Black Hole Entropy in the Causal Set Approach

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BLACK HOLE ENTROPY IN THE CAUSAL SET APPROACH

A thesis
presented in partial fulfillment of requirements
for the degree of Master of Science
in the Department of Physics and Astronomy
The University of Mississippi

by

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ABSTRACT

Einstein’s field equations delineate the intricate dance between the geometry of spacetime and the energy-momentum. It is astonishing to realize how spacetime geometry is influenced by the presence of matter and energy and how this in turn affects the dynamics of matter in the spacetime background. Matter and energy are quantum mechanical in nature but the geometry of spacetime is a mathematical theory of a 4-dimensional pseudo-Riemannian manifold. Causal set theory is one of the approaches to reconcile quantum mechanics and gravity (an emergent property of spacetime geometry), and therefore it is an approach to quantum gravity. In this approach, the spacetime continuum is assumed to be discrete at very small scales (of Planckian order). The discrete points in a causal set can be seen as a continuum of spacetime if they can be embedded in a manifold such that the causal structure is preserved. In this regard, a manifold can be constructed by embedding a causal set preserving causal information between the neighboring points.

In this thesis, the area-like scaling of a black hole entropy is studied in the causal set approach and the possibility of the horizon-crossing links as a source of black hole entropy is examined in detail. We calculate the number of horizon-crossing links for a Schwarzschild black hole using a computer simulation and observe a linear scaling with the horizon area. Later, we impose a maximal/minimal condition to calculate the number of surface-crossing links close to a particular spacelike hypersurface for a generalized $N$-dimensional flat black holes. Excellent linearity between the number of horizon-crossing links and the area of the horizon is observed upon computer simulation as well as numerical integration using MATLAB.
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I have always been fascinated by the ease and brevity with which Dr. Bombelli explains complicated topics in physics. He has taught me that a true understanding of physics does not merely come from studying the mathematical structure the theory is built upon but by carefully considering the premises and analyzing the logical argumentation that leads to that particular mathematical foundation. Throughout this research project, he has taught me how to ‘think’ right and how to ask ‘meaningful’ questions which in turn has enriched my understanding of many aspects of quantum gravity. I am grateful for his support, guidance, and encouragement that has led to the completion of this thesis. I am thankful to He Liu for helping me with MATLAB coding and discussing many properties of Schwarzschild spacetime. I am equally thankful to Nauman Ibrahim, Santosh Bhandari, and Coleman Irby for discussing my work and enriching me with new ideas. Finally, I would like to thank my family for always being supportive of my work and providing me a strong foundation to flourish my pursuit of knowledge in theoretical physics.

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CHAPTER 1

INTRODUCTION

1.1 Entropy

Imagine that there is a box of volume $V$ and this box has only one electron inside it. We now bring a piece of a magnet close to this box such that the magnetic field is pointing along the positive $z$-axis inside the box. What are the possible states of the electron spin inside this box? The electron can be in either spin up or spin down state. We know what state it is in once we make a measurement. If we now put 1000 electrons inside this box, we would need to know the spin orientation of each of these 1000 electrons to completely specify the spin state of the whole box. We increased the information content of the box by putting in more electrons. Is there a limit to how much information this box of volume $V$ can hold?

Imagine now that instead of just electrons we put many atoms of hydrogen, helium, oxygen, nitrogen, and some free electrons inside this box. We then heat this box such that it is now at a certain temperature $T$. We have increased the information content of this box again. How much information do we need to completely specify the state this box is in? Well, this depends on how deep we can probe. We would need to know where each of the atoms is and what momentum they have, what energy levels each of the electrons is in, what states each quark is in, if the electrons are exchanging photons with the atoms, and how quarks are exchanging bosons with other quarks, how electrons and quarks are interacting with the Higgs field, etc. The deeper we wish to probe, the more information we would need to completely specify the state of the box. If we wish to look at an infinitely small length scale, would we need infinite information? Does this also mean that this box can, in principle, hold an infinite amount of information? It appears so if we keep gravity completely out of the picture. However, once we let gravity be a part of the interaction between the
atoms, this is not true.

So far we have established that increasing the number of particles inside the box increases its information content. Can we keep adding particles forever? We cannot do this while holding the volume of the box constant. If we have added so many particles that now adding one more can turn this box into a black hole, we cannot then add more particles (information) and not increase the volume of the black hole. It then appears that the maximum information a region of volume $V$ can contain is not infinite but is bounded.

What does all of this have to do with entropy? How is the entropy of this box then related to the black hole area? Entropy is the measure of information content a system has that we are unaware of or are ignorant of. For a box with a single electron which can be in the spin-up or spin-down state, the information this box has is 1 bit (2 states of the electron). We do not know what state this electron is in unless we make a measurement and therefore the entropy of this box is 1 bit. If instead of only spin, we also wish to know what position this electron is in and what momentum it has inside this box, we would need more information. If we were to increase the volume of this box, the electron can then be in new places which were not accessible to it before. Therefore, we would need still more information. This is true in general. If we were to increase the volume of the box the particles can be in, we increase the information content that we require to specify the state of the box and therefore we increase the entropy. It is quite natural to see how the entropy of a box scales with its volume. What does entropy have to do with the area of the box?

The volume-like scaling of entropy holds when we are dealing with normal day-to-day physics. In extreme situations, this does not hold. If we have added so many particles inside the same box of volume $V$ (imagine the box was spherical to start with) that it has now turned into a black hole, adding more particles increases the size of the black hole such that the entropy increase of the black hole is proportional to the increase in surface area and not volume$^1$. This strange area-like scaling behavior of entropy for black holes was first shown by Bekenstein in 1973$^2$.

At what point of the process (adding particles in the box) did the volume-like scaling of entropy change to an area-like relation? The most logical explanation for this behavior would
be that there had always been both volume and area (and possibly other dimension-dependent) terms for entropy. At small density, the volume term dominates, and when the density becomes humongous, the area term dominates. This has to do with gravity. If the force of gravitation is out of the picture, there is no area-like relation. In this sense, gravity is putting a constraint as to how much information (possible quantum interaction between particles/fields) a region of spacetime may have. Black hole entropy, therefore, hints at the dependence of quantum mechanical states of particles (fields) and the force of gravitation. To better understand this dependence, we would require a quantum theory of gravity.

Entropy is an intuitive notion when dealing with particles confined in a box. What does the entropy of a black hole mean? What are the possible microstates (perhaps as a function of a few extremely specified parameters such as mass, angular momentum, and charge) the black hole can be in which gives rise to this entropy? It does not seem logical to imagine particles flying here and there with a certain value of momentum inside a black hole. If they could do this they’d have to travel faster than light when they’re traveling in a direction toward the black hole surface and they might as well come out of it which violates the very definition of a black hole (nothing can come out). If particles can only move towards the central singularity (not away from it) for a Schwarzschild black hole then they must be confined there (if we believe they still exist once they get inside). Between this location (maybe singularity, \( r = 0 \), but we do not know what this means) and the surface, there is nothing but an empty region of spacetime. Does black hole entropy have to do with the information this immovable ‘thing’ at \( r = 0 \) has? Is there something more? If we had a theory of quantum gravity, maybe we could have answers to questions like these.

In the following chapters, we shall explore what information black holes might be hiding from an external observer and how this information content (entropy) scales like the area of the horizon. We shall do this in the causal set theoretical model of spacetime.
1.2 Quantum gravity

Einstein’s groundbreaking concepts on gravity have led to a paradigm shift in our understanding of the fundamental structure of the universe, opening up new avenues for exploration. The conventional and simplistic view of the world was rendered obsolete. Subsequent experimentation has lent support to Einstein’s theory of gravity, yet it falls short of providing a comprehensive understanding of reality. In particular, the theory fails to hold up in areas of extremely intense gravitational fields, such as the black hole centers. Physicists have since sought a universal theory, but unfortunately, none has been developed to date.

General Relativity (G.R) is extremely successful in describing the behavior of planets around their host stars, the formation of a galaxy from colossal gas clouds, the dynamics between two stars or two galaxies, the evolution of galaxies on local clusters, and so on. General Relativity very precisely describes physics on a large scale. However, to understand why atoms are stable, the behavior of electrons inside an atom, and what their energy levels are or to study the atomic spectra, General Relativity cannot be used. We need a new theory to study things and their behavior on a very small scale. This is the theory of quantum mechanics (Q.M). It is then natural to ask if one can construct a theory that applies at all scales, one that can describe the behavior of atoms and also that of galaxies.

One of the reasons why G.R and Q.M are extremely successful in their respective domains is that the quantum effects become negligible when we are dealing with a system of a large number of particles (a planet, star, or a galaxy) and gravitational effects become negligible at atomic scales. What if gravity was so strong that even at very small scales its effects were significant? This actually happens in the interior of the remnants of massive dead stars called black holes. In places like this, we need a theory of Q.M and G.R woven together concisely and beautifully to describe the physical observations. Such a theory is the theory of quantum gravity (Q.G).

At a first glance, this does not seem that difficult of a task. As Lee Smolin puts it, a clever physicist immediately sees two paths to construct a theory a Q.G. (1) Take the theory of Q.M as it is and modify it to include gravity. (2) Take the theory of G.R. as it is and modify it to include
quantum mechanical effects. Physicists were divided as to which path to take to develop this new theory. The group of like-minded physicists that took the first path developed string theory and the group that took the second approach developed the theory of loop quantum gravity. However, there were a few others who did not take the obvious paths. If Q.M and G.R are both incomplete, why not start with something much more fundamental, the premises that we know are correct with 100 percent certainty, and from it try to construct a new theory that can describe physics at extremely small and extremely large scales? Among the group of physicists that took this obscure path, one group built a theory of gravity based on nothing but causality information between pairs of events and they called it the Causal Set Theory (CST). In the following chapters, we shall look at what this theory has to say and what new predictions made using this theory agree with what we already know. In particular, we shall show that using the CST, one can observe the area-like scaling of black hole entropy.

1.3 **Causal set theory as an approach to quantum gravity**

Is there a limit to how small a volume in three-dimensional space can be? If I take a cup of water of volume $V$ and divide it infinitely many times, would the resulting infinitesimal volume element still be three-dimensional? What about time? Can we divide time (say an interval of a second) infinitely and observe changes that might take place in this infinitesimal time interval? Are there elementary events? The simplest possible change that can happen? If we assume that space and time are continuous then, in principle, we may divide them indefinitely. There would then be no fundamental unit of time or space. The real universe does not have to be like that. The seemingly continuous nature of material surrounding us in our everyday lives hides underneath a world of discrete elementary particles. Understanding the true nature of the material world involves probing into scales much smaller than the scale we deal with every day. This has already taught us an important lesson. In a similar manner, if we probe into events that take place in extremely small time intervals (of Planckian order), would we be able to discern fundamental events? The theory of causal sets assumes that the world is composed of these fundamental elements (or events)
an elementary unit of change. These elements in themselves possess no fundamental property but carry the information of causal relation with other such elements. At the most fundamental level, the continuum approximation of spacetime, therefore, becomes a patch of locally finite partially ordered sets – posets or “causal sets”, whose elements carry the information of whether they are causally related to other elements in the causal set or not\(^3\).

For a particular causal set to approximate a spacetime continuum, the causal set must be embeddable in the spacetime continuum with every causal relation between each pair of elements preserved. This means that there has to be a way to map the causal set elements into the manifold such that the relation between the elements in the causal set is reflected as a causal relation between elements (or events) in the manifold. The locally finite structure of a causal set limits its cardinality and therefore the elements are countable. The volume of a spacetime region is then nothing but the cardinality of the causal set that represents such a patch\(^4\).
CHAPTER 2

CAUSAL SETS

2.1 Definition

The causal set theory was proposed by Luca Bombelli, Rafael Sorkin, Joohan Lee, and David Meyer in 1987\(^3\). According to the causal set theory, the structure of spacetime is discrete at the most fundamental level. Spacetime is composed of discrete events that are related to one another, which gives rise to the causal structure in spacetime. A causal set is a set \(X\) with a relation \(\prec\) among its elements which must have three important basic properties:

1. **Irreflexive**: \(x \not\prec x\) \(\forall x \in X\)
2. **Transitive**: \(x \prec y\) and \(y \prec z\) \(\Rightarrow x \prec z\) \(\forall x, y, z \in X\)
3. **Locally finite**: \(I[x, y] = \{z | x \prec z \prec y\}\) is finite, \(\forall x, y \in X\)

The causal information between each pair of discrete spacetime events in the causal set can be encoded in a matrix with elements 0s and 1s called the relations matrix \(R\). If any two events in the causal set are causally related, the corresponding relations matrix entry is 1. If they are not causally related, the relations matrix entry is 0. Also, any element in the causal set cannot be causally related to itself and therefore all entries \(R_{ii}\) are 0 by default,

\[
R_{ij} = \begin{cases} 
1 & \text{when } x_i \prec x_j \\
0 & \text{otherwise.} 
\end{cases}
\]

In the example of Figure 2.1, the causal set elements are numbered in the order of increasing time labels. In this sense, for such a time-ordered 10-element causal set embedded in a manifold on which there is a coordinate patch that covers all of the manifold, the first element is the one that has the smallest coordinate time label and the tenth element is the one that has the largest coordinate
time label. The first element precedes every element in this causal set (except itself) and therefore all the entries (except $R_{11}$) of the first row of the relations matrix are 1s. Also, given $x_i < x_j$ and $i < j$, $R_{ii} = R_{jj} = 0$, $R_{ij} = 1$, and $R_{ji} = 0 \forall i, j$. This makes the relations matrix $R$ upper triangular.

A relations matrix contains causal information about every pair of elements and encodes the geometry of the manifold if the causal set is embeddable down to the length scale determined by the embedding density. Therefore, the relations matrix can be used to obtain the dimension and the curvature information of the spacetime manifold. Figure 2.1 shows a causal set with 10 elements. A relations matrix $R$ that corresponds to this causal set is shown right below Figure 2.1.

Figure 2.1: A causal diamond with 10 elements embedded in a flat 2D manifold. The elements are labeled in the order of increasing coordinate time.

In order to obtain the relations matrix for a particular causal set, each element is checked for the causal relation with every other element in the causal set, as described above. For this particular causal set, the relations matrix is shown below
Once we obtain the relations matrix for a particular causal set, we may construct out of it a new matrix, the link matrix $L_{ij}$, which has the property,

$$L_{ij} = \begin{cases} 
1 & \text{when } x_i < x_j \text{ and } R_{ik}R_{kj} = 0, \forall i < k < j \\
0 & \text{otherwise.}
\end{cases}$$

A link matrix for the causal set shown in Figure 2.1 is

$$L_{ij} = \begin{bmatrix}
0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}.$$

The non-zero element of the relations matrix ($R_{ij} \neq 0$) tells us that there exists a timelike curve from event $i$ to event $j$, or that event $j$ is inside the future light cone of event $i$. In the simple language of the causal set, it means $i < j$. The non-zero entries of link matrices ($L_{ij} \neq 0$) are a bit tricky. They mean that event $i$ and event $j$ are causally related, but there are no other events causally between events $i$ and $j$. $L_{ij} = 1$ are special, in the sense that a ‘cause’ emanating at event $i$ influences event $j$ uniquely. By this we mean, the information that event $j$ receives due to a ‘cause’ originating at event $i$ can only come from event $i$ and no other events because even if there are other events that can be influenced by event $i$ (which would then carry the information from $i$ to $j$), they are not timelike related to event $j$ and therefore the causal influence is unique. If a causal set is an
underlying fundamental discrete structure of spacetime at very small length scales, it is natural to imagine that the total number of links that cross the event horizon is an obvious source of black hole entropy. In the following chapters, we explore this possibility.

2.2 Sprinkling on a causal manifold

A causal manifold is one in which we know which manifold points precede which other ones. In order to obtain the relations matrix as shown above for a specific causal set, one can randomly scatter causal set elements inside a \( n \)-dimensional causal diamond with maximal and minimal elements \( x \) and \( y \) respectively. A causal diamond is simply \( J^+(x) \cap J^-(y) \), where \( J^+(x) \) is the causal future of the event \( x \) and \( J^-(y) \) is the causal past of the event \( y \). Also, an element \( x \) of a causal set is said to be maximal (minimal) if the region bounded by the future (past) light cone of this event and the spacelike hypersurface is empty (contains no other elements). The process of obtaining a causal set by randomly scattering the spacetime points on a metric manifold, the density of points dictated by the spacetime metric, is called sprinkling. In any 2-D conformally flat spacetime, if the causal diamond is centered at the origin and time ranges from \( t_{\text{min}} \) to \( t_{\text{max}} \), the space coordinate can range from

\[
x_{\text{min}} = \frac{t_{\text{min}} - t_{\text{max}}}{2} \quad \text{to} \quad x_{\text{max}} = \frac{t_{\text{max}} - t_{\text{min}}}{2}.
\]

(2.1)

This can be understood intuitively. Imagine that an event happens at the location \((0, t_{\text{min}})\). The maximum spatial influence that this event can make in the time \( t_{\text{min}} \) is the distance \( ct_{\text{min}} \) and \(-ct_{\text{min}} \) from the origin to the right and to the left respectively. Therefore space can range from \(-t_{\text{min}}\) to \( t_{\text{min}} \) in the units where \( c = 1 \). For a causal diamond centered at the origin in two dimensions, this is mirrored in (1). Now the main objective is to obtain random points inside the square around a causal diamond with the use of a random number generator. This random number \( r \in [0, 1] \) can then be translated to a random number \( R \) between \((t_{\text{max}}, t_{\text{min}})\) or \((x_{\text{max}}, x_{\text{min}})\). Each mapping of a pair of random numbers into a point \((x, t)\) inside the square around a causal diamond can then be
checked for adherence to causality with the maximal and minimal element \((0, t_{\text{min}})\) and \((0, t_{\text{max}})\). Finally, keeping only the points that are causally related to the maximal and minimal elements yields the desired causal set.

2.2.1 Minkowski space

The flat 3+1 dimensional spacetime manifold is often called Minkowski space. It has the Lorentzian signature of \((-1, 1, 1, 1)\) and therefore the spacetime interval is \(ds^2 = -dt^2 + dx^2 + dy^2 + dz^2\). Figure 2.2 below shows a causal diamond with 200 elements sprinkled on 2D flat spacetime.

![Causal Diamond](image)

**Figure 2.2:** A causal diamond on a flat 2D space with 200 elements. The minimal and the maximal elements are not causally related to the points outside the diamond and therefore the blue points outside the red diamond are disregarded.

As discussed earlier, a computer random number generator was used to obtain the points inside the square centered around the causal diamond. In 2D flat spacetime, the differential volume element is, \(dv = dx dt\). It can be seen from the volume element that the volume of the causal diamond is independent of time and space label and is constant. This yields the relation,

\[
t = r (t_{\text{max}} - t_{\text{min}}) + t_{\text{min}}. \tag{2.2}
\]
In a similar way, the random number can be used to obtain a coordinate point along the spatial axis $x$,

$$ x = r \left( x_{\text{max}} - x_{\text{min}} \right) + x_{\text{min}}. $$

(2.3)

Equations (2.2) and (2.3) are quite easy to understand. Our main goal is to obtain a number between $t_{\text{min}}$ and $t_{\text{max}}$ or $x_{\text{min}}$ and $x_{\text{max}}$ using $r \in [0, 1]$. Since $r$ is generated randomly and can be anywhere between 0 and 1, multiplying $r$ with the time interval $(t_{\text{max}} - t_{\text{min}})$ yields a number anywhere between 0 and $(t_{\text{max}} - t_{\text{min}})$. Now, adding $t_{\text{min}}$ to this number generates a number anywhere between $t_{\text{min}}$ and $t_{\text{max}}$ which was our goal all along. A different random number is used to find $x$ between $x_{\text{min}}$ and $x_{\text{max}}$ using the same reasoning as before. Once a coordinate point $(x, t)$ is generated inside the square, it can be checked for causality relation with the maximal and minimal points: $(t - t_{\text{min}})^2 > x^2$ and $(t - t_{\text{max}})^2 > x^2$, which is to say that the time separation should be greater than the spatial separation in the units where $c = 1$. Now, only the points that lie inside the causal diamond are kept and the other points are disregarded as shown in Figure 2.2.

2.2.2 Schwarzschild space

To sprinkle random events in Schwarzschild spacetime, we begin by writing the line element in familiar form,

$$ ds^2 = - \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\Omega^2. $$

(2.4)

where, $d\Omega^2 = d\theta^2 + r^2 \sin^2 \theta d\phi^2$.

The volume element is, therefore,

$$ \sqrt{-g} = r^2 \sin \theta. $$

(2.5)

We see that the volume element is a function of angle and radius. If the total number of elements of a causal set tells us about the volume of the manifold, we expect more elements (or ‘events’) at a larger value of radius and towards the equator ($\sin \theta = 1$). We now want to find the coordinates $(t, r, \theta, \phi)$ such that $t$ is between $t_{\text{min}}$ and $t_{\text{max}}$, $r$ is between $r_{\text{min}} = 0$ and $r_{\text{max}}$, $\theta$ is between 0 and $\pi$, and $\phi$ can be anything. This method is then used to understand the causal structure of the Schwarzschild spacetime.
and $\phi$ is between 0 and $2\pi$, such that the volume measure of Schwarzschild spacetime is respected. The volume element is independent of $t$ and $\phi$ and we may distribute time and azimuthal angle uniformly in $[t_{\text{min}}, t_{\text{max}}]$, and $[0, 2\pi]$, i.e.,

$$t = t_{\text{min}} + N(t_{\text{max}} - t_{\text{min}}),$$  \hspace{1cm} (2.6)$$

$$\phi = 2\pi N.$$  \hspace{1cm} (2.7)$$

Where $N \in (0, 1)$ is a computer-generated random number with a uniform probability distribution.

How do we generate the radial and the angular coordinates $r$ and $\theta$? We want to generate coordinates $(t, r, \theta, \phi)$ such that there are many causal set elements (or ‘events’) at larger values of $r$ and toward the equator. One way to make this intuitive is by drawing an analogy between $N$ and probability. Imagine a sphere with radius $r_{\text{max}}$ in 3-dimensional space. If $r \in [0, r_{\text{max}}]$, what is the probability that a randomly generated event is inside the volume of radius $r$? It is the ratio of volumes of a sphere of radius $r$ to the total volume of the sphere (of radius $r_{\text{max}}$). This way, we can find the coordinate $r$ as a function of $N \in (0, 1)$, which we have here interpreted as a probability.

$$N = \frac{\text{Volume of sphere of radius } r}{\text{Volume of sphere of radius } r_{\text{max}}} = \frac{r^3}{r_{\text{max}}^3}$$

$$r = \sqrt[3]{N} \cdot r_{\text{max}}$$  \hspace{1cm} (2.8)$$

We can do the exact same thing to find coordinates in $\theta$. We equate the probability with the ratio of the volume of the spherical cone of angle $\theta$ to the volume of the sphere. In doing so we obtain,

$$N = \frac{\text{Volume of spherical cone of angle } \theta}{\text{Volume of the sphere}} = \frac{1 - \cos \theta}{2}$$

$$\theta = \arccos(1 - 2N).$$  \hspace{1cm} (2.9)$$

In this way, we map four computer-generated random numbers, $N \in (0, 1)$, to four coordinates $(t, r, \theta, \phi)$. In this thesis, we generate events ranging from a hundred to a thousand inside
Schwarzschild geometry, find the causal relationship between every pair, and compute the number of links that cross the horizon using a link matrix.
Black holes became an exciting and interesting area of research after Einstein crafted his groundbreaking theory of relativity that described spacetime structure (i.e., the metric) as a dynamic quantity that can be bent and curved due to the influence of energy-momentum. Black holes described regions of spacetime with infinite curvature and infinite mass density at the singularity. The idea that black holes are thermodynamic objects and that there is an analogy between black hole mechanics and thermodynamics started when Roger Penrose published his paper on energy extraction from Kerr black holes. Kerr black holes are rotating black holes that can be completely characterized by their mass and angular momentum. Penrose had shown that it was possible to send a particle of total energy $E$, such that it avoids getting inside the horizon and slingshots from a region between the horizon and the surface of infinite redshift. This region for Kerr black holes is called the ergosphere. This particle when it comes out of the ergosphere would have an energy $E' > E$. The increase in energy of the particle comes from the rotational energy of the Kerr black hole and the angular momentum of this black hole decreases by a tiny amount.

If the rotational energy of a Kerr black hole could be extracted by the Penrose process, this would mean that a Kerr black hole could be transformed into a Schwarzschild (non-rotating and chargeless) black hole after we send enough particles that all of its angular momentum is extracted. The resulting Schwarzschild black hole would have a lower mass because the rotational mass-energy is lost. This mass of a Kerr black hole after all of its rotational mass-energy is lost is called the irreducible mass $M_{ir}$. Conversely, we could also send particles of mass $m$ and angular momentum $\tilde{l}$ into a Schwarzschild black hole of mass $M_{ir}$ and make it Kerr. Is the process Schwarzschild-Kerr-Schwarzschild a reversible process? Demetrious Christodoulou studied such
processes and concluded that one can approach arbitrarily closely to reversible transformations that increase or decrease the rotational contribution to the square of the mass and that there exist no processes that will decrease the irreducible mass $M_{ir}$, i.e., $dM_{ir} \geq 0$. He also derived an equation that relates the irreducible mass $M_{ir}$ of a black hole to the area of the event horizon $A$, i.e.,

$$A = 16\pi M_{ir}^2. \tag{3.1}$$

A simple differentiation of equation 3.1 results in

$$dM_{ir} = \frac{1}{32\pi M_{ir}} dA. \tag{3.2}$$

Imagine we have a non-rotating and chargeless black hole of mass $M_{ir}$ and we send a particle of mass $m$ into this black hole. What is the change in the internal energy of this black hole? One can easily guess that $dE = dM_{ir} = m$ ($c = 1$). What change will the addition of this particle bring in the black hole? Will there be a volume increase? Note that volume is not a well-defined notion when dealing with black holes. Will there be an area increase? From 3.2, we see that just like $dM_{ir} \geq 0$ for any processes involving black hole transformation, so is $dA \geq 0$. This reminds us of the never decreasing property of entropy for a closed system. For a more general Kerr–Newman black hole of mass $M$, charge $Q$, and angular momentum $\bar{L}$, we have

$$dM = \Theta d\alpha + \Omega \cdot d\bar{L} + \Phi dQ \tag{3.3}$$

where

$\alpha = A/4\pi$ is called the “rationalized area”,

$\Theta = (r_+ - r_-)/4\alpha$,

$r_\pm = M \pm (M^2 - Q^2 - a^2)^{1/2}$, $\bar{a} = \bar{L}/M$,

$\Omega = \bar{a}/\alpha$,

$\Phi = Qr_+/\alpha$. 

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Equation 3.3 resembles the first law of thermodynamics,

\[ dE = TdS - PdV. \] (3.4)

Comparing 3.3 and 3.4, one can easily see that \( \Omega \cdot d\tilde{L} + \Phi dQ \) is like work done on a black hole by an external agent in increasing its angular momentum and charge by an amount \( d\tilde{L} \) and \( dQ \) respectively\(^2\). Following this analogy, \( \Theta \) looks like a black hole analog of temperature and the area of a black hole resembles the entropy. This resemblance was not attributed a deeper meaning at first because black holes were thought to be objects at 0 K temperature that did not radiate. However, with Hawking’s discovery of Hawking radiation, the analogy between thermodynamics and black hole mechanics was considered seriously. Hawking later showed that the ratio of entropy to the area of the horizon is \( 1/4 \) \((G = c = \hbar = 1)^9\).

Entropy in general is a function of the number of quantum states a physical system may occupy. If one knew in principle everything there is to know about the system, then there would be nothing more to know about and the entropy would be zero. Stationary black holes are simple objects that can be described by only three parameters; mass, angular momentum, and electric charge. What are the microstates of a black hole? Why is entropy proportional to the area of the event horizon? We shall see in the following chapters that if we consider a loss of causal correlation between spacetime elements for an observer who is outside the event horizon in the causal set formalism, this scales linearly with the size of the event horizon. Such correlations between the causal set elements, which we call ‘links’, are then interpreted as a possible source of black hole entropy.

3.1 **Schwarzschild spacetime**

Schwarzschild spacetime is like a favorite playground for many physicists as it provides a background to test the theories of gravity and quantum mechanics or to find a way in which these two may be related. Our goal is to generate random events in this spacetime and find the causal
relationship between every pair. We then check if the total number of links crossing the event horizon can be interpreted as a possible source of black hole entropy by studying how it scales in relation to other parameters like the horizon area, time duration, and the total number of events.

In a 4-dimensional flat Minkowski spacetime, given two events \( E_1(t_1, x_1, y_1, z_1) \) and \( E_2(t_2, x_2, y_2, z_2) \), it is easy to figure out whether they are timelike related or not. This is not the case for a spacetime with curved geometry. In general, to check whether two events are causally related or not, we can imagine null geodesics emanating from the first event (by first we mean that this event occurs earlier in time than the second event, i.e. \( t_1 < t_2 \)) and intersecting the worldline where the second event takes place later in time. If the fastest arriving null geodesic intersects the worldline of the second event at a time later than \( t_2 \), then these events cannot be causally related.

In order to find out the causal relation between pairs of events sprinkled on a Schwarzschild space, we refer to the paper by Song He and David Rideout\(^\text{10}\) and therefore we avoid the long and tedious discussion of finding causal relations in this thesis.

This was our first attempt to study how the number of links changes by changing the size of the event horizon. A closed-form analytical result for the expected number of links in Schwarzschild geometry is really difficult to calculate (if not impossible) because the light cone structure bends and tilts as we approach the horizon from the outside. We therefore only have computer simulation results. Here, we do not impose a maximal and minimal condition for the events below and above the hypersurface (as we shall do later) but rather count the number of horizon-crossing links between two hypersurfaces separated by an interval \( \Delta t \). We start by generating events in the Schwarzschild spacetime whose coordinates have the distribution given by equations (2.6), (2.7), (2.8), and (2.9) and we sort these events on the basis of their time coordinate. We then separate these events into two categories: (1) Events that are inside the event horizon and (2) Events that are outside the event horizon. We then find the causal relationships between pairs of events that are (1) both inside the horizon, (2) both outside the horizon, and (3) one inside and the other outside the horizon. Finding causal relations between events in Schwarzschild spacetime can quickly become complicated and messy and we believe that this categorization of events makes coding much easier. Finally, we
compose the three relations matrices. The resulting overall relations matrix then has the causal information about every pair of events in a Schwarzschild spacetime. From this relations matrix, we then obtain the link matrix. The non-zero entries of the link matrix that correspond to the first event being outside the horizon and the second event being inside the horizon, represent a loss of one bit of causal information. We calculate the number of such links and find how it changes with the total number of events, the time duration, and the area of the event horizon. Without the maximal/minimal condition, the number of links tends to increase with the volume of the region where we sprinkle events.

**Number of links vs. time**

It is fairly obvious that in the causal set formalism, the entropy of the black hole, which we take to be proportional to the number of links crossing the horizon, increases if more events are causally related across the horizon. For two events to be timelike related, the temporal separation between them must be larger than their spatial separation. If \( N \) events are sprinkled inside a bounded spacetime region with a small time duration, the events might not be sufficiently separated in time to possess a timelike relation with each other. Only the events that are really close in space will have a causal relation. Therefore, for a fixed number of events sprinkled inside a bounded region of Schwarzschild spacetime, one can expect the number of causally related pairs to increase with the time separation between the initial and final hypersurfaces. This will in turn increase the non-zero entries of the relations matrix and the link matrix, meaning an increase in entropy. However, entropy should be independent of the size of the sprinkling region. But, we see that in causal set formalism, if we sprinkle events in a spatially bounded region with a very small time duration (\( t \approx 0 \)), the entropy approaches zero. In order to make the number of horizon-crossing links independent of time interval, we later impose maximal/minimal conditions on two events and only calculate the number of links close to a particular spacelike hypersurface.

To test the dependence of entropy (interpreted as the number of horizon-crossing links) on the time interval between two spacelike hypersurfaces, we sprinkled 50 events in a bounded
Schwarzschild region with $0 \leq r \leq 3$ (in units where $G$, $M$, and $c = 1$), and $0 \leq t \leq 1$ ($\Delta t = 1$). By fixing $M = 1$ we have set the event horizon at $r_s = 2$. We calculated the number of horizon-crossing links for this configuration a hundred times and took the average. Keeping the number of events and the spatial extent fixed, i.e. $0 \leq r \leq 3$, we increased the time duration by 0.01, i.e., $0 \leq t \leq 1.01$ ($\Delta t = 1.01$), calculated the entropy a hundred times, and averaged. We repeat this process until $0 \leq t \leq 10$ ($\Delta t = 10$). Again, we take $0 \leq t \leq 10$ ($\Delta t = 10$), $0 \leq t \leq 10.1$ ($\Delta t = 10.1$), $0 \leq t \leq 10.2$ ($\Delta t = 10.2$), and so on until $0 \leq t \leq 100$ ($\Delta t = 100$). Similarly, we take $\Delta t = 100$, $\Delta t = 101$, $\Delta t = 102$ and so on when $0 \leq t \leq 100$, $0 \leq t \leq 101$, $0 \leq t \leq 102$ and so on until $0 \leq t \leq 1000$. Finally, we increase the time interval in increments of 10 when $0 \leq t \leq 10000$. We choose this configuration to avoid crowding of data points on the semilog scale as shown in Figure 3.1. We found that the entropy increases with the time duration as we discussed above. However, the total number of links seems to reach a maximum value for a specific time duration, decrease slowly, and eventually saturate if the time duration is taken to be extremely large as can be seen in Figure 3.1 when $1000 \leq \Delta t \leq 10000$. This means that the entropy for a fixed number of events sprinkled on a spatially bounded region say $0 \leq r \leq r_{\text{max}}$, but unbounded in time, i.e., $0 \leq t < \infty$ is finite and approaches a constant for $\Delta t \to \infty$.

The number of surface-crossing links tends to reach a maximum value for a particular time interval and then decrease as can be seen in Figure 3.1. On further increasing the time interval, the number of links tends to approach a constant value. On increasing the time interval (starting from $\Delta t = 0$), the number of horizon-crossing links increases because more events will have causal relations as discussed above. On further increasing the time interval, the causal relation between pair of events that was a link before now becomes a relation (a new event shall come inside the causal diamond of this pair). This explains the dip in the number of links as seen in Figure 3.1. On further increasing the time interval, the rate at which new relation is created (from earlier links) is the same as the rate at which new links are created (from earlier relations). This equilibrium between the creation and destruction of links tends to saturate the number of surface-crossing links.
Figure 3.1: Each dot in this figure is an average number of horizon crossing links taken from 100 causal sets calculated for 50 events sprinkled in the region $0 \leq r \leq 3$ and varying in time. It can be seen that the entropy increases on increasing the temporal extent of the sprinkling and reaches a maximum and then decreases until it saturates for large time intervals.

**Number of links vs. number of elements**

By increasing the total number of events, we increase the total number of pairs that might be causally related to one another. It is quite straightforward to think that increasing the number of elements inside a bounded spacetime region, keeping every other parameter fixed, should increase the total number of links crossing the horizon too.
We have seen from Figure 3.1 that the total number of links that cross the horizon tends to saturate only for a large time duration. We have therefore taken the temporal bound to be $0 \leq t \leq 1000$ so that we may study the dependence of entropy on the total number of causal set elements at saturation. Our spatial bound is still $0 \leq r \leq 3$, and the event horizon is at $r_s = 2$. We start by sprinkling 100 elements in this region and find the entropy. We repeat this process a hundred times and take the average entropy. We then sprinkle 200 elements, 300 elements, and so on up to 1000 elements, and find the average entropy each time. Notice that we do not change our temporal or spatial bounds where we are sprinkling events, we only increase the number of events.

**Number of links vs. area of the horizon**

We finally check if the number of links that cross the horizon scales linearly with the area of the horizon. Keeping the number of events and the temporal and spatial extent of sprinkling fixed,
increasing the area of the event horizon (changing where the horizon is) increases the number of events that fall inside the horizon. This also increases the number of events that are inside the horizon and close to it spatially, which then increases the number of causal relations these events inside the horizon may have with the events that are outside. We expect the number of links to rise, and this is shown in Figure 3.3.

We sprinkled 1000 elements in a spacetime region bounded spatially by \(0 \leq r \leq 5\) and temporally by \(0 \leq t \leq 1000\). Here, we have not imposed the maximal/minimal condition on the events inside and outside the horizon. We are however counting all links that cross the horizon for a given time interval and therefore we chose a large temporal bound to give sufficient room for the number of links to saturate. We started with the location of the horizon put at \(r_s = 2\), calculated entropy a hundred times for this configuration, and averaged. We then changed the location of the horizon to \(r_s = 2.1\), \(r_s = 2.2\), and so on up to \(r_s = 3\). We calculate the number of links that cross the horizon for each value of the horizon area a hundred times and take the average. The result we have obtained is shown in Figure 3.3.
3.2 Expected number of links in two-dimensional flat spacetime

Imagine a region in two-dimensional spacetime such that the events inside this region cannot have a causal influence on the events that occur outside. They may however be causally influenced by the events that occur outside this one-way membrane which we call the horizon. In this sense, the region that is causally bounded by a horizon in two dimensions resembles a 2D black hole. We may call these objects flat black holes. Although a realistic black hole curves the spacetime structure, here we will assume that the background spacetime structure is flat and the light cone structure does not tilt towards the horizon as we approach the horizon from outside and we just want to see how the causal information that passes through the surface representing a one-way horizon at a particular spacelike hypersurface scales with the size of the horizon. To ensure that
only events that are close to the hypersurface contribute to the lost causal information, we impose
the condition that the event below the hypersurface and outside the horizon be maximal for the
causal set elements that are below the hypersurface. This means that there are no other events in
the region bounded by the intersection of the future light cone of this event and the hypersurface.
Similarly, we impose a second condition that the event that occurs inside the horizon be minimal,
meaning there are no other events in the region bounded by the intersection of the past light cone
of this event and the hypersurface.

Imagine a bounded volume region in a two-dimensional space of volume $V_0$ such that $N$
eles are sprinkled randomly and uniformly inside it. The number density is $\rho = N/V_0$. Imagine
now a smaller volume region $V$ inside $V_0$. The probability that there will be exactly $n$ events in a
volume $V$ when $V_0$ is made infinitely big ($N \to \infty$) keeping the number density fixed is then given
by the Poisson distribution, i.e., $P(N, n) = (\rho V)^n e^{-\rho V}/n!$, where $N$ is the number of events in the
volume $V_0$ and $\rho$ is the fundamental density which is assumed to be of Planckian magnitude $^{11}$.

![Figure 3.4: Events sprinkled randomly and uniformly inside a two-dimensional Minkowski space. The horizon is at $x = -H$ and $x = H$ and region 1, region 2, and region 3 are the regions where events occur outside the horizon and below the $t = 0$ hypersurface.](image)

Two events $x$ and $y$ are linked iff they are timelike related ($x \prec y$) and there are no
other events in the volume region causally between them i.e., $J(x, y) = J^+(x) \cap J^-(y)$ is empty.
Also, event $x$ is maximal iff $J^+(x) \cap J^-(H)$ is empty and event $y$ is minimal iff $J^-(y) \cap J^+(H)$ is empty, where $H$ is the event horizon. The probability that there is exactly one event in the infinitesimal volume elements $dV_x$ and $dV_y$ can be obtained by expanding $P(N, n)$ in powers of $\rho V$ and keeping only the first-order term with $V = dV$, and is $\rho dV_x$ and $\rho dV_y$ respectively. This is because $P(N, 1) = (\rho dV_x)e^{-\rho dV_x} = (\rho dV_x)(1 - \rho dV_x + \cdots) \approx \rho dV_x$. The probability that event $x$ is maximal and event $y$ is minimal is $e^{-\rho V(x)}$ and $e^{-\rho V(y)}$, where $V(x)$ ($V(y)$) is the volume of the region bounded by the future (past) light cone of event $x$ ($y$) and the spacelike hypersurface. Then, the probability that the events $x$ and $y$ are linked is $\rho^2 e^{-\rho(V(x)+V(y))}dV_x dV_y$ and the expected number of such pairs is given by\(^5\)

\[
\langle n \rangle = \int_D \rho^2 e^{-\rho(V(x)+V(y))}dV_x dV_y. \tag{3.5}
\]

Here, $D$ is the region of integration for events $x$ and $y$. If the event $x$ is in region 1 as shown in Figure 3.4 then the region of integration for this event in space is from $H$ to $\infty$ and in time from $(-x + H)$ to 0 as shown in Figure 3.5. If the location of event $x$ is $(x_1, t_1)$, then the region of integration of event $y$ is the region bounded by the future light cone of event $(x_1, t_1)$ and the black hole. Also, the volume of the region bounded by the future light cone of event $x$ $(x_1, t_1)$ and the $t = 0$ axis in two dimensions is the area of the triangle of height $t_1$ and base $2t_1$ and therefore $V(x) = t_1^2$. Similarly, for an event $y$ $(x_2, t_2)$, $V(y) = t_2^2$ in two–dimensional spacetime. In the calculations that follow, we shall assume that there is exactly one event on average inside the volume region $l_N^p$, where $N$ is the spacetime dimension and $l_p$ is assumed to be Planck length, therefore we keep $\rho = 1$.  

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Links from the elements in region 3

\[ \langle n_3 \rangle = \int_H^\infty dx_1 \int_{x_1-H}^0 dt_1 \int_{-H}^H dx_2 \int_0^\infty dt_2 \ e^{-t_1^2-t_2^2} \]
\[ = \int_H^\infty dx_1 \int_{x_1-H}^0 dt_1 \ e^{-t_1^2} \sqrt{\pi} H \]
\[ = \sqrt{\pi} H \int_H^\infty dx_1 (1 - \text{erf}(x_1 + H)) \cdot \frac{\sqrt{\pi}}{2} \]
\[ = \frac{\pi H}{2} \lim_{C \to \infty} \left[ (C - H) - (C + H) \text{erf}(C + H) + 2H \text{erf}(2H) - \frac{e^{-(C+H)^2}}{\sqrt{\pi}} + \frac{e^{-4H^2}}{\sqrt{\pi}} \right] \]
\[ = \frac{\pi H}{2} \left[ 2H(\text{erf}(2H) - 1) + \frac{e^{-4H^2}}{\sqrt{\pi}} \right]. \]

Links from the elements in region 1

\[ \langle n_1 \rangle = \int_H^\infty dx_1 \int_0^0 dt_1 \int_{-H}^H dx_2 \int_{-x_2+x_1+t_1}^\infty dt_2 \ e^{-t_1^2-t_2^2} \]
\[ = \int_H^\infty dx_1 \int_0^0 dt_1 \int_{-H}^H dx_2 \ e^{-t_1^2} \frac{\sqrt{\pi}}{2} (1 - \text{erf}(x_1 + t_1 - x_2)) \]
\[ = \sqrt{\pi} H \int_H^\infty dx_1 \int_0^0 dt_1 e^{-t_1^2} - \int_H^\infty dx_1 \int_{-H}^{x_1+H} dt_1 \int_0^0 dx_2 \ e^{-t_1^2} \frac{\sqrt{\pi}}{2} \text{erf}(x_1 + t_1 - x_2) \]
\[ = \lim_{C \to \infty} \frac{\pi H}{2} \left[ (C - H) \text{erf}(C - H) - \frac{1}{\sqrt{\pi}} \right] - \frac{\sqrt{\pi}}{2} \int_H^\infty dx_1 \int_0^0 dt_1 e^{-t_1^2} \left[(x_1 + t_1 + H) \right. \]
\[ \text{erf}(x_1 + t_1 + H) - (x_1 + t_1 - H) \text{erf}(x_1 + t_1 - H) + \left. \frac{e^{-(x_1+t_1+H)^2} - e^{-(x_1+t_1-H)^2}}{\sqrt{\pi}} \right]. \]

In region 1, \( x_1 \geq H, -x_1 + H \leq t_1 \leq 0 \), and \( 2H \text{erf}(2H) \leq [(x_1 + t_1 + H) \text{erf}(x_1 + t_1 + H) - (x_1 +}


\( t_1 - H \) \( \text{erf}(x_1 + t_1 - H) \) \( \leq 2H \). Therefore, the integral is

\[
\langle n_1 \rangle \leq \lim_{C \to \infty} \frac{\pi H}{2} \left[ (C - H) \text{erf} (C - H) - \frac{1}{\sqrt{\pi}} \right] - \frac{\sqrt{\pi}}{2} \int_H^\infty dx_1 \int_{-x_1 + H}^0 dt_1 e^{-t_1^2} \left[ 2H \text{erf}(2H) + \frac{e^{-(x_1+t_1+H)^2} - e^{-(x_1+t_1-H)^2}}{\sqrt{\pi}} \right] 
\]

\[
\approx \lim_{C \to \infty} \frac{\pi H}{2} \left[ (C - H) \text{erf} (C - H) - \frac{1}{\sqrt{\pi}} \right] - \lim_{C \to \infty} \frac{\pi H}{2} \left[ (C - H) \text{erf} (C - H) - \frac{1}{\sqrt{\pi}} \right] - \frac{\sqrt{\pi}}{4\sqrt{2}} \int_H^\infty dx_1 \frac{e^{-(x_1+H)^2}}{2} \left[ \text{erf} \left( \frac{x_1 + H}{\sqrt{2}} \right) - 2e^{2Hx_1} \text{erf} \left( \frac{x_1 - H}{\sqrt{2}} \right) + \text{erf} \left( \frac{x_1 - 3H}{\sqrt{2}} \right) \right] 
\]

\[
\leq \frac{\sqrt{\pi}}{4\sqrt{2}} \int_H^\infty dx_1 \frac{e^{-(x_1+H)^2}}{2} \left[ 2e^{2Hx_1} \text{erf} \left( \frac{x_1 - H}{\sqrt{2}} \right) - \text{erf} \left( \frac{x_1 + H}{\sqrt{2}} \right) + 1 \right] 
\]

\[
= \frac{\sqrt{\pi}}{4\sqrt{2}} \left[ \frac{\sqrt{\pi}}{2\sqrt{2}} - \frac{\sqrt{\pi}}{2\sqrt{2}} (1 - \text{erf}^2 \sqrt{2}H) + \frac{\sqrt{\pi}}{\sqrt{2}} (1 - \text{erf} \sqrt{2}H) \right].
\]

The last term in the second line cannot be integrated analytically. But, the function \( e^{-(x_1+H)^2/2} \) is centered around \( x_1 = -H \). When \( x_1 = -H, \text{erf} \left( (x_1 - 3H)/\sqrt{2} \right) \leq -1, \forall \ H > 1. \)

**Links from the elements in region 2**

For an event that falls in Region 2, the limits of integration for the second event inside the horizon can be divided into three sub-regions as shown in Figure 3.5,

\[
\langle n_2 \rangle = \int_H^\infty dx_1 \int_{-x_1 - H}^{-x_1+H} dt_1 \left[ \text{Region}(A) + \text{Region}(B) + \text{Region}(C) \right] 
\]

\[
= \int_H^\infty dx_1 \int_{-x_1 - H}^{-x_1+H} dt_1 \left[ \int_{-H}^{H+x_1+t_1} dx_2 \int_{-x_2 + x_1 + t_1}^{H+x_1+t_1} dt_2 e^{-t_1^2 - t_2^2} + \int_{-H}^{H+x_1+t_1} dx_2 \int_0^{H+x_1+t_1} dt_2 e^{-t_1^2 - t_2^2} 
\]

\[
+ \int_{-H}^{H} dx_2 \int_{H+x_1+t_1}^{\infty} dt_2 e^{-t_1^2 - t_2^2} \right].
\]
**Figure 3.5:** For an event \((x_1, t_1)\) inside Region 2, the region of integration for the events \((x_2, t_2)\) occurring in the portion of the future of \((x_1, t_1)\) that is inside the horizon can be subdivided into Regions A, B, and C.

**Region A**

\[
\langle n_{2A} \rangle = \int_{-\infty}^{\infty} \left( \int_{-\infty}^{-x_1+H} dx_1 \int_{x_1}^{x_1+H} dt_1 \int_{-\infty}^{-H} dx_2 \int_{x_2}^{H+x_1+t_1} dt_2 \right) e^{-t_1^2} e^{-t_2^2} \\
= \int_{-\infty}^{\infty} dx_1 \int_{-x_1+H}^{x_1+H} dt_1 \int_{-\infty}^{-H} dx_2 \sqrt{\pi} e^{-t_1^2} \left( \text{erf}(x_1 + t_1 + H) + \text{erf}(x_2 - x_1 - t_1) \right) \\
= \frac{\sqrt{\pi}}{2} \int_{-\infty}^{\infty} dx_1 \int_{-x_1+H}^{x_1+H} dt_1 e^{-t_1^2} \left[ \frac{1}{\sqrt{\pi}} - \frac{e^{-(x_1+t_1+H)^2}}{\sqrt{\pi}} \right] \\
= \frac{\sqrt{\pi}}{2} \int_{-\infty}^{\infty} dx_1 \left[ \text{erf}(x_1 + H) - \text{erf}(x_1 - H) - \frac{e^{-(x_1+H)^2}}{2 \sqrt{2}} (\text{erf}(\frac{x_1+H}{\sqrt{2}}) - \text{erf}(\frac{x_1-3H}{\sqrt{2}})) \right] \\
\leq \frac{\sqrt{\pi}}{2} \left[ \frac{1}{2} \left( 1 - e^{-4H^2} \right) \frac{1}{\sqrt{\pi}} + 2H(1 - \text{erf}(2H)) \right] + \frac{\sqrt{\pi}(\text{erf}^2(\sqrt{2}H) - 1)}{8} + \frac{\sqrt{\pi}(1 - \text{erf}(\sqrt{2}H))}{8}.
\]
Region B

\[ \langle n_{2B} \rangle = \int_{H}^{\infty} dx_1 \int_{-x_1-H}^{-x_1+H} dt_1 \int_{x_1+H}^{H} dx_2 \int_{0}^{H+x_1+t_1} dt_2 e^{-t_1^2-t_2^2} \]

\[ = \int_{H}^{\infty} dx_1 \int_{-x_1-H}^{-x_1+H} dt_1 \int_{x_1+H}^{H} dx_2 \frac{\sqrt{\pi}}{2} e^{-t_1^2} \text{erf}(x_1 + t_1 + H) \]

\[ \leq \int_{H}^{\infty} dx_1 \int_{-x_1-H}^{-x_1+H} dt_1 \int_{x_1+H}^{H} dx_2 \frac{\sqrt{\pi}}{2} e^{-t_1^2} \]

\[ = \frac{\sqrt{\pi}}{2} \int_{H}^{\infty} dx_1 \int_{-x_1-H}^{-x_1+H} dt_1 (H - x_1 - t_1) e^{-t_1^2} \]

\[ = \frac{\sqrt{\pi}}{2} \int_{H}^{\infty} dx_1 \left[ \frac{\sqrt{\pi}(H - x_1)}{2} \left( \text{erf}(H - x_1) + \text{erf}(H + x_1) \right) + \left( \frac{e^{-(H-x_1)^2}}{2} - \frac{e^{-(H+x_1)^2}}{2} \right) \right] \]

\[ = \frac{\pi}{4} \left[ \frac{1}{2\sqrt{\pi}} e^{-2(x_1^2+H^2)} ((x_1 - H)e^{(x_1+H)^2} + (3H - x_1)e^{(x_1-H)^2}) \right. \]

\[ \left. - \frac{1}{4}(2(x_1 - 3H)(x_1 + H) - 1) \right] \]

\[ \left. \text{erf}(x_1 + H) - 2(x_1 - H)^2 - 1 \text{erf}(x_1 - H) \right) \]

\[ \lim_{H \to \infty} \int_{H}^{\infty} dx_1 \left[ \frac{\sqrt{\pi}}{2} \left( \frac{\pi}{8} \text{erf}(2H) \right) \right] + \frac{\pi}{8} \text{erf}(2H) \]

\[ = \pi \text{erf}(2H) - \frac{\sqrt{\pi}H}{4} e^{-4H^2} - \frac{\pi}{2}H^2(\text{erf}(2H) - 1). \]

Region C

\[ \langle n_{2C} \rangle = \int_{H}^{\infty} dx_1 \int_{-x_1-H}^{-x_1+H} dt_1 \int_{H}^{H+x_1+t_1} dx_2 \int_{0}^{H+x_1+t_1} dt_2 e^{-t_1^2-t_2^2} \]

\[ = \int_{H}^{\infty} dx_1 \int_{-x_1-H}^{-x_1+H} dt_1 \sqrt{\pi} H e^{-t_1^2} \]

\[ (1 - \text{erf}(x_1 + t_1 + H)). \]

It is not possible to do an explicit analytical integration for the second term. But, for the events outside the horizon at \( x = H, x_1 \geq H, \) and \( \text{erf}(x_1 + t_1 + H) \geq \text{erf}(2H + t_1). \) We may then write,

\[ \langle n_{2C} \rangle \leq \int_{H}^{\infty} dx_1 \int_{-x_1-H}^{-x_1+H} dt_1 \sqrt{\pi} H e^{-t_1^2} \]

\[ (1 - \text{erf}(2H + t_1)). \]

The function \( e^{-t_1^2} \) is centered around \( t_1 = 0. \) When, \( t_1 = 0, \) for \( H \gg 1, \) \( \text{erf}(2H + t_1) \approx 1. \) Therefore, the whole integral must converge to 0.
We now have an analytical expression for the expected number of horizon-crossing links in two-dimensional flat spacetime. We compare this with the result obtained from computer simulations. We take a region in two-dimensional flat spacetime such that $-7 \leq x \leq 7$ and $-7 \leq t \leq 7$. We select 196 events randomly in this region such that the density is unity and calculate the links crossing the horizon at the $t = 0$ hypersurface and average over 500 such sprinklings. The expected number of links that cross the horizon is independent of the size of the black hole in two dimensions and is bounded between, $0 \leq \langle n \rangle_{2D} \leq (\pi + 2)/4$. The expected number of links crossing the horizon calculated theoretically (by evaluating the integrals for expected value $\langle n \rangle$) agrees well with computer simulation (computer sprinkling) and the result obtained from numerical integration as shown in Figure 3.6. In two dimensions, the horizon does not have a “size” and therefore the number of horizon-crossing links is independent of the size of the black hole, which makes intuitive sense.

![Figure 3.6](image)

**Figure 3.6:** Number of links crossing the horizon in two-dimensional flat spacetime averaged over 500 sprinklings where the number of events $N = 196$.

It can be seen from Figure 3.6 that the number of horizon-crossing links obtained from computer simulation is always less than the theoretical expectation calculated analytically and the numerical integration done in MATLAB when averaged over many sprinklings. A need to put
a spatial and temporal bound to do an actual simulation is the reason for this discrepancy. In reality, space and time can be infinite and there will be some contributions coming from regions outside of our simulation bounds. The numerical integration and an analytical expression for the expectation become the same when $H \gg 1$ as Figure 3.6 shows. When $0 \leq H \leq 1$, we have dropped many integration terms when finding the analytical expression for the expected number of links. Therefore, the discrepancy between numerical integration and the analytical result appears when $0 \leq H \leq 1$. Also, the error bars in Figure 3.6 indicate the standard deviation from the mean number of links for 500 sprinklings. The standard deviation is large because we have only taken 196 elements to calculate the number of surface-crossing links. We choose 196 elements in order to maintain a unit density.

3.3 Three and four-dimensional flat spacetimes

3.3.1 Expected number of links in three dimensions

In three dimensions, the region of integration for the first event is the region outside the event horizon and below the spacelike hypersurface. This region can be subdivided into three regions as shown in Figure 3.7. Region 1 is the region bounded by the plane ($t = 0$) and the big cone. Region 2 is the region bounded by the two cones and Region 3 is the region outside the cylinder and below the small cone as shown in Figure 3.7. In reality, the plane, the cylinder, and the two cones all extend to infinity, and Figure 3.7 only shows the regions with spatial and temporal bounds imposed for visual aid.

Imagine now that an event occurs below the $t = 0$ plane, outside the horizon $r = H$, and in the first quadrant. The future light cone of this event intersects the cylinder after a certain time interval. Any causal influence that this event (referred to as Event 1) might have on the events inside the black hole is irretrievable causal information. The region of causal influence that Event 1 might have inside the black hole is the volume region bounded by the intersection of the future light cone of Event 1 and the cylinder as shown in Figure 3.8 below. Given a random event $(x_1, y_1, t_1)$, such that $\sqrt{x_1^2 + y_1^2} \geq H$ and $t_1 \leq 0$, we have to find the volume of the region bounded by the future
Figure 3.7: The cylinder represents a two-dimensional black hole evolving in time. The event horizon is at $r = H$ (cylindrical surface) and the region where events occur outside the horizon and below the $t = 0$ plane has been subdivided into three regions. These events can have a causal influence on the events that occur inside the horizon and above the $t = 0$ plane.

light cone of this event and the cylinder. To achieve this, imagine the future light cone of this event as being composed of infinite circles of radius $r = \Delta t \ (c = 1)$ stacked one over another. This is shown in Figure 3.9. The intersection volume is then the area of the intersection of two circles $x^2 + y^2 = H^2$ and $(x - x_1)^2 + (y - y_1)^2 = (t - t_1)^2$ integrated over time such that, $t_{max} \geq t \geq t_{min}$.

Also, notice that the intersection area of the two circles is coordinate-independent and only depends on how far the first event is from the event horizon, i.e., spatial distance from the horizon to Event 1 = $\sqrt{x_1^2 + y_1^2} - H = r_1 - H$, where $r_1$ is the distance from the center of the coordinate system to Event 1. Light has to travel this distance to just touch the horizon, i.e., $t_{min} - t_1 = r_1 - H$. Similarly, $t_{max} - t_1 = r_1 + H$. If we now find the intersection area of two circles when $t_{max} \geq t \geq t_{min}$, we can integrate this area over time to find the volume. The equations of the two circles are;
Figure 3.8: Future light cone of Event 1 \((x_1, y_1, t_1)\) intersecting the cylinder.

\[
x^2 + y^2 = H^2
\]  
(3.6)

and,

\[
(x - r_1)^2 + y^2 = (t_2 - t_1)^2
\]  
(3.7)

Notice that we have rotated our coordinate system such that Event 1 is on the \(x\)-axis and relabelled the coordinate \(t\) as \(t_2\). This is shown in Figure 3.10 below. This does not change the intersection area. We may now solve equations 3.6 and 3.7 to get

\[
x_{int} = \frac{1}{2r_1} (r_1^2 + H^2 - (t_2 - t_1)^2)
\]  
(3.8)

and

\[
y_{int} = \frac{1}{2r_1} \sqrt{4r_1^2H^2 - (r_1^2 + H^2 - (t_2 - t_1)^2)^2}
\]  
(3.9)
Figure 3.9: Top view of Figure 3.8 where the $t$ axis is pointing out of the page. The light cone of Event 1 intersects the horizon at $t_{\text{min}}$ and $t_{\text{max}}$. The area of the intersection of two circles between $t_{\text{min}}$ and $t_{\text{max}}$ integrated over time is the volume of the region bounded by the future light cone of Event 1 and the cylinder.

Figure 3.10: The area of the intersection of two circles is $2(A_1 + A_2)$. A coordinate transformation has been made such that Event 1 is on the $x$ axis. This does not change the intersection area but makes it easier to calculate it.
The intersection area is then, \( A = 2(A_1 + A_2) \), where

\[
A_1 = \int_{r_1+t_1-t_2}^{r_1+t_1+t_2} \sqrt{(t_2 - t_1)^2 - (x - r_1)^2} \, dx
\]  

(3.10)

and

\[
A_2 = \int_{\frac{1}{2}(r_1^2 + H^2 - (t_2 - t_1)^2)}^{H} \sqrt{H^2 - x^2} \, dx.
\]  

(3.11)

Equations 3.10 and 3.11 can be integrated analytically, and we obtain

\[
A = \frac{1}{4r_1^2}(H^2 - r_1^2 - (t_2 - t_1)^2)\sqrt{4r_1^2(t_2 - t_1)^2 - (H^2 - r_1^2 - (t_2 - t_1)^2)^2}
\]

\[+ (t_2 - t_1)^2 \arctan \left( \frac{H^2 - r_1^2 - (t_2 - t_1)^2}{\sqrt{4r_1^2(t_2 - t_1)^2 - (H^2 + r_1^2 - (t_2 - t_1)^2)^2}} \right) \]

\[- \frac{1}{4r_1^2}(H^2 + r_1^2 - (t_2 - t_1)^2)\sqrt{4r_1^2H^2 - (H^2 + r_1^2 - (t_2 - t_1)^2)^2} \]

\[- H^2 \arctan \left( \frac{H^2 + r_1^2 - (t_2 - t_1)^2}{\sqrt{4r_1^2H^2 - (H^2 + r_1^2 - (t_2 - t_1)^2)^2}} \right) + \frac{\pi H^2}{2} \]

\[+ \frac{\pi(t_2 - t_1)^2}{2}.
\]  

(3.12)

The volume of the region bounded by the future light cone of Event 1 \((r_1, 0, t_1)\) and the cylinder \(r = H\), between \(t_{max} \geq t_2 \geq t_{min}\) is then:

\[
V_{int} = \int_{t_{min}}^{t_{max}} A \, dt.
\]

where \(t_{min} = r_1 + t_1 - H\) and \(t_{max} = r_1 + t_1 + H\).
The intersection area depends on the location of Event 1 \((r_1, t_1)\) and the time \((t = t_2)\), between \(t_{\text{min}}\) and \(t_{\text{max}}\), at which we are calculating the intersection area and therefore is written as a function of \(r_1, t_1,\) and \(t_2\). Also, the volume of a right cone of height ‘\(t\)’ in three dimensions is \(\pi r^3 / 3\). The absolute sign is used in equation 3.13 to ensure that the volume is always positive. Also, notice that at first look, equation 3.13 looks dimensionally inconsistent. The subtlety is that we have assumed the density of elements to be unity \(\rho = 1\), and therefore multiplying equation 3.13 by \(\rho^2\) makes it unitless as it should be.

The number of links that cross the event horizon at a spacelike hypersurface \(t = 0\) such that the first event is maximal in the region \(r \geq H\) and \(t \leq 0\) and the second event is minimal in the region \(r \leq H\) and \(t \geq 0\) in three-dimensional spacetime is therefore;

\[
\langle n \rangle = \langle n_1 \rangle + \langle n_2 \rangle + \langle n_3 \rangle. \tag{3.16}
\]
Integrals 3.13, 3.14, and 3.15 were evaluated numerically in MATLAB with the size of the horizon ranging from 100 to 200 in increments of 1. A graph of the number of horizon-crossing links and the area of the event horizon is shown in Figure 3.11. An excellent linear fit was obtained with a slope of 0.885 which verifies the linearity between the number of surface-crossing links and the area of the event horizon in three-dimensional spacetime.

![Graph showing the number of surface-crossing links plotted as a function of the area of the event horizon in three-dimensional spacetime. The horizon ranges from 100 to 200 with unit increments. The linearity between the number of links and the surface area is clearly evident.](image)

**Figure 3.11:** The number of surface-crossing links plotted as a function of the area of the event horizon in three-dimensional spacetime. The horizon ranges from 100 to 200 with unit increments. The linearity between the number of links and the surface area is clearly evident.

### 3.3.2 Expected number of links in four dimensions

It is not possible to draw spacetime regions in four dimensions and therefore we will have to rely on our imagination. Imagine a black hole of surface area $4\pi H^2$ evolving in time. We wish to calculate the number of horizon-crossing links at the hypersurface $t = 0$ just like we did in two and three-dimensional spacetimes. We impose similar constraints on events, i.e., the event below this hypersurface $t \leq 0$ and outside the horizon $r = \sqrt{x^2 + y^2 + z^2} \geq H$ is maximal and the event above the hypersurface $t \geq 0$ and inside the horizon $r = \sqrt{x^2 + y^2 + z^2} \leq H$ is minimal. This means that the volume of the region in the future of event $(r_1, t_1)$ and satisfying $r_1 \geq H$ and $t_1 \leq 0$ is empty.
and the volume of the region bounded by the past light cone of event \((r_2, t_2)\) such that \(r_2 \leq H\) and \(t_2 \geq 0\) is empty. The probability that these regions contain no elements is \(e^{-\rho V(r_1, t_1) - \rho V(r_2, t_2)}\), where \(V(r_1, t_1)\) is the volume of the region bounded by the future light cone of the event located at \((r_1, t_1)\) and the hypersurface \(t = 0\) and \(V(r_2, t_2)\) is the volume of the region bounded by the past light cone of the event located at \((r_2, t_2)\) and the hypersurface \(t = 0\). In four dimensions the volume of a right cone of height \(t\) is \(\pi t^4/3\) which can be obtained by integrating spheres of radius \(r\) from \(r = 0\) to \(r = t\) over time. Also, we keep \(\rho = 1\) just like we did before meaning that there is exactly one element on average inside the smallest possible volume region of four-dimensional spacetime.

Imagine that there is an event at \((x_1, y_1, z_1, t_1)\) such that \(\sqrt{x_1^2 + y_1^2 + z_1^2} \geq H\) and \(t_1 \leq 0\). Imagine that this event corresponds to turning on a light source such that at \(t = t_1 + \delta t\), the light propagates and covers a volume of radius \(c\delta t\). After a certain time interval, this spherical shell of light just touches the event horizon. When this happens, the light would have traveled a distance of \(r_1 - H\), where \(r_1 = \sqrt{x_1^2 + y_1^2 + z_1^2}\). We may call this time \(t_{\text{min}}\) where, \(t_{\text{min}} - t_1 = r_1 - H\) and similarly, \(t_{\text{max}} - t_1 = r_1 + H\). At any time between \(t_{\text{min}}\) and \(t_{\text{max}}\) this shell of light intersects the black hole and we may find the volume of the intersection of these two spheres. This volume integrated from \(t_{\text{min}}\) to \(t_{\text{max}}\) gives the volume of the intersection of the future light cone of this event and the cylinder \(x^2 + y^2 + z^2 = H^2\) in four-dimensional spacetime. The intersection volume is independent of the angular variables and only depends on how far this event is from the horizon. Therefore, we may relabel the coordinates of this event \((x_1, y_1, z_1, t_1)\) as \((r_1, 0, 0, t_1)\). The equations of the two spheres are

\[
x^2 + y^2 + z^2 = H^2 \tag{3.17}
\]

and

\[
(x - r_1)^2 + y^2 + z^2 = (t_2 - t_1)^2. \tag{3.18}
\]

Notice that we have relabelled the coordinate \(t\) to \(t_2\). This is shown in Figure 3.10. This does not
change the intersection area. We may now solve equations 3.17 and 3.18 to get

\[ x_{int} = \frac{1}{2r_1}(r_1^2 + H^2 - (t_2 - t_1)^2). \]  

The intersection volume of the two spheres is shown in Figure 3.12. The volume of a spherical cap of height \( h \) with radius \( R \) is given by

\[ V(R, h) = \frac{\pi h^2 (3R - h)}{3}. \]

We see from Figure 3.12 that, \( h_1 = (t_2 - t_1) - (r_1 - x_{int}) \) and \( h_2 = H - x_{int} \). We may now find the volume of these caps and the intersection volume is \( V = V_1 + V_2 \).

\[ V(cap \ 1) = V((t_2 - t_1), h_1) = \frac{\pi h_1^2 (3(t_2 - t_1) - h_1)}{3} \]
\[ = \frac{\pi}{3} ((t_2 - t_1) + x_{int} - r_1)^2 (3(t_2 - t_1) - (t_2 - t_1) - x_{int} + r_1) \]
\[ = \frac{\pi}{24r_1^3} (H - r_1 + (t_2 - t_1))^2 (H + r_1 - (t_2 - t_1))^2 (r_1^2 + (t_2 - t_1)^2 - H^2 + 4r_1(t_2 - t_1)). \]

Similarly,

\[ V(cap \ 2) = V(H, h_2) = \frac{\pi h_2^2 (3H - h_2)}{3} \]
\[ = \frac{\pi}{3} (H - x_{int})^2 (3H - H + x_{int}) \]
\[ = \frac{\pi}{24r_1^3} (H - r_1 + (t_2 - t_1))^2 (-H + r_1 + (t_2 - t_1))^2 (r_1^2 - (t_2 - t_1)^2 + H^2 + 4r_1H). \]

The volume of the region bounded by the intersection of these two spheres is the volume of these two caps added together as shown in Figure 3.12. To calculate the four-dimensional volume bounded by the intersection of the future light cone of event \((r_1, t_1)\) and the cylinder, we integrate
Figure 3.12: Top view of the intersection of two spheres where $t$ and $z$ coordinates are suppressed. The positive $t$ axis is out of the page. The volume of the intersection is the volume of the two caps $V_1$ and $V_2$ added together.

The intersection volume between two spheres from $t_{\text{min}}$ to $t_{\text{max}}$, where,

$$V(3D) = V_1 + V_2 = \frac{\pi}{12r_1}(H - r_1 + (t_2 - t_1))^2(r_1^2 + 2r_1(t_2 - t_1) - 3(t_2 - t_1)^2 + 2r_1H + 6H(t_2 - t_1) - 3H^2).$$

(3.20)

The volume of the region bounded by the future light cone of Event 1 $(r_1, t_1)$ and the cylinder $r = H$, between $t_{\text{max}} \geq t_2 \geq t_{\text{min}}$ is then,

$$V_{\text{int}}(4D) = \int_{t_{\text{min}}}^{t_{\text{max}}} V(3D) \, dt.$$ 

where $t_{\text{min}} = r_1 + t_1 - H$ and $t_{\text{max}} = r_1 + t_1 + H$. 

Links from elements in Region 1

\[
\langle n_1 \rangle = \int_H^\infty 4\pi r_1^2 dr_1 \int_{-r_1+H}^{0} dt_1 e^{-\frac{\pi r_1^4}{3}} \int_{r_1+t_1+H}^{r_1+t_1+H} dt_2 V(r_1, t_1, t_2) e^{-\frac{\pi r_1^4}{3}} \\
+ \int_H^\infty 4\pi r_1^2 dr_1 \int_{-r_1+H}^{0} dt_1 e^{-\frac{\pi r_1^4}{3}} \int_{r_1+t_1+H}^{\infty} dt_2 \frac{4}{3} \pi H^3 e^{-\frac{\pi r_1^4}{3}}. \tag{3.21}
\]

The intersection volume depends on the location of Event 1 \((r_1, t_1)\) and the time \((t = t_2)\), between \(t_{\text{min}}\) and \(t_{\text{max}}\), at which we are calculating it and therefore is written as a function of \(r_1, t_1, \text{and } t_2\).

Links from elements in Region 2

\[
\langle n_2 \rangle = \int_H^\infty 4\pi r_1^2 dr_1 \int_{-r_1-H}^{0} dt_1 e^{-\frac{\pi r_1^4}{3}} \int_{r_1+t_1+H}^{r_1+t_1+H} dt_2 V(r_1, t_1, t_2) e^{-\frac{\pi r_1^4}{3}} \\
+ \int_H^\infty 4\pi r_1^2 dr_1 \int_{-r_1-H}^{0} dt_1 e^{-\frac{\pi r_1^4}{3}} \int_{r_1+t_1+H}^{\infty} dt_2 \frac{4}{3} \pi H^3 e^{-\frac{\pi r_1^4}{3}}. \tag{3.22}
\]

Links from elements in Region 3

\[
\langle n_3 \rangle = \int_H^\infty 4\pi r_1^2 dr_1 \int_{-\infty}^{-r_1-H} dt_1 e^{-\frac{\pi r_1^4}{3}} \int_{r_1+t_1+H}^{\infty} dt_2 \frac{4}{3} \pi H^3 e^{-\frac{\pi r_1^4}{3}}. \tag{3.23}
\]

The integrals in equations 3.21, 3.22, and 3.23 were evaluated numerically using MATLAB, and the results are shown in Figures 3.13 and 3.14. It can be seen that when the size of the horizon is small, there is no area-like scaling of the number of horizon-crossing links. In Figure 3.13 (left), the size of the horizon was changed from 0 to 0.3 in increments of 0.01. Figure 3.13 (right) is a plot where the size of the horizon ranges from 100 to 200 with unit increments. The relation is linear to an excellent approximation with a slope of 1.363. This agrees with the computer simulation in Figure 3.16.
Figure 3.13: The number of surface crossing links plotted as a function of the area of the event horizon. For small black holes \((H \ll 1)\), no linearity is observed (left). The horizon ranges from 0 to 0.3 with an increment of 0.01 (left) and from 100 to 200 with a unit increment (right). When the size of the horizon is big \((H \gg 1)\), the number of links scales linearly with the surface area (right).

The ratio of entropy to the area of the event horizon for a realistic black hole in four-dimensional spacetime is \(1/4^2\). Entropy is just a measure of the area of the event horizon and is independent of the time interval between two spacelike hypersurfaces in the continuum. The closest analogy of a spacelike hypersurface in causal set formalism is the set of elements such that they are maximal below the hypersurface and minimal above the hypersurface for the causal set elements that are below and above the hypersurface respectively. In the continuum, all the elements inside and outside the event horizon at a particular spacelike hypersurface are spacelike related. If we want this to be reflected in the causal set approach, we have to impose a strict constraint on Event 1, such that this event has to be spacelike related to the horizon and has to be a maximal event below the \(t = 0\) hypersurface and outside the horizon. We are then only allowed to take contributions to the number of horizon-crossing links coming from the region 1 (see Figure 3.5). Doing so gives a slope of 0.261 as shown in Figure 3.14 which is really close to the continuum ratio of entropy and area. A ratio different than 0.25 is expected because, on a flat geometry as opposed to the curved geometry of realistic black holes, the number of horizon-crossing links is different. Can this, however, just be a coincidence?
Figure 3.14: The number of surface crossing links from region 1 plotted as a function of the area of the event horizon. The size of the horizon ranges from 10 to 50 in unit increments.

Expected number of links in $N + 2$ dimensions

We may now generalize the result we have obtained for two, three, and four-dimensional flat black holes in $N + 2$ dimensions, where $N \geq 1$. This should be quite straightforward, i.e.,

Links from elements in Region 1

$$\langle n_1 \rangle = \int_H^\infty A(N, r_1) dr_1 \int_{-r_1 + H}^{0} dt_1 e^{-V(N + 2, t_1)} \int_{r_1 + t_1 + H}^{r_1 + t_1 - H} dt_2 V_{int}(r_1, t_1, t_2) e^{-V(N + 2, t_2)}$$

$$+ \int_H^\infty A(N, r_1) dr_1 \int_{-r_1 + H}^{0} dt_1 e^{-V(N + 2, t_1)} \int_{r_1 + t_1 + H}^{\infty} dt_2 V(N, H) e^{-V(N + 2, t_2)}. \quad (3.24)$$

Links from elements in Region 2

$$\langle n_2 \rangle = \int_H^\infty A(N, r_1) dr_1 \int_{-r_1 - H}^{-r_1 + H} dt_1 e^{-V(N + 2, t_1)} \int_{0}^{r_1 + t_1 + H} dt_2 V_{int}(r_1, t_1, t_2) e^{-V(N + 2, t_2)}$$

$$+ \int_H^\infty A(N, r_1) dr_1 \int_{-r_1 - H}^{-r_1 + H} dt_1 e^{-V(N + 2, t_1)} \int_{r_1 + t_1 + H}^{\infty} dt_2 V(N, H) e^{-V(N + 2, t_2)}. \quad (3.25)$$
Links from elements in Region 3

\[
\langle n_3 \rangle = \int_H^\infty A(N,r_1) dr_1 \int_{-\infty}^{-r_1+H} dt_1 e^{-V(N+2,t_1)} \int_0^\infty dt_2 V(N,H) e^{-V(N+2,t_2)}. \quad (3.26)
\]

where

\( A(N,r_1) = (2\pi^{(N+1)/2}/\Gamma((N + 1)/2)) r_1^N \) is the surface area of a \( N \)-Sphere of radius \( r_1 \),

\( V(N,H) = (\pi^{(N+1)/2}/\Gamma((N + 3)/2)) H^{N+1} \) is the volume of a \( N \)-Sphere of radius \( H \),

\( V(N+2,t_1) = (\pi^{(N+1)/2}/((N + 2)\Gamma((N + 3)/2))) t_1^{N+2} \) is the volume of a \( N + 2 \)-dimensional right cone of height \( t_1 \),

\( V_{int}(r_1,t_1,t_2) \) is the intersection volume of two \( N \)-Spheres of radius \( H \) and \( t_2 - t_1 \), of which one is at the origin and the other is at a distance \( r_1 \) from the origin.

3.4 Computer simulation results

In three and four-dimensional flat spacetime, we generate \( N \) random events inside a bounded region of volume \( V \) such that the density is unity and we find the relations matrix, from which we evaluate the link matrix. We then find the number of links crossing the horizon at a particular hypersurface. In Figures 3.15 and 3.16, we have a plot of the number of links that cross the horizon in three and four-dimensional spacetime against the size of the horizon.

In three-dimensional flat spacetime, we generate 1000 events randomly in a volume region bounded by \(-5 \leq x \leq 5, -5 \leq y \leq 5, \) and \(-5 \leq t \leq 5\). We change the size of the horizon from \( H = 1.5 \) to \( H = 2 \) in increments of 0.1. We then count the number of horizon-crossing links close to the \( t = 0 \) hypersurface by imposing maximal and minimal conditions on the linked events at each value of the horizon. The number of surface crossing links averaged over 200 sprinklings in three dimensions is shown in Figure 3.15. A linear relation between the size of the horizon (circumference of a circle in two-dimensional flat space) and the number of links was observed from computer simulation.
Figure 3.15: Number of links crossing the horizon in three-dimensional flat spacetime averaged over 200 sprinklings where the number of events $N = 1000$.

Figure 3.16: Number of links crossing the horizon in four-dimensional flat spacetime averaged over 120 sprinklings where the number of events $N = 1296$. 

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Similarly, in four-dimensional flat spacetime, we sprinkle 1296 (to keep unit density) events in the volume region bounded by \(-3 \leq x \leq 3, \ -3 \leq y \leq 3, \ -3 \leq z \leq 3, \) and \(-3 \leq t \leq 3\). The size of the horizon was changed from \(H = 1\) to \(H = 1.6\) in increments of 0.1 and for each value of \(H\), the number of links was calculated and averaged over 120 sprinklings and plotted as shown in Figure 3.16. A linear relation between the surface area and the number of links was observed.
CHAPTER 4

CONCLUSION

It took years of devoted research and thousands of failed experiments before we finally accepted the discrete view of the “atomic” world around us. We see the world as being continuous rather than discrete because the length scale at which we live our everyday lives is humongous as compared to the length scale of an atom. The claim that not only atoms and all the elementary particles have a discrete nature encoded in them but also the stage in which they play around (spacetime continuum), is rather a difficult notion to comprehend and is a revolutionary one. The field of “Causal Sets” was established with the goal to discretize the spacetime continuum such that the force of gravity (and possibly the dynamics of energy and matter in relation to gravity) would become emergent from a vast network of relationships between these fundamental structures of spacetime. The majority of this thesis was centered on whether we can use CST to make predictions about the behavior of systems at excruciatingly intense gravitational fields. In particular, we looked at the area-like scaling of the entropy of black holes in the causal set approach.

The irretrievable causal information (interpreted here as entropy) for an observer outside the black hole was found to scale with the area of the horizon in the causal set approach. In two-dimensional flat spacetime, the number of links crossing the horizon was solved analytically and was found to be a constant as expected. This was also in agreement with computer simulations in 2D. In three and four-dimensional flat spacetimes, an average of many computer simulations showed a linear scaling of the number of links with the area of the horizon. Similarly, in Schwarzschild geometry, a linear relationship between the black hole area and the number of links crossing the horizon was observed from averaging over many simulations. The consistency of area-like scaling of black hole entropy observed in the causal set approach hints at a discrete structure of the seeming
continuum nature of spacetime. A possible future work is to formulate the growth dynamics of causal sets and study if the area-like scaling of entropy is still preserved.


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