate from the strategies of the previous work of Goldston and Fujii to obtain more precise input from the primes, which is necessary for Theorem 1.4.1.

Lemma 6.3.1 (Asymptotic estimate of $H_i - G_i$)**.** *Assume RH, let $T \geq 4$, and let $0 < \delta \ll \log T$. Fix $0 < \beta \leq 1$ such that $x = T^{\beta}$. Define the function $c(v)$ as in* (1.14)*. Then, as $T \to \infty$,*

(a) $$H_1 - G_1 = \frac{T}{2}\left\{\log\log T + \gamma_0 + \sum_{m=2}^{\infty}\sum_{p\geq 2}\frac{1}{p^m}\left(\frac{1}{m^2}-\frac{1}{m}\right) + \log\beta - \int_0^1 \alpha g(\alpha)^2 \mathrm{d}\alpha\right\} + O\left(\frac{T\log\log T}{\log T}\right)$$

(b) $$H_2 - G_2 = T\left\{\int_0^{2\pi\delta\beta}\frac{1-\cos u}{u}\mathrm{d}u + c\left(\frac{2\pi\delta}{\log T}\right) - \int_0^1 \alpha[1-\cos(2\pi\delta\beta\alpha)]g^2(\alpha)\mathrm{d}\alpha\right\} + o(T).$$

Proof. Let $T > 3$, and $0 < \delta \ll \log T$. Fix $0 < \beta \leq 1$ such that $x = T^{\beta}$. First, we consider the difference of $H_1 - G_1$. Recall the function $g(v)$ defined in (2.4), which also appears in the sums over the zeros of $\zeta(s)$. Note by Lemma 2.2.1, we have $vg(v) = 1 - f(v)$. This implies that

$$v^2 g^2(v) = f^2(v) - 2f(v) + 1. \tag{6.8}$$

Then writing the difference of Lemma (6.2.2) part (a) and Lemma (6.2.1) part (a) using (6.8) gives

$$\begin{aligned} H_1 - G_1 &= -\frac{T}{2}\sum_{n\leq x}\frac{\Lambda^2(n)}{n\log^2 n}\left(f^2\left(\frac{\log n}{\log x}\right) - 2f\left(\frac{\log n}{\log x}\right) - 1 + 1\right) \\ &\quad + O\left(\frac{x\log\log x\log T}{\log^2 x}\right) \\ &= \frac{T}{2}\sum_{n\leq x}\frac{\Lambda^2(n)}{n\log^2 n} - \frac{T}{2\log^2 x}\sum_{n\leq x}\frac{\Lambda^2(n)}{n}g^2\left(\frac{\log n}{\log x}\right) + O\left(\frac{T\log\log T}{\log T}\right) \\ &= S_1 + S_2 + O\left(\frac{T\log\log T}{\log T}\right) \end{aligned}$$

as $T \to \infty$. For S_1, by Lemma 4.1.3 we know

$$\begin{aligned} S_1 &= \frac{T}{2}\left(\log\log x + \gamma_0 + \sum_{m=2}^{\infty}\sum_{p}\left(\frac{1}{m^2} - \frac{1}{m}\right)\frac{1}{p^m} + O\left(\frac{1}{\log x}\right)\right) \\ &= \frac{T}{2}\left(\log\log T + \log\beta + \gamma_0 + \sum_{m=2}^{\infty}\sum_{p}\left(\frac{1}{m^2} - \frac{1}{m}\right)\frac{1}{p^m}\right) \\ &\quad +O\left(\frac{T}{\log T}\right). \end{aligned}$$

Next, for S_2, recall that $g(v)$ is uniformly bounded for $v \in [0,1]$. Also, by using Lemma 4.1.1 and partial summation, we simplify S_2 so that

$$S_2 = -\frac{T}{2}\int_0^1 \alpha\ g^2(\alpha)\ \mathrm{d}\alpha + O\left(\frac{T}{\log^2 T}\right).$$

Notice that we can extended the range of integration in the above integral to 0 using the fact that $g(v)$ is bounded. By combining our estimates for S_1 and S_2, we conclude that

$$\begin{aligned} H_1 - G_1 = \frac{T}{2}\Bigg\{&\log\log T + \gamma_0 + \sum_{m=2}^{\infty}\sum_{p\geq 2}\left(\frac{1}{m^2} - \frac{1}{m}\right)\frac{1}{p^m} + \log\beta - \int_0^1 \alpha g^2(\alpha)\mathrm{d}\alpha\Bigg\} \\ &+ O\left(\frac{T\log\log T}{\log T}\right). \end{aligned}$$

Hence the proof of part (a) is complete.

Similarly for part (b), we use (6.8) in the representation of $H_2 - G_2$, which results in

$$
\begin{aligned}
H_2 - G_2 = &-T \sum_{n \le x} \frac{\Lambda^2(n)}{n \log^2 n} \left[1 - \cos\Big(\tfrac{2\pi\delta \log n}{\log T}\Big)\right]\Big(f^2\Big(\frac{\log n}{\log x}\Big) - 2f\Big(\frac{\log n}{\log x}\Big) + 1\Big) \\
&+ T \sum_{n \le x} \frac{\Lambda^2(n)}{n \log^2 n} \left[1 - \cos\Big(\tfrac{2\pi\delta \log n}{\log T}\Big)\right] + o(T) \\
= &-T \sum_{n \le x} \frac{\Lambda^2(n)}{n \log^2 n} \left[1 - \cos\Big(\tfrac{2\pi\delta \log n}{\log T}\Big)\right] \frac{\log^2 n}{\log^2 x} g^2\Big(\frac{\log n}{\log x}\Big) \\
&+ T \sum_{n \le x} \frac{\Lambda^2(n)}{n \log^2 n} \left[1 - \cos\Big(\tfrac{2\pi\delta \log n}{\log T}\Big)\right] + o(T),
\end{aligned} \tag{6.9}
$$

as $T \to \infty$. For the first term on the right-hand side of (6.9), by Lemma 4.1.1, summation by parts, and integration by parts we obtain

$$
\begin{aligned}
\frac{-T}{\log^2 x} \sum_{n \le x} \frac{\Lambda^2(n) g^2\big(\frac{\log n}{\log x}\big)\left[1 - \cos\big(\frac{2\pi\delta \log n}{\log T}\big)\right]}{n} = &-T \int_0^1 \alpha \left[1 - \cos(2\pi\delta\beta\alpha)\right] g^2(\alpha) \mathrm{d}\alpha \\
&+ O\left(\frac{T}{\log x}\right).
\end{aligned} \tag{6.10}
$$

To estimate the second term on the right-hand side of (6.9), we use Lemma 4.1.8, which states that

$$
M(y) := \sum_{n \le y} \Lambda^2(n) = y \log y - y + E(y),
$$

where $E(y)$ is defined in (1.13) with $E(y) = O\left(\sqrt{y} \log^3 y\right)$ assuming RH. We will use summation by parts with the measure $dM(y)$. For this, we let $1 < \ell < 2$ be a parameter. We anticipate that we will eventually take $\ell \to 1^+$. Also, denote

$$
r := r(T) = \frac{2\pi\delta}{\log T} \text{ and } U(y) := \frac{1 - \cos(r \log y)}{y \log^2 y}.
$$

Then, the sum in the second term on the right-hand side of (6.9) is

$$\sum_{n\le x} \frac{\Lambda(n)^2}{n\log^2 n}[1-\cos(r\log n)] = \int_{\ell-}^{x} U(y)\mathrm{d}M(y) = U(y)M(y)\Big|_{\ell-}^{x} - \int_{\ell}^{x} M(y)U'(y)\mathrm{d}y,$$

where

$$U'(y) = \frac{r\log y\sin(r\log y) - (1-\cos(r\log y))(\log y+2)}{y^2\log^3 y}.$$

Integrating term by term with the change of variables $u = r\log y$, we find that

$$\begin{aligned} &T\sum_{n\le x}\frac{\Lambda(n)^2}{n\log^2 n}[1-\cos(r\log n)] \\ &= T\Bigg\{\int_{r\log\ell}^{r\log x}\frac{1-\cos u}{u}\mathrm{d}u - r\int_{r\log\ell}^{r\log x}\frac{\sin u}{u}\mathrm{d}u + r\int_{r\log\ell}^{\infty}\frac{1-\cos u}{u^2}\mathrm{d}u \\ &\quad + r^2\int_{r\log\ell}^{\infty}\frac{\sin u}{u^2}\mathrm{d}u - 2r^2\int_{h\log\ell}^{\infty}\frac{1-\cos u}{u^3}\mathrm{d}u \\ &\quad + \int_{\ell}^{\infty}\frac{E(y)}{y^2\log^3 y}[-r\log y\sin(r\log y) + (1-\cos(r\log y))(\log y+2)]\mathrm{d}y\Bigg\} \\ &\quad + o(T), \end{aligned} \tag{6.11}$$

where we use that $E(y) \ll y$ as $y\to\infty$ to extend the last integral above to infinity, up to an error term. Now we estimate the sum of four of the integrals on the right-hand side of (6.11), which we define below. Let

$$A := -r\int_{r\log\ell}^{r\log x}\frac{\sin u}{u}\mathrm{d}u + r\int_{r\log\ell}^{\infty}\frac{1-\cos u}{u^2}\mathrm{d}u + r^2\int_{r\log\ell}^{\infty}\frac{\sin u}{u^2}\mathrm{d}u - 2r^2\int_{r\log\ell}^{\infty}\frac{1-\cos u}{u^3}\mathrm{d}u. \tag{6.12}$$

We choose to express these integrals only in terms of the sine integral and cosine integrals, which are defined as in (1.16). Note that $\mathrm{Si}(\infty) = \frac{\pi}{2}$ and $\mathrm{Ci}(\infty) = 0$.

Additionally, we have the estimate

$$\mathrm{Si}(x) = \frac{\pi}{2} + O\left(\frac{1}{x}\right), \tag{6.13}$$

uniformly for any $x > 0$. We can verify the following antiderivatives:

$$\begin{aligned}
\int \frac{1-\cos u}{u}\,\mathrm{d}u &= \log(u) - \mathrm{Ci}(u),\\
\int \frac{1-\cos u}{u^2}\,\mathrm{d}u &= \mathrm{Si}(u) - \frac{1}{u} + \frac{\cos u}{u},\\
\int \frac{\sin(u)}{u^2}\,\mathrm{d}u &= \mathrm{Ci}(u) - \frac{\sin(u)}{u},\\
\int \frac{1-\cos u}{u^3}\,\mathrm{d}u &= \frac{\mathrm{Ci}(u)}{2} - \frac{1}{2u^2} + \frac{\cos u}{2u^2} - \frac{\sin(u)}{2u}.
\end{aligned}$$

By inserting these calculations into (6.12), we obtain the following:

$$A = \frac{r\pi}{2} - r\mathrm{Si}(r\log x) + (1-\cos(r\log \ell))\left(\frac{\log \ell - 1}{\log^2 \ell}\right).$$

By (6.13), note that $\frac{r\pi}{2} - r\mathrm{Si}(r\log x) = O(\frac{1}{\log x})$, uniformly in r. This implies

$$A = (1-\cos(r\log \ell))\left(\frac{\log \ell - 1}{\log^2 \ell}\right) + O\left(\frac{1}{\log x}\right).$$

Inserting this estimate for A into (6.11) gives

$$\begin{aligned}
&T\sum_{n\le x}\frac{\Lambda(n)^2}{n\log^2 n}[1-\cos(r\log n)]\\
&\quad = T\Bigg\{\int_{r\log \ell}^{r\log x}\frac{1-\cos u}{u}\mathrm{d}u + (1-\cos(r\log \ell))\left(\frac{\log \ell - 1}{\log^2 \ell}\right)\\
&\qquad + \int_{\ell}^{\infty}\frac{E(y)}{y^2\log^3 y}[-r\log y\sin(r\log y) + (1-\cos(r\log y))(\log y + 2)]\mathrm{d}y\Bigg\}\\
&\qquad + O\left(\frac{T}{\log x}\right).
\end{aligned} \tag{6.14}$$

Now, we let $\ell \to 1^+$. Note that

$$\lim_{\ell \to 1}(1 - \cos(r \log \ell))\frac{\log \ell - 1}{\log^2 \ell} = -\frac{r^2}{2}.$$

Additionally, since $E(y) = y - y\log y$ for all $1 \leq y < 2$ we know that in this range

$$\frac{E(y)}{y^2 \log^3 y}[-r\log y \sin(r\log y) + (1 - \cos(r\log y))(\log y + 2)] = \frac{r^2}{2} + O(r^2(y-1)).$$

This shows that the second integral is absolutely convergent on $(1, \infty)$. Therefore, recalling that $r = \frac{2\pi\delta}{\log T}$ and $x = T^\beta$, we may let $\ell \to 1^+$ in (6.14) to find that the second term on the right-hand side of (6.9) is

$$T\sum_{n\leq x}\frac{\Lambda(n)^2\left[1-\cos\left(\frac{2\pi\delta\ \log n}{\log T}\right)\right]}{n\log^2 n} = T\left\{\int_0^{2\pi\delta\beta}\frac{1-\cos u}{u}\mathrm{d}u + c\left(\frac{2\pi\delta}{\log T}\right)\right\} + O\left(\frac{T}{\log T}\right), \quad (6.15)$$

where $c(v)$ is defined as in (1.14). By combining (6.15) and (6.10), we complete the proof of Lemma 6.3.1.

□

7 CONTRIBUTION FROM THE ZEROS

In this chapter, we estimate expressions R_i, as defined in (5.2). In order to recover and estimate these expressions, we first need to handle one of the differences between our work and the work of Goldston [12]. As mentioned previously in Section 2.2, the function $h(v)$ is unbounded near $v = 0$. This implies that $|h[(t-\gamma)\log x]|$ is large whenever an ordinate γ of a zero of $\zeta(s)$ is close to t. Due to this fact, the arguments of Montgomery and Goldston do not apply directly in this case. However, we have already shown in Lemma 2.2.2 that $h \in L^1$, in particular that the singularity is integrable at the origin. Using this fact, we give two lemmas that allow us to estimate this sum for a sequence of T tending to infinity. We show that in every interval of length 1, there is a choice of T for which we can estimate R_i.

Lemma 7.0.1. *Assume RH. For $4 \le x \le T$, $\gamma \ne t$, and $\tau = |t| + 2$, we have*

$$\sum_{\gamma} |h[(t-\gamma)\log x]| = \sum_{|t-\gamma| \le \frac{1}{\log x}} |h[(t-\gamma)\log x]| + O(\log \tau). \tag{7.1}$$

Lemma 7.0.2. *Assuming RH, for $4 \le x \le T$, and $t \in [0, T]$ we have*

$$\sum_{\substack{\gamma \\ \gamma \notin [0,T]}} |h[(t-\gamma)\log x]| = \sum_{\gamma \in I} |h[(t-\gamma)\log x]| + O\Big(\Big[\tfrac{1}{T-t+1} + \tfrac{1}{T+1}\Big]\log T\Big), \tag{7.2}$$

where $I = \{\gamma : T < \gamma \le T + \frac{1}{\log x}\}$.

The proofs of Lemmas 7.0.1 and 7.0.2 are technical, but are similar to the corresponding estimates in Goldston [12, Eq. 3.4, Eq 3.6]. This is a modification of an argument originally due to Montgomery [22, pg. 187]. In Goldston's work, there are only the big-O terms on the right-hand sides of the above lemmas, and we do not need to consider separately the zeros near t. This is because in Goldston's case, for $S(t)$, the analogue of his function h in his sums is bounded. We follow Goldston's argument until he uses the assumption that his function is bounded. For this reason, we keep the unbounded terms on the right-hand side without estimating them. These correspond to the terms where t is near an ordinate γ of a zero of $\zeta(s)$.

Proof of Lemma 7.0.1. Consider the series of the function $|h[(t-\gamma)\log x]|$ over the ordinates, γ, such that $0 < \gamma \leq T$ and $\gamma \neq t$. Observe, by the definition of $h(v)$ in (2.5), that if $|v| > 1$ then $|h(v)| \ll \frac{1}{v^2}$. Note that if $|v| \leq 1$, then $|t-\gamma| \leq \frac{1}{\log x}$, and so only a finite number of zeros give us difficulty. Splitting the series under such conditions gives

$$\sum_{\gamma} |h[(t-\gamma)\log x]| = \sum_{|t-\gamma| \leq \frac{1}{\log x}} |h[(t-\gamma)\log x]| + \sum_{|t-\gamma| > \frac{1}{\log x}} |h[(t-\gamma)\log x]|. \tag{7.3}$$

Notice for the second sum on the right-hand side of (7.3), we can bound the function h such that $|h[(t-\gamma)\log x]| \ll \frac{1}{((t-\gamma)\log x)^2}$ on the interval $|t-\gamma| > \frac{1}{\log x}$. Furthermore, using results from Titchmarsh [32], Montgomery [22, pg. 187] proved that

$$\sum_{\gamma} \frac{1}{1+(t-\gamma)^2} \ll \log \tau.$$

Therefore,

$$\sum_{|t-\gamma| > \frac{1}{\log x}} |h[(t-\gamma)\log x]| \ll \frac{1}{\log^2 x} \sum_{|t-\gamma| > \frac{1}{\log x}} \frac{1}{(t-\gamma)^2} \ll \frac{1}{\log^2 x} \sum_{|t-\gamma| > \frac{1}{\log x}} \frac{\log^2 x}{1+(t-\gamma)^2} \ll \log \tau.$$

Plugging this estimate back into (7.3) completes the proof. ∎

Proof of Lemma 7.0.2. Assume RH, let $\epsilon > 0$ with $\epsilon = \frac{1}{\log x}$, and take $4 \leq x \leq T$. We consider cases on $t \in [0, T]$ for the sum

$$S := \sum_{\substack{\gamma \\ \gamma \notin [0,T]}} |h[(t-\gamma)\log x]|.$$

First, suppose that $\epsilon \leq t \leq T - \epsilon$. By definition of $h(v)$ in (2.5), recall for all $|v| > 1$, we have that $|h(v)| \ll \frac{1}{v^2}$. Then $\gamma \notin [0, T]$ implies that $|t - \gamma| > \frac{1}{\log x}$. Consequently,

$$|h[(t-\gamma)\log x]| \ll \frac{1}{(t-\gamma)^2 \log^2 x}.$$

Again, by the fact that there are $O(\log t)$ zeros in any given interval $[t, t+1]$, for $\epsilon \leq t \leq T - \epsilon$, the sum S reduces to

$$\begin{aligned}
S &\ll \frac{1}{\log^2 x} \sum_{k=T}^{\infty} \sum_{\gamma \in [k,k+1]} \frac{1}{(t-\gamma)^2} \\
&\ll \frac{1}{\log^2 x} \frac{\log T}{(t-T)^2} + \frac{1}{\log^2 x} \sum_{k=T+1}^{\infty} \frac{\log k}{(t-k)^2} \\
&\ll \frac{1}{\log^2 x} \int_{T+1}^{\infty} \frac{\log u}{(u-t)^2} \mathrm{d}u \\
&\ll \frac{1}{\log^2 x} \left[+ \frac{\log(T+1)}{t} + \frac{\log(T+1)}{T-t+1} - \frac{\log((T+1)-t)}{t} \right] \\
&\ll \left[\frac{1}{T+1-t} + \frac{1}{t+1} \right] \log T.
\end{aligned}$$

Next, suppose that $0 \leq t < \epsilon$. Since there are no zeros of $\zeta(s)$ near 0, we know that $|t-\gamma| > \frac{1}{\log x}$. Thus, using similar logic to that in the first case, we have

$$\begin{aligned} S &\ll \frac{1}{\log^2 x} \sum_{k=T}^{\infty} \sum_{\gamma \in [k,k+1]} \frac{1}{(-\gamma)^2} \\ &\ll \frac{1}{\log^2 x} \frac{\log T}{T^2} + \frac{1}{\log^2 x} \sum_{k=T+1}^{\infty} \frac{\log(k)}{k^2} \\ &\ll \frac{1}{\log^2 x} \int_{T+1}^{\infty} \frac{\log u}{u^2} \mathrm{d}u \\ &\ll \frac{1}{\log^2 x} \left[\frac{1}{T+1} + \frac{\log(T+1)}{T+1} \right] \\ &\ll \left[1 + \frac{1}{T+1} \right] \log T, \end{aligned}$$

which is the approximation we expect if t is close to 0. Finally, suppose $T-\epsilon < t \leq T$. Recall that $\epsilon = \frac{1}{\log x}$, and $|h(v)|$ is unbounded at $v = 0$. This means that $|h(t-\gamma)\log x|$ is large whenever

$$t - \frac{1}{\log x} \leq \gamma \leq t + \frac{1}{\log x}.$$

Since $T - \epsilon < t \leq T$, this means that $|h(t-\gamma)\log x|$ is large for γ such that

$$T - \epsilon - \frac{1}{\log x} = T - \frac{2}{\log x} < t - \frac{1}{\log x} \leq \gamma \leq t + \frac{1}{\log x} \leq T + \frac{1}{\log x}.$$

However, because $\gamma \notin [0,T]$, it must be the case that $T < \gamma \leq T + \frac{1}{\log x}$. Let $I = \{\gamma \ : \ T < \gamma \leq T + \frac{1}{\log x}\}$. Consequently

$$S = \sum_{\gamma \in I} |h[(t-\gamma)\log x]| + \sum_{\substack{\gamma \\ \gamma > T + \frac{1}{\log x}}} |h[(t-\gamma)\log x]| \tag{7.4}$$

whenever $T-\epsilon < t \le T$. Using similar logic to that in the first case, for the second sum on the right-hand side of (7.4), we have

$$\begin{aligned}\sum_{\substack{\gamma \\ \gamma > T + \frac{1}{\log x}}} |h[(t-\gamma)\log x]| &\ll \frac{1}{\log^2 x}\sum_{k=T+1}^{\infty}\frac{\log(k)}{(T-k)^2} \\ &\ll \frac{1}{\log^2 x}\int_{T+1}^{\infty}\frac{\log u}{(u-T)^2}\mathrm{d}u \\ &\ll \left[\frac{\log(T+1)}{T}+\log(T+1)\right]\frac{1}{\log^2 x} \\ &\ll \left[\frac{1}{T+1}+1\right]\log T,\end{aligned}$$

for $T-\epsilon < t \le T$. Therefore, for all $t \in [0,T]$, we have

$$S = \sum_{\gamma\in I}|h[(t-\gamma)\log x]| + O\left(\left[\frac{1}{T-t+1}+\frac{1}{T+1}\right]\log T\right), \tag{7.5}$$

which completes the proof. ☐

7.1 Unbounded discontinuities

In this section, our goal is to express R_i as a sum over pairs of zeros of $\zeta(s)$ and then apply Montgomery's pair correlation method to estimate R_i. The arguments of Montgomery and Goldston consist of localizing the sum in question to zeros within the interval [0, T], and then extending the integral in the definition of R_i in (5.2) to infinity, up to small errors. However, due to the unbounded discontinuity of our weight function h at the origin, we modify the aformentioned arguments using Lemmas 7.0.1 and 7.0.2. We then introduce a sequence of T_n's for which the following results will hold. The idea of using such a sequence is classical (for instance, see [8, Ch.17]). Since $N(T+1)-N(T) \ll \log T$, by the pigeonhole principle, for every

$n \in \mathbb{N}$ we can find a sequence $\{T_n\}$ satisfying

$$n \leq T < n+1 \text{ and } |\gamma - T_n| \gg \frac{1}{\log n}. \tag{7.6}$$

In this way, we obtain similar results to Goldston on a sequence of points tending to infinity, despite the unbounded discontinuty of our function h.

Lemma 7.1.1. *Let $T \in \{T_n\}$, where T_n satisfies* (7.6)*. Define k as in* (2.7) *and R_i as in* (5.2)*. For $4 \leq x \leq T$ and $0 < \delta \ll \log T$, we have*

(a) $R_1 = \dfrac{\pi^2}{\log x} \displaystyle\sum_{0<\gamma,\gamma' \leq T} \widehat{k}[(\gamma - \gamma') \log x] + O(\sqrt{T} \log^2 T)$,

(b) $R_2 = \dfrac{2\pi^2}{\log x} \displaystyle\sum_{0<\gamma,\gamma' \leq T} \left\{ \widehat{k}[(\gamma - \gamma') \log x] - \widehat{k}\left[\left(\gamma - \gamma' - \frac{2\pi\delta}{\log T}\right) \log x\right] \right\}$
$\qquad + O(\sqrt{T} \log^2 T)$.

The proofs of parts (a) and (b) are very similar. Part (b) is proved using a comparable argument to that used in part (a). Consequently, we only prove part (a).

Proof. Let $T \in T_n$, $4 \leq x \leq T$, and $0 < \delta \ll \log T$. We use logic similar to an argument in Montgomery's work [22, pg.187]. Recall the definition of R_1 from (5.2). We use Lemmas 7.0.1 and 7.0.2 to restrict the interval of zeros considered in the definition of R_1 to $\gamma, \gamma' \in [0, T]$. Then by expanding the integral, we rewrite R_1 as

$$
\begin{aligned}
R_1 &= \sum_{0<\gamma,\gamma'\le T} \int_1^T h[(t-\gamma)\log x]\ h[(t-\gamma')\log x]\,\mathrm{d}t \\
&\quad + O\Bigg(\int_1^T \sum_{\gamma\in I} |h[(t-\gamma)\log x]| \sum_{|t-\gamma'|\le \frac{1}{\log x}} |h[(t-\gamma')\log x]|\,\mathrm{d}t \Bigg) \\
&\quad + O\Bigg(\int_1^T \sum_{\gamma\in I} |h[(t-\gamma)\log x]| \log t\,\mathrm{d}t \Bigg) + O\Bigg(\log T \int_1^T \Big[\tfrac{1}{T-t+1} + \tfrac{1}{T+1}\Big] \log t\,\mathrm{d}t \Bigg) \\
&\quad + O\Bigg(\log T \int_1^T \Big[\tfrac{1}{T-t+1} + \tfrac{1}{T+1}\Big] \sum_{|t-\gamma'|\le \frac{1}{\log x}} |h[(t-\gamma')\log x]|\,\mathrm{d}t \Bigg), \qquad (7.7)
\end{aligned}
$$

where $I = \{\gamma : T < \gamma \le T + \frac{1}{\log x}\}$. Integrating the third error term on the right-hand side of (7.7) gives,

$$
\log T \int_1^T \Big[\tfrac{1}{T-t+1} + \tfrac{1}{T+1}\Big] \log t\,\mathrm{d}t \ll \log T \int_1^T \frac{\log t}{t}\,\mathrm{d}t \ll \log^3 T.
$$

Since $h \in L^1$, we use multiple places throughout this proof that

$$
\int_J |h[(t-\gamma')\log x]|\mathrm{d}x = \frac{1}{\log x} \int_{J'} |h(u)|\mathrm{d}u \ll \frac{1}{\log x} \int_{\mathbb{R}} |h(u)|\mathrm{d}u \ll \frac{1}{\log x},
$$

where J is any subset of $\mathbb{R}$ and J' is the subset obtained after the variable change. Using the facts that $h \in L^1$ and $|I| < 1$, the second error term on the right-hand side of (7.7) reduces to

$$
\begin{aligned}
\int_1^T \sum_{\gamma\in I} |h[(t-\gamma)\log x]| \log t\,\mathrm{d}t &\ll \log T \sum_{\gamma\in I} \int_{-\infty}^{\infty} |h[(t-\gamma)\log x]|\,\mathrm{d}t \\
&\ll \frac{\log T}{\log x} \sum_{\gamma\in I} 1 \\
&\ll \frac{\log^2 T}{\log x},
\end{aligned}
$$

since there are $O(\log T)$ zeros with $\gamma \in I$. Similarly, the fourth error term on the right-hand side of (7.7) yields

$$\log T \int_{1}^{T} \Big[\frac{1}{T-t+1} + \frac{1}{T+1}\Big] \sum_{|t-\gamma'|\le \frac{1}{\log x}} |h[(t-\gamma')\log x]| \, \mathrm{d}t$$

$$= \int_{1}^{T} \frac{\log T}{T-t+1} \sum_{|t-\gamma'|\le \frac{1}{\log x}} |h[(t-\gamma')\log x]| \, \mathrm{d}t + \int_{1}^{T} \frac{\log T}{T+1} \sum_{|t-\gamma'|\le \frac{1}{\log x}} |h[(t-\gamma')\log x]| \, \mathrm{d}t$$

$$= S_1 + S_2.$$

Since $h \in L^1$, after a variable change, we estimate S_2 as follows:

$$S_2 \ll \frac{\log T}{T\log x} \sum_{0\le\gamma'\le \frac{1}{\log x}+T} \int_{-\infty}^{\infty} |h(u)| \mathrm{d}u \ll \frac{\log T}{T\log x} \sum_{0\le\gamma'\le \frac{1}{\log x}+T} 1 \ll \frac{\log^2 T}{\log x}.$$

By far, one of the most delicate calculations in the proof is the estimation of S_1. We introduce a parameter H to split the range of integration for S_1, as follows:

$$S_1 = \log T \int_{1}^{T-H} \frac{1}{T-t+1} \sum_{|t-\gamma'|\le \frac{1}{\log x}} |h[(t-\gamma')\log x]| \, \mathrm{d}t$$

$$+ \log T \int_{T-H}^{T} \frac{1}{T-t+1} \sum_{|t-\gamma'|\le \frac{1}{\log x}} |h[(t-\gamma')\log x]| \, \mathrm{d}t$$

$$\ll \frac{\log T}{H+1} \sum_{0\le\gamma'\le T-H+\frac{1}{\log x}} \int_{1}^{T-H} |h[(t-\gamma')\log x]| \, \mathrm{d}t$$

$$+ \log T \sum_{T-H-\frac{1}{\log x}\le\gamma'\le T+\frac{1}{\log x}} \int_{T-H}^{T} |h[(t-\gamma')\log x]| \, \mathrm{d}t$$

$$\ll \frac{\log T}{H+1} \sum_{0\le\gamma'\le T-H+\frac{1}{\log x}} 1 \quad + \quad \log T \sum_{T-H-\frac{1}{\log x}\le\gamma'\le T+\frac{1}{\log x}} 1$$

$$\ll \frac{\left(T+\frac{1}{\log x}\right)\log^2\left(T+\frac{1}{\log x}\right)}{H+1} + \left(H+\frac{1}{\log x}\right)\log^2\left(T+\frac{1}{\log x}\right).$$

To balance these two error terms, we choose $H = \sqrt{T}$. Therefore,

$$S_1 \ll \sqrt{T} \log^2 \left(T + \tfrac{1}{\log x} \right) \ll \sqrt{T} \log^2 T.$$

By combining the estimates for S_1 and S_2, the third error term on the right-hand side of (7.7) yields

$$\log T \int_1^T \left[\tfrac{1}{T-t+1} + \tfrac{1}{T+1} \right] \sum_{|t-\gamma'| \leq \frac{1}{\log x}} |h[(t-\gamma') \log x]| \, \mathrm{d}t \ll \sqrt{T} \log^2 T. \tag{7.8}$$

For the first error term of (7.7), we again split the range of integration and find that

$$\begin{aligned} &\int_1^T \sum_{\gamma \in I} |h[(t-\gamma) \log x]| \sum_{|t-\gamma'| \leq \frac{1}{\log x}} |h[(t-\gamma') \log x]| \, \mathrm{d}t \\ &= \int_1^{T-1} \sum_{\gamma \in I} |h[(t-\gamma) \log x]| \sum_{|t-\gamma'| \leq \frac{1}{\log x}} |h[(t-\gamma') \log x]| \, \mathrm{d}t \\ &\quad + \int_{T-1}^{T} \sum_{\gamma \in I} |h[(t-\gamma) \log x]| \sum_{|t-\gamma'| \leq \frac{1}{\log x}} |h[(t-\gamma') \log x]| \, \mathrm{d}t \\ &= \Sigma_1 + \Sigma_2, \end{aligned}$$

say. For $\gamma \in I$ and $t \in [1, T-1]$, we know $|h[(t-\gamma) \log x]| \ll \frac{1}{(t-\gamma)^2 \log^2 x}$. Since $h \in L^1$, by an argument similar to the proof of Lemma 7.0.2, a variable change, and the bound from (7.8), we see that

$$\begin{aligned} \Sigma_1 &\ll \int_1^{T-1} \sum_{\gamma \in I} \frac{1}{(t-\gamma)^2 \log^2 x} \sum_{|t-\gamma'| \leq \frac{1}{\log x}} |h[(t-\gamma') \log x]| \, \mathrm{d}t \\ &\ll \int_1^{T-1} \left[\frac{1}{T-t+1} + \frac{1}{T+1} \right] \log T \sum_{|t-\gamma'| \leq \frac{1}{\log x}} |h[(t-\gamma') \log x]| \, \mathrm{d}t \\ &\ll \sqrt{T} \log^2 T, \end{aligned}$$

Because $T \in \{T_n\}$, we know that $|\gamma - T| \gg \frac{1}{\log T}$. Thus for $t \in I$, we have that $T-1 \le t \le T$ and $T < \gamma \le T + \frac{1}{\log x}$ implies $|t-\gamma| \gg \frac{1}{\log T}$. Since $|I| < 1$ and $\beta > 0$ is fixed, we know

$$\sum_{\gamma \in I} |h[(t-\gamma)\log x]| \ll \sum_{\gamma \in I} \left| h\left(\frac{\log x}{\log T}\right)\right| \ll |h(\beta)| \sum_{\gamma \in I} 1 \ll \log T.$$

Hence, because γ' is contained in an interval of size less than 1, it follows that

$$\begin{aligned}
\Sigma_2 &= \int_{T-1}^{T} \sum_{\gamma \in I} |h[(t-\gamma)\log x]| \sum_{|t-\gamma'| \le \frac{1}{\log x}} |h[(t-\gamma')\log x]| \, \mathrm{d}t \\
&\ll \log T \int_{T-1}^{T} \sum_{|t-\gamma'| \le \frac{1}{\log x}} |h[(t-\gamma')\log x]| \, \mathrm{d}t \\
&\ll \frac{\log T}{\log x} \sum_{T-1+\frac{1}{\log x} \le \gamma' \le T + \frac{1}{\log x}} \int_{-\infty}^{\infty} |h(u)| \mathrm{d}u \\
&\ll \frac{\log^2 T}{\log x}
\end{aligned}$$

for all $T \in \{T_n\}$. Hence combining our estimates for Σ_1 and Σ_2 gives

$$\int_{1}^{T} \sum_{\gamma \in I} |h[(t-\gamma)\log x]| \sum_{|t-\gamma'| \le \frac{1}{\log x}} |h[(t-\gamma')\log x]| \, \mathrm{d}t = \Sigma_1 + \Sigma_2 \ll \sqrt{T}\log^2 T.$$

Therefore, R_1 is confined to $\gamma, \gamma' \in [0, T]$ with an added error of $O(\sqrt{T}\log^2 T)$. Similarly, we extend the range of integration to $(-\infty, \infty)$ with the same error. Thus,

$$R_1 = \sum_{0 < \gamma, \gamma' \le T} \int_{-\infty}^{\infty} h[(t-\gamma)\log x] \; h[(t-\gamma')\log x] \, \mathrm{d}t + O(\sqrt{T}\log^2 T).$$

We now use the properties of $h(v)$ discussed in Chapter 2 to simplify our expression for R_1. Using (2.1), the fact that $h \in L^1$ and even, and the substitution $u = (t-\gamma')\log x$ with $a = (\gamma-\gamma')\log x$ gives

$$\begin{aligned} R_1 &= \sum_{0<\gamma,\gamma'\le T} \int_{-\infty}^{\infty} h((t-\gamma')\log x - (\gamma-\gamma')\log x)\ h[(t-\gamma')\log x]\,\mathrm{d}t \\ &\quad + O(\sqrt{T}\log^2 T) \\ &= \frac{1}{\log x}\sum_{0<\gamma,\gamma'\le T} \int_{-\infty}^{\infty} h(a-u)\ h(u)\,\mathrm{d}u + O(\sqrt{T}\log^2 T) \\ &= \frac{1}{\log x}\sum_{0<\gamma,\gamma'\le T} h * h(a) + O(\sqrt{T}\log^2 T). \end{aligned}$$

Note that convolution here is well-defined since $\widehat{h} \in L^2$, and therefore $h \in L^2$ by Plancherel's theorem. Furthermore, using Lemma 2.2.2, we know $h \in L^1$ implies $\widehat{h*h} = \hat{h}^2$ and $\widehat{h} \in L^2$ implies $k(\xi) = \frac{1}{\pi^2}\widehat{h}(\xi)^2 \in L^1(\mathbb{R})$. Hence k also has a well-defined Fourier transform. Thus, by Lemma 2.2.2, (2.2), (2.7), and properties of Fourier Transform, we have

$$\begin{aligned} R_1 &= \frac{1}{\log x}\sum_{0<\gamma,\gamma'\le T} \widehat{(\widehat{h}^2)}(a) + O(\sqrt{T}\log^2 T) \\ &= \frac{\pi^2}{\log x}\sum_{0<\gamma,\gamma'\le T} \hat{k}[(\gamma-\gamma')\log x] + O(\sqrt{T}\log^2 T), \end{aligned}$$

as claimed. □

7.2 A modified pair correlation approach

The next step is to introduce the weight function $w(u)$, from (1.6), by writing each R_i in Lemma 7.1.1 in terms of Montgomery's function $F(\alpha)$. Often, the weight $w(u)$ can be added or dropped from sums over pairs of zeros up to a small error, and this can be done for R_1 as in Goldston's work. However, for R_2, when the shift $\delta \gg \log T$

is large, the error term is no longer negligible. Therefore, in the following lemma, we are naturally led to incorporate the weight in a different way, and then use the function $F_\delta(\alpha)$ introduced by Chan instead of Montgomery's $F(\alpha)$ for R_2.

Lemma 7.2.1. *Assume RH and define k as in* (2.7). *Fix $0 < \beta \le 1$, and choose $x = T^\beta$. For $T \ge 4$, $0 < \delta \ll \log T$, and $T \in \{T_n\}$, where T_n satisfies* (7.6), *we have*

$$\text{(a)}\ R_1 = \frac{T}{(2\beta)^2} \int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big) F(\alpha) \mathrm{d}\alpha + O\left(\frac{T}{\log T}\right);$$

$$\text{(b)}\ R_2 = \frac{T}{2\beta^2} \int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big) \left[F(\alpha) - F_\delta(\alpha)\right] \mathrm{d}\alpha + O\left(\frac{T}{\log T}\right).$$

Proof. The proofs of the expressions in parts (a) and (b) are proved using similar methods, but the proof of part (b) is more involved. For this reason, we only work out part (b). Let $T \in \{T_n\}$, fix $0 < \beta \le 1$, and choose $x = T^\beta$ for $T \ge 4$. Also, recall that k is the function defined in (2.7). Then by Lemma 2.2.3, we have that $\widehat{k}(y) \ll \min(1, \frac{1}{y^2})$. From this estimate we introduce the weight function $w(u)$, defined in (1.6), into the sum over zeros

$$\sum_{0<\gamma,\gamma' \le T} \widehat{k}[(\gamma - \gamma') \log x]$$

using the following argument. First, we consider the difference

$$D_1 := \sum_{0<\gamma,\gamma' \le T} \widehat{k}\left((\gamma - \gamma) \log x\right) - \sum_{0<\gamma,\gamma' \le T} \widehat{k}\left((\gamma - \gamma) \log x\right) w(\gamma - \gamma').$$

Using (1.3) and the fact that there are $O(\log t)$ zeros in any given interval $[t, t+1]$, using an argument of Goldston [12, pg. 161], we notice that

$$D_1 = \sum_{0<\gamma,\gamma' \le T} \widehat{k}\left((\gamma - \gamma) \log x\right) (1 - w(\gamma - \gamma')) \ll \frac{1}{\log^2 x} \sum_{0<\gamma' \le T} \sum_{\gamma} \frac{1}{4 + (\gamma - \gamma')^2} \ll T.$$

Thus, by (3.1) and definition of Fourier transform, we have

$$\begin{aligned}\sum_{0<\gamma,\gamma'\le T}\hat{k}\left((\gamma-\gamma)\log x\right) &= \sum_{0<\gamma,\gamma'\le T}\hat{k}\left((\gamma-\gamma)\log x\right)w(\gamma-\gamma')+O(T)\\ &= \int_{-\infty}^{\infty}k(u)\sum_{0<\gamma,\gamma'\le T}T^{-2\pi\beta iu(\gamma-\gamma')}w(\gamma-\gamma')\mathrm{d}u+O(T)\\ &= \frac{T\log T}{(2\pi)^2\beta}\int_{-\infty}^{\infty}k\Big(\frac{\alpha}{2\pi\beta}\Big)F(\alpha)\mathrm{d}\alpha+O(T). \qquad (7.9)\end{aligned}$$

Next we shift the weight function $w(u)$ by a factor of $\frac{2\pi\delta}{\log T}$ for $0<\delta\ll\log T$. We then introduce the shifted weight $w(u)$ into the sum

$$\sum_{0<\gamma,\gamma'\le T}\widehat{k}\left[\left(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\right)\log x\right],$$

and we consider the difference

$$D_2:=\widehat{k}\Big[\Big(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\Big)\log x\Big]-\widehat{k}\Big[\Big(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\Big)\log x\Big]w\Big(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\Big).$$

By (1.3), since there are $O(\log t)$ zeros in any given interval $[t,t+1]$, and since $\frac{2\pi\delta}{\log T}\ll 1$ we have

$$\begin{aligned}D_2 &= \sum_{0<\gamma,\gamma'\le T}\widehat{k}\Big[\big(\gamma-\gamma'-\tfrac{2\pi\delta}{\log T}\big)\log x\Big]\Big(1-w\Big(\gamma-\gamma'-\tfrac{2\pi\delta}{\log T}\Big)\Big)\\ &\ll \frac{1}{\log^2 x}\sum_{0<\gamma'\le T}\sum_{\gamma}\frac{1}{4+(\gamma-\gamma'-\frac{2\pi\delta}{\log T})^2}\\ &\ll \frac{1}{\log^2 x}\sum_{0<\gamma'\le T}\sum_{\gamma}\frac{1}{4+(\gamma-\gamma')^2}\\ &\ll T.\end{aligned}$$

Recall by (1.12) that

$$F_\delta(\alpha) = F_\delta(\alpha,\, T) = \frac{2\pi}{T\log T}\sum_{0<\gamma,\,\gamma'\le T} T^{i\alpha\left(\gamma-\gamma'-\frac{2\pi\delta}{\log T}\right)} w\Big(\gamma-\gamma'-\tfrac{2\pi\delta}{\log T}\Big).$$

Thus, by (3.1) and definition of Fourier transform, we have

$$\begin{aligned}
&\sum_{0<\gamma,\gamma'\le T} \widehat{k}\Big[\big(\gamma-\gamma'-\tfrac{2\pi\delta}{\log T}\big)\log x\Big] \\
&\quad = \sum_{0<\gamma,\gamma'\le T} \hat{k}\left(\big(\gamma-\gamma'-\tfrac{2\pi\delta}{\log T}\big)\log x\right) w\Big(\gamma-\gamma'-\tfrac{2\pi\delta}{\log T}\Big) + O(T) \\
&\quad = \int_{-\infty}^{\infty} k(u) \sum_{0<\gamma,\gamma'\le T} T^{-2\pi\beta iu(\gamma-\gamma'-\frac{2\pi\delta}{\log T})} w(\gamma-\gamma'-\tfrac{2\pi\delta}{\log T})\mathrm{d}u + O(T) \\
&\quad = \frac{T\log T}{(2\pi)^2\beta}\int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)F_\delta(\alpha)\mathrm{d}\alpha + O(T).
\end{aligned} \tag{7.10}$$

By Lemma 7.1.1 part (b), we combine (7.9) and (7.10) to yield

$$\begin{aligned}
R_2 &= \frac{2\pi^2}{\log x}\sum_{0<\gamma,\gamma'\le T}\Big\{\widehat{k}[(\gamma-\gamma')\log x] - \widehat{k}\left[\big(\gamma-\gamma'-\tfrac{2\pi\delta}{\log T}\big)\log x\right]\Big\} + O(\sqrt{T}\log^2 T) \\
&= \frac{2\pi^2}{\beta\log T}\left[\frac{T\log T}{(2\pi)^2\beta}\int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)\left[F(\alpha)-F_\delta(\alpha)\right]\mathrm{d}\alpha + O\left(T\right)\right] + O(\sqrt{T}\log^2 T) \\
&= \frac{T}{2\beta^2}\int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)\left[F(\alpha)-F_\delta(\alpha)\right]\mathrm{d}\alpha + O\Big(\frac{T}{\log T}\Big),
\end{aligned}$$

which completes the proof of part (b). The proof of part (a) follows analogously. □

7.3 Estimating R_i

Using Lemma 7.2.1 and the properties of $F(\alpha)$ and $F_\delta(\alpha)$ from Chapter 3, we again choose $x = T^\beta$ and proceed to estimate R_i.

Lemma 7.3.1 (Asymptotic estimate of R_i)**.** *Assume RH, fix $0 < \beta \leq 1$, and let g be defined in (2.4). For $T \geq 4$, $x = T^\beta$, $0 < \delta \ll \log T$, and $T \in \{T_n\}$, where T_n satisfies (7.6), we have*

(a) $$R_1 = \frac{T}{2}\left\{\int_0^1 v\ g^2\,(v)\,\mathrm{d}v - \log\beta + \frac{\log^2 2}{2\beta^2} + \int_1^\infty \frac{F(\alpha)}{\alpha^2}\mathrm{d}\alpha\right\} + o(T).$$

(b) $$R_2 = T\left\{\int_0^1 v\,g^2\,(v)\left[1 - w\left(\frac{2\pi\delta}{\log T}\right)\cos(2\pi\delta v\beta)\right]\mathrm{d}v - \log\beta \right.$$
$$\left. - w\left(\frac{2\pi\delta}{\log T}\right)\int_{2\pi\delta\beta}^{2\pi\delta}\frac{\cos u}{u}\mathrm{d}u + \frac{1}{2}\int_1^\infty \frac{2F(\alpha) - F_\delta(\alpha) - F_{-\delta}(\alpha)}{\alpha^2}\mathrm{d}\alpha\right\} + o(T),$$

where the term of $o(T)$ is of size $O\left(\frac{T\sqrt{\log\log T}}{\sqrt{\log T}}\right)$.

Proof. Let $T \in \{T_n\}$, fix $0 < \beta \leq 1$, and choose $x = T^\beta$ for $T \geq 4$. Recall $k(u)$ is a piecewise function, as defined in (2.7), so we consider have to consider two separate ranges for u. Since $F(\alpha)$ and $k(u)$ are both even and nonnegative functions, by Theorem 3.1.1, we find that

$$\begin{aligned}
\int_{-\infty}^\infty k\left(\frac{\alpha}{2\pi\beta}\right)F(\alpha)\mathrm{d}\alpha &= 2\int_0^\beta k\left(\frac{\alpha}{2\pi\beta}\right)\left[\alpha + o(1) + T^{-2\alpha}\log T(1+o(1))\right]\mathrm{d}\alpha \\
&+ 2\int_\beta^1 \left(\frac{\beta}{\alpha}\right)^2\left[\alpha + o(1) + T^{-2\alpha}\log T(1+o(1))\right]\mathrm{d}\alpha \\
&+ 2\int_1^\infty \left(\frac{\beta}{\alpha}\right)^2 F(\alpha)\mathrm{d}\alpha. \qquad (7.11)
\end{aligned}$$

For the second integral on the right-hand side in (7.11), because β is fixed

$$2\int_\beta^1 \left(\frac{\beta}{\alpha}\right)^2\left[\alpha + o(1) + T^{-2\alpha}\log T(1+o(1))\right]\mathrm{d}\alpha = -2\beta^2\log\beta + o(1). \qquad (7.12)$$

Similarly, the first integral on the right-hand side of (7.11) is

$$2\int_0^\beta k\Big(\frac{\alpha}{2\pi\beta}\Big)\left[\alpha + o(1) + T^{-2\alpha}\log T(1+o(1))\right]\mathrm{d}\alpha$$

$$= 2\log T\left[\left.\frac{k\left(\frac{\alpha}{2\pi\beta}\right)}{-2\log T T^{2\alpha}}\right|_0^\beta + \int_0^\beta \frac{1}{4\pi\beta\log T}k'\left(\frac{\alpha}{2\pi\beta}\right)T^{-2\alpha}\mathrm{d}\alpha\right]$$

$$+2\int_0^\beta \alpha\ k\Big(\frac{\alpha}{2\pi\beta}\Big)\mathrm{d}\alpha + o(1), \tag{7.13}$$

since $0 < \beta \leq 1$, k is uniformly bounded, and the length of the interval is finite. Using the fact that $k(0) = \log^2 2$ from (2.8), straightforward but technical manipulations show that the first term on the right-hand side of (7.13) is

$$2\log T\left[\left.\frac{-k\left(\frac{\alpha}{2\pi\beta}\right)}{2T^{2\alpha}\log T}\right|_0^\beta + \int_0^\beta \frac{T^{-2\alpha}}{4\pi\beta\log T}k'\left(\frac{\alpha}{2\pi\beta}\right)\mathrm{d}\alpha\right] = \log^2 2 + o(1).$$

Finally, the second term on the right-hand side of (7.13) yields

$$2\int_0^\beta \alpha\ k\Big(\frac{\alpha}{2\pi\beta}\Big)\mathrm{d}\alpha = 2\int_0^\beta \alpha\ g^2\left(\frac{\alpha}{\beta}\right)\mathrm{d}\alpha = 2\beta^2\int_0^1 v\ g^2(v)\,\mathrm{d}v.$$

By combining the above terms, the first integral on the right-hand side of (7.11) is

$$2\int_0^\beta k\Big(\frac{\alpha}{2\pi\beta}\Big)\left[\alpha + o(1) + \frac{\log T}{T^{2\alpha}}(1+o(1))\right]\mathrm{d}\alpha = 2\beta^2\int_0^1 vg^2(v)\mathrm{d}v + \log^2 2 + o(1). \tag{7.14}$$

Inputting the estimates (7.12) and (7.14) into (7.11) yields

$$\int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)F(\alpha)\mathrm{d}\alpha = 2\beta^2\int_0^1 vg^2(v)\mathrm{d}v - 2\beta^2\log\beta + \log^2 2 + 2\beta^2\int_1^\infty \frac{F(\alpha)}{\alpha^2}\mathrm{d}\alpha + o(1). \tag{7.15}$$

Finally inputing (7.15) into the respentation for R_1 in Lemma 7.2.1 concludes the proof of part (a). For part (b) we recall from Lemma 7.2.1 that

$$R_2 = \frac{T}{2\beta^2}\left\{\int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)F(\alpha)\mathrm{d}\alpha - \int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)F_\delta(\alpha)\mathrm{d}\alpha\right\} + O\left(\frac{T}{\log T}\right).$$

Splitting the second integral on the right-hand side using Theorem 3.2.1 for F_δ yields

$$-\frac{T}{2\beta^2}\int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)F_\delta(\alpha)\mathrm{d}\alpha = -\frac{T}{2\beta^2}\int_{0}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)[F_\delta(\alpha)+F_{-\delta}(\alpha)]\mathrm{d}\alpha. \tag{7.16}$$

Next, we divide the integral over the intervals $(0,\,\beta)$, $(\beta,\,1)$, and $(1,\,\infty)$, and apply (3.2.1). Since $T^{i\alpha\frac{2\pi\delta}{\log T}} + T^{-i\alpha\frac{2\pi\delta}{\log T}} = 2\cos(2\pi\alpha\delta)$ and $k(0) = \log^2 2$, we obtain:

$$\begin{aligned}&-\frac{T}{2\beta^2}\int_{0}^{\beta} k\Big(\frac{\alpha}{2\pi\beta}\Big)[F_\delta(\alpha)+F_{-\delta}(\alpha)]\mathrm{d}\alpha\\ &\qquad = -\frac{T\log^2 2}{2\beta^2} - w\left(\frac{2\pi\delta}{\log T}\right)\int_0^1 vg(v)^2\cos(2\pi\delta v\beta)\mathrm{d}v + o(T),\end{aligned}$$

and

$$-\frac{T}{2\beta^2}\int_{\beta}^{1} k\Big(\frac{\alpha}{2\pi\beta}\Big)[F_\delta(\alpha)+F_{-\delta}(\alpha)]\mathrm{d}\alpha = -w\left(\frac{2\pi\delta}{\log T}\right)\int_{2\pi\beta\delta}^{2\pi\delta}\frac{\cos u}{u}\mathrm{d}u + o(T).$$

By combining the above integrals, we have that

$$\begin{aligned}&-\frac{T}{2\beta^2}\int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)F_\delta(\alpha)\mathrm{d}\alpha\\ &\qquad = -T\left\{\frac{\log^2 2}{2\beta^2} + w\left(\frac{2\pi\delta}{\log T}\right)\int_0^1 vg(v)^2\cos(2\pi\delta v\beta)\mathrm{d}v\right.\\ &\qquad\quad \left. + w\left(\frac{2\pi\delta}{\log T}\right)\int_{2\pi\beta\delta}^{2\pi\delta}\frac{\cos u}{u}\mathrm{d}u + \frac{1}{2}\int_1^\infty \frac{F_\delta(\alpha)+F_{-\delta}(\alpha)}{\alpha^2}\mathrm{d}\alpha\right\}\\ &\qquad\quad + o(T).\end{aligned} \tag{7.17}$$

From the proof of part (a), we know that

$$\frac{T}{2\beta^2}\int_{-\infty}^{\infty} k\Big(\frac{\alpha}{2\pi\beta}\Big)F(\alpha)\,\mathrm{d}\alpha = T\left[\int_0^1 v\ g^2(v)\,\mathrm{d}v - \log\beta + \frac{\log^2 2}{2\beta^2} + \int_1^{\infty}\frac{F(\alpha)}{\alpha^2}\mathrm{d}\alpha\right] + o(T). \tag{7.18}$$

By adding (7.17) and (7.18) together, our asymptotic formula for R_2 reduces to

$$\begin{aligned} R_2 =& T\left[\int_0^1 v\ g^2(v)\left(1 - w\Big(\frac{2\pi\delta}{\log T}\Big)\cos(2\pi\delta v\beta)\right)\mathrm{d}v - \log\beta \right.\\ &\left. - w\Big(\frac{2\pi\delta}{\log T}\Big)\int_{2\pi\beta\delta}^{2\pi\delta}\frac{\cos u}{u}\mathrm{d}u + \frac{1}{2}\int_1^{\infty}\frac{2F(\alpha) - F_\delta(\alpha) - F_{-\delta}(\alpha)}{\alpha^2}\mathrm{d}\alpha\right] + o(T), \end{aligned}$$

which completes the proof. □

7.4 Proofs of Theorems 1.3.1, 1.4.1, and 1.5.1

Finally, in this section we explain how the two main results of this thesis follow from the results in Chapters 5, 6, and 7. For $T \in \{T_n\}$, the proof of Theorem 1.3.1 follows from inputting part (a) of Lemmas 6.3.1 and 7.3.1 into the representation formula for $|\log\zeta(1/2+it)|$, which we proved in part (a) of Proposition 5.1.2. Some of the integrals in these results are over the interval $[1,T]$, but these can easily be extended to $[0,T]$ since

$$\int_0^1 \log^2|\zeta(\tfrac{1}{2}+it)|\ dt \ll 1.$$

In particular, Theorem 1.3.1 holds for all $T \in \{T_n\}$ such that $T \geq 4$. We now extend this result to hold for all $T \geq 4$.

Assume $T_n \le T \le T_{n+1}$. Because the integrand in Theorem 1.3.1 is positive, we know that

$$\int_0^{T_n} \log^2 \left|\zeta\left(\tfrac{1}{2}+it\right)\right| \,\mathrm{d}t \le \int_0^{T} \log^2 \left|\zeta\left(\tfrac{1}{2}+it\right)\right| \,\mathrm{d}t \le \int_0^{T_{n+1}} \log^2 \left|\zeta\left(\tfrac{1}{2}+it\right)\right| \,\mathrm{d}t$$

Moreover, because both T_n and T_{n+1} are at most 1 away from T and Theorem 1.3.1 holds for T_n and T_{n+1}, we know that

$$\int_0^{T_n} \log^2 \left|\zeta\left(\tfrac{1}{2}+it\right)\right| \,\mathrm{d}t = \frac{T_n}{2}\log\log T_n + aT_n = \frac{T}{2}\log\log T + aT + o(T),$$

and

$$\int_0^{T_{n+1}} \log^2 \left|\zeta\left(\tfrac{1}{2}+it\right)\right| \,\mathrm{d}t = \frac{T_{n+1}}{2}\log\log T_{n+1} + aT_{n+1} = \frac{T}{2}\log\log T + aT + o(T).$$

Therefore, it follows that

$$\int_0^{T} \log^2 \left|\zeta\left(\tfrac{1}{2}+it\right)\right| \,\mathrm{d}t = \frac{T}{2}\log\log T + aT + o(T),$$

which completes the proof of Theorem 1.3.1 for all $T \ge 4$.

For the proof of Theorem 1.4.1, when we input part (b) of Lemmas 6.3.1 and 7.3.1 into part (b) of Proposition 5.1.2, we get

$$\begin{aligned}
&\int_1^T \left[\log\left|\zeta\left(\frac{1}{2}+it+i\frac{2\pi\delta}{\log T}\right)\right| - \log\left|\zeta(\frac{1}{2}+it)\right|\right]^2 \mathrm{d}t \\
&\quad = T\left\{\int_0^{2\pi\delta\beta} \frac{1-\cos u}{u}\mathrm{d}u - w\left(\frac{2\pi\delta}{\log T}\right)\int_{2\pi\delta\beta}^{2\pi\delta} \frac{\cos u}{u}\mathrm{d}u - \log\beta \right. \\
&\qquad + \int_0^1 v\; g^2(v)\cos(2\pi\delta v\beta)\left(1-w\left(\frac{2\pi\delta}{\log T}\right)\right)\mathrm{d}v \\
&\qquad \left. + c\left(\frac{2\pi\delta}{\log T}\right) + \frac{1}{2}\int_1^\infty \frac{2F(\alpha)-F_\delta(\alpha)-F_{-\delta}(\alpha)}{\alpha^2}\mathrm{d}\alpha \right\} + o(T).
\end{aligned} \tag{7.19}$$

Because our results hold independently of our choice of β, there should be no β dependence in our final result. We then combine the first three terms on the right-hand side of (7.19) to yield

$$\begin{aligned}
&\int_0^{2\pi\delta\beta} \frac{1-\cos u}{u}\mathrm{d}u - w\left(\frac{2\pi\delta}{\log T}\right)\int_{2\pi\delta\beta}^{2\pi\delta} \frac{\cos u}{u}\mathrm{d}u - \log\beta \\
&\quad = \int_0^{2\pi\delta\beta} \frac{1-\cos u}{u}\mathrm{d}u + w\left(\frac{2\pi\delta}{\log T}\right)\int_{2\pi\delta\beta}^{2\pi\delta} \frac{1-\cos u}{u}\mathrm{d}u - w\left(\frac{2\pi\delta}{\log T}\right)\int_{2\pi\delta\beta}^{2\pi\delta} \frac{1}{u}\mathrm{d}u - \log\beta \\
&\quad = \int_0^{2\pi\delta} \frac{1-\cos u}{u}\mathrm{d}u + \left(w\left(\frac{2\pi\delta}{\log T}\right) - 1\right)\int_{2\pi\delta\beta}^{2\pi\delta} \frac{1-\cos u}{u}\mathrm{d}u + \left(w\left(\frac{2\pi\delta}{\log T}\right) - 1\right)\log\beta \\
&\quad = \int_0^{2\pi\delta} \frac{1-\cos u}{u}\mathrm{d}u - \left(w\left(\frac{2\pi\delta}{\log T}\right) - 1\right)\int_{2\pi\delta\beta}^{2\pi\delta} \frac{\cos u}{u}\mathrm{d}u \\
&\quad = \int_0^{1} \frac{1-\cos(2\pi\delta\alpha)}{\alpha}\mathrm{d}\alpha + O\left(\frac{\delta}{\log^2 T}\right).
\end{aligned}$$

Next we consider the integral involving $g^2(v)$ on the right-hand side of (7.19). Let $\ell(v) = vg^2(v)$. Then using integration by parts, we see that

$$\begin{aligned}
&\int_0^1 v\ g^2(v)\cos(2\pi\delta v\beta)\left(1 - w\left(\frac{2\pi\delta}{\log T}\right)\right)\mathrm{d}v \\
&\quad = \left(1 - w\left(\frac{2\pi\delta}{\log T}\right)\right)\left[\ell(v)\,\frac{\sin(2\pi\delta v\beta)}{2\pi\delta\beta}\bigg|_0^1 - \int_0^1 \frac{\ell'(v)\sin(2\pi\delta v\beta)}{2\pi\delta\beta}\mathrm{d}v\right] \\
&\quad \ll \frac{\delta}{\log^2 T}.
\end{aligned}$$

Combining these simplifed expressions together gives

$$\int_1^T \left[\log\left|\zeta\left(\frac{1}{2}+it+i\frac{2\pi\delta}{\log T}\right)\right| - \log\left|\zeta\left(\frac{1}{2}+it\right)\right|\right]^2 \mathrm{d}t$$
$$= T\left\{\int_0^1 \frac{1-\cos(2\pi\delta\alpha)}{\alpha}\,\mathrm{d}\alpha + \frac{1}{2}\int_1^\infty \frac{2F(\alpha)-F_\delta(\alpha)-F_{-\delta}(\alpha)}{\alpha^2}\,\mathrm{d}\alpha\right\}$$
$$+ T\,c\left(\frac{2\pi\delta}{\log T}\right) + o(T);$$

We then extend the range of integration to [0,T] since

$$\int_0^1 \left[\log\left|\zeta\left(\frac{1}{2}+it+i\frac{2\pi\delta}{\log T}\right)\right| - \log\left|\zeta\left(\frac{1}{2}+it\right)\right|\right]^2 \mathrm{d}t \ll 1.$$

This completes the proof of Theorem 1.4.1 for $T \in \{T_n\}$. Since the integrand is nonnegative, this result can be extended to all $T \geq 4$ using the same argument as above.

With the proofs of Theorems 1.3.1 and 1.4.1 completed, we are ready to prove the new case of Berry's conjecture in Theorem 1.5.1.

Proof of Theorem 1.5.1. Let $\delta \asymp \log T$. We want to show that

$$\pi^2\int_0^T \left[S\left(t+\frac{2\pi\delta}{\log T}\right) - S(t)\right]^2 \mathrm{d}t = T\left[\sum_{n\leq T}\frac{\Lambda^2(n)}{n\log^2 n}\left(1-\cos\left(\frac{2\pi\delta\,\log n}{\log T}\right)\right)+1\right]+o(T).$$

First, note that by taking $x=T$ and $\beta=1$ in (6.15) and using a change of variables in the integral, we have that

$$T\sum_{n\leq T}\frac{\Lambda(n)^2}{n\log^2 n}\left[1-\cos\left(\frac{2\pi\delta\,\log n}{\log T}\right)\right] = T\left\{\int_0^1 \frac{1-\cos 2\pi\delta u}{u}\mathrm{d}u + c\left(\frac{2\pi\delta}{\log T}\right)\right\}$$
$$+ O\left(\frac{T}{\log T}\right),$$

where $c(v)$ is defined in (1.14). Therefore, to prove part (b) of Conjecture 1, by Theorem 1.4.1, it is enough to show that

$$\frac{1}{2}\int_{1}^{\infty} \frac{2F(\alpha) - F_{\delta}(\alpha) - F_{-\delta}(\alpha)}{\alpha^2}\, \mathrm{d}\alpha = 1 + o(1).$$

By Conjecture 2, we have

$$\frac{1}{2}\int_{1}^{\infty} \frac{2F(\alpha) - F_{\delta}(\alpha) - F_{-\delta}(\alpha)}{\alpha^2}\, \mathrm{d}\alpha = \int_{1}^{\infty} \frac{1 - \cos(2\pi\delta\alpha)\, w\left(\frac{2\pi\delta}{\log T}\right)}{\alpha^2}\, \mathrm{d}\alpha + o(1).$$

We see this by noting that Conjecture 2 applies uniformly in compact sets. We then integrate from 1 to M, and let $M \to \infty$. First note that

$$\int_{1}^{\infty} \frac{1}{\alpha^2}\, \mathrm{d}\alpha = 1.$$

Now, integrating by parts with the substitution $u = \frac{1}{\alpha^2} w\left(\frac{2\pi\delta}{\log T}\right)$ and $\mathrm{d}v = \cos(2\pi\delta\alpha)$, we find that

$$\begin{aligned}\int_{1}^{\infty} \frac{\cos(2\pi\delta\alpha)\, w\left(\frac{2\pi\delta}{\log T}\right)}{\alpha^2}\, \mathrm{d}\alpha &= \frac{w\left(\frac{2\pi\delta}{\log T}\right)}{\alpha^2} \frac{\sin(2\pi\delta\alpha)}{2\pi\delta}\Bigg|_{1}^{\infty} + \int_{1}^{\infty} \frac{2\sin(2\pi\alpha\delta)}{2\pi\delta\,\alpha^2} w\left(\frac{2\pi\delta}{\log T}\right) \mathrm{d}\alpha \\ &= O\left(\frac{1}{\delta}\right) \\ &= O\left(\frac{1}{\log T}\right),\end{aligned}$$

as claimed. Therefore, Berry's conjecture holds. □

BIBLIOGRAPHY

[1] M. V. Berry, *Semiclassical formula for the number variance of the Riemann zeros*, Nonlinearity **1** (1988), no. 3, 399–407.

[2] E. B. Bogomolny and J. P. Keating, *Gutzwiller's trace formula and spectral statistics: beyond the diagonal approximation*, Physical review letters **77** (1996), no. 8, 1472.

[3] E. Carneiro and V. Chandee and A. Chirre and M. B. Milinovich, *On Montgomery's pair correlation conjecture: a tale of three integrals*, arXiv preprint arXiv:2108.09258 (2021).

[4] E. Carneiro and V. Chandee and M. B. Milinovich, *Bounding $S(t)$ and $S\_1(t)$ on the Riemann hypothesis*, Mathematische Annalen **356** (2013), no. 3, 939–968.

[5] T. H. Chan, *Pair correlation of the zeros of the Riemann zeta-function in longer ranges*, Acta Arith. **115** (2004), no. 2.

[6] A. Chirre and O. E. Quesada-Herrera, *The second moment of $S_n(t)$ on the Riemann hypothesis*. Preprint. Available as `https://arxiv.org/abs/2006.08503`.

[7] J. B. Conrey and N. C. Snaith, *Applications of the L-functions ratios conjectures*, Proceedings of the London Mathematical Society **94** (2007), no. 3, 594–646.

[8] H. Davenport, *Multiplicative number theory*, 2nd ed. (H. L. Mongomery, ed.), Vol. 74, Springer-Verlag, 1980.

[9] A. Fujii, *On the distribution of the zeros of the Riemann zeta-function in short intervals*, Proc. Japan Acad., Ser. A **66** (1990), no. 3, 75–79.

[10] ———, *On the distribution of the zeros of the Riemann zeta-function in short intervals*, Bull. Am. Math. Soc. **81** (1975), no. 1, 139–142.

[11] ———, *On the zeros of Dirichlet L-functions. II (with corrections to "On the zeros of Dirichlet L-functions. I" and the subsequent papers)*, Transactions of the Am. Math. Soc. **267** (1981), 33–40.

[12] D. A. Goldston, *On the function $S(T)$ in the theory of the Riemann zeta-function*, J. of Number Theory **27** (1987), no. 2, 149–177.

[13] D. A. Goldston and H. L. Montgomery, *Pair correlation of zeros and primes in short intervals*, Analytic number theory and Diophantine problems, 1987, pp. 183–203.

[14] S. M. Gonek, *An explicit formula of Landau and its applications to the theory of the zeta function*, Contemp.Math. **143** (1993), 395–413.

[15] I. S. Gradshteyn and I.M. Ryzhik, *Table of integrals, series, and products*, 7th ed. (A. Jeffrey and D. Zwillinger, ed.), Elsevier Academic Press, 2007.

[16] D. R. Heath-Brown, *Prime number theory and the Riemann zeta-function*, London Math. Soc. Lecture Note Series **322** (2005), 1–33.

[17] E. Landau, *Handbuch der Lehre von der Verteilung der Primzahlen*, 2nd ed., Chelsea Publishing Co., 1953.

[18] J. E. Littlewood, *On the zeros of the Riemann zeta-function*, Math. Proc. Cambridge Philos. Soc. **22** (1924), no. 3, 295–318.

[19] ———, *On the Riemann zeta-function*, Proc. London Math. Soc. **24** (1925), no. 2, 175–201.

[20] H. M. Bui and S. J. Lester and M. B. Milinovich, *On Balazard, Saias, and Yors equivalence to the Riemann Hypothesis*, Journal of Mathematical Analysis and Applications **409** (2014), no. 1, 244–253.

[21] H. L. Montgomery, *Early Fourier Analysis*, Vol. 22, American Math. Soc., 2014.

[22] ———, *The pair correlation of zeros of the zeta function*, Proc. Symp. Pure Math. **24** (1973), 181–193.

[23] H. L. Montgomery and R. C. Vaughan, *Hilbert's inequality*, J. London Math. Soc **2** (1974), no. 8, 73–82.

[24] ———, *Multiplicative number theory I: Classical theory*, Cambridge Univ. Press, 2007.

[25] J. Mueller, *Arithmetic equivalent of essential simplicity of zeta zeros*, Trans. Amer. Math. Soc. **275** (1983), no. 1, 175–183.

[26] A. M. Odlyzko, *On the distribution of spacings between zeros of the zeta function*, Mathematics of Computation **48** (1987), no. 177, 273–308.

[27] B. Riemann, *Ueber die anzahl der primzahlen unter einer gegebenen grosse*, Ges. Math. Werke und Wissenschaftlicher Nachlaß **2** (1859), 145–155.

[28] A. Selberg, *On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip* $0 < t < T$, Avh. Norske. Vid. Akad. Oslo., 1944.

[29] ———, *Contribution to the theory of the Riemann zeta-function*, Arch. Math. Naturvid. **48** (1946), no. 5, 88-155.

[30] ———, *Old and new conjectures and results about a class of Dirichlet series*, Proc. of the Amalfi Conf. on Analytic Number Theory (Maiori, 1989), 1992, pp. 47–63.

[31] E. M. Stein and R. Shakarchi, *Fourier analysis: an introduction* **1** (2011).

[32] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed. (D. R. Heath-Brown, ed.), Oxford Univ. Press, 1986.

[33] ———, *On the remainder in the formula for $N(T)$, the number of zeros of $\zeta(s)$ in the strip* $0 < t < T$, Proc. Lond. Math. Soc. (2) **27** (1928), 449–458.

[34] K. M. Tsang, *The distribution of the values of the Riemann zeta-function*, Princeton Univ., 1984.

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