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Introduction to Actuarial Science (concluded)

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Introduction to Actuarial Science*

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(Concluded)

Bond Prices

The price at which a bond will sell is affected by the nominal interest rate, the mortgaged security, the financial standing of the issuing company and the probability of being able to sell the bond if occasion requires.

Bonds rarely sell on the market at par—usually a premium is added or a discount deducted. Quotations are made "on a basis" or "at a price." When bonds are sold at a price other than par the effective interest rate differs from the nominal or coupon rate. Thus if a bond is issued at a discount, the principal borrowed is really less than par. Moreover, the borrower pays not only the interest coupons but also the discount for the use of the borrowed money. Hence the effective rate on the loan is greater than the nominal rate. If the bond sells at a premium, the principal borrowed is more than par; and since the borrower does not have to pay back the premium at maturity, the premium is really a deduction from the interest. Hence the effective rate is less than the nominal rate.

Quotations "on a basis" state the effective rate to be earned. The price, above or below par, may then be found in a bond table, or computed. Thus a 5 year 6% bond of $100.00 bought on a 5% basis would cost $104.38 as shown in the following table—or a 5 year 5% bond of $100.00 bought on a 6% basis would cost $95.73.

5 years

Per cent. per annum	Interest payable semi-annually 3%	3½%	4%	4½%	5%	6%	7%
4.75	92.29	94.49	96.70	98.90	101.10	105.51	109.91
4.80	92.08	94.28	96.48	98.68	100.88	105.28	109.58
4.875	91.77	93.96	96.16	98.35	100.55	104.94	109.33
4.90	91.66	93.86	96.05	98.25	100.44	104.83	109.21

5 years

Per cent. per annum	Interest payable semi-annually 3%	3½%	4%	4½%	5%	6%	7%
5.	91.25	93.44	95.62	97.81	100.00	104.38	108.75
5.10	90.83	93.02	95.20	97.38	99.56	103.93	108.29
5.125	90.73	92.91	95.09	97.27	99.45	103.82	108.18
5.20	90.42	92.60	94.78	96.95	99.13	103.48	107.84
5.25	90.22	92.39	94.57	96.74	98.91	103.26	107.61
5.30	90.01	92.18	94.35	96.53	98.70	103.04	107.38
5.375	89.71	91.87	94.04	96.21	98.37	102.71	107.04
5.40	89.61	91.77	93.94	96.10	98.27	102.60	106.93
5.50	89.20	91.36	93.52	95.68	97.84	102.16	106.48
5.625	88.70	90.85	93.00	95.16	97.31	101.61	105.92
5.75	88.20	90.34	92.49	94.63	96.78	101.07	105.37
5.875	87.70	89.84	91.98	94.12	96.26	100.53	104.81
6.	87.20	89.34	91.47	93.60	95.73	100.00	104.27

Computing the Premium

When the effective rate is less than the nominal rate, the premium may be computed by a method based on the following reasoning:

Assume that the par of the bond is $1,000.00, the time 5 years, the nominal rate 6% a year payable semi-annually, and that the bond is to be purchased on a 5% basis. The conditions are such that the purchaser is satisfied with 5% a year; therefore, if the bond bore coupons of 2½% or $25.00 payable each six months the bond would presumably sell at par. In other words, the purchaser would pay par, $1,000.00, for the right to receive par at maturity and interest of $25.00 semi-annually. But since the coupons are $30.00 each, he must also pay for the right to receive this extra $5.00 each six months. This semi-annual payment of $5.00 is an annuity, and the purchaser will pay its present value discounted at the effective interest rate of 2½% per period thus:

.781198=present value of 1 @ 2½% in 10 periods

.218802÷.025=8.75208 present value of annuity of 1

8.75208×5=43.76040 present value of annuity of 5, the premium.

The method may be stated as follows:

Compute the interest for 1 period at the nominal rate on par	$30.00
" " " " 1 " " " effective " " "	25.00
Find the difference, which is one rent of an annuity	5.00

Find the present value of this annuity at the effective rate.

Another method of computing the cost of a bond at a premium is based on the following reasoning:

The purchaser will pay the present value of the benefits to be received, which are ordinarily:

Par at maturity;
The interest coupons.

The present value of these benefits will be computed at the effective rate. Applied to the preceding illustration, the computation by this method would be:

Present value of par:	
.781198, present value of 1 due 10 periods hence at 2½%	
.781198×1000=	781.198
Present value of coupons:	
.218802 (compound discount)÷.025=8.75208 present value of annuity of 1	
30.00 (coupons) × 8.75208=	262.562
Total (as computed above)	1,043.760

The interest and amortization of premium on this bond may be scheduled as follows:

AMORTIZATION—BOND AT A PREMIUM

Cost			1,043.76
1st interest date:	Coupon	30.00	
	Interest: 2½% of 1043.76	26.09	3.91
	Carrying value		1,039.85
2nd " "	Coupon	30.00	
	Interest: 2½% of 1039.85	26.00	4.00
	Carrying value		1,035.85
3rd " "	Coupon	30.00	
	Interest: 2½% of 1035.85	25.90	4.10
	Carrying value		1,031.75

4th " "	Coupon		30.00	
	Interest: 2½% of 1031.75		25.79	4.21
	Carrying value			1,027.54
5th " "	Coupon		30.00	
	Interest: 2½% of 1027.54		25.69	4.31
	Carrying value			1,023.23
6th " "	Coupon		30.00	
	Interest: 2½% of 1023.23		25.58	4.42
	Carrying value			1,018.81
7th " "	Coupon		30.00	
	Interest: 2½% of 1018.81		25.47	4.53
	Carrying value			1,014.28
8th " "	Coupon		30.00	
	Interest: 2½% of 1014.28		25.36	4.64
	Carrying value			1,009.64
9th " "	Coupon		30.00	
	Interest: 2½% of 1009.64		25.24	4.76
	Carrying value			1,004.88
10th " "	Coupon		30.00	
	Interest: 2½% of 1004.88		25.12	4.88
	Par—payable at maturity			1,000.00

Or the schedule may be shown thus:

AMORTIZATION—BOND AT A PREMIUM

End of period	Coupon	Effective interest	Premium written off	Carrying value
				1,043.76
1	30.00	26.09	3.91	1,039.85
2	30.00	26.00	4.00	1,035.85
3	30.00	25.90	4.10	1,031.75
4	30.00	25.79	4.21	1,027.54
5	30.00	25.69	4.31	1,023.23
6	30.00	25.58	4.42	1,018.81

7	30.00	25.47	4.53	1,014.28
8	30.00	25.36	4.64	1,009.64
9	30.00	25.24	4.76	1,004.88
10	30.00	25.12	4.88	1,000.00
	300.00	256.24	43.76	

The second method of computing the price, described above, is the better one to use when bonds are repayable at a premium.

Illustration: what price, to net 5%, should be paid for a 5½% twenty-year bond of $1,000.00, repayable with a bonus of 5%?

Present value of 1050 due at maturity:

.3724306, present value of 1 at 2½% in 40 periods.

.3724306×1050=	391.052
Present value of coupons:	
.6275694÷.025=25.102776 present value of annuity of 1	
25.102776×27.50=	690.326
Total	1,081.378

COMPUTING THE DISCOUNT

The two methods described for computing the price of a bond sold at a premium may also be used when the bond is sold at a discount.

To illustrate: at what price should a $1,000.00 5 year 5% bond be sold to net the investor 6%? Interest is payable semi-annually.

By the first method the difference between the interest on par at the effective rate and at the nominal rate is computed—the discount is the present value of an annuity for the number of interest periods, each rent of which is the difference in the interest at the two rates. This annuity is discounted at the effective interest rate.

Effective rate: 3% on 1,000=	30.00
Nominal rate: 2½% on 1,000=	25.00
Difference:	5.00

.744094 is the present value of 1 at 3% due 10 periods hence:
.255906÷.03=8.5302 present value of annuity of 1.
8.5302×5=42.6510, the discount.
1,000.00—42.65=957.35, the price.

By the second method the present values of par and coupons are computed at the effective rate:

Present value of par:	
.744094×1000=	744.094
Present value of coupons:	
.255906÷.03=8.5302	
8.5302×25=	213.255
Total:	957.349

The reduction of this discount may be scheduled thus:

SCHEDULE OF AMORTIZATION—BOND AT DISCOUNT

First period:		
Cost:		957.35
Interest: 3% of 957.35	28.72	
Coupon	25.00	3.72
Carrying value		961.07
Second period:		
Interest: 3% of 961.07	28.83	
Coupon	25.00	3.83
Carrying value		964.90
Third period:		
Interest: 3% of 964.90	28.95	
Coupon	25.00	3.95
Carrying value		968.85
Fourth period:		
Interest: 3% of 968.85	29.07	
Coupon	25.00	4.07
Carrying value		972.92
Fifth period:		
Interest: 3% of 972.92	29.19	
Coupon	25.00	4.19
Carrying value		977.11

Sixth period:		
Interest: 3% of 977.11	29.31	
Coupon	25.00	4.31
Carrying value		981.42
Seventh period:		
Interest: 3% of 981.42	29.44	
Coupon	25.00	4.44
Carrying value		985.86
Eighth period:		
Interest: 3% of 985.86	29.58	
Coupon	25.00	4.58
Carrying value		990.44
Ninth period:		
Interest: 3% of 990.44	29.71	
Coupon	25.00	4.71
Carrying value		995.15
Tenth period:		
Interest: 3% of 995.15	29.85	
Coupon	25.00	4.85
Carrying value		1000.00

Or thus:

SCHEDULE OF AMORTIZATION

Period	Effective interest	Coupon	Discount	Carrying value
				957.35
1	28.72	25.00	3.72	961.07
2	28.83	25.00	3.83	964.90
3	28.95	25.00	3.95	968.85
4	29.07	25.00	4.07	972.92
5	29.19	25.00	4.19	977.11
6	29.31	25.00	4.31	981.42
7	29.44	25.00	4.44	985.86
8	29.58	25.00	4.58	990.44
9	29.71	25.00	4.71	995.15
10	29.85	25.00	4.85	1000.00
Total	292.65	250.00	42.65	

Purchases at Intermediate Dates

When bonds are sold between interest dates, the customary method of computing the price is as follows:

Determine the price which would have been paid at the last preceding interest date. Determine the price which would have been paid at the next succeeding interest date. Find the difference in these prices. This difference is the premium or discount which would be amortized for the entire period in which the purchase was made.

Determine the fraction of the period expired between the last preceding interest date and the date of purchase.

Multiply the difference in prices (premium or discount for whole period) by the fraction of the period expired. The product is the premium or discount for the fractional period.

Deduct such premium for the fractional period from the price at the last preceding interest date, or add the discount for the fractional period.

To the result thus obtained add the accrued interest on par at the nominal rate.

Illustration—bond at premium: what price should be paid for a $100.00 bond due in 6 years and 2 months, bearing 6% and bought on a 5% basis plus accrued interest?

Value 6½ years to maturity (per bond table)	105.49
" 6 " " " " " "	105.13
Difference—premium amortized in 6 months	.36
Multiply by fraction of period expired—4 months	⅔
Premium amortized in 4 months	.24
Price 6½ years to maturity	105.49
Deduct premium for 4 months	.24
Value 6 years, 2 months before maturity (flat)	105.25
Interest for 4 months on $100.00 at 6%	2.00
Price including interest	107.25

At the next interest date the bond will be written down to $105.13, its value at that date as shown by the bond table.

Illustration—bond at a discount: what price should be paid for a $100.00 bond due in 8 years and 1 month, bearing 4% and bought on a 5½% basis plus accrued interest?

Value 8 years to maturity (per bond table)	90.40
" 8½ " " " " " "	89.92
Difference—discount amortized in 6 months	.48
Multiply by fraction of period expired—5 months	5/6
Discount amortized in 5 months	.40
Add value 8½ years before maturity	89.92
Add accrued interest at 4% on $100 for 5 months	1.67
Price including interest	91.99

Optional Redemption

When a bond or other obligation gives the debtor the option of paying the debt before maturity, this right must be taken into consideration in determining the price to be paid if the purchase is to be made at a premium or a discount.

If the debtor has the right to redeem at par before maturity, the purchase price should be computed on the assumption that the right will be exercised if the bond is purchased at a premium. The reason for the assumption can be shown by a comparison of prices in a bond table.

A 6% bond payable in 20 years bought on a 5% basis should cost 112.55.

A 6% bond payable in 15 years bought on a 5% basis should cost 110.47.

Now if the bond is payable in twenty years with an option to redeem in fifteen years the purchaser may be buying a bond with only fifteen years to run; and he should pay for it on the supposition that it will be paid at the optional date.

On the other hand if the bond is to be purchased at a discount, he should assume that it will not be paid until maturity. A bond table shows that

A 5% bond payable in 15 years bought on a 6% basis should cost 90.20; a 5% bond payable in 20 years bought on a 6% basis should cost 88.44.

If the purchaser pays 90.20 for the bond and it is not paid for twenty years, he will not earn 6% on his investment. In fact, he will earn a little less than 5⅞%.

If the debtor must pay a premium in order to redeem the bond before maturity, the purchaser should assume that the option will not be exercised in case the bond is to be sold at a discount. It was shown in the preceding paragraph that a 5% bond sold on a 6% basis would sell for $90.20 if redeemable at par at the end of 15 years. If redeemable at a premium, it would sell for a still higher price. But a purchaser would be unwise to pay this higher price when there was a possibility that the bond would run the full twenty years. He should buy on the assumption that the option will not be exercised. If it is exercised his rate of earning will be more than 6%.

But if the bond is to be purchased at a premium, and if the debtor must pay a premium to redeem the bond before maturity, the purchaser cannot assume that the option will be exercised, nor can he assume that it will not be exercised. The advantage to the debtor arising from the payment at par on an optional maturity date may vanish if he has to pay a premium if he redeems before maturity. Whether or not it will be advantageous will depend on the amount of the premium. Therefore, the purchaser should compute the price to be paid on the given basis if the bond runs to maturity, and the price to be paid if the option to redeem at a premium is exercised; and he should then pay the lower price.

To illustrate: on a 5% basis what should be paid for a $1,000.00 6% bond, due in 20 years, with a privilege of redemption in 15 years at 110?

A bond table shows the value on a 5% basis of a 20 year 6% bond to be $1,125.50.

The value if the option is exercised could be computed thus:

Value of 1,100 in 15 years:

Present value of 1 at 2½% due 30 periods hence	.476742685
Multiply by	1100
	524.4169535

Value of coupons:

Present value of 1, as above		.476742685
Compound discount		.523257315
.523257315÷.02½=20.9302926, present value of annuity of 1		
20.9302926×30=627.908778		
Value of par and premium	524.4169	
Value of coupons	627.9087	
Total	1152.3256	

The price, on the assumption that the bond runs to maturity, is $1,125.50. On the assumption that it is paid at the optional date, the price is $1,152.33. The purchaser should assume that the option will not be exercised and pay $1,125.50.

If the premium to be paid at the optional redemption date is not too large, it may still be desirable to exercise the option.

To illustrate: what price should be paid for the bond in the preceding illustration if the optional redemption price is 101 instead of 110?

Price if option is not exercised: $1,125.50, as above.

Price if option is exercised:

Present value of 1010 in 15 years:	
Present value of 1 at 2½% due 30 periods hence	.476742685
Multiply by	1010
	481.5101
Present value of coupons—as above	627.9087
Total	1,109.4188

This price, $1,109.42, should be paid because it is less than $1,125.50.

Computing the Rate on Bonds Sold at Premium or Discount

When a bond is purchased on the basis of an effective rate other than the nominal rate, it is a simple matter to compute the price to be paid; but when the bond is purchased at a price not listed in the bond tables, it is by no means an easy matter to compute the price. In fact, there is no mathematical formula which can be applied to determine the rate exactly. The rate can be approximated in several ways, two of which will be explained.

A 25 year 6% $100 bond, interest payable semi-annually, is purchased for $110.38. What is the effective rate?

This price of $110.38 is shown by a bond table to be the price on a 5.25% basis. The bond table shows:

	25 years 6%
5.20%	111.12
5.25	110.38
5.30	109.64

But let us assume that the 110.38 is not shown by the table and that the nearest values are

On a 5.20 basis	111.12
On a 5.30 "	109.64

Then .10 of one per cent. difference in the rate causes 1.48 difference in price.

Price on a 5.20 basis	111.12
Price given in illustration	110.38
Difference	.74

Then to find the approximate rate, add to 5.20%

$$\frac{74}{148} \text{ of } .10 \text{ of } 1\%$$

$$\frac{74}{148} \text{ of } .10 = .05$$

$$5.20\% + .05\% = 5.25\% \text{ the rate.}$$

This rate happens to be exactly correct, but it is very unusual to obtain exact results by interpolation.

The rate may be approximated by the following formulas when bond tables are not available for interpolation:

Bond at a premium:

$$r = \frac{2\,(I - Pr)}{n\left(C + P + \frac{Pr}{n}\right)}$$

Bond at a discount:

$$r = \frac{2\,(I + D)}{n\left(C + P - \frac{D}{n}\right)}$$

The symbols used are:

r—effective rate per period
I—total interest on bonds
Pr—premium
D—discount
C—cost (par + premium; or par — discount).
n—number of interest periods.

Applying the first formula to the illustration:

$$r = \frac{2\ (150-10.38)}{50\left(110.38+100+\frac{10.38}{50}\right)} = 2.65\%\text{ per period}$$

or 5.30% per annum.

It will be noted that this result is much less exact than the one obtained by interpolation—still it is useful when one has no bond table and desires to obtain a rough approximation of the rate.

Illustration of bond at a discount: a 10 year 5% $100 bond is bought at 96.94. Interest is payable semi-annually. What is the approximate effective rate?

$$r = \frac{2\ (50+3.06)}{20\left(96.94+100-\frac{3.06}{20}\right)} = 2.692\%\text{ per period}$$

or 5.384% per annum.

The true effective rate is 5.40%.

This method produces a rate which is too large on bonds sold at a premium and too small on bonds sold at a discount. The error is due to the fact that the formula is based on arithmetical progression, while the amortized premium or discount does not increase or decrease periodically in an arithmetical progression.

Depreciation

Two depreciation methods—the annuity and sinking fund methods—involve compound interest. When the annuity method is used, the investment in the depreciating asset is dealt with as if it were an investment in an annuity. The periodical depreciation charges are analogous to rents and must be large enough to exhaust the cost of the asset, or the cost less residual value, and also provide for the interest. In other words, the charge to operations for depreciation must provide for credits to interest, for interest on the gradually diminishing investment, and for credits to the depreciation reserve. The amount of the credit to interest is computed by multiplying the carrying value of the asset (cost less reserve at beginning of period) by the interest rate. The credit to the reserve is the difference between the charge to depreciation and the credit to interest.

When there is no scrap value, the formula for computing the periodical depreciation, is

$$d = \frac{c}{P}$$

In this formula:

d=periodical depreciation
c=cost of asset
P=present value of annuity of 1.

Illustration: what is the annual depreciation on an asset costing $5,000 which will have no value at the end of five years, if depreciation is to be computed by the annuity method using a rate of 5%? Present value at 5% of 1 due 5 periods hence is .783526166.

Then .216473834÷.05=4.32947668, or P.

$$d = \frac{5000}{4.32947668} = 1,154.87.$$

The annual depreciation entries may be tabulated thus:

Year	Debit depreciation	Credit interest	Credit reserve	Carrying value
				5,000.00
1	1,154.87	250.00	904.87	4,095.13
2	1,154.87	204.76	950.11	3,145.02
3	1,154.87	157.25	997.62	2,147.40
4	1,154.87	107.37	1,047.50	1,099.90
5	1,154.87	55.00	1,099.87	.03

When there is a scrap value, the formula is:

$$d = \frac{c - (s \times p)}{P}$$

In which the new symbols are s, representing scrap value and p, representing present value of 1.

Assuming that the asset in the preceding illustration will have a residual value of $2,000 at the end of 5 years, what should be the annual depreciation?

The present value of 1 at 5% due in 5 periods, was stated in the illustration to be .783526166; and the present value of an annuity of 1 for 5 periods at 5% was computed above, as 4.32947668. Then

$$d = \frac{5000-(2000\times.783526166)}{4.32947668} = 792.92.$$

The annual depreciation entries may be tabulated thus:

Year	Debit depreciation	Credit interest	Credit reserve	Carrying value
				5,000.00
1	792.92	250.00	542.92	4,457.08
2	792.92	222.85	570.07	3,887.01
3	792.92	194.35	598.57	3,288.44
4	792.92	164.42	628.50	2,659.94
5	792.92	133.00	659.92	2,000.02

The sinking fund method is based on the assumption that a fund is created at compound interest to equal the total depreciation. If a fund is created, the contribution is computed in accordance with the formula already stated and explained, namely:

$$S.F.C. = S.F. \div A$$

Since the total fund required is the difference between the cost and the scrap value of the asset, the formula for determining the periodical contribution to the fund is

$$S.F.C. = \frac{c-s}{A}$$

Illustration: what annual contribution should be made to a fund on a 5% basis, compounded annually, to provide for an asset costing $5,000.00 and expected to have a residual value of $2,000 at the expiration of 5 years? And what should be the annual entries for the fund and the depreciation reserve? $1.05^5 = 1.276282$.

$$A = .276282 \div .05 = 5.52564$$

$$S.F.C. = \frac{5000-2000}{5.52564} = 542.92$$

Table of Fund Entries

Year	Debit fund	Credit interest	Credit cash	Balance fund
1	542.92		542.92	542.92
2	570.07	27.15	542.92	1,112.99
3	598.57	55.65	542.92	1,711.56
4	628.50	85.58	542.92	2,340.06
5	659.92	117.00	542.92	2,999.98

The reserve should keep pace with the fund so that at the end of the anticipated life of the asset, the fund and the reserve each will equal the total depreciation. Therefore, the amount charged each year to the fund, as shown in the "debit fund" column, should be charged to depreciation and credited to the reserve for depreciation.