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On Intersections of Long Cycles and Paths in k-Connected Graphs

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ON INTERSECTIONS OF LONG CYCLES AND PATHS IN k-CONNECTED GRAPHS

DISSERTATION

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ABSTRACT

Interest in the cardinality of the intersection of two longest cycles is inspired by Scott Smith, who conjectured that in a k-connected graph, two longest cycles meet in at least k vertices. Grötschel and Nemhauser, Grötschel, and Stewart and Thompson proved Smith's Conjecture for $2 \leq k \leq 8$, and for general k, Chen, Faudree, and Gould proved that in a k-connected graph, two longest cycles meet in at least $c_0 k^{3/5}$ vertices, where $c_0 \approx 0.2615$. In this dissertation, we study the intersection of two long cycles or two long paths passing through a specified linear forest subgraph in which each component is a path or empty set.

Let G be a k-connected graph ($k \geq 2$), F be a linear forest subgraph of G with at most $k-1$ vertices, and $c_F(G)$ be the length of a longest cycle containing F. We pose a more general conjecture than Smith's Conjecture which states that if C and D are cycles of a k-connected graph G containing F such that $|C| + |D| \geq 2c_F(G) - 1$, then C and D must meet in at least k common vertices. In Chapter 3, we prove this conjecture for $2 \leq k \leq 6$. In Chapter 4, we extend Chen, Faudree, and Gould's result and give a lower bound for the intersection of two long cycles in a k-connected graph. In Chapter 5, we give a lower bound on the intersection of two long cycles containing a linear forest in a k-connected graph. Finally, in Chapter 6, as consequences of the main theorems regarding cycles, we provide analogous path results.

DEDICATION

To the loved ones I cannot thank.

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First and foremost, I'd like to thank God for the ability and opportunity to pursue this work.

While I have worked hard and spent a considerable amount of time towards this dissertation, I have not been alone in these efforts. I sincerely thank my advisor Dr. Haidong Wu for all that he has done for me during the past six years. His commitment to my progress and the development of this project is a testament to his passion for mathematics and scholarship. His encouragement, wisdom, and direction were all vital to my success. I am forever grateful for his help and friendship.

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Any celebration of this dissertation is a celebration of the incredible team mentioned above.

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1 INTRODUCTION

1.1 Notation and Definitions

In this dissertation, we generally follow the definitions and notations of West [24]. A graph G is a vertex set $V(G)$, an edge set $E(G)$, and a relation that associates two vertices with each edge. We call these vertices the endpoints of that edge. A loop is an edge whose endpoints are the same vertex. Multiple edges are edges that share the same endpoints. A simple graph is a graph that has no loops or multiple edges. We say that two vertices are adjacent if they are the endpoints of the same edge. If vertex v is an endpoint of edge e, then v and e are incident. The degree of a vertex v in a graph G, denoted $d_G(v)$ or $d(v)$, is the number of edges incident to v, except that each loop at v counts twice. We denote the maximum degree by $\Delta(G)$, the minimum degree by $\delta(G)$, and say G is regular if $\Delta(G) = \delta(G)$. G is k-regular if the common degree is k. A graph G is called bipartite if $V(G)$ is the union of two disjoint independent sets, called partite sets of G. A subgraph of G is a graph H such that $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and the relation associating two vertices with an edge in G is the same relation in H. A complete graph, denoted K_n, is a simple graph whose n vertices are pairwise adjacent. A complete bipartite graph, denoted $K_{r,s}$, is a simple bipartite graph such that two vertices are adjacent if and only if they are in different partite sets, with one partite set having r vertices and the other having s vertices.

A path is a simple graph whose vertices can be ordered so that two vertices are adjacent if and only if they are consecutive in the list. The two vertices of a path that are only adjacent to one other vertex are the endpoints of the path. All other vertices are called internal vertices. Two paths are parallel if they have the same endpoints and are internally disjoint. We denote a path between two vertices u and v as a $[u, v]$-path. The path resulting by removing the two endpoints of a path P is called the truncation of P, denoted by $\overline{P}$. Note that a truncated path $\overline{P}$ may be empty if P is a single edge. We use $l(P)$ to denote the length of P.

A cycle is a graph with an equal number of vertices and edges whose vertices can be placed around a circle so that two vertices are adjacent if and only if they appear consecutively along the circle. We call a cycle a longest cycle when it has the maximum number of edges for cycles in the graph, and denote that number, also called the circumference of G, by $c(G)$. We call a cycle a longest cycle containing some specified vertex v when it has the maximum number of edges for cycles in the graph containing v, and denote that number by $c_v(G)$. We similarly define the length of longest paths and longest paths containing v, denoting them by $p(G)$ and $p_v(G)$ respectively. A Hamiltonian cycle is a spanning cycle, that is, a cycle containing every vertex in the graph. Likewise, a Hamiltonian path is a spanning path. If P and Q are paths which are internally disjoint from each other and have at least one endpoint in common, then $P \cup Q$ denotes the concatenation of P and Q, and this concatenation may either be a path or a cycle. For two vertices u and v on a path or cycle X, a $[u, v]$-segment of X is the path from u to v on X (when X is a cycle, there are two $[u, v]$-segments).

A linear forest is a subgraph in which every component is a path, including the trivial path (a single vertex) or the empty set. Thus the empty set is also considered a (trivial) linear forest. We define $c_F(G)$ as the length of a longest cycle containing a specified linear forest F. Similarly, $p_F(G)$ is the length of a longest path containing a specified linear forest F. A cycle or path X that passes through F means that $V(F) \subseteq V(X)$ and $E(F) \subseteq E(X)$.

1.2 Graph Connectivity

We say that G is connected if it has a $[u, v]$-path whenever $u, v \in V(G)$. Otherwise G is disconnected. The components of a graph are its maximal connected subgraphs. A cut-edge or cut-vertex of a graph is an edge or vertex whose deletion increases the number of components. An articulation set is a set of vertices whose removal results in a disconnected graph or the graph with one vertex (similarly a separating set or vertex cut of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one component). The connectivity of a graph G, denoted $\kappa(G)$, is the minimum size of a vertex set S such that $G - S$ is disconnected or only has a single vertex. We say a graph is k-connected if its connectivity is at least k. We say that the intersection of two cycles C and D is $V(C) \cap V(D)$ and that the intersection of two paths P and Q is $V(P) \cap V(Q)$.

Connectivity in graphs is an important topic to study in graph theory and it has wide ranging applications, including research on the intersection of long paths and cycles. The most famous result on connectivity is Menger's Theorem.

Theorem 1.2.1. *(Menger [1927]) If x, y are vertices of a graph G and $xy \notin E(G)$, then the minimum size of an x, y-cut equals the maximum number of pairwise internally disjoint x, y-paths.*

This notion of pairwise internally disjoint paths between two vertices is extremely useful in constructing proofs. For instance, if we assume a graph is 2-connected, then the minimum size of an x, y-cut is at least 2, and thus we can assume there exists 2 pairwise internally disjoint x, y-paths.

A famous application of Menger's Theorem is the Fan Lemma, attributed to Dirac. Given a vertex x and a set U of vertices, an x, U-fan is a set of paths from x to U such that any two of them share only the vertex x (see Figure 1.1).

Theorem 1.2.2. *(Dirac [1960]) A graph is k-connected if and only if it has at least $k + 1$ vertices and, for every choice of x, U with $|U| \geq k$, it has an x, U-fan of size k.*

Dirac also showed the following, which links the number of vertices in a cycle to the connectivity of the graph.

Theorem 1.2.3. *[4] Let G be a k-connected graph, where $k \geq 2$, and let X be a set of k vertices of G. Then there is a cycle in G containing every vertex of X.*

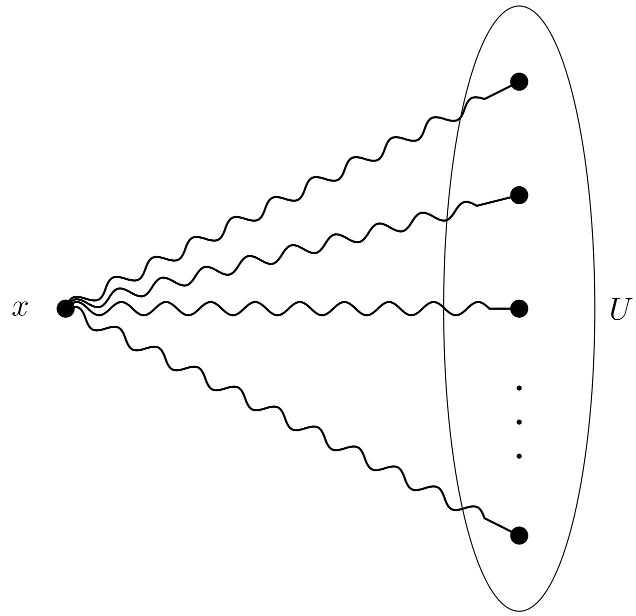


Figure 1.1: $x, U - fan$

Additionally, Dirac proved an important result on the minimum length of a longest cycle in a graph, dependent on the minimum degree of the graph.

Theorem 1.2.4. *[4] Let G be a 2-connected graph of minimum degree δ on n vertices, where $n \geq 3$. Then G contains either a cycle of length at least 2δ or a Hamiltonian cycle.*

By taking the cycle in Lemma 1.2.4 and removing an edge, we obtain an analogous corollary for paths, where G contains either a path of length at least 2δ or a path of length $n-1$. Further extending Lemma 1.2.4 is the following lemma of Grötschel.

Theorem 1.2.5. *[7] Let G be a 2-connected graph of minimum degree δ on n vertices, where $n \geq 3$. Then G contains either a cycle of length at least 2δ or a Hamiltonian cycle which passes through a given vertex.*

This is an improvement on Dirac's famous result as Grötschel showed that there exists such a cycle given by Dirac that passes through any specified vertex of the graph. We use this to prove results on cycles and paths containing a specified vertex.

Noting that the path we construct from the cycle now in Lemma 1.2.5 contains the same vertices, we may obtain the following similar corollary for paths.

Corollary 1.2.6. *Let G be a 2-connected graph of minimum degree δ on n vertices, where $n \geq 3$. Then G contains either a path of length at least 2δ or a path of length at least $n-1$ which passes through a given vertex.*

The next result of Hu, Tian, and Wei gives a general improvement of the last result. Recall that a cycle or path X that passes through F (alternatively the edges and vertices of F, $E(F) \cup V(F)$) means that $V(F) \subseteq V(X)$ and $E(F) \subseteq E(X)$.

Let $\sigma_k(G) = min\left\{ \sum_{i=1}^{k} d_G(v_i) : \{v_1, v_2, ..., v_k\} \text{ is an independent set of } G \right\}$.

Theorem 1.2.7. *[13] Let $k \geq 3$, $m \geq 0$, and $0 \leq s \leq k-3$. Let G be a $(m+k-1)$-connected graph and let F be a linear forest subgraph with m edges and s isolated vertices. Then G has a cycle of length $\geq min\{|V(G)|, \frac{2}{k}\sigma_k(G) - m\}$ passing through $E(F) \cup V(F)$.*

The following is a corollary to Theorem 1.2.7 that we will use in many of our proofs involving cycles containing a linear forest.

Corollary 1.2.8. *Let G be a k-connected graph and F be a linear forest subgraph with l edges, t isolated vertices, and $l+t \leq k-2$. Then G has a cycle of length at least $min\{|V(G)|, 2\delta - l\}$ passing through F.*

Egawa et al. [5] provide this useful result similar to Theorems 1.2.4 and 1.2.5 which we also use for many of our proofs involving cycles containing a linear forest.

Theorem 1.2.9. *[5] Let G be a k-connected graph, $k \geq 2$, with minimum degree δ and with at least 2δ vertices. Let X be a set of k vertices of G. Then G has a cycle C of length at least 2δ such that every vertex of X is on C.*

1.3 Some Extremal Results

The following lemmas and corollaries in this section are some extremal results that are useful for our proofs in later chapters.

Theorem 1.3.1. *[6] Every sequence of* n^2+1 *real numbers contains a monotone subsequence of length* $n+1$.

A generalization of the theorem from Erdös and Szekeres [6] gives

Lemma 1.3.2. *[1, Lemma 3] Let* Σ *be a set of* n *permutations of a sequence of* S *of* 2^{2^n} *elements. Then there is a subsequence* (a, b, c) *of* S *on which each permutation* $\sigma \in \Sigma$ *is monotonic (that is, either* $\sigma(a) < \sigma(b) < \sigma(c)$ *or* $\sigma(a) > \sigma(b) > \sigma(c)$*).*

Theorem 1.3.3. *[14] Let* $G \subseteq K_{n,n}$ *be a bipartite graph. Then* G *contains* $K_{s,t}$ *as a subgraph if*

$$e(G) \geq (s-1)^{1/t}(n-t+1)n^{1-1/t} + (t-1)n.$$

Setting $t = 3$ and $s = 257$ gives

Corollary 1.3.4. *Let* $G \subseteq K_{n,n}$ *where* $n \geq 3$*. Then* G *contains* $K_{3,257}$ *if*

$$e(G) \geq \sqrt[3]{256}(n-2)n^{2/3} + 2n.$$

2 MAIN PROBLEMS AND REVIEW OF LITERATURE

2.1 Smith's Conjecture and Main Problems

Current interest in the number of vertices contained in the intersection of two longest cycles is inspired by the conjecture of Scott Smith:

Conjecture 2.1.1. *(See [1]) In a k-connected graph, $k \geq 2$, two longest cycles meet in at least k vertices.*

Smith's Conjecture is proven to be true for $2 \leq k \leq 8$.

Theorem 2.1.2. *(Smith's Conjecture for small k) Let G be a k-connected graph where $2 \leq k \leq 8$. If C and D are longest cycles of G, then C and D meet in at least k vertices.*

This conjecture is proven for $k \in \{2, 3\}$ by Grötschel and Nemhauser [9], for $k \in \{4, 5, 6\}$ by Grötschel [8], and for $k \in \{7, 8\}$ by Stewart and Thompson [23]. Grötschel [8] reported that the conjecture is probably proven up to ten.

A lower bound on the intersection of two longest cycles for general k is given by Chen, Faudree, and Gould [1].

Theorem 2.1.3. *[1, Theorem 2] If G is a k-connected graph, then any two different longest cycles meet in at least $c_0 k^{3/5}$ vertices, where $c_0 = 1/(\sqrt[3]{256} + 3)^{3/5} \approx 0.2615$.*

Similar to the conjecture of Smith, Hippchen made the following conjecture.

Conjecture 2.1.4. *[12] If G is k-connected, then any two paths in G must meet in at least k vertices.*

Gutiérrez and Valqui prove this Hippchen's Conjecture for $k \leq 6$ [15], and Gutiérrez provides a lower bound on the intersection for general k [10]. In [3], it is proven that Hippchen's conjecture is true for $k \geq \frac{n+2}{5}$.

The current strategy for the proofs of Smith's Conjecture for small k rely on results related to the intersection as an articulation set, which we explore in detail in the following section. Related conjectures have been proven for small k as well (See [12, 15, 18, 19, 22]), and we discuss these briefly later in this chapter. For further reading, see the survey paper, [21].

The main focus of this dissertation is the following conjecture which generalizes Smith's conjecture, as we consider long cycles relative to the longest cycles containing a linear forest. Note that the longest cycle containing F may be much shorter than the longest cycle of G in general. Recall that a linear forest is a subgraph in which every component is a path, including the trivial path (a single vertex) or the empty set, and that $c_F(G)$ as the length of a longest cycle containing a specified linear forest, F. We make the following conjecture:

Conjecture 2.1.5. *Let G be a k-connected graph ($k \geq 2$) and F be a linear forest subgraph with at most $k-1$ vertices. If C and D are longest cycles of G containing F then C and D must meet in at least k common vertices.*

In fact, we believe the following, stronger conjecture to be true.

Conjecture 2.1.6. *Let G be a k-connected graph ($k \geq 2$) and F be a linear forest subgraph with at most $k-1$ vertices. If C and D are cycles of G containing F such that $|C| + |D| \geq 2c_F(G) - 1$, then C and D must meet in at least k common vertices.*

In this dissertation, we prove the above conjecture for $2 \leq k \leq 6$, and provide a bound for general k. We note that Conjecture 2.1.5 implies a special conjecture for two cycles containing a specified vertex when F is equal to a single vertex, and implies Smith's Conjecture when F is the empty set.

As a final note in this section, we pose a similar path version to Conjecture 2.1.5.

Conjecture 2.1.7. *Let G be a k-connected graph ($k \geq 2$) and F be a linear forest subgraph with at most $k-1$ vertices. If P and Q are longest paths of G containing F then P and Q must meet in at least k common vertices.*

2.2 Articulation Sets in Longest Cycles and Proof of Smith's Conjecture for Small k

Smith's conjecture is still an open problem for general k, but in 1984, Grötschel and Nemhauser gave a proof of this conjecture for $k \in \{2, 3\}$ using part (b) of the following theorem. Recall that an articulation set is a set of vertices whose removal results in a disconnected graph or the graph with a single vertex. In the following four theorems, multiple edges are allowed, but loops are not allowed.

Theorem 2.2.1. *[9, Theorem 4.2] Let C_1 and C_2 be two longest cycles of G whose intersection is the set $\{u, v\}$. Suppose $C_1 = P_1 \cup Q_1$ and $C_2 = P_2 \cup Q_2$, where P_1, P_2, Q_1, Q_2 are internally disjoint $\{u, v\}$-paths. Then*
(a) paths P_1, P_2, Q_1, Q_2 have the same length (which implies that $|C_1| = |C_2|$ is even).
(b) $\{u, v\}$ is an articulation set of G and every truncated path $\overline{P}_1, \overline{P}_2, \overline{Q}_1, \overline{Q}_2$ obtained from P_1, P_2, Q_1, Q_2 by removing the two endpoints u and v belongs to a different component of $G - \{u, v\}$.

They also showed that if two longest cycles meet in exactly two vertices, that those two vertices are contained in every longest cycle extending the proposition listed below.

Proposition 2.2.2. *[9, Proposition 4.1]*
(a) Every pair of longest cycles containing v of G meet in at least two vertices.
(b) If two longest cycles meet in exactly two vertices, then these two vertices are not adjacent on either of the two cycles.

Theorem 2.2.3. *[9, Theorem 4.3] Suppose two longest cycles C_1 and C_2 of G meet in exactly two vertices, say u and v.*

(a) Then all longest cycles of G contain u and v, but not the edge uv.

(b) For every longest cycle C of G, the two pieces of $C - \{u, v\}$ in $G - \{u, v\}$ belong to different components of $G - \{u, v\}$.

In the same year, Grötschel gave a proof of Smith's Conjecture for $k \in \{4, 5, 6\}$, which follows from the next result.

Theorem 2.2.4. *[8, Theorem 1.2] Let $k \in \{3, 4, 5\}$ and let $G = [V, E]$ be a 2-connected graph with at least $k + 1$ vertices. Suppose that C and D are two different longest cycles meeting in a set W of exactly k vertices. Then:*

(a) W is an articulation set of G.

(b) In the case $k = 3$, the paths obtained by removing W from C and D are in different components of $G - W$.

In [8], Grötschel gives a counterexample for Theorem 2.2.4 when $k = 6$ (See Figure 2.1). Here, the graph has circumference six and two different cycles with length six (1234561 and 1234651) meeting in six vertices such that the removal of these vertices leaves a connected graph. Grötschel however conjectures that a restricted version could be true for $k \in \{6, 7\}$, with the restriction being that the length of the longest cycle in G must be at least $k+1$. This was proven by Stewart and Thompson [23], listed below, and is tight due to a counterexample for $k = 8$ (the Petersen graph, see Figure 2.2) provided by Grötschel [8]. Here, the cycles 1238079451 and 6805432796 intersect in $k = 8$ vertices, both have length $k + 1 = 9$, and the removal of the intersection vertices leaves the edge 16, thus the intersection is not an articulation set.

Theorem 2.2.5. *[23] Let $k \in \{6, 7\}$ and let G be a graph whose circumference is at least $k + 1$. Suppose that C and D are distinct longest cycles of G meeting in a set W of exactly k vertices. Then W is an articulation set of G.*

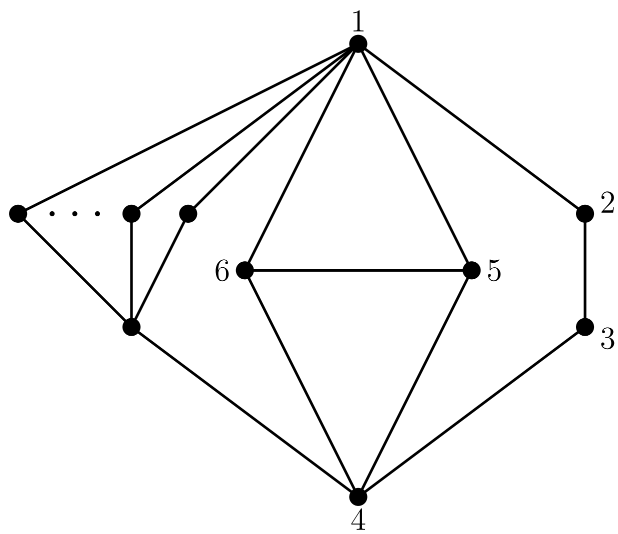


Figure 2.1: Counterexample to Theorem 2.2.4 when $k = 6$

Using Lemma 1.2.4, the 2-connected graph contains either a Hamiltonian cycle or a cycle of length at least $2\delta \geq 2k$, which satisfies the circumference of at least $k+1$. Thus, an application of Theorem 2.2.5 implies Smith's Conjecture for $k \in \{7, 8\}$.

2.3 Bounds for the Intersection of Long Cycles

Though Smith's conjecture for general k is still open, as mentioned above in Theorem 2.1.3, Chen, Faudree, and Gould [1] found a lower bound for the number of vertices in this intersection that improves on a theorem by Burr and Zamfirescu (See [1]).

Many previous results consider longest cycles or longest paths, but McGuinness shows that if two cycles, not necessarily longest, are long enough, they will intersect in at least two vertices.

Theorem 2.3.1. *[17, Theorem 1.2] Suppose that G is a k-connected graph where $k \geq 2$ having circumference $c \geq 2k$. If C_1 and C_2 are a pair of cycles of G such that $|V(C_1)| + |V(C_2)| \geq 2c - 2k + 3$, then C_1 and C_2 intersect in at least two vertices.*

Wu extends this by showing that in a highly connected graph, two sufficiently long cycles will intersect many times, and gives a path version as well.

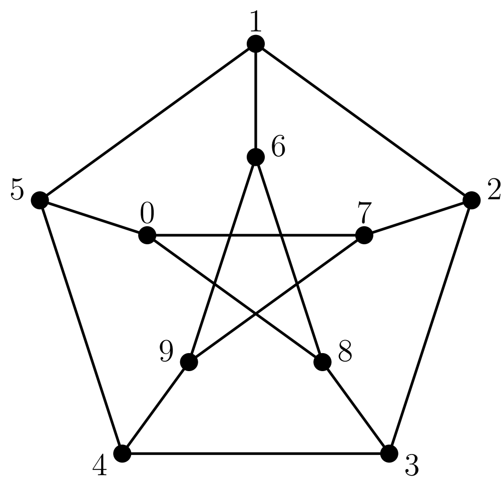


Figure 2.2: Petersen graph, counterexample to Theorem 2.2.5 when $k = 8$

Theorem 2.3.2. *[25] Let G be a k-connected graph for $s \geq 3$ and $k \geq ds^2$ where $d = (3 + \sqrt{\frac{7}{13}})^2 \approx 13.9413$. Suppose C and D are cycles of G such that $|V(C)| + |V(D)| \geq 2c(G) - \frac{c\sqrt{k}}{s} + 12$ where $c = 2\sqrt{\frac{13}{7}} \approx 3.7337$. Then C and D meet in at least $s + 1$ common vertices.*

Theorem 2.3.3. *[25] Let G be a k-connected graph where $k \geq 2710$. Suppose C and D are cycles such that $|V(C)| + |V(D)| \geq 2c(G) - c\sqrt[6]{k} + 12$ where $c = 2\sqrt{\frac{13}{7}} \approx 3.7337$. Then C and D meet in at least $\lfloor \sqrt[3]{k} \rfloor + 1$ common vertices.*

Chen, Chen, and Liu pose a conjecture related to Smith's Conjecture, and prove it for $k \leq 6$.

Conjecture 2.3.4. *[2] Let G be a k-connected graph, $k \geq 2$, and let C and D be two cycles of G. Then there exist two cycles C^* and D^* such that $V(C^*) \cup V(D^*) \supseteq V(C) \cup V(D)$ and $|V(C^*) \cap V(D^*)| \geq k$.*

Additionally, they provide a bound to their conjecture for general k.

Theorem 2.3.5. *[2] Let G be a k-connected graph, $k \geq 2$, and let C and D be two cycles of G. Then there exist two cycles C^* and D^* such that $V(C^*) \cup V(D^*) \supseteq V(C) \cup V(D)$ and $|V(C^*) \cap V(D^*)| \geq \frac{k^{3/5}}{(\sqrt[3]{256}+3)^{3/5}}$.*

2.4 Other Related Conjectures

Smith's conjecture has also been generalized to matroids by McMurray, Reid, Wei, and Wu. For Matroid definitions and notation, see Oxley [20].

Conjecture 2.4.1. *[19] If C and D are largest circuits of a k-connected matroid M with at least $2(k-1)$ elements, then $r(C \cup D) \leq r(C) + r(D) - k + 1$.*

This conjecture is true for $k = 2$ [19] and $k = 3$ [16]. Noting that this conjecture's dual leads to a statement of cocircuits, which when restricting that dual to graphic matroids relates to graphic bonds. A bond is a minimal nonempty edge-cut, and $\omega(G)$ is the number of components in a graph G.

Conjecture 2.4.2. *[18] If C and D are largest bonds of a k-connected graph G, then $\omega(G - (C \cup D)) \geq k + 2 - |C \cap D|$.*

This conjecture has been proven for $k \leq 6$ in [18]. In [22], Sheppardson provides two lower bounds for $\omega(G - (C \cup D))$ when $k \geq 7$ as well.

Theorem 2.4.3. *[22] If C and D are distinct bonds of a k-connected graph G, $k \geq 7$, with C a largest bond and $|D| \geq |C| - 1$, then $\omega(G - (C \cup D)) \geq \frac{1}{4}(k + 14 - \frac{3}{2}|C \cap D|)$.*

For small k, Sheppardson improves the bound above.

Theorem 2.4.4. *[22] Let C and D be distinct bonds of a k-connected graph G with C a largest bond and $|D| \geq |C| - 1$. If $\alpha \in \mathbb{R}$, $0 \leq \alpha \leq 16$, and $4 \leq k \leq 40 - 2\alpha$, then $\omega(G - (C \cup D)) \geq \frac{1}{5}(2k + \alpha - 3|C \cap D|)$.*

In Chapter 3, we examine the intersection of two long cycles containing a linear forest as an articulation set, and prove Conjecture 2.1.6 for $2 \leq k \leq 6$. In Chapter we 4, we extend Theorem 2.1.3 by improving the bound and not requiring the cycles to be longest. In Chapter 5, we provide a lower bound on the intersection of two long cycles containing a linear forest for general k. In Chapter 6, as consequences of the main results, we obtain similar results for long paths containing a linear forest.

3 INTERSECTION OF LONG CYCLES CONTAINING A LINEAR FOREST AS AN ARTICULATION SET

3.1 Introduction

In this chapter we extend the results of Grötschel and Nemhauser [9] and Grötschel [8] regarding the intersection of two cycles as an articulation set, and raise and partially prove the following conjectures related to Smith's Conjecture.

Conjecture 3.1.1. *Let G be a k-connected graph ($k \geq 2$) and F be a linear forest subgraph of G with at most $k-1$ vertices. If C and D are longest cycles of G containing F, then C and D must meet in at least k common vertices.*

Smith's Conjecture is identical to the above conjecture when $F = \emptyset$. We believe that Smith's Conjecture may be modified to consider two cycles whose sum is one edge short of two longest cycles.

Conjecture 3.1.2. *Let G be a k-connected graph ($k \geq 2$). If C and D are cycles of G such that $|C| + |D| \geq 2c(G) - 1$, then C and D must meet in at least k common vertices.*

In fact, we believe the following, stronger conjecture to be true, considering two long cycles passing through a linear forest. Recall $c_F(G)$ is the length of a longest cycle containing the linear forest F, and if P is a path, we use $|P|$ to denote the path length of P.

Conjecture 3.1.3. *Let G be a k-connected graph ($k \geq 2$) and F be a linear forest subgraph of G with at most $k-1$ vertices. If C and D are cycles of G containing F such that $|C| + |D| \geq 2c_F(G) - 1$, then C and D must meet in at least k common vertices.*

When F is a single vertex v, we use $c_v(G)$ to denote $c_F(G)$.

Our work in this chapter extends Theorems 2.2.1 and 2.2.4, which state that in a 2-connected graph G with at least $k+1$ vertices, the intersection of two longest cycles is an articulation set of G when $k = 2$ and when $k \in \{3, 4, 5\}$, respectively. We also extend Theorem 2.2.3 which states that if two longest cycles of a graph G contain exactly two vertices in their intersection, then those two vertices are contained in every longest cycle of G. We show that the intersection of two long cycles may still be an articulation set under certain conditions. Finally, we apply our results to partially prove Conjecture 3.1.3.

Recall that an articulation set is a set of vertices whose removal results in a disconnected graph or the graph with one vertex. In this chapter, we assume that G is loopless, but multiple edges are allowed. Also recall that a simplified graph of G, denoted by $si(G)$, is isomorphic to the graph obtained from G with loops and multiple edges deleted. Also recall that the path resulting by removing the two endpoints of a path P is called the truncation of P, denoted by $\overline{P}$, and that a truncated path $\overline{P}$ may be empty if P is a single edge. If P and Q are paths which are internally disjoint from each other and have at least one endpoint in common, then $P \cup Q$ denotes the concatenation of P and Q, and this concatenation may either be a path or a cycle.

3.2 Main Results

In this section we state the main results. The first two results study the intersection of two cycles when the intersection has at most two vertices. The third result studies the intersection of two cycles with three, four, or five vertices. Finally, the fourth result proves Conjecture 3.1.3 for $2 \leq k \leq 6$.

We define $K_{2,m}^{+}$ as the graph obtained from $K_{2,m}$ with an edge connecting the two vertices in the partite set of size two.

Theorem 3.2.1. *Let G be a 2-connected graph and C and D be two cycles of G passing through a specified linear forest subgraph F, where $|V(F)| \leq 2$, and F contains no edge, such* *that* $|C| + |D| \geq 2c_F(G) - 1$. *Then:*

(i) $|V(C) \cap V(D)| \geq 2$,

(ii) if $V(C) \cap V(D) = \{u, v\}$ *and* $c_F(G) \geq 5$, *then* uv *is not an edge on* C *or* D,

(iii) if $V(C) \cap V(D) = \{u, v\}$ *and* $c_F(G) = 3$, *then* $si(G) \cong K_3$ *and* G *is obtained from a triangle by adding at least one multiple edge, and*

(iv) if $V(C) \cap V(D) = \{u, v\}$ *and* $c_F(G) = 4$, *then either (1)* $si(G) \cong K_{2,m}^{+}$ *for* $m \geq 3$ *or (2)* $si(G) \cong K_{2,m}$ *for* $m \geq 4$.

In the next result, we consider the case where the linear forest has no edge and has at most two vertices. The case when F is a single edge is straightforward, and will be discussed later.

Theorem 3.2.2. *Let* C *and* D *be two different cycles of a* 2*-connected graph* G *meeting in exactly two vertices,* u *and* v, $|V(G)| \geq 4$, *and* F *be a linear forest subgraph with no edges and* $V(F) \subseteq \{u, v\}$. *Assume that* $|C| + |D| \geq 2c_F(G) - 1$ *and* $C = P_1 \cup P_2$ *and* $D = Q_1 \cup Q_2$ *where* P_i *and* Q_i $(i = 1, 2)$ *are* $[u, v]$*-segments of* C *and* D *respectively. Suppose without loss of generality that* $|P_1| \leq |P_2|$ *and* $|Q_1| \leq |Q_2|$. *Then the following hold:*

(a) $\{u, v\}$ *is an articulation set of* G *if* $c_F(G) \geq 4$,

(b) if $|C| = |D| = c_F(G)$, *then* $|P_1| = |P_2| = |Q_1| = |Q_2|$,

(c) if $|C| = |D| - 1$, *then either (i)* $c_F(G)$ *is even,* $|P_2| = |Q_1| = |Q_2|$, *and* $|P_1| = |P_2| - 1$, *or (ii)* $c_F(G)$ *is odd, and* $|P_1| = |P_2| = |Q_1| = |Q_2| - 1$,

(d) the nonempty truncated paths $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, *and* $\overline{Q}_2$ *obtained by removing* $\{u, v\}$ *from* P_1, P_2, Q_1, *and* Q_2 *are in different components of* $G - \{u, v\}$,

(e) every cycle R *containing* F *with at least* $c_F(G) - 1$ *vertices must contain both* u *and* v *if* $c_F(G) \geq 4$,

(f) if R *is a longest cycle passing through* F *and* $c_F(G) \geq 4$, *then the two segments of* $R - \{u, v\}$ *in* $G - \{u, v\}$ *belong to different components of* $G - \{u, v\}$, *and*

(g) if $|C| = |D| = c_F(G) \geq 5$, *then the conclusion of (f) is true for any cycle* R *passing through* F *with length at least* $c_F(G) - 1$.

Our third result for this chapter generalizes Theorem 2.2.4 and extends Theorem 3.2.2 for intersections of two long cycles containing a specified linear forest of size three, four, and five vertices.

Theorem 3.2.3. *Let $k \in \{3, 4, 5\}$ and let G be a 2-connected graph with at least $k+1$ vertices and F be a linear forest subgraph with at most $k-1$ vertices. Suppose that C and D are two different cycles containing F meeting in a set W of exactly k vertices such that $|C| + |D| \geq 2c_F(G) - 1$. Then the following are true:*

(i) For $k = 3, 4$, assume $c_F(G) \geq k+2$ if $|C| + |D| = 2c_F(G) - 1$, and $c_F(G) \geq k+1$ if $|C| + |D| = 2c_F(G)$. Then W is an articulation set of G.

(ii) In the case above for $k = 3$, the paths obtained by removing W from C and D are in different components of $G - W$.

(iii) For $k = 5$ and $c_F(G) \geq 7$, W is an articulation set of G.

(iv) Moreover, when $|V(F)| \leq 1$ and $k = 3, 4$, W is always an articulation set.

For $k = 5$ and $c_v(G) = k + 1 = 6$, Figure 3.1 shows a counterexample, where $|C| + |D| = 2c_v(G) - 1$, C and D meet in a set W of exactly five vertices, but W is not an articulation set.

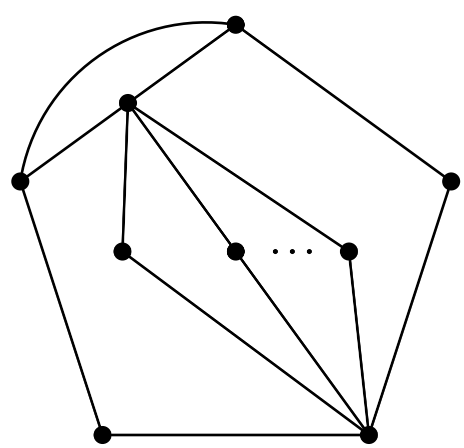


Figure 3.1: Two cycles intersecting in 5 vertices with $c_v(G) = 6$

Also note that the theorem is untrue if G is not 2-connected. In our first counter example, suppose we have two cycles intersecting in $k = 4$ vertices with $c_v(G) = 5$. The deletion of the intersection vertices leaves one connected component (here a tree, see Figure 3.2 (a)). Our second counter example similarly shows the theorem is not true but for two cycles intersecting in $k = 5$ vertices with $c_v(G) = 6$ (see Figure 3.2 (b)).

(a) (b)

Figure 3.2: Two cycles intersecting in four vertices with $c_v(G) = 5$, and intersecting in six vertices with $c_v(G) = 6$, respectively, and G not 2-connected

The condition $c_F(G) \geq k + 2$ is natural as we will see in the proof of our next main theorem. As a consequence of Theorems 3.2.1, 3.2.2, and 3.2.3, we obtain the following main result, which proves Conjecture 3.1.3 for $2 \leq k \leq 6$.

Theorem 3.2.4. *Let G be a k-connected graph where $k \in \{2, 3, 4, 5, 6\}$, and F be a specified linear forest subgraph of G with at most $k - 1$ vertices. If C and D are cycles containing F such that $|C| + |D| \geq 2c_F(G) - 1$, then C and D must meet in at least k common vertices.*

We note that the above theorem also implies Conjectures 3.1.1 and 3.1.2 when $k \in \{2, 3, 4, 5, 6\}$. All proofs will be delayed until section 3.4.

3.3 Some Lemmas for Long Cycles

In this section, we provide lemmas that are used in the proofs of our main results in the next section.

Proposition 3.3.1. *Let G be a 2-connected graph and F be a set containing at most one vertex of G. Then the following hold:*

(i) $c_F(G) = 3$ *if and only if* $si(G) = K_3$.

(ii) $c_F(G) = 4$ *if and only if* $si(G) \cong K_4, K_{2,m},$ *or* $K^+_{2,m}$ *for* $m \geq 2$.

Note that [8] showed that $c(G) = 4$ if and only if G is one of the graphs in the above proposition. Therefore, we need only consider the case for $c_F(G) = 3$ or $c_F(G) = 4$ for $|V(F)| = 1$.

Proof. (Proof of Proposition 3.3.1) One direction is straightforward, so we prove the other direction. (i) Suppose that $c_F(G) = 3$, and uvw is a cycle of length three containing F.

If there is another vertex $t \notin S = \{u, v, w\}$, then as G is 2-connected, there are two internally disjoint (t, S)-paths. Now it is easily seen that $c_F(G) \geq 4$, a contradiction. Thus $si(G) \cong K_3$.

(ii) Suppose $c_F(G) = 4$ where $F = \{v\}$. Then, there is a 4-cycle C containing v, and suppose $C = v_1v_2v_3v_4v_1$ where $v \in \{v_1, v_2, v_3, v_4\}$. If $|V(G)| = 4$, then $si(G) = K_{2,2}$, $K^+_{2,2}$, or K_4, and the theorem holds. So we may assume that $|V(G)| \geq 5$. For any $t \notin V(C)$, as G is 2-connected, there are at least two internally disjoint $(t, V(C))$-paths, S_1 and S_2, with end vertices $v_i, v_j \in V(C)$. As $c_v(G) = 4$, v_i and v_j cannot be consecutive vertices in C, otherwise we obtain a cycle of length at least five, a contradiction. Thus without loss of generality, we have that either $v_i = v_1$ and $v_j = v_3$, or $v_i = v_2$ and $v_j = v_4$. Suppose the latter is true. As $c_v(G) = 4$, it is easily seen both S_1 and S_2 have length one. Then, $\{v_1, v_3, t_1\}$ must be an independent set, otherwise we would obtain a cycle of length five passing through v, a contradiction. Now for any $t_2 \notin V(C)$ and $t_2 \neq t_1$, then by the above argument either $t_2v_2, t_2v_4 \in E(G)$ or $t_2v_1, t_2v_3 \in E(G)$. The latter case gives a cycle such that $c_v(G) \geq 6$, a contradiction (see Figure 3.3). Additionally, it is easily seen that $v_1v_3 \notin E(G)$, but v_2v_4 may possibly be an edge of G. We conclude that t_2 may only be such that it is adjacent to v_2 and v_4. Moreover, $V - \{v_2, v_4\}$ is independent. Thus $si(G) \cong K_{2,m}$ or $K^+_{2,m}$ for $m \geq 3$ if $|V(G)| \geq 5$. Thus the proposition holds.

□

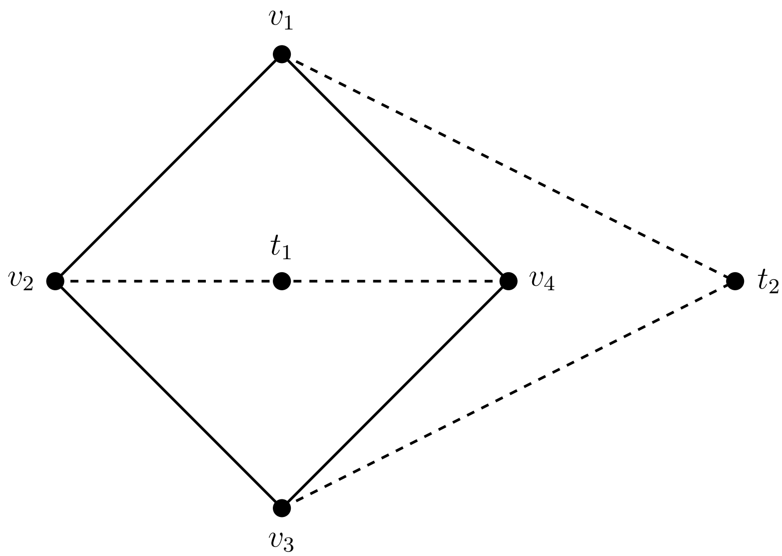


Figure 3.3: $c_v(G) = 4$ if and only if $si(G) \cong K_4, K_{2,n}$, or $K^+_{2,n}$ for $n \geq 2$

If C and D are two cycles which both contain the vertices u and v and if P is a $[u, v]$-segments of C and Q a $[u, v]$-segment of D, then P and Q are called parallel if P is internally disjoint from D and Q internally disjoint from C. We say that P and Q are parallel paths on the cycles C and D. In the following, if e is a common edge of two cycles C and D, we consider (e, e) to be a trivial parallel path.

Lemma 3.3.2. *Let C and D be cycles of a 2-connected graph G passing through a linear forest subgraph F such that $|C| + |D| \geq 2c_F(G) - 1$. Assume without loss of generality that $|C| \geq |D|$. Suppose (P_i, Q_i), $1 \leq i \leq t$, are parallel paths of C and D. Then*
(i) $|P_i| \geq |Q_i|$ for all $i \in \{1, ..., t\}$,
(ii) for at most possibly one i, say $i = 1$, we have that $|P_1| - |Q_1| \leq 1$, and $|P_i| = |Q_i|$ for all $2 \leq i \leq k$.

Proof. As C is a longest cycle containing F and $|C| + |D| \geq 2c_F(G) - 1$, we deduce that $|P_i| \geq |Q_i|$ for all $i \in \{1, ..., t\}$, otherwise, if $|P_i| < |Q_i|$ for some $i \in \{1, 2, ..., t\}$, we obtain a longer cycle than C containing F by replacing P_i by Q_i. First we show that $|P_i| - |Q_i| \leq 1$ for all $1 \leq i \leq t$. For contradiction, suppose $|P_i| - |Q_i| \geq 2$ for some i. Then $|P_i| \geq |Q_i| + 2$. As P_i and Q_i are parallel segments, we may construct D' such that D' contains every segment of D except Q_i, but replaces Q_i by P_i. Then clearly, D' still contains F, and $|D'| \geq |D| + 2$. As $|D| \geq c_F(G) - 1$, we have that $|D'| \geq c_F(G) - 1 + 2 = c_F(G) + 1 > c_F(G)$, a contradiction.

Now suppose that there are at least two i, say $i = 1, 2$, such that $|P_i| - |Q_i| \geq 1$. If $|C| + |D| = c_F(G)$, then it is easy to see that $|P_i| = |Q_i|$ for all $i \in \{1, ..., t\}$, a contradiction. Thus we assume that $|C| = |D| + 1 = c_F(G)$, and all P_i belong to C, and all Q_i belong to D for $i \in \{1, ..., t\}$. Now suppose $C = P \cup P_1 \cup P_2$ and $D = Q \cup Q_1 \cup Q_2$ where P and Q are unions of segments of C and D respectively. Note that $P \cup Q_1 \cup Q_2$ and $Q \cup P_1 \cup P_2$ are both cycles containing F. If $|P| < |Q|$, then $|Q \cup P_1 \cup P_2| > |P \cup P_1 \cup P_2| = |C| = c_F(G)$, a contradiction. Thus $|P| \geq |Q|$. Then we have that $|D| = |Q \cup Q_1 \cup Q_2| \leq |P| + |P_1| - 1 + |P_2| - 1 = |C| - 2$, a contradiction. □

Lemma 3.3.3. *Let C and D be two cycles of G containing a linear forest subgraph F such that $|C| + |D| \geq 2c_F(G) - a$. Let P be a segment on C and Q be a segment on D such that P and Q have u as one endpoint, P is internally disjoint from D, and Q is internally disjoint from C. Then G contains no $[\overline{P}, \overline{Q}]$-path internally disjoint from C and D such that its length is greater than $\frac{a}{2}$.*

Proof. Let P_1 be the union of remaining segments of C such that $C = P \cup P_1$, and similarly let Q_1 be the union of remaining segments of Q such that $D = Q \cup Q_1$. Let R be a path connecting an internal vertex x of P to an internal vertex y of Q such that R is internally disjoint from C and D. Write P and Q now such that $P = P' \cup P''$ and $Q = Q' \cup Q''$, separated by x and y respectively (see Figure 3.4). Then

$$D' = Q_1 \cup P' \cup R \cup Q''$$

$$C' = P_1 \cup Q' \cup R \cup P''$$

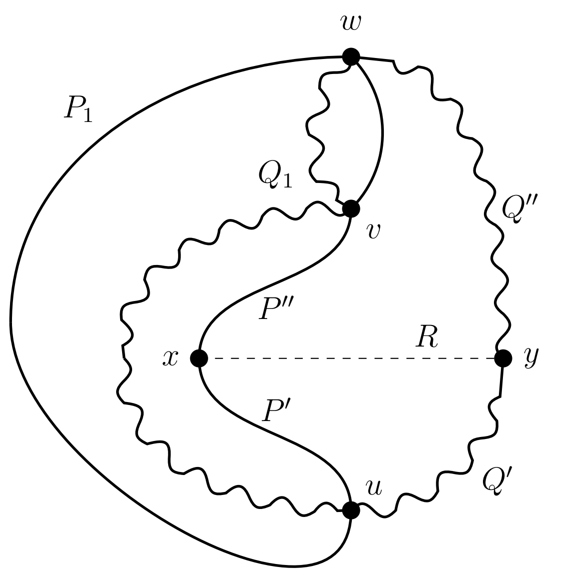


Figure 3.4: The Graph in Lemma 3.3.3

are cycles, both containing F, as we only replaced P' by $Q' \cup R$ in C and Q' by $P' \cup R$ in D, and P is internally disjoint from D, and Q is internally disjoint from C. Now,

$$|C'| + |D'| = |C| + |D| + 2|R| \geq 2c_F(G) - a + 2|R|.$$

As $|R| > \frac{a}{2}$, $|C'| + |D'| > 2c_F(G)$, a contradiction.

□

Corollary 3.3.4. *Let C and D be two cycles of G containing a linear forest subgraph F such that $|C| + |D| \geq 2c_F(G) - 1$. Let P be a segment on C and Q be a segment on D such that P and Q have u as one endpoint, P is internally disjoint from D, and Q is internally disjoint from C. Then G contains no $[\overline{P}, \overline{Q}]$-path R internally disjoint from C and D.*

Lemma 3.3.5. *Let C and D be two cycles of a graph G containing a linear forest subgraph F such that $|C| + |D| \geq 2c_F(G) - a$ and let $C = P_1 \cup P_2 \cup P_3$ and $D = Q_1 \cup Q_2 \cup Q_3$ be concatenations of three paths respectively. Suppose that P_1 and Q_1 are parallel, and P_2*

and Q_2 are internally disjoint from D and C respectively. Then G contains no $[\overline{P}_1, \overline{P}_2]$-, $[\overline{P}_1, \overline{Q}_2]$-, $[\overline{Q}_1, \overline{Q}_2]$-, and no $[\overline{Q}_1, \overline{P}_2]$-paths R internally disjoint from C and D such that $|R| > \frac{a}{2}$.

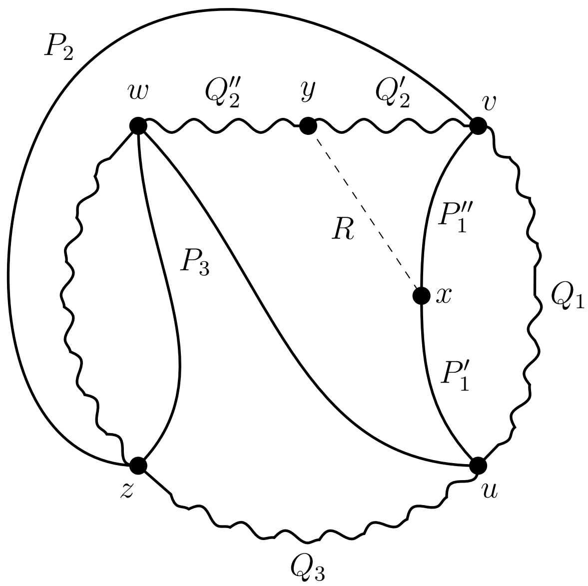


Figure 3.5: The Graph in Lemma 3.3.5

Proof. By Lemma 3.3.3, there is no $(\overline{P}_1, \overline{Q}_1)$-path internally disjoint from $C \cup D$ with $|R| > \frac{a}{2}$. We show that there does not exist a $[\overline{P}_1, \overline{Q}_2]$-path internally disjoint from C and D such that $|R| > \frac{a}{2}$, and the others follow by symmetry. Suppose R is a path connecting $x \in V(\overline{P}_1)$ and $y \in V(\overline{Q}_2)$ such that R is internally disjoint from C and D and $|R| > \frac{a}{2}$. Let $P_1 = P_1' \cup P_1''$ and $Q_2 = Q_2' \cup Q_2''$, separated by x and y respectively (see Figure 3.5). Then $C = P_1' \cup P_1'' \cup P_2 \cup P_3$ and $D = Q_1 \cup Q_2' \cup Q_2'' \cup Q_3$. Now,

$$C_1 = P_1' \cup R \cup Q_2' \cup P_2 \cup P_3$$

$$D_1 = Q_1 \cup P_1'' \cup R \cup Q_2'' \cup Q_3$$

are cycles, both containing F, as in C, we only replace P_1'' by $R \cup Q_2'$, and in D, we only replace Q_2' by $R \cup P_1''$, and P_1 is internally disjoint from D, and Q_2 is internally disjoint from

C, respectively. Now,

$$|C_1| + |D_1| = |C| + |D| + 2|R| \geq 2c_F(G) - a + 2|R| > 2c_F(G) - a + a = 2c_F(G),$$

a contradiction. □

Corollary 3.3.6. *Let C and D be two cycles of a graph G containing a linear forest subgraph F, such that $|C| + |D| \geq 2c_F(G) - 1$ and let $C = P_1 \cup P_2 \cup P_3$ and $D = Q_1 \cup Q_2 \cup Q_3$ be concatenations of three paths respectively. Suppose that P_1 and Q_1 are parallel, and P_2 and Q_2 are internally disjoint respectively from D and C. Then G contains no $[\overline{P}_1, \overline{P}_2]$-, $[\overline{P}_1, \overline{Q}_2]$-, $[\overline{Q}_1, \overline{Q}_2]$-, and no $[\overline{Q}_1, \overline{P}_2]$-paths, R, internally disjoint from C and D.*

Lemma 3.3.7. *Let P and Q be parallel paths on two cycles C and D respectively, both containing a linear forest subgraph F of a graph G, such that $|C| + |D| \geq 2c_F(G) - 1$. Let S be a segment on one of the cycles disjoint from the other. If there is a $[\overline{P}, S]$-path R (respectively a $[\overline{Q}, S]$-path) internally disjoint from $C \cup D$, then there is no $[\overline{Q}, S]$-path R' (respectively no $[\overline{P}, S]$-path) internally disjoint from C and D.*

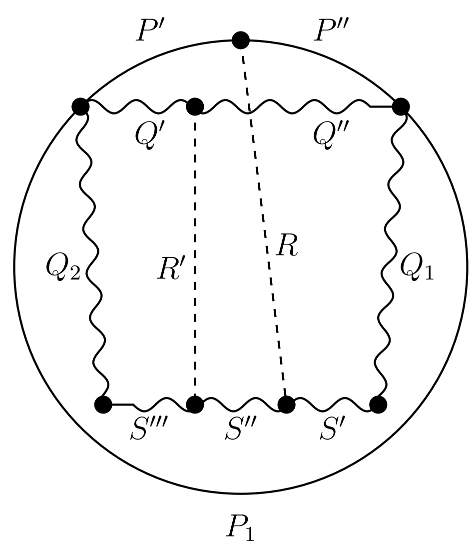


Figure 3.6: The Graph in Lemma 3.3.7

Proof. Suppose S is a segment on D and that there exists both a $[\overline{P}, S]$-path R and a $[\overline{Q}, S]$-path R' internally disjoint from $C \cup D$. Note that by Corollary 3.3.4, R is internally disjoint from R'. Then, we may assume $S = S' \cup S'' \cup S'''$, $P = P' \cup P''$, $Q = Q' \cup Q''$, $C = P \cup P_1$, and $D = Q \cup Q_1 \cup S \cup Q_2$ (see Figure 3.6). Now, consider the cycles C_1 and C_2, such that

$$C_1 = P_1 \cup P'' \cup R \cup S'' \cup R' \cup Q'$$
$$C_2 = P_1 \cup Q'' \cup R' \cup S'' \cup R \cup P'.$$

Note that $V(\overline{P}) \cap V(\overline{Q}) = \emptyset$, so F is a subgraph of P_1. Therefore, both C_1 and C_2 contain F. By Lemma 3.3.2, $|P| \leq |Q| + 1$. Moreover, $|P_1| \geq |Q_1 \cup S \cup Q_2| - 2$, as otherwise, $|C| = |P_1| + |P| \leq |Q_1 \cup S \cup Q_2| - 3 + |Q| + 1 = |Q_1 \cup S \cup Q_2 \cup Q| - 2 = |D| - 2$, a contradiction.

Thus,

$$|C_1| + |C_2| = |P_1| + |P| + |P_1| + |Q| + 2(|R| + |R'| + |S''|) \geq |C| + |D| - 2 + 4 \geq 2c_F(G) + 1;$$

a contradiction. □

3.4 Proof of the Main Results

Proof. (Proof of Theorem 3.2.1)

Case 1: $|V(F)| = 0$. Then $c_F(G) = c(G)$. If $|C| + |D| = 2c(G)$, then both C and D are longest cycles, and parts (i) and (ii) of the theorem follow from [9, Proposition 4.1], and parts (iii) and (iv) follow from [8, Theorem 2]. Thus we assume $F = \emptyset$ and $|C| + |D| = 2c(G) - 1$, say $|C| = c(G)$ and $|D| = c(G) - 1$. If C and D do not meet, then as G is 2-connected, there are two vertex-disjoint $(V(C), V(D))$-paths R_1 and R_2 which divide C into $C' \cup C''$ and divide D into $D' \cup D''$. Suppose without loss of generality, $|C'| \geq |C''|$ and $|D'| \geq |D''|$.

Then $C_1 = C' \cup R_1 \cup D' \cup R_2$ is a cycle of G such that $|C_1| \geq \frac{c(G)}{2} + \frac{c(G)-1}{2} + 2 > c(G) + 1$, a contradiction.

If C and D meet at exactly one vertex v, then as $G - v$ is connected, there is a $(V(C) - v, V(D) - v)$-path R. Suppose v and R divide C into $C' \cup C''$, and divide D into $D' \cup D''$. Let $C_1 = C' \cup R \cup D'$ and $C_2 = C'' \cup R \cup D''$. Then both are cycles of G such that $|C_1| + |C_2| = |C| + |D| + 2|R| \geq 2c(G) + 1$, a contradiction. Therefore $|V(C) \cap V(D)| \geq 2$. Now note that (ii) is clearly true for $c(G) \geq 4$ in this case, (iii) is also clearly true, and (iv) is true due to [8] (see Proposition 3.3.1 (ii)).

Case 2: $1 \leq |V(F)| \leq 2$.

(i): If $|V(C) \cap V(D)| = 1$, let $V(C) \cap V(D) = \{v\} = F$. Then as G is 2-connected, there exists a path S internally disjoint from $C \cup D$ connecting $V(C) - v$ and $V(D) - v$. Suppose the path divides $C - v$ into $P_1 \cup P_2$ and $D - v$ into $Q_1 \cup Q_2$, where P_1, P_2, Q_1, and Q_2 are all paths starting from v. Now, let $C_1 = P_1 \cup Q_1 \cup S$ and $D_1 = P_2 \cup Q_2 \cup S$. Then $v \in V(C_1) \cap V(D_1)$ and $|C_1| + |D_1| = |C| + |D| + 2|S| \geq 2c_F(G) + 1$, a contradiction. Thus $|V(C) \cap V(D)| \geq 2$.

(ii): Suppose without loss of generality that $uv \in E(C)$. Then C_1 is divided into $P_1 \cup P_2$ such that both are $[u, v]$-segments, $P_2 = uv$, and D is divided into two $[u, v]$-segments Q_1 and Q_2 (let $|Q_1| \leq |Q_2|$). As $c_F(G) \geq 5$, $|P_1| \geq 3$, and $|Q_2| \geq 3$. Then $|P_1 \cup Q_2| \geq |P_1| + 3 = (|P_1| + 1) + 2 \geq (c_F(G) - 1) + 2 = c_F(G) + 1$, a contradiction.

(iii): As $c_F(G) = 3$, then it is obvious that $|C| + |D| = 2c_F(G) - 1$. Assume, $|D| = 3$ and $|C| = 2$ such that $D = uvwu$ and $C = uvu$. If there exists another vertex t adjacent to one of u, v, w, say u, as G is 2-connected, t must be adjacent to another vertex, say w, but then we obtain a cycle $R = uvwtu$ which implies $c_F(G) \geq 4$, a contradiction. Thus no other vertex may exist, only multiple edges could be added, and $si(G) \cong K_3$.

(iv): First, suppose that $|C| = 3$ and $|D| = 4$ such that $C = vuav$ and $D = vbucv$. If ab, ac, or bc is an edge of G, we obtain a cycle of length five containing F, a contradiction as $c_F(G) = 4$. So $ab, ac, bc \notin E(G)$. Now, if there exists a vertex t such that $t \notin S =$

$V(C) \cup V(D)$, then as G is 2-connected, there exists two internally disjoint (t, S)-paths P_1 and P_2 with endpoints $s_1, s_2 \in S$. If s_1 and s_2 are adjacent edges, then we can clearly reroute through t and obtain a cycle of length at least five, a contradiction as $c_F(G) = 4$. So suppose s_1 and s_2 are not adjacent. Now, if $\{s_1, s_2\} = \{u, v\}$, then $P_1 = tu$ and $P_2 = tv$. As t is arbitrary, it is easily seen that $G \cong K^+_{2,m}$ for some $m \geq 3$. So suppose $\{s_1, s_2\} \subseteq \{a, b, c\}$. Without loss of generality suppose $s_1 = a$ and $s_2 = b$ such that P_1 is a $[t, a]$-path and P_2 is a $[t, b]$-path. Then we obtain a new cycle $R = tP_1avubP_2t$ which implies $c_F(G) \geq 5$, a contradiction. Thus $si(G) \cong K^+_{2,m}$ for $m \geq 3$.

Now suppose that $c_F(G) = 4 = |C| = |D|$ and suppose $C = ubvau$ and $D = ucvdu$. Let $S = V(C) \cup V(D)$. Note that $\{a, b, c, d\}$ must be an independent set, as otherwise we obtain a cycle of length five containing $\{u, v\}$, a contradiction as $c_F(G) = 4$. If there exists a vertex $t \notin S$, then as G is 2-connected, there exists two internally disjoint (t, S)-paths P_1 and P_2 with endpoints $s_1, s_2 \in S$. If $\{s_1, s_2\} = \{u, v\}$, then $P_1 = tu$ and $P_2 = tv$ as $c_F(G) = 4$. So suppose $\{s_1, s_2\} \subseteq \{a, b, c, d\}$. Without loss of generality suppose $s_1 = a$ and $s_2 = b$. Then we obtain a new cycle $R = tP_1aucvbP_2t$ which implies $c_F(G) \geq 6$, a contradiction. Similarly, we can show that the case where $s_1 \in \{u, v\}$ and $s_2 \in \{a, b, c, d\}$ is impossible. Thus if there is any $t \in S$, then $tu, tv \in E(G)$. Now, it is easily seen that the only other possible edge is uv. Thus $G \cong K_{2,m}$ or $K^+_{2,m}$ for some $m \geq 4$. This completes the proof of Theorem 3.2.1. □

Proof. (Proof of Theorem 3.2.2) If $c_F(G) = 3$ or 4, the theorem follows from Theorem 3.2.1 (iii) and (iv). So we assume $c_F(G) \geq 5$ from now on. We defer the proof of part (a) until after we present the proofs of parts (b), (c), and (d).

Part (b): Suppose either $|P_1| \neq |P_2|$ or $|Q_1| \neq |Q_2|$, and without loss of generality say $|P_1| < |P_2|$. Then $P_2 \cup Q_2$ is a cycle containing F with size $|P_2 \cup Q_2| > \frac{1}{2}c_F(G) + \frac{1}{2}c_F(G) = c_F(G)$, a contradiction.

Part (c): Suppose $|C| = c_F(G) - 1$ and $|D| = c_F(G)$.

Case (1): $c_F(G) = 2a$ is even. Recall $C = P_1 \cup P_2$, $D = Q_1 \cup Q_2$, and $|C| = |D| - 1$. Then $|Q_2| \geq a$. As $|C| = 2a - 1$, if $|P_1| \leq a - 2$, then $|P_2| \geq a + 1$. Now note that $P_2 \cup Q_2$

is a cycle containing F such that $|P_2 \cup Q_2| \geq a + 1 + a = 2a + 1$, a contradiction. Thus $|P_1| \geq a-1$. As $|P_1| \leq |P_2|$, we have that $|P_1| = a-1$ and $|P_2| = a$. Additionally, if $|Q_2| > a$, then again $|P_2 \cup Q_2| > 2a$, a contradiction. Thus $|Q_1| = |Q_2| = a$.

Case (2): $c_F(G) = 2a-1$ is odd. Then we have that $|C| = 2a-2$ and $|D| = 2a-1$. Thus $|Q_2| \geq a$, so $|P_2| \leq a-1$, otherwise $|P_2 \cup Q_2| > 2a-1$. Therefore $|P_1| \leq a-1$, and thus $|P_1| = |P_2| = a-1$. We also have that $|Q_2| \leq a$, otherwise $|P_2 \cup Q_2| \geq a+1+a-1 = 2a$, a contradiction, as $P_2 \cup Q_2$ is a cycle containing F. Thus $|P_1| = |P_2| = |Q_1| = a-1$ and $|Q_2| = a$. Note that in this case, if there exists a cycle R passing through F with size $c_F(G)$, then $V(D) \cap V(R) \neq \{u, v\}$. Otherwise, if $R = R_1 \cup R_2$, where R_i is the $[u, v]$-segment of R such that $|R_1| \leq |R_2|$, then $R_2 \cup Q_2$ is a cycle passing through F such that $|R_2 \cup Q_2| \geq a + a = 2a$, a contradiction as $c_F(G) = 2a-1$.

Part (d): The proof is simple when $|C| = |D| = c_F(G)$, so we will only prove the case when $|C| = |D| - 1$.

Case (1): $c_F(G) = 2a$ is even. By (c), $|P_2| = |Q_1| = |Q_2| = a$ and $|P_1| = a-1$. Suppose now that some of the nonempty truncated paths $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, and $\overline{Q}_2$ are in the same component of $G - \{u, v\}$, say a path S internally disjoint from $C \cup D$ connecting $\overline{P}_1$ and $\overline{Q}_1$ such that $P_1 = P_1' \cup P_1''$ and $Q_1 = Q_1' \cup Q_1''$ with P_1' and Q_1' containing u as an end vertex. Let $C_1 = Q_2 \cup Q_1' \cup S \cup P_1''$ and $C_2 = Q_2 \cup P_1' \cup S \cup Q_1''$ be cycles both passing through F. Then $|C_1| + |C_2| = |P_1| + |Q_1| + 2|S| + 2|Q_2| \geq a - 1 + a + 2 + 2a = 4a + 1 > 2c_F(G)$, a contradiction. Thus there is no such path and $\overline{P}_1$ and $\overline{Q}_1$ are in different components of $G - \{u, v\}$. Similarly, we have that no such path between any two of the nonempty truncated paths which may exist and that all of $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, and $\overline{Q}_2$ (if nonempty) are in different components of $G - \{u, v\}$.

Case (2): $c_F(G) = 2a-1$ is odd. If $c_F(G) = 3$, by Theorem 3.2.1, $si(G) \cong K_3$, and the result is trivially true. So we assume $c_F(G) \geq 5$ from now on. By (c), $|P_1| = |P_2| = |Q_1| = a-1$ and $|Q_2| = a$. Suppose now that some of the nonempty truncated paths $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, and $\overline{Q}_2$ are in the same component of $G - \{u, v\}$, say a path S internally disjoint from $C \cup D$

connecting $\overline{Q}_1$ and $\overline{Q}_2$ such that $Q_1 = Q_1' \cup Q_1''$ and $Q_2 = Q_2' \cup Q_2''$ with Q_1' and Q_2' containing u as an end vertex. Let $D_1 = P_1 \cup Q_1' \cup S \cup Q_2''$ and $D_2 = P_1 \cup Q_1'' \cup S \cup Q_2'$ be cycles both passing through F. Then $|D_1|+|D_2| = 2|P_1|+|Q_1|+|Q_2|+2|S| \geq 2(a-1)+(a-1)+a+2 = 4a-1 > 2c_F(G)$, a contradiction. Thus there is no such path and $\overline{Q}_1$ and $\overline{Q}_2$ are in different components of $G - \{u, v\}$. Similarly, we have that no such path between any two of the nonempty truncated paths which may exist and that all of $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, and $\overline{Q}_2$ (if nonempty) are in different components of $G - \{u, v\}$.

Part (a): As $c_F(G) \geq 4$, by (b) and (c) $\overline{P}_2, \overline{Q}_1, \overline{Q}_2 \neq \emptyset$. Now (a) follows.

Part (e): Here, we prove the case when $|C|+|D| = 2c_F(G)-1$ and omit the similar proof when $|C| + |D| = 2c_F(G)$. Suppose R is a cycle passing through F with $|R| \geq c_F(G) - 1$. If $c_F(G) = 4$, then $G \cong K^+_{2,m}$ for some $m \geq 3$ or $G \cong K_{2,m}$ for some $m \geq 4$, and the conclusion holds. So suppose $c_F(G) \geq 5$. If $|V(F)| = 2$, then there is nothing needed to prove, so assume $V(F) \leq \{v\}$. Now, suppose $u \notin V(R)$, then by (d), $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, and $\overline{Q}_2$ must be in different components of $G-\{u, v\}$. Thus $R-v$ can only be in exactly one component which contains, say, $\overline{P}_1$. Note that by (b) and (c), all $\overline{P}_i, \overline{Q}_i \neq \emptyset$ for $i = 1, 2$. However, note that $P_2 \cup Q_2$ is a cycle containing F such that $|P_2 \cup Q_2| = c_F(G)$, and now, R and $P_2 \cup Q_2$ have at most one common vertex v, a contradiction to Theorem 3.2.1 as $|R|+|P_2 \cup Q_2| \geq 2c_F(G)-1$. Thus (e) holds.

Part (f): By (e), R must contain both u and v. If $c_F(G) = 4$, the conclusion is easily verified by Theorem 3.2.1. So we assume $c_F(G) \geq 5$. Thus $\overline{P}_1, \overline{P}_2, \overline{Q}_1, \overline{Q}_2 \neq \emptyset$.

Case (1): $c_F(G)$ is even. By (b) and (c), $|P_2| = |Q_1| = |Q_2| = a$, $|P_1| = a - 1$ or a, and $c_F(G)) = 2a$. Note that $R-\{u, v\}$ has two segments R_1 and R_2. By (d), two of $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, and $\overline{Q}_2$, by symmetry, say either $\overline{P}_2$ and $\overline{Q}_1$, or $\overline{P}_1$ and $\overline{Q}_1$, are in different components of $G - \{u, v\}$ from the components containing $\overline{R}_1$ and $\overline{R}_2$, respectively. We assume the former is true, and note that the proof for the latter case is similar. Without loss of generality, suppose $|R_1| \leq |R_2|$. If $|R_2| \geq a + 1$, then $Q_1 \cup R_2$ is a cycle containing F such that $|Q_1 \cup R_2| \geq 2a + 1$, a contradiction. So, $|R_2| \leq a$. Thus $|R_1| = |R_2|$ as $|R| = 2a$. Now, let

$C_1 = R_1 \cup Q_1$ and $C_2 = R_2 \cup P_2$. Then both cycles pass through F, and $|V(C_1) \cap V(C_2)| = 2$ with $|C_1| = |D_1| = c_F(G)$. By (d), $\overline{R}_1$ and $\overline{R}_2$ are in different components of $G - \{u, v\}$.

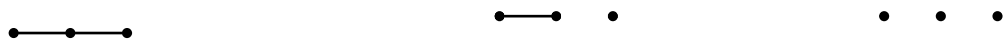
Case (2): $c_F(G)$ is odd. By (c), $|P_1| = |P_2| = |Q_1| = a - 1$ and $|Q_2| = a$. By (d) and symmetry, we may assume that either $\overline{P}_2$ and $\overline{Q}_1$, or $\overline{P}_1$ and $\overline{Q}_2$ are in different components of $G - \{u, v\}$ from the components containing $\overline{R}_1$ and $\overline{R}_2$ respectively. We assume the former is true, and note that the proof for the latter case is similar. As $|R| = |R_1| + |R_2| = c_F(G) = 2a - 1$, and $|R_1| \leq |R_2|$, we have that $|R_2| \geq a$. Indeed, $|R_2| = a$, and thus $|R_1| = a - 1$, otherwise $|R_2| \geq a + 1$, then $Q_1 \cup R_2$ is a cycle passing through F such that $|Q_1 \cup R_2| \geq a - 1 + a + 1 = 2a > c_F(G)$, a contradiction. Now suppose $\overline{R}_1$ and $\overline{R}_2$ belong to the same component of $G - \{u, v\}$. Then there exists an $(\overline{R}_1, \overline{R}_2)$-path S in $G - \{u, v\}$. By our assumption, S is disjoint from $\overline{P}_2 \cup \overline{Q}_1$. Note that the end vertices of S divide R_1 into two segments R_1' and R_1'', and R_2 into two segments R_2' and R_2'', where R_1' and R_2' have u as an end vertex. Let $C_1 = R_1' \cup S \cup R_2'' \cup P_2$ and $C_2 = R_1'' \cup S \cup R_2' \cup Q_1$ be cycles passing through F. Then $|C_1| + |C_2| = |R_1| + |R_2| + 2|S| + |P_2| + |Q_1| \geq 2a - 1 + 2 + 2(a - 1) = 4a - 1 = 2c_F(G) + 1$, a contradiction. Thus (f) holds.

Part (g): As $|C| = |D| = c_F(G)$, by (b), $c_F(G) = 2a$ is even and $C = P_1 \cup P_2$, $D = Q_1 \cup Q_2$ such that $|P_1| = |P_2| = |Q_1| = |Q_2| = a$. As $|R| \geq c_F(G) - 1$, by (e), R must pass through both u and v. Now, $G - \{u, v\}$ has two segments R_1 and R_2, $|R_1| \leq |R_2|$. By (d), two of $\overline{P}_1$, $\overline{P}_2$, $\overline{Q}_1$, and $\overline{Q}_2$ are in different components of $G - \{u, v\}$ from the components containing $\overline{R}_1$ and $\overline{R}_2$ respectively, say $\overline{P}_2$ and $\overline{Q}_1$. Thus, $|R_1| = a - 1$ and $|R_2| = a$. Additionally, $C_1 = R_1 \cup P_2$ and $D_1 = R_2 \cup Q_1$ are cycles containing F of length $c_F(G) - 1$ and $c_F(G)$, respectively, and $V(C_1) \cap V(D_1) = \{u, v\}$. By (d), $\overline{R}_1$ and $\overline{R}_2$ are in different components of $G - \{u, v\}$. This completes the proof of Theorem 3.2.2. □

Next we prove Theorem 3.2.3.

Proof. (Proof of Theorem 3.2.3) We may assume without loss of generality that $|C| \geq |D|$ and W is not an articulation set.

(a) $k = 3$: We show (i) and (ii) both hold in this case. Suppose that $W = V(C) \cap V(D) = \{v_1, v_2, v_3\}$. In this case, F is a subgraph of a path of length two. The following in Figure 3.7 are just three examples of F.



(a) (b) (c)

Figure 3.7: Examples of F when $|V(F)| = 3$

In (a), C and D have at least two trivial parallel paths. In (b), C and D have at least one trivial parallel path. In the proof below, we will create new cycles, each containing all three common vertices and all possible trivial parallel paths in F. Therefore, the new cycles still contain F.

We can write C and D as concatenations of three paths such that $C = P_1 \cup P_2 \cup P_3$ and $D = Q_1 \cup Q_2 \cup Q_3$, and P_i and Q_i are parallel $[v_i, v_{i+1}]$-segments where i is read modulo 3 (see Figure 3.8). Note that in the rest of the proof when $|P_i| = |Q_i| = 1$ for some $i \in \{1, 2, 3\}$, it is possible that $P_i = Q_i$, which is a single edge. As $c_F(G) \geq k + 2 = 5$ when $|C| + |D| = 2c_F(G) - 1$, and $c_F(G) \geq k + 1 = 4$ when $|C| + |D| = 2c_F(G)$, we deduce that $|P_i| \geq 2$ for some $i \in \{1, 2, 3\}$, and $|Q_j| \geq 2$ for some $j \in \{1, 2, 3\}$.

Case (a.1): There exists an $i \in \{1, 2, 3\}$ such that $|P_i| \geq 2$ and $|Q_i| \geq 2$, say $i = 1$. By Lemma 3.3.2 (ii), $|P_1| - |Q_1| \leq 1$. By Corollary 3.3.4, there does not exist a $[\overline{P}_1, \overline{Q}_1]$-path internally disjoint from C and D. Note now that each of the paths P_2, P_3, Q_2, and Q_3 satisfy the assumptions of Corollary 3.3.6 with respect to the parallel paths P_1 and Q_1. Thus, by Corollary 3.3.6, there does not exist a $[\overline{P}_1, \overline{P}_2]$-, $[\overline{P}_1, \overline{P}_3]$-, $[\overline{P}_1, \overline{Q}_2]$-, or $[\overline{P}_1, \overline{Q}_3]$-path internally disjoint from C and D. Thus, the component of $G - W$ containing $\overline{P}_1$ does not contain any of the (non-empty) paths $\overline{P}_2$, $\overline{P}_3$, $\overline{Q}_1$, $\overline{Q}_2$, or $\overline{Q}_3$. Thus, we have that $\overline{P}_1$ and $\overline{Q}_1$ are in different components of $G - W$, and W is an articulation set of G, a contradiction.

Case (a.2): For all $i \in \{1, 2, 3\}$, $min\{|P_i|, |Q_i|\} = 1$. By Lemma 3.3.2, we may assume that $|P_2| = |P_3| = |Q_2| = |Q_3| = 1$ and $|Q_1| = |P_1| - 1 = 1$. In this case $c_F(G) = 4$,

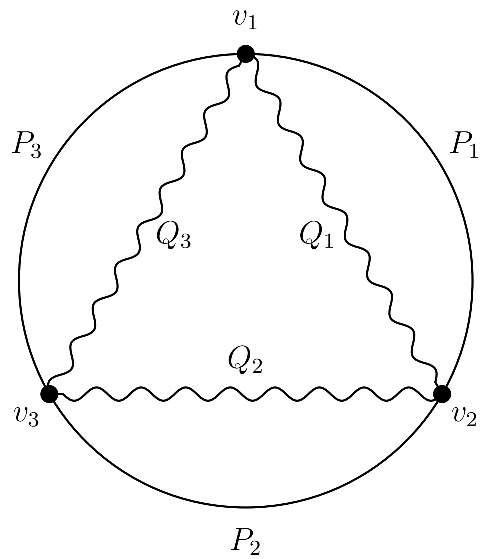


Figure 3.8: Two cycles with intersection size $k = 3$ and P_i parallel to Q_i for $i \in \{1, 2, 3\}$

and $|C| + |D| = 2c_F(G) - 1 = 7$, and we are in part (iv) of Theorem 3.2.3. Thus we assume $|V(F)| \leq 1$. Suppose $F = \{v\} \subseteq W$. Note that it is possible that $P_2 = Q_2$ and $P_3 = Q_3$. Suppose that $\overline{P}_1 = \{u\}$. If W is not an articulation set, then there exists a vertex $t \notin \{u\} \cup W$ such that $ut \in E(G)$. As G is 2-connected, there is a (t, W)-path in $G - u$ (see Figure 3.9). Regardless of what the end vertex of this path is in W, it is easy to see that G has a new cycle C_1 passing through F and the edge ut such that $|C_1| \geq 5 = c_F(G) + 1$. This contradiction completes the case for $k = 3$.

(b) $k = 4$: Suppose $W = \{u_1, u_2, u_3, u_4\}$. Let $D = Q_1 \cup Q_2 \cup Q_3 \cup Q_4$ where Q_i is a $[u_i, u_{i+1}]$-segment of D for $i \in \{1, 2, 3\}$, and Q_4 is a $[u_4, u_1]$-segment. In this case F is a subgraph of the path P_3. The following graphs in Figure 3.10 list all possible cases of F where $W = V(F)$ has four vertices, just note that $|V(F)|$ could be less than four.

In our next proof, all of the new cycles will contain W, and each trivial parallel path of C and D, thus containing F as well. As $c_F(G) \geq k + 2 = 6$ when $|C| + |D| = 2c_F(G) - 1$ and $c_F(G) \geq k + 1 = 5$ when $|C| + |D| = 2c_F(G)$, we conclude that $|C|, |D| \geq 5$, and thus there are $i, j \in \{1, 2, 3, 4\}$ such that $|P_i| \geq 2$ and $|Q_j| \geq 2$.

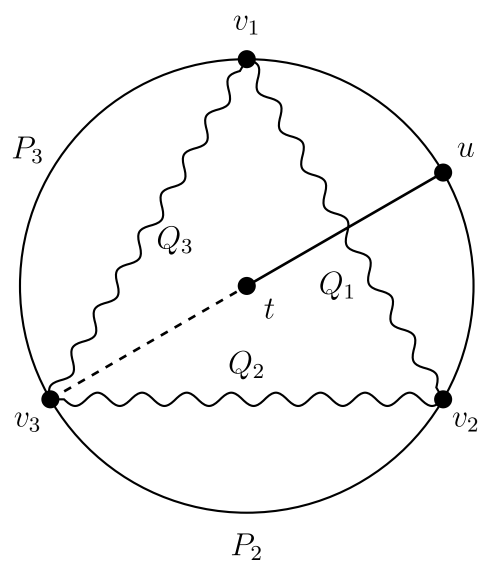


Figure 3.9: The Graph in Case a.2

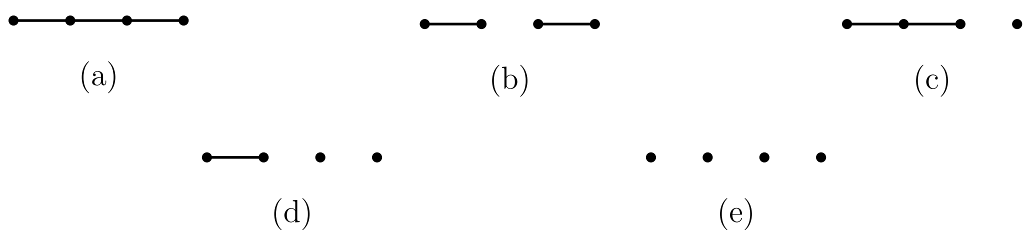


Figure 3.10: Examples of F when $|V(F)| = 4$

Case (b.1): Each segment of D is parallel to a segment of C, say P_i is parallel to Q_i for all $i \in \{1, 2, 3, 4\}$ (see Figure 3.11). Note that it is possible that $P_i = Q_i$ if $|P_i| = |Q_i| = 1$. First, assume that there is an $i \in \{1, 2, 3, 4\}$ such that $|P_i|, |Q_i| \geq 2$, say $i = 1$.

By Lemma 3.3.2 (ii) and Corollaries 3.3.4 and 3.3.6, we have that there does not exist a $[\overline{P}_1, \overline{Q}_1]$-, $[\overline{P}_1, \overline{Q}_2]$-, $[\overline{P}_1, \overline{Q}_4]$-, $[\overline{P}_1, \overline{P}_2]$-, or $[\overline{P}_1, \overline{P}_4]$-path internally disjoint from C and D. If $\overline{P}_3$ and $\overline{Q}_3$ are empty, then $\overline{Q}_1$ and $\overline{P}_1$ are in different components of $G - W$, a contradiction as W is not an articulation set by assumption. If $\overline{P}_3$ or $\overline{Q}_3$ is nonempty, then by Lemma 3.3.7, if there exists a $[\overline{P}_1, \overline{Q}_3]$-path internally disjoint from C and D, there cannot exist a

$[\overline{Q}_1, \overline{Q}_3]$-path internally disjoint from C and D. Moreover, there does not exist a $[\overline{P}_3, \overline{Q}_3]$-path internally disjoint from C and D. Thus $\overline{P}_1$ and $\overline{Q}_1$ are in different components of $G - W$, a contradiction.

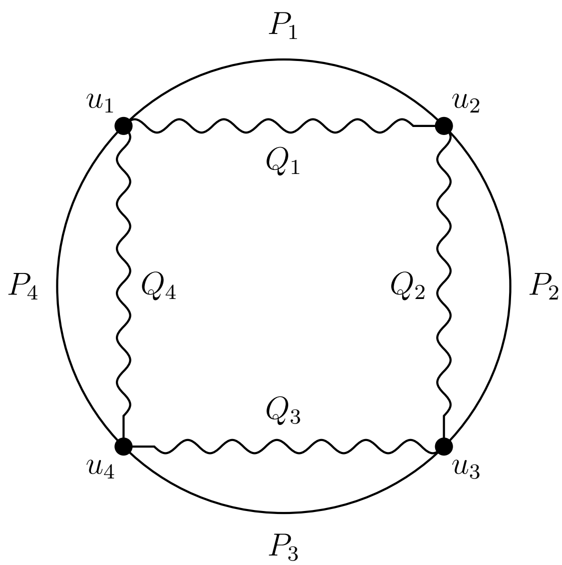


Figure 3.11: Two cycles with intersection size $k = 4$ and P_i parallel to Q_i for $i \in \{1, 2, 3, 4\}$

Now we assume that for all $i \in \{1, 2, 3, 4\}$, either $|P_i| = 1$ or $|Q_i| = 1$. By Lemma 3.3.2, we may assume that $|P_1| = 2$, $|Q_1| = 1$, and $|P_i| = |Q_i| = 1$ for all $i \in \{2, 3, 4\}$. Thus $c_F(G) = 5 = k + 1$, and $|C| + |D| = 2c_F(G) - 1$. Thus we are in part (iv) of Theorem 3.2.3, and $F \subseteq \{v\} \subseteq W$. Now suppose $\overline{P}_1 = \{u\}$. As W is not an articulation set, there exists a vertex $t \notin \{u\} \cup W$ such that $ut \in E(G)$. As G is 2-connected, $G - u$ has a (t, W)-path S internally disjoint from $C \cup D$ ending at some $u_i \in W$. Since $c_F(G) = 5$, $i \neq 1, 2$ as we would obtain a cycle containing v of length at least 6 by replacing either the edge $u_1 u$ on C with $u_1 Stu$, or the edge $u_2 u$ on C with $u_2 Stu$. Now, by symmetry, we assume $i = 3$ (see Figure 3.12). Then, $C_1 = utSu_3u_4u_1u_2u$ is a cycle containing F such that $|C_1| \geq 6$, a contradiction.

Case (b.2): Two segments of C are parallel to segments of D respectively, say P_1 is parallel to Q_1 and P_3 is parallel to Q_3 (see Figure 3.13). As $c_F(G) = |C| \geq k + 1 = 5$, for some $i \in \{1, 2, 3, 4\}$, $|P_i| \geq 2$.

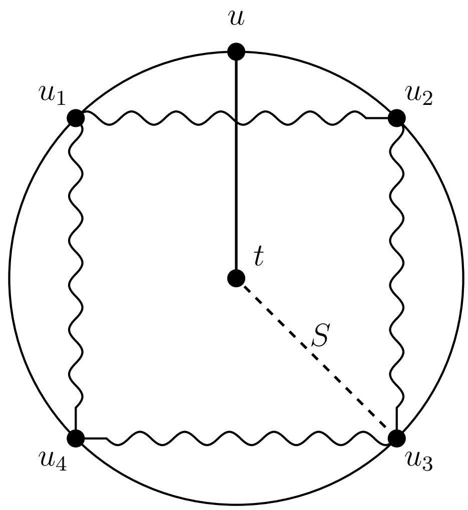


Figure 3.12: The Graph in Case b.1 where $\overline{P}_1 = \{u\}$

Case (b.2.1): Assume that either $|P_2|$ or $|P_4|$ has length at least 2, say $|P_2| \geq 2$. By Corollaries 3.3.4 and 3.3.6, there is no $(\overline{P}_2, \overline{Q}_i)$-path internally disjoint from $C \cup D$ for $i \in \{1, 2, 3, 4\}$, and no $(\overline{P}_2, \overline{P}_1)$-path or $(\overline{P}_2, \overline{P}_3)$-path internally disjoint from $C \cup D$. Next we show that there is no $(\overline{P}_2, \overline{P}_4)$-path internally disjoint from $C \cup D$. Note that if $\overline{P}_4 = \emptyset$, then this is certainly true. So assume the contrary and that there is such a path R dividing P_2 into two segments P_2' and P_2'', and dividing P_4 into two segments P_4' and P_4'', such that P_2' is adjacent to u_2 and P_4' is adjacent to u_4 (see Figure 3.13). Then $D_1 = Q_1 \cup P_2' \cup R \cup P_4' \cup Q_3 \cup Q_4$ and $D_2 = P_1 \cup Q_2 \cup P_3 \cup P_2'' \cup R \cup P_4''$ are both cycles of G. As both D_1 and D_2 contain all vertices of W, and the only possible edges of F are u_1u_2 (in this case $P_1 = Q_1 = u_1u_2$) and u_3u_4 (in this case $P_3 = Q_3 = u_3u_4$). Thus both D_1 and D_2 pass through F. Moreover, $|D_1| + |D_2| = |C| + |D| + 2|R| \geq 2c_F(G) + 1$, a contradiction. Thus, the components containing $\overline{P}_2$ in $G - W$ contain none of the other nonempty truncated paths, a contradiction as W is not an articulation set.

Now, consider (iv) and consider the case when all other truncated paths are empty. Thus, $|P_i| = 1$ for $i \in \{1, 3, 4\}$ and $|Q_i| = 1$ for all $i \in \{1, 2, 3, 4\}$. Then $F \subseteq \{v\} \subseteq W$. Note now that $|D| = 4$ and $|C| = 5$, and $|P_2| = 2$. Assume that $\overline{P}_2 = \{u\}$. As $G - W$ is not an articulation set, there exists a vertex $t \notin W \cup \{u\}$ such hat $ut \in E(G)$. As G is 2-connected,

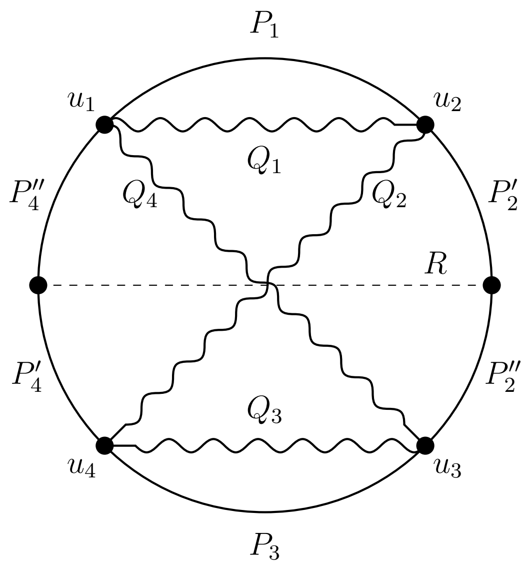


Figure 3.13: The Graph in Case b.2.1

$G-u$ has a (t,W)-path S_1 ending at some u_i internally disjoint from $C \cup D$. As $c_F(G)=5$, $i \neq 2,3$, as we would obtain a cycle containing F of length at least six by replacing either the edge u_2u on C with u_2S_1tu, or the edge u_3u on C with u_3S_1tu. By symmetry we may assume that $i=1$ (see Figure 3.14). Now, $C_1 = utS_1u_1u_2u_4u_3u$ is a cycle of length at least six passing through F, a contradiction.

Case (b.2.2): Suppose that neither $|P_2|$ nor $|P_4|$ has length at least 2, thus $|P_2|=|P_4|=1$. As $c_F(G) \geq 5$, we may assume then that $|P_1| \geq 2$. If $|Q_1| \geq 2$, or both $|P_3| \geq 2$ and $|P_4| \geq 2$, then using the same proof as in case (b.1), we deduce that $\overline{P}_1$ and $\overline{Q}_1$ (or $\overline{P}_3$ and $\overline{Q}_3$) are in different components of $G-W$, a contradiction as W is not an articulation set.

Thus, we may assume $|P_1|=2$ and $|Q_1|=1$. By Lemma 3.3.2, $|P_3|=|Q_3|$. Now, $c_F(G)=5$ and $|D|=4$. Thus we are in part (iv) again, and $F \subseteq \{v\} \subseteq W$. If $G-W$ is connected with more than one vertex, then there exists a vertex $t \notin W \cup \{u\}$ such that $ut \in E(G)$. As G is 2-connected, there exists a (t,W)-path S_2 in $G-u$ which is internally disjoint from $C \cup D$. As $c_F(G)=5$, $i \neq \{1,2\}$, otherwise we would obtain a cycle containing F of length at least six by replacing either the edge u_1u on C with u_1S_2tu, or the edge u_2u on C with u_2S_2tu. By symmetry, assume $i=3$ (see Figure 3.15). Then there exists a cycle

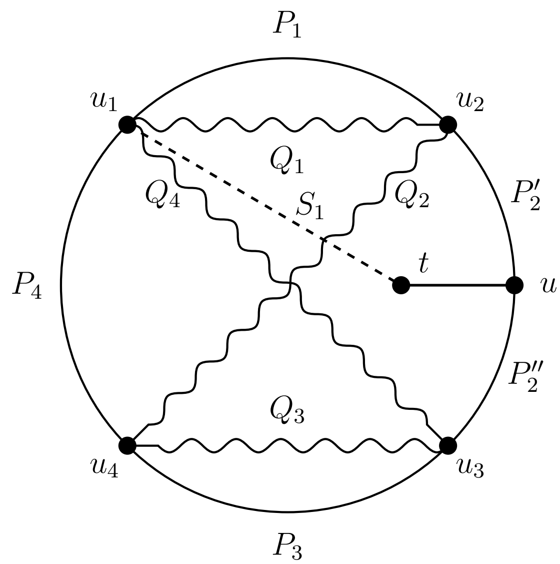


Figure 3.14: The Graph in Case b.2.1 where $|D| = 4$ and $|C| = 5$

$C_2 = uu_2u_1u_4u_3S_2tu$ passing through F with length at least $6 = c_F(G) + 1$, a contradiction. Thus the proof for $k = 4$ part (i) is complete.

(c) $k = 5$. Again here we assume $|C| \geq |D|$. Now, note that we assume that $c_F(G) \geq 7$. Let $W = \{u_1, u_2, u_3, u_4, u_5\}$ be the intersection of two cycles C and D containing the linear forest subgraph F with at most five vertices such that $|C| + |D| \geq 2c_F(G) - 1$. Then, without loss of generality, C and D can be written as concatenations of five paths, $C = P_1 \cup P_2 \cup P_3 \cup P_4 \cup P_5$ and $D = Q_1 \cup Q_2 \cup Q_3 \cup Q_4 \cup Q_5$ respectively. Relabeling as necessary, P_i is the $[u_i, u_{i+1}]$-segment of C for $1 \leq i \leq 4$, and P_5 is the $[u_5, u_1]$-segment. In this case, the linear forest F is a subgraph of P_5. When $|V(F)| = 5$, F is one of the graphs in Figure 3.16.

In the following proof, all new cycles we create will contain all vertices of W, therefore containing all vertices of F. Moreover, these cycles will also contain each trivial parallel path of C and D, thus contain F as well.

Then, there are four cases of how C and D can intersect: $(c.1)$ each segment P_i is parallel to Q_i for $1 \leq i \leq 5$, $(c.2)$ three pairs of P_i and Q_i segments are parallel, $(c.3)$ two pairs of P_i and Q_i segments are parallel, and $(c.4)$ no pairs of P_i and Q_i segments are parallel. Note

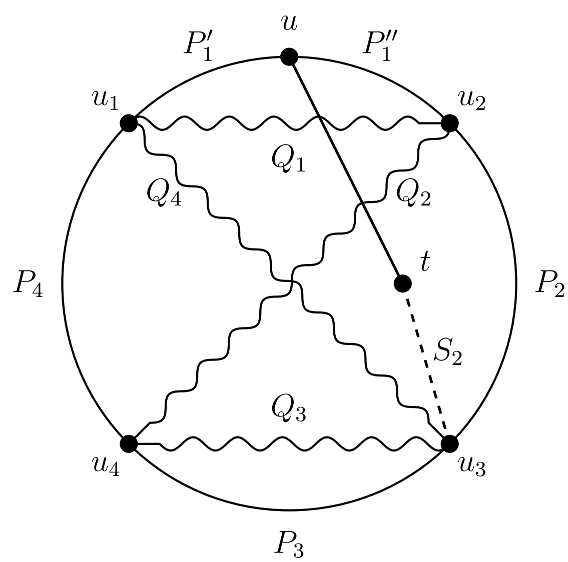


Figure 3.15: The Graph in Case b.2.2

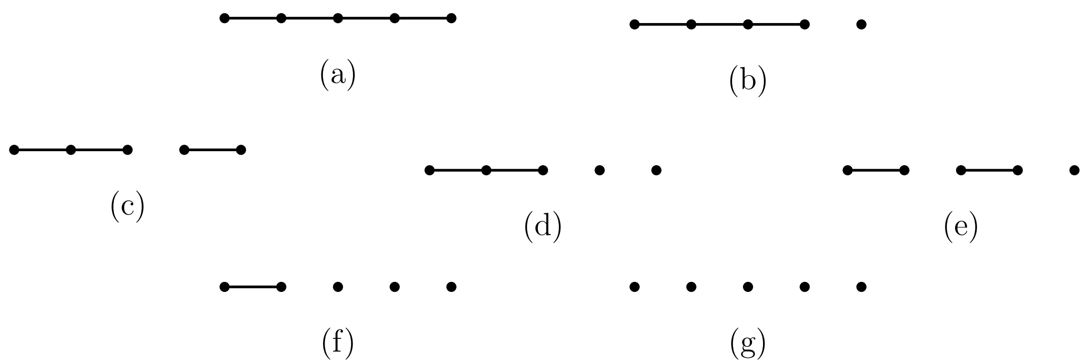


Figure 3.16: Examples of F when $|V(F)| = 5$

that four parallel segments would force all five to be parallel, and one parallel segment would force at least a second pair of segments to be parallel, so these are not cases to be considered. Again, we proceed by supposing W is not an articulation set, and prove the subcases here by contradiction. Note here again if $|P_i| = |Q_i| = 1$, it is possible that $P_i = Q_i$ is a single edge when (P_i, Q_i) is a trivial parallel pair.

As $c_F(G) \geq 7$ and $|C| \geq |D|$ with $|C| + |D| \geq 2c_F(G) - 1$, we have that $|C| = c_F(G) \geq 7$ and $|D| \geq c_F(G) - 1 \geq 6$. Thus there are P_i and Q_j for $i, j \in \{1, 2, 3, 4, 5\}$ such that $|P_i| \geq 2$ and $|Q_j| \geq 2$.

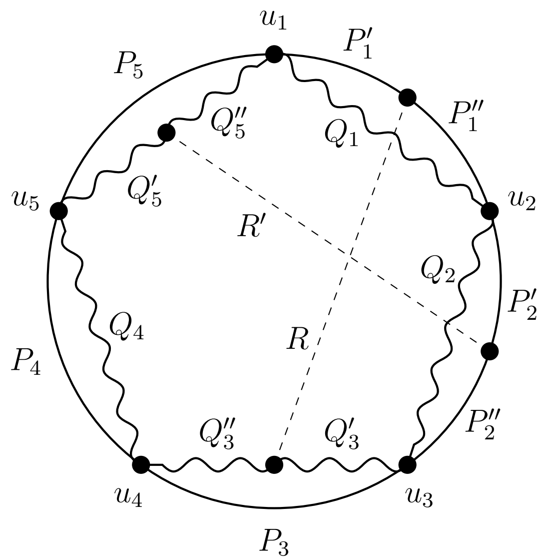


Figure 3.17: The Graph in Case c.1.1

Case (c.1): Suppose that (P_i, Q_i) are parallel for all $i = 1, 2, 3, 4, 5$.

Case (c.1.1): Assume there is an i such that $|P_i| \geq 2$ and $|Q_i| \geq 2$, say $i = 1$. By Corollaries 3.3.4 and 3.3.6, there does not exist a $[\overline{P}_1, \overline{Q}_1]$-, $[\overline{P}_1, \overline{Q}_2]$-, $[\overline{P}_1, \overline{Q}_5]$-, $[\overline{P}_1, \overline{P}_2]$-, or $[\overline{P}_1, \overline{P}_5]$-path internally disjoint from C and D. By symmetry, suppose there exists a $[\overline{P}_1, \overline{Q}_3]$-path, R, internally disjoint from C and D (see Figure 3.17). First assume that (P_2, Q_2) is not a trivial parallel segment in F (recall that the trivial segment in F means $P_2 = Q_2 = u_2u_3 \in E(F)$). Then

$$C_1 = R \cup Q_3' \cup P_3 \cup P_4 \cup P_5 \cup Q_1 \cup P_1''$$

is a cycle in G containing F. Now, $|C_1| \leq |C|$ and $|Q_1| \geq |P_1| - 1$ imply that $|R \cup Q_3' \cup P_1''| \leq |P_2| + 1$. Thus $|P_2| \geq 2$. As $G - W$ is connected, there exists a path from $\overline{P}_2$ to another segment. However, we have that $\overline{P}_2$ may only possibly be connected to $\overline{P}_4$, $\overline{P}_5$, $\overline{Q}_4$, or $\overline{Q}_5$. By symmetry, suppose there exists a $[\overline{P}_2, \overline{Q}_5]$-path, R', internally disjoint from C and D (see

Figure 3.17). Then, we get two cycles, both containing F such that

$$C' = P_1' \cup R \cup Q_3' \cup P_3 \cup P_4 \cup Q_5' \cup R' \cup P_2' \cup Q_1$$
$$D' = Q_5'' \cup R' \cup P_2'' \cup Q_2 \cup P_1'' \cup R \cup Q_3'' \cup Q_4 \cup P_5.$$

Note though that both cycles contain all vertices of F and contain every possible trivial (P_i, Q_i)-path, thus contain F. Moreover,

$$|C'| + |D'| = |C| + |D| + 2|R| + 2|R'| \geq 2c_F(G) - 1 + 4 = 2c_F(G) + 3 > 2c_F(G),$$

a contradiction.

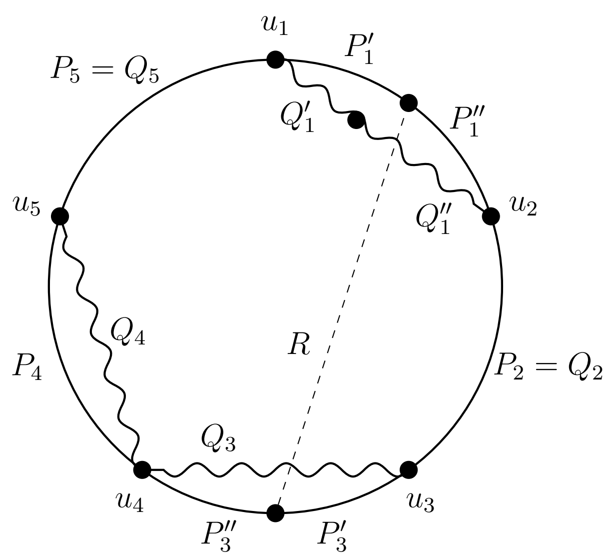


Figure 3.18: The Graph in Case c.1.1 where $P_2 = Q_2$ is an edge of F

Now assume that $P_2 = Q_2 = u_2u_3$ is an edge of F. By symmetry, we can assume that $P_5 = Q_5 = u_5u_1$ is also an edge of F. As $\overline{P_1} \neq \emptyset$ and $\overline{Q_1} \neq \emptyset$, there is a $(\overline{P_1}, \overline{P_3})$, $(\overline{P_1}, \overline{P_4})$, $(\overline{P_1}, \overline{Q_3})$, or $(\overline{P_1}, \overline{Q_4})$-path, say, $(\overline{P_1}, \overline{P_3})$-path R, internally disjoint from $C \cup D$ (see Figure 3.18). By Corollaries 3.3.4 and 3.3.6, and Lemma 3.3.7, there is no $(\overline{P_3}, \overline{P_4})$, $(\overline{P_3}, \overline{Q_4})$, $(\overline{P_3}, \overline{Q_3})$, or $(\overline{P_3}, \overline{Q_1})$-path internally disjoint from $C \cup D$. Thus $\overline{P}_1$ and $\overline{Q}_1$ are in different components of $G - W$, a contradiction. This completes the proof of case (c.1.1).

Case (c.1.2): For all $i \in \{1, 2, 3, 4, 5\}$, either $|P_i| = 1$ or $|Q_i| = 1$. As $c_F(G) \geq 7$, there is a P_i, say P_1, such that $|P_1| \geq 2$. By Lemma 3.3.2, $|P_i| = |Q_i| = 1$ for $i = 2, 3, 4, 5$ and $|Q_1| = |P_1| - 1 = 2$. Thus, $c_F(G) = 6$, contradicting the assumption that $c_F(G) \geq 7$.

Case (c.2): Suppose without loss of generality that (P_1, Q_1), (P_3, Q_3), and (P_4, Q_4) are parallel. As noted earlier, $|P_i| \geq 2$ for some $i \in \{1, 2, 3, 4, 5\}$. Thus we have that one of the following cases is true: case (c.2.1): $|P_1| \geq 2$, case (c.2.2): $|P_2| \geq 2$ or $|P_5| \geq 2$, say $|P_2| \geq 2$, or case (c.2.3): $|P_3| \geq 2$ or $|P_4| \geq 2$, say $|P_3| \geq 2$.

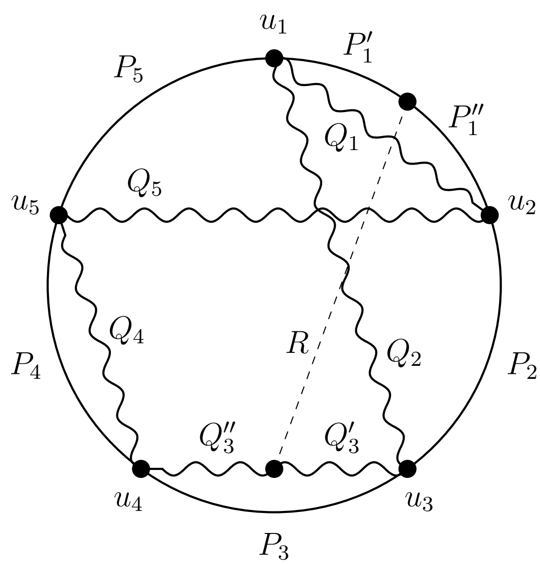


Figure 3.19: The Graph in Case c.2.1

Case (c.2.1): Suppose $|P_1| \geq 2$. As $\overline{P}_1 \neq \emptyset$ and $\overline{Q}_j \neq \emptyset$ for some j, there is a $(\overline{P}_1, X)$-path in $G - W$ internally disjoint from $C \cup D$, where X is some truncated path other than $\overline{P}_1$. Corollaries 3.3.4 and 3.3.6 show that G contains no $[\overline{P}_1, \overline{Q}_1]$, $[\overline{P}_1, \overline{Q}_2]$, $[\overline{P}_1, \overline{Q}_5]$, $[\overline{P}_1, \overline{P}_2]$, $[\overline{P}_1, \overline{P}_5]$-paths internally disjoint from C and D. Now, we show that $\overline{P}_1$ is in its own component of $G - W$ by showing there are no $[\overline{P}_1, \overline{P}_3]$, $[\overline{P}_1, \overline{P}_4]$, $[\overline{P}_1, \overline{Q}_3]$, $[\overline{P}_1, \overline{Q}_4]$-paths as well. By symmetry and for contradiction, suppose there is a $[\overline{P}_1, \overline{Q}_3]$-path, R, internally disjoint from C and D such that R splits P_1 into $P_1' \cup P_1''$ and Q_3 into $Q_3' \cup Q_3''$ (see Figure 3.19). Then we obtain two cycles, C' and D', both containing F, as they contain all vertices of W and contain the only possible common edge in F (when $u_4u_5 = P_4 = Q_4$ is an edge of

F) such that

$$C' = P_1' \cup R \cup Q_3' \cup P_3 \cup P_4 \cup Q_5 \cup Q_1$$

$$D' = P_5 \cup Q_4 \cup Q_3'' \cup R \cup P_1'' \cup P_2 \cup Q_2.$$

Note now that

$$|C'| + |D'| = |C| + |D| + 2|R| \geq 2c_F(G) - 1 + 2|R| = 2c_F(G) + 1 > 2c_F(G)$$

which is a contradiction.

Case (c.2.2): Suppose $|P_2| \geq 2$. By case (c.2.1) we may assume $|P_1| = |Q_1| = 1$. By Corollaries 3.3.4 and 3.3.6, there is no $[\overline{P}_2, \overline{P}_1]$, $[\overline{P}_2, \overline{P}_3]$, $[\overline{P}_2, \overline{Q}_1]$, $[\overline{P}_2, \overline{Q}_2]$, $[\overline{P}_2, \overline{Q}_3]$, or $[\overline{P}_2, \overline{Q}_5]$-path internally disjoint from C and D. Note that there may exist a $[\overline{P}_2, \overline{Q}_4]$, $[\overline{P}_2, \overline{P}_4]$, or $[\overline{P}_2, \overline{P}_5]$-path internally disjoint from C and D. We show in each case however that we reach a contradiction.

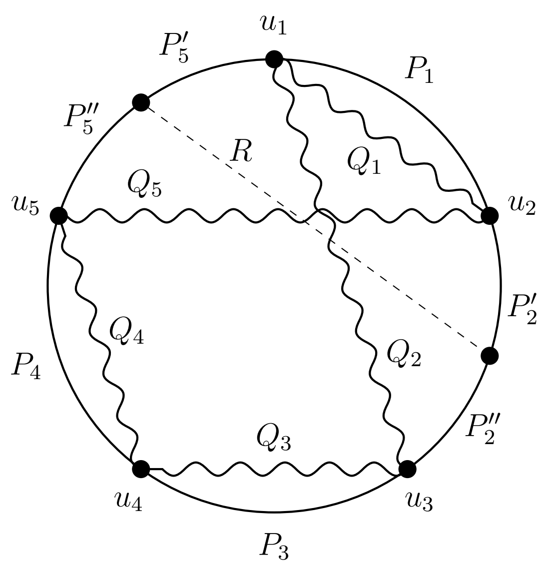


Figure 3.20: The Graph in Case c.2.2 (i)

(i) Suppose there exists a $[\overline{P}_2, \overline{P}_5]$-path, R, internally disjoint from C and D such that R splits P_2 into $P_2' \cup P_2''$ and P_5 into $P_5' \cup P_5''$ (see Figure 3.20). Then we obtain two cycles

$$C' = P_1 \cup P_2' \cup R \cup P_5'' \cup P_4 \cup P_3 \cup Q_2$$
$$D' = Q_1 \cup Q_5 \cup Q_4 \cup Q_3 \cup P_2'' \cup R \cup P_5',$$

such that both C' and D' contain F as both contain all vertices of W. Moreover, if e is an edge of F, then $e = u_1u_2$ (in this case $P_1 = Q_1 = e$), $e = u_3u_4$ (in this case $P_3 = Q_3 = e$), or $e = u_4u_5$ (in this case $e = P_4 = Q_4$). Clearly C' and D' contain all possible edges of F. Moreover,

$$|C'| + |D'| = |C| + |D| + 2|R| \geq 2c_F(G) - 1 + 2|R| \geq 2c_F(G) + 1 > 2c_F(G)$$

which is a contradiction.

Note that $|P_i| \geq |Q_i|$ for $i = 1, 3, 4$ and at most one of the inequalities could occur. If $|P_3| \geq 2$ or $|Q_3| \geq 2$, say $|P_3| \geq 2$, then by the above argument, $\overline{P}_2$ and $\overline{P}_3$ must be in different components of $G - W$. To see this, any $[\overline{P}_2, \overline{P}_3]$-path S in $G - W$ must meet $\overline{P}_4$ or $\overline{Q}_4$, say $\overline{P}_4$. However, there is no $[\overline{P}_3, \overline{P}_4]$-path or $[\overline{P}_3, \overline{Q}_4]$-path internally disjoint from $C \cup D$, and no $[\overline{P}_3, \overline{Q}_3]$-path, and no $[\overline{P}_4, \overline{Q}_4]$, $[\overline{P}_4, \overline{Q}_5]$, $[\overline{P}_4, \overline{P}_5]$, $[\overline{P}_4, \overline{P}_1]$, $[\overline{P}_4, \overline{Q}_1]$-path internally disjoint from $C \cup D$. Therefore, once reaching $\overline{P}_4$, S can only reach $\overline{Q}_2$, then possibly $\overline{Q}_4$, but will never reach $\overline{P}_3$ in $G - W$. This is a contradiction. Thus we may now assume $|P_1| = |Q_1| = |P_3| = |Q_3| = 1$.

(ii) Now suppose there exists a $[\overline{P}_2, \overline{Q}_4]$-path or a $[\overline{P}_2, \overline{P}_4]$-path R, say the former, internally disjoint from C and D such that R splits P_2 into $P_2' \cup P_2''$ and Q_4 into $Q_4' \cup Q_4''$ (see Figure 3.21). The latter case is similar unless $\overline{Q}_4 = \emptyset$. Note that $|P_4| \geq |Q_4| \geq 2$, and as $G - W$ is connected, there must exist a path from $\overline{P}_4$ to another segment. By Corollaries 3.3.4 and 3.3.6, there is no $[\overline{P}_4, \overline{P}_3]$, $[\overline{P}_4, \overline{P}_5]$, $[\overline{P}_4, \overline{Q}_3]$, $[\overline{P}_4, \overline{Q}_4]$, or $[\overline{P}_4, \overline{Q}_5]$-path, and by Lemma 3.3.7, there is no $[\overline{P}_4, \overline{P}_2]$-path. Additionally, by the fact that $|P_1| = |Q_1| = 1$, there

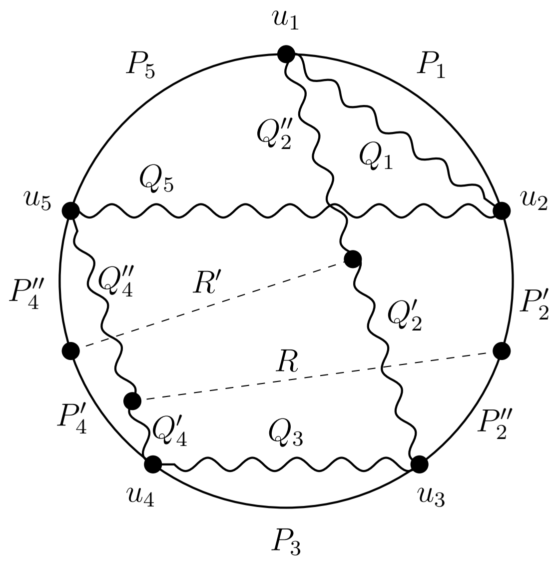


Figure 3.21: The Graph in Case c.2.2 (ii)

is no $[\overline{P}_1, \overline{Q}_1]$-path. Thus there may only be a $[\overline{P}_4, \overline{Q}_2]$-path, R', internally disjoint from C and D splitting P_4 into $P_4' \cup P_4''$, and Q_2 into $Q_2' \cup Q_2''$ (see Figure 3.21). Then we obtain two cycles C' and D' such that

$$C' = P_1 \cup P_2' \cup R \cup Q_4' \cup P_3 \cup Q_2' \cup R' \cup P_4'' \cup P_5$$
$$D' = Q_1 \cup Q_5 \cup Q_4'' \cup R \cup P_2'' \cup Q_3 \cup P_4' \cup R' \cup Q_2''.$$

Now, both C' and D' contain all vertices of W, and contain all possible edges of F (the only possibilities occur when $u_1u_2 = P_1 = Q_1$, and/or $u_3u_4 = P_3 = Q_3$). Moreover,

$$|C'| + |D'| = |C| + |D| + 2|R| + 2|R'| \geq 2c_F(G) - 1 + 4 = 2c_F(G) + 3 > 2c_F(G),$$

which is a contradiction.

In the case that there is a $[\overline{P}_2, \overline{P}_4]$-path but $\overline{Q}_4 = \emptyset$, by Lemma 3.3.2, it must be true that $|P_4| = 2$ and $|Q_4| = 1$ as $|P_4| \leq |Q_4| + 1$. Now $|C| \geq 7$, thus $|Q_2| \geq 2$ or $|Q_5| \geq 2$. If $\overline{Q}_5 = \emptyset$, we show $\overline{P}_2 \cup \overline{P}_4$ and $\overline{Q}_5$ must be in different components of $G - W$. By Lemma

3.3.5, noting that $|P_1| = |Q_1| = |P_3| = |Q_3| = |Q_4| = 1$, any $[\overline{P}_2 \cup \overline{P}_4, \overline{Q}_5]$-path must meet $\overline{P}_5$, which contradicts (c.2.1).

If $\overline{Q}_5 = \emptyset$ but $\overline{Q}_2 \neq \emptyset$, there is no $[\overline{P}_2, \overline{P}_5]$-path by (c.2.1), any $[\overline{P}_2, \overline{Q}_2]$-path must meet $\overline{P}_4$. So there is a $[\overline{P}_2, \overline{P}_4]$-path R which divides P_2 into $P_2' \cup P_2''$, a $[\overline{Q}_2, \overline{P}_4]$-path R' which divides Q_2 into $Q_2' \cup Q_2''$, and P_4 is divided into $P_4' \cup P_4''$ (as $|P_4| = 2$, $|P_4'| = |P_4''| = 1$). Note R and R' only meet at a vertex in $\overline{P}_4$ as there is no $[\overline{P}_2, \overline{Q}_2]$-path internally disjoint from $C \cup D$ (see Figure 3.22).

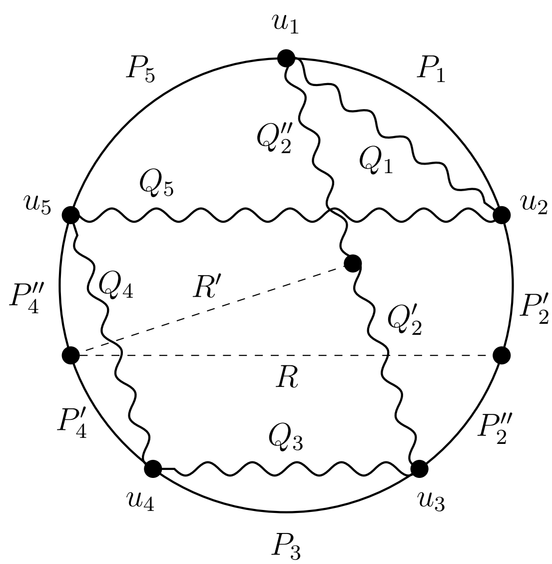


Figure 3.22: The Graph in Case c.2.2 where R and R' meet at a vertex in $\overline{P}_4$

Now, let C_2 and D_2 be two cycles such that

$$C_2 = P_1 \cup P_2' \cup R \cup R' \cup Q_2' \cup P_3 \cup Q_4 \cup P_5$$

$$D_2 = Q_1 \cup Q_5 \cup Q_4 \cup Q_3 \cup P_2'' \cup R \cup R' \cup Q_2''.$$

Then, $V(W) \subseteq V(C_2) \cap V(D_2)$ and C_2 and D_2 contain all possible edges of F (the only possibilities occur when $u_1u_2 = P_1 = Q_1$, and/or $u_3u_4 = P_3 = Q_3$ are edges of F). Moreover, $|C_2| + |D_2| \geq |C| + |D| - 1 + 2|R| + 2|R'| \geq 2c_F(G) - 1 - 1 + 4 = 2c_F(G) + 2 > 2c_F(G)$, a contradiction.

This concludes the case (c.2.2).

Case (c.2.3): Suppose $|P_3| \geq 2$. By the above argument, we may assume that $|P_1| = |Q_1| = |P_2| = 1$. By symmetry, we may also assume $|P_5| = 1$. By Corollaries 3.3.4 and 3.3.6, there is no $[\overline{P}_3, \overline{Q}_2]$, $[\overline{P}_3, \overline{Q}_3]$, $[\overline{P}_3, \overline{Q}_4]$, $[\overline{P}_3, \overline{P}_2]$, or $[\overline{P}_3, \overline{P}_4]$-path internally disjoint from C and D. Additionally, as $|P_1| = |Q_1| = 1$, there can be no $[\overline{P}_3, \overline{Q}_1]$ or $[\overline{P}_3, \overline{P}_1]$-path internally disjoint from C and D. The remaining possibilities for a path are a $[\overline{P}_3, \overline{P}_5]$-path or a $[\overline{P}_3, \overline{Q}_5]$-path internally disjoint from $C \cup D$. As $|P_5| = 1$, there is no such $[\overline{P}_3, \overline{P}_5]$-path. Thus we need only consider the case where there exists a $[\overline{P}_3, \overline{Q}_5]$-path internally disjoint from $C \cup D$ (see Figure 3.23). However, note that this case is equivalent to case (c.2.2) (ii) when $|P_2| \geq 2$. This completes the case for $|P_2| \geq 2$.

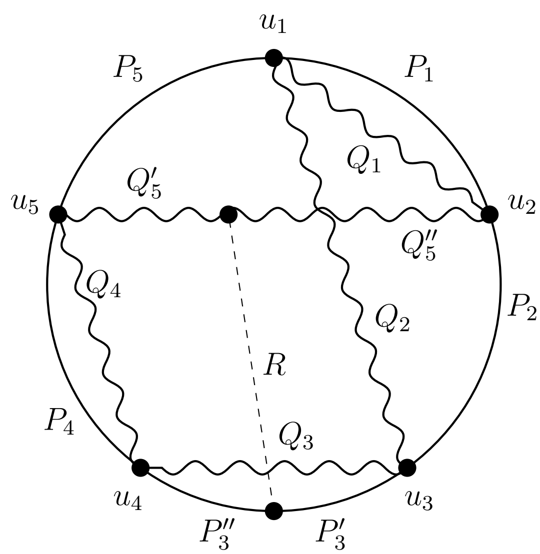


Figure 3.23: The Graph in Case c.2.3

Case (c.3): Suppose there are two parallel segments, say (P_2, Q_2) and (P_5, Q_5). We have three subcases: (c.3.1) $|P_1| \geq 2$, (c.3.2) $|P_2| \geq 2$ or $|P_5| \geq 2$, say $|P_2| \geq 2$, and (c.3.3) $|P_3| \geq 2$ or $|P_4| \geq 2$, say $|P_3| \geq 2$.

Case (c.3.1): Suppose $|P_1| \geq 2$. As $c_F(G) \geq 7$, at least one of $\overline{Q}_i \neq \emptyset$ for some $i \in \{1, 2, 3, 4, 5\}$. As $G - W$ is connected, there is a path from $\overline{P}_1$ to some other truncated path internally disjoint from $C \cup D$. By Corollaries 3.3.4 and 3.3.6, there is no $[\overline{P}_1, \overline{P}_2]$, $[\overline{P}_1, \overline{P}_5]$,

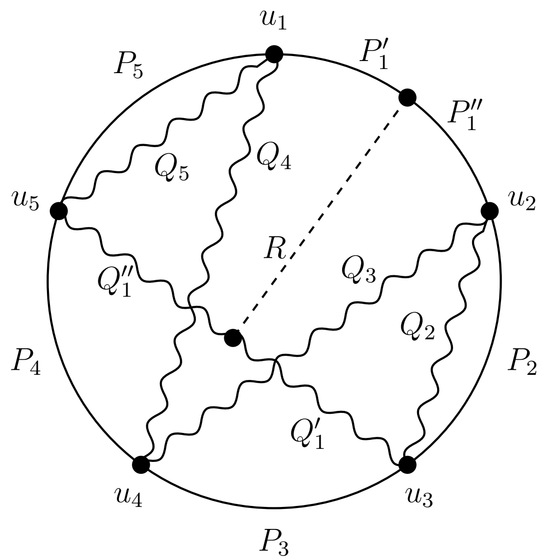


Figure 3.24: The Graph in Case c.3.1 where R is a $[\overline{P}_1, \overline{Q}_1]$-path

$[\overline{P}_1, \overline{Q}_2]$, $[\overline{P}_1, \overline{Q}_3]$, $[\overline{P}_1, \overline{Q}_4]$, or $[\overline{P}_1, \overline{Q}_5]$-path internally disjoint from $C \cup D$. Thus, there may only exist a $[\overline{P}_1, \overline{Q}_1]$, $[\overline{P}_1, \overline{P}_3]$, or $[\overline{P}_1, \overline{P}_4]$-path. By symmetry, assume that there is a $[\overline{P}_1, \overline{Q}_1]$ or $[\overline{P}_1, \overline{P}_3]$-path internally disjoint from $C \cup D$.

If R is a $[\overline{P}_1, \overline{Q}_1]$-path dividing P_1 into $P_1' \cup P_1''$ and Q_1 into $Q_1' \cup Q_1''$ (see Figure 3.24), then we obtain two new cycles C_1 and D_1 such that

$$C_1 = R \cup P_1' \cup P_5 \cup P_4 \cup Q_3 \cup Q_2 \cup Q_1'$$

$$D_1 = R \cup P_1'' \cup P_2 \cup P_3 \cup Q_4 \cup Q_5 \cup Q_1''.$$

Thus, $W \subseteq V(C_1) \cap V(D_1)$, and both contain all possible edges of F (the only possibilities occur when $u_2u_3 = P_2 = Q_2$, and/or $u_5u_1 = P_5 = Q_5$ are edges of F). Moreover, $|C_1|+|D_1| = |C| + |D| + 2|R| \geq 2c_F(G) - 1 + 2 = 2c_F(G) + 1 > 2c_F(G)$, a contradiction.

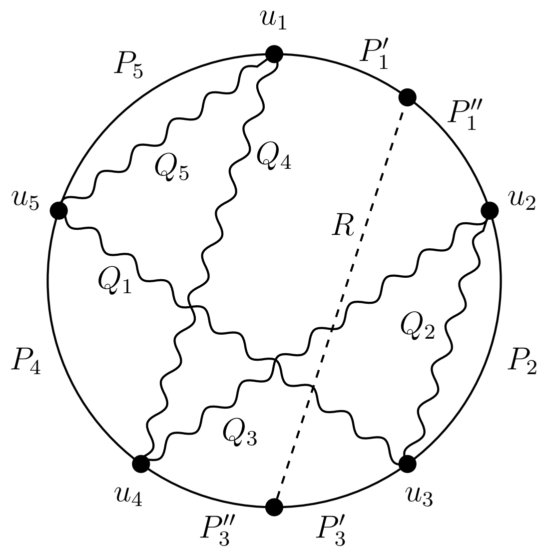


Figure 3.25: The Graph in Case c.3.1 where R is a $[\overline{P}_1, \overline{P}_3]$-path

If R is a $[\overline{P}_1, \overline{P}_3]$-path dividing P_1 into $P_1' \cup P_1''$ and P_3 into $P_3' \cup P_3''$ (see Figure 3.25), then we obtain two new cycles C_2 and D_2 such that

$$C_2 = R \cup P_1' \cup P_5 \cup P_4 \cup Q_3 \cup P_2 \cup P_3'$$

$$D_2 = R \cup P_1'' \cup Q_2 \cup Q_1 \cup Q_5 \cup Q_4 \cup P_3''.$$

Thus, again, $W \subseteq V(C_2) \cap V(D_2)$ and both contain all possible edges of F (the only possibilities occur when $u_2u_3 = P_2 = Q_2$, and/or $u_5u_1 = P_5 = Q_5$ are edges of F). Moreover, $|C_2| + |D_2| = |C| + |D| + 2|R| \geq 2c_F(G) - 1 + 2 = 2c_F(G) + 1 > 2c_F(G)$, a contradiction. This completes case (c.3.1), and we may assume that $|P_1| = 1$ from now on.

Case (c.3.2): Suppose $|P_2| \geq 2$. By Corollaries 3.3.4 and 3.3.6, there is no $[\overline{P}_2, \overline{P}_1]$, $[\overline{P}_2, \overline{P}_3]$, $[\overline{P}_2, \overline{Q}_1]$, $[\overline{P}_2, \overline{Q}_2]$, or $[\overline{P}_2, \overline{Q}_3]$-path internally disjoint from $C \cup D$. As $G - W$ is connected, there exists a path in $G - W$ connecting $\overline{P}_2$ and some other truncated segment which is internally disjoint from $C \cup D$. Note that $\overline{Q}_i \neq \emptyset$ for some $i \in \{1, 2, 3, 4, 5\}$ as $|D| \geq c_F(G) - 1 \geq 6$.

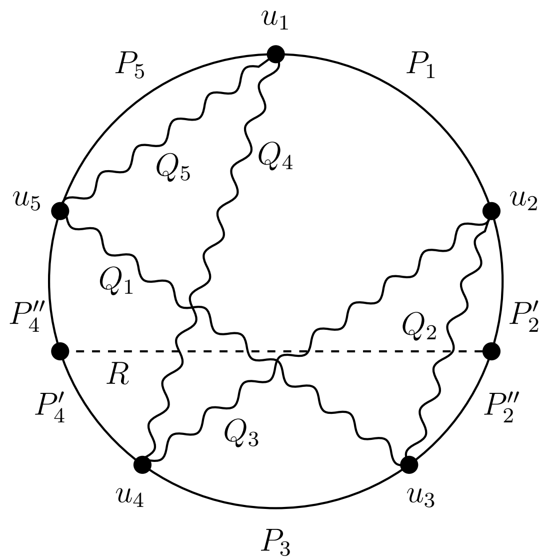


Figure 3.26: The Graph in Case c.3.2 (i)

(i) Assume there is a $[\overline{P}_2, \overline{P}_4]$-path R dividing P_2 into $P_2' \cup P_2''$ and P_4 into $P_4' \cup P_4''$ (see Figure 3.26). Then we obtain two new cycles C_1 and D_1 such that

$$C_1 = R \cup P_4' \cup P_3 \cup Q_1 \cup P_5 \cup P_1 \cup P_2'$$
$$D_1 = R \cup P_4'' \cup Q_5 \cup Q_4 \cup Q_3 \cup Q_2 \cup P_2''.$$

Thus, $W \subseteq V(C_1) \cap V(D_1)$. Additionally, C_1 and D_1 contain P_5 and Q_5 respectively, and thus C_1 and D_1 contain the only possible edge of F (when $u_5u_1 = P_5 = Q_5$ is an edge of F). Moreover, $|C_1| + |D_1| = |C| + |D| + 2|R| \geq 2c_F(G) - 1 + 2 = 2c_F(G) + 1 > 2c_F(G)$, a contradiction. By symmetry, there is no $[\overline{Q}_2, \overline{P}_4]$-path internally disjoint from $C \cup D$.

(ii) Assume there is a $[\overline{P}_2, \overline{Q}_4]$-path R in $G - W$ internally disjoint from $C \cup D$ dividing P_2 into $P_2' \cup P_2''$ and Q_4 into $Q_4' \cup Q_4''$ (see Figure 3.27). Then we obtain two new cycles C_2 and D_2 such that

$$C_2 = R \cup P_2' \cup Q_2 \cup P_3 \cup P_4 \cup P_5 \cup Q_4''$$
$$D_2 = R \cup P_2'' \cup Q_1 \cup Q_5 \cup P_1 \cup Q_3 \cup Q_4'.$$

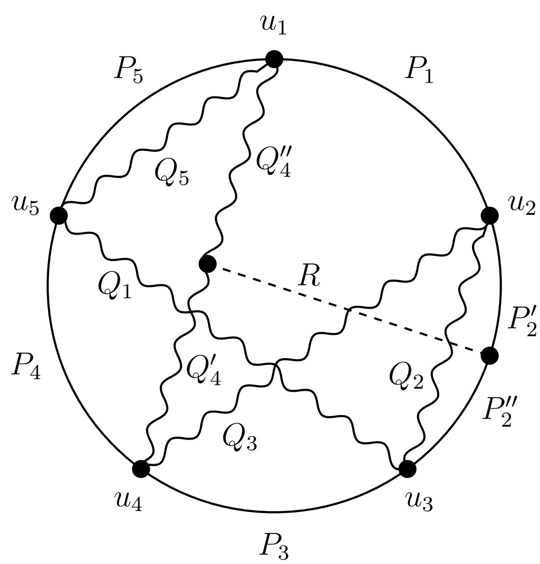


Figure 3.27: The Graph in Case c.3.2 (ii)

Thus, $W \subseteq V(C_2) \cap V(D_2)$, C_2 and D_2 contain P_5 and Q_5 respectively, and thus C_2 and D_2 contain the only possible edge of F (when $u_5u_1 = P_5 = Q_5$ is an edge of F). Moreover, $|C_2| + |D_2| = |C| + |D| + 2|R| \geq 2c_F(G) - 1 + 2 = 2c_F(G) + 1 > 2c_F(G)$, a contradiction. Additionally by symmetry, there is no $[\overline{Q}_2, \overline{Q}_4]$-path internally disjoint from $C \cup D$ in $G - W$.

(iii) Assume there is a $[\overline{P}_2, \overline{P}_5]$ or $[\overline{P}_2, \overline{Q}_5]$-path R in $G - W$ internally disjoint from $C \cup D$, say the former by symmetry. Suppose then that R divides P_2 into $P_2' \cup P_2''$ and P_5 into $P_5' \cup P_5''$ (see Figure 3.28). Then we obtain two new cycles C_3 and D_3 such that

$$C_3 = R \cup P_2'' \cup Q_2 \cup Q_3 \cup P_4 \cup Q_5 \cup P_5''$$

$$D_3 = R \cup P_2' \cup P_1 \cup Q_4 \cup P_3 \cup Q_1 \cup P_5'.$$

Note that in this case, F has no edges, and $V(F) \subseteq W$. Moreover, $W \subseteq V(C_3) \cap V(D_3)$, and $|C_3| + |D_3| = |C| + |D| + 2|R| \geq 2c_F(G) - 1 + 2 = 2c_F(G) + 1 > 2c_F(G)$, a contradiction. This completes case (c.3.2).

Case (c.3.3): Suppose $|P_3| \geq 2$. By (c.3.1) and (c.3.2) we may assume that $|P_1| = |P_2| = |Q_2| = |P_5| = |Q_5| = 1$. As $\overline{Q}_j \neq \emptyset$ for some $j \in \{1, 2, 3, 4, 5\}$, and $G - W$ is connected, there

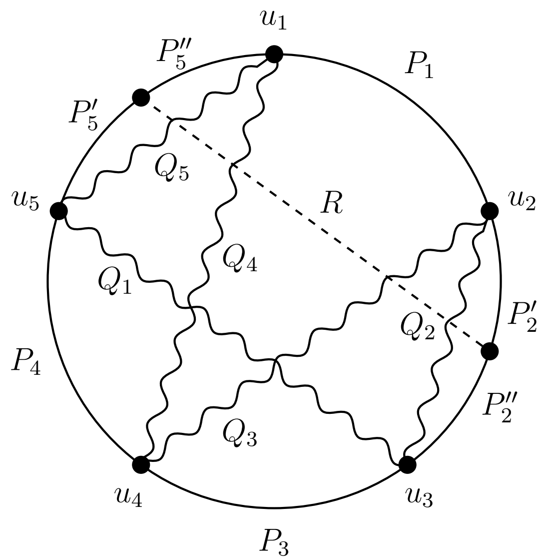


Figure 3.28: The Graph in Case c.3.2 (iii)

exists a path in $G-W$ from $\overline{P}_3$ to some other truncated segment which is internally disjoint from $C \cup D$. By Corollaries 3.3.4 and 3.3.6, the only possible paths are $[\overline{P}_3, \overline{P}_1]$, $[\overline{P}_3, \overline{P}_5]$, and $[\overline{P}_3, \overline{Q}_5]$. However, this is not possible as $|P_1| = |P_5| = |Q_5| = 1$. This completes the proof for the case (c.3).

Case (c.4): No parallel segments exist. Then, F has no edges and $V(F) \subseteq W$. As $c_F(G) \geq 7$, $|P_i| \geq 2$ for some $i \in \{1,2,3,4,5\}$. As $\overline{Q}_j \neq \emptyset$ for some $j \in \{1,2,3,4,5\}$ and $G-W$ is connected, there exists a $[\overline{P}_i, \overline{Q}_j]$-path R for some truncated path $\overline{Q}_j \neq \emptyset$, which is internally disjoint from $C \cup D$ in $G-W$. We may assume that $i = 1$. By Corollary 3.3.4, there is no $[\overline{P}_1, \overline{Q}_j]$-path ($i = 1,3,4,5$) internally disjoint from $C \cup D$. Thus $j = 2$. Suppose then that R divides P_1 into $P'_1 \cup P''_1$ and Q_2 into $Q'_2 \cup Q''_2$ (see Figure 3.29).

Then we obtain two new cycles C_1 and D_1 such that

$$C_1 = R \cup Q'_2 \cup P_3 \cup Q_4 \cup Q_3 \cup P_5 \cup P'_1$$

$$D_1 = R \cup P''_1 \cup P_2 \cup Q_1 \cup Q_5 \cup P_4 \cup Q''_2.$$

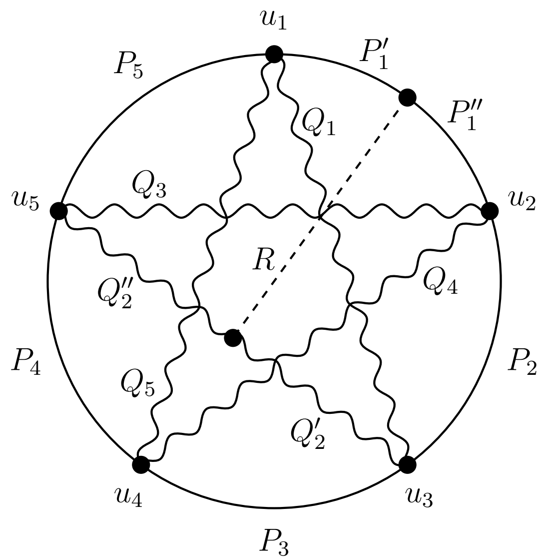


Figure 3.29: The Graph in Case c.4

Thus, $W \subseteq V(C_3) \cap V(D_3)$, and therefore both C_3 and D_3 contain F, and $|C_3| + |D_3| = |C| + |D| + 2|R| \geq 2c_F(G) - 1 + 2 = 2c_F(G) + 1 > 2c_F(G)$, a contradiction. This completes case (c.4), and thus the proof of (iii).

Part (iv): In this case, $F \subseteq \{v\}$. For $k = 3$, the case for $c_F(G) \geq k + 2 = 5$ has been covered in (i) and (ii). If $c_F(G) = 3$, then by Proposition 3.3.1, $si(G) \cong K_3$, and the theorem is true. If $c_F(G) = 4$, then by Proposition 3.3.1 again, $si(G) \cong K_{2,m}, K_{2,m}^{+}(n \geq 2)$, or K_4. It is easily verified that the theorem is true. For $k = 4$, as $|V(G)| \geq k + 1 = 5$, we deduce from Proposition 3.3.1 that if $c_F(G) \leq 4$, then $si(G) \cong K_{2,m}$ or $K_{2,m}^{+}$ for some $m \geq 3$. But then C and D would meet in at most three common vertices. Then $c_F(G) \geq 5$, which has been covered in (i) and (ii). This completes the proof of (iv) and thus the proof of Theorem 3.2.3. □

Proof. (Proof of Theorem 3.2.4) Suppose that G is a k-connected graph for $2 \leq k \leq 6$, and C and D are two cycles containing a linear forest subgraph F with at most $k - 1$ vertices. Thus $|E(F)| \leq k - 2$. By Corollary 1.2.8 and Theorem 1.2.9, $c_F(G) \geq min\{|V(G)|, 2\delta(G) - |E(F)|\}$. Hence, $c_F(G) \geq min\{|V(G)|, k + 2\}$.

Therefore, if $|V(G)| < k+2$, then G has a Hamiltonian cycle containing F, and the theorem is clearly true. So we assume $c_F(G) \geq k+2$.

(i) $k = 2$, $|V(F)| \leq k-1 = 1$. So we may assume $|V(C)| \cap V(D)| \leq 1$. As $|C| + |D| \geq 2c_F(G) - 1$, by Theorem 3.2.1, $|V(C) \cap V(D)| \geq 2$.

(ii) $k = 3$. Then $|V(F)| \leq 2$, and $|E(F)| \leq 1$. Thus either F is a single edge $e = uv$, or $|E(F)| = 0$, and $|V(F)| \leq 2$. If $|V(F)| = 2$, then $|V(C) \cap V(D)| \geq 2$. If $|V(F)| \leq 1$, by Theorem 3.2.1, we also have $|V(C) \cap V(D)| \geq 2$. Thus either $F \subseteq \{u, v\}$ or $F \cong K_2$ with $V(F) = \{u, v\}$. If $G \cong K_4$, it is easily checked that the theorem holds. Thus we assume $|V(G)| \geq 5$, thus $c_F(G) \geq k+2 = 5$. Assume $V(C) \cap V(D) = \{u, v\}$.

If $E(F) = \emptyset$, we deduce that $\{u, v\}$ is an articulation set by Theorem 3.2.2, a contradiction as G is 3-connected. If F is an edge $e = uv$, then as G is 3-connected, $G - \{u, v\}$ is connected, thus there is a $(V(C) - \{u, v\}, V(D) - \{u, v\}$-path R internally disjoint from $C \cup D$. R divides the $[u, v]$-segment of $P = C - e$ into $P = P' \cup P''$, and the $[u, v]$-segment of $Q = D - e$ into $Q = Q' \cup Q''$, as shown in Figure 3.30.

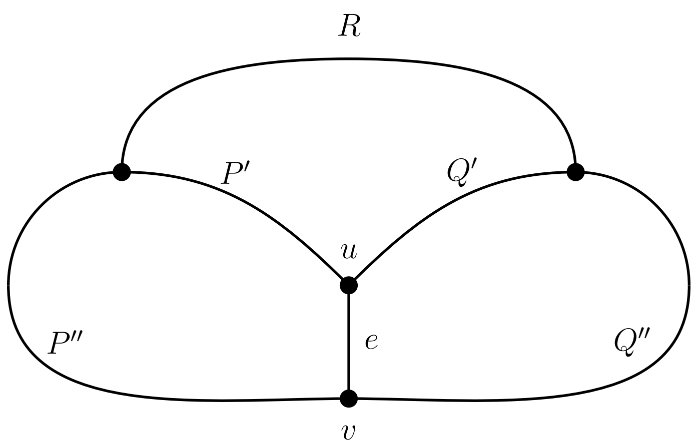


Figure 3.30: Cycles C and D Containing F as a single edge, e

Then, both $C_1 = P' \cup R \cup Q'' \cup uv$ and $D_1 = Q' \cup R \cup P'' \cup uv$ pass through $e = uv$ and $|C_1| + |D_1| = |C| + |D| + 2|R| \geq 2c_F(G) + 1$, a contradiction. Thus $|V(C) \cap V(D)| \geq 3$.

(iii) Now assume $4 \leq k \leq 6$. Suppose that $t = |V(C) \cap V(D)| \leq k-1$. By (ii), $t \geq 3$. Thus $3 \leq t \leq k-1 = 5$. Moreover, $c_F(G) \geq k+2 \geq t+3$. Therefore, by Theorem 3.2.3,

$W = V(C) \cap V(D)$ is an articulation set with exactly t vertices ($t \leq k - 1$). This is a contradiction as G is k-connected. This completes the proof of Theorem 3.2.4 $\square$

4 INTERSECTION OF LONG CYCLES IN k-CONNECTED GRAPHS

4.1 Introduction

Recall that Theorem 2.1.3 states that if G is a k-connected graph, then any two different longest cycles meet in at least $c_0 k^{3/5}$ vertices, where $c_0 = 1/(\sqrt[3]{256} + 3)^{3/5} \approx 0.2615$. In this chapter, we improve the result of Theorem 2.1.3 in two ways. The first is that we do not require the cycles to be longest, and the second is that we improve the constant in the bound.

Also recall that $c(G)$ is the length of a longest cycle in G. Our main results are listed below. In the first, we only require that the sum of the length of our two cycles must have at least $2c(G) - 7$ edges. In the second, we have a stronger restriction in that the sum of the length of our two cycles must be at least $2c(G) - 3$, in exchange for a tighter bound than the first.

Theorem 4.1.1. *Let C and D be cycles of a k-connected graph G where $k \geq 11$, such that $|C| + |D| \geq 2c(G) - 7$. Then $|V(C) \cap V(D)| \geq ck^{3/5}$ where $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx$* 0.2968.

Theorem 4.1.2. *Let C and D be cycles of a k-connected graph ($k \geq 2$) G such that $|C| + |D| \geq 2c(G) - 3$. Then, $|V(C) \cap V(D)| \geq c_1 k^{3/5}$ where $c_1 = 1/(\sqrt[3]{256} + 0.02481)^{3/5} \approx 0.3291$.*

In Section 2, we will prove some lemmas and in Section 3, we prove the main results of this chapter.

4.2 Preliminary Results

Here, we present some lemmas that help prove our main results. We obtain two different lower bounds for the intersection of two long cycles, each corresponding to the minimum length of the sum of our two cycles.

Our first bound is dependent on showing that $k - 9m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$.

Lemma 4.2.1. *If $m < ck^{3/5}$ where $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$, then*

$$k - 9m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m.$$

Proof. Let $m < ck^{3/5}$ and let $c = 1/(\sqrt[3]{256}+t)^{3/5}$, where $t > 0$ is a to be determined constant. Note that

$$m < ck^{3/5} \implies k > (1/c)^{5/3}m^{5/3}$$
$$c = 1/(\sqrt[3]{256} + t)^{3/5} \implies (1/c)^{5/3} = \sqrt[3]{256} + t.$$

Combining the above, we obtain

$$k > (\sqrt[3]{256} + t)m^{5/3} = \sqrt[3]{256}m^{5/3} + tm^{5/3}.$$

Now, we need to show that $k - 9m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$, which is equivalent to

$$k \geq \sqrt[3]{256}(m-2)m^{2/3} + 11m.$$

Thus, it suffices to show that $\sqrt[3]{256}m^{5/3}+tm^{5/3} \geq \sqrt[3]{256}(m-2)m^{2/3}+11m$. This is equivalent to

$$tm^{5/3} \geq \sqrt[3]{256}(m^{5/3} - 2m^{2/3}) + 11m - \sqrt[3]{256}m^{5/3}$$

$$tm^{5/3} \geq \sqrt[3]{256}m^{5/3} - 2\sqrt[3]{256}m^{2/3} + 11m - \sqrt[3]{256}m^{5/3}$$

$$tm^{5/3} \geq -2\sqrt[3]{256}m^{2/3} + 11m$$

$$tm \geq -2\sqrt[3]{256} + 11m^{1/3}$$

$$tm - 11m^{1/3} + 2\sqrt[3]{256} \geq 0.$$

For $x = m^{1/3}$, it then suffices to show that $f(x) = tx^3 - 11x + 2\sqrt[3]{256} \geq 0$.

Note that $f(x) = tx^3 - 11x + 2\sqrt[3]{256}$ $(x \geq 0)$ obtains its absolute minimum at $x = \sqrt{\frac{11}{3t}}$, and $f(\sqrt{\frac{11}{3t}}) = t(\frac{11}{3t})^{3/2} - 11(\sqrt{\frac{11}{3t}}) + 2\sqrt[3]{256} = \sqrt{11^3}/(3\sqrt{3}\sqrt{t}) - (11\sqrt{11})/(\sqrt{3}\sqrt{t}) + 2\sqrt[3]{256}$. We need $t > 0$ such that $f(\sqrt{\frac{11}{3t}}) > 0$, which gives us $\sqrt{11^3}/(3\sqrt{3}\sqrt{t}) - 3(11\sqrt{11})/(3\sqrt{3}\sqrt{t}) + 2\sqrt[3]{256} > 0$. Thus $\sqrt{t} > (11\sqrt{11})/(3\sqrt{3}\sqrt[3]{256})$ which gives us that $t > (11\sqrt{11}/(3\sqrt{3}\sqrt[3]{256}))^2$, thus $t > 1.222703009$. Choosing $t = 1.22271 > ((11\sqrt{11})/(3\sqrt{3}\sqrt[3]{256}))^2$ gives us $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5}$ as a constant such that $f(x) > 0$ for all $x \geq 0$. Thus $f(m^{\frac{1}{3}}) > 0$, which implies $k - 9m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$. □

We now give a similar lemma for our second bound, which is dependent here on showing that $k - m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$.

Lemma 4.2.2. *If $m < c_1 k^{3/5}$ where $c_1 = 1/(\sqrt[3]{256} + 0.02481)^{3/5} \approx 0.3291$, then*

$$k - m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m.$$

Proof. Let $m < c_1 k^{3/5}$ and let $c_1 = 1/(\sqrt[3]{256} + t)^{3/5}$, where $t > 0$ is a to be determined constant. Note that

$$m < c_1 k^{3/5} \implies k > (1/c_1)^{5/3} m^{5/3}$$
$$c_1 = 1/(\sqrt[3]{256} + t)^{3/5} \implies (1/c_1)^{5/3} = \sqrt[3]{256} + t.$$

Combining the above, we obtain

$$k > (\sqrt[3]{256} + t)m^{5/3} = \sqrt[3]{256}m^{5/3} + tm^{5/3}.$$

Now, we need to show that $k - m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$, which is equivalent to

$$k \geq \sqrt[3]{256}(m-2)m^{2/3} + 3m.$$

Thus, it suffices to show that $\sqrt[3]{256}m^{5/3} + tm^{5/3} \geq \sqrt[3]{256}(m-2)m^{2/3} + 3m$. This is equivalent to

$$tm^{5/3} \geq \sqrt[3]{256}(m^{5/3} - 2m^{2/3}) + 3m - \sqrt[3]{256}m^{5/3}$$
$$tm^{5/3} \geq \sqrt[3]{256}m^{5/3} - 2\sqrt[3]{256}m^{2/3} + 3m - \sqrt[3]{256}m^{5/3}$$
$$tm^{5/3} \geq -2\sqrt[3]{256}m^{2/3} + 3m$$
$$tm \geq -2\sqrt[3]{256} + 3m^{1/3}$$
$$tm - 3m^{1/3} + 2\sqrt[3]{256} \geq 0.$$

For $x = m^{1/3}$, it then suffices to show that $f(x) = tx^3 - 3x + 2\sqrt[3]{256} \geq 0$.

Note that $f(x) = tx^3 - 3x + 2\sqrt[3]{256}$ $(x \geq 0)$ obtains its absolute minimum at $x = \sqrt{1/t}$, and $f(\sqrt{1/t}) = t(1/t)^{3/2} - 3(1/\sqrt{t}) + 2\sqrt[3]{256} = 1/\sqrt{t} - 3/\sqrt{t} + 2\sqrt[3]{256} = -2/\sqrt{t} + 2\sqrt[3]{256}$. We need $t > 0$ such that $f(\sqrt{1/t}) > 0$, which gives us $-2/\sqrt{t} + 2\sqrt[3]{256} > 0$. Thus $2\sqrt[3]{256} > 2/\sqrt{t}$. Thus $\sqrt{t} > 1/\sqrt[3]{256}$. Thus $t > (1/\sqrt[3]{256})^2$, and thus $t > 0.024803141$. Choosing t then to be

$t = 0.02481 > 1/\sqrt[3]{256}$ gives us $c_1 = 1/(\sqrt[3]{256} + 0.02481)^{3/5}$ as a constant such that $f(x) > 0$ for all $x \geq 0$. Thus $f(m^{\frac{1}{3}}) > 0$, which implies $k - m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$. □

Our next pair of lemmas are elementary.

Lemma 4.2.3. *If $k \geq 7$ and $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$, then $k - 6 \geq ck^{3/5}$.*

Proof. Let $f(k) = k - 6 - ck^{\frac{3}{5}}$. Then $f'(k) = 1 - \frac{3c}{5}k^{-\frac{2}{5}} > 0$ for all $k \geq 1$. Thus $f(k)$ is increasing for all $k \geq 1$. In particular, for $k \geq 7$, $f(k) \geq f(7) = 7 - 6 - c \cdot 7^{\frac{3}{5}} > 0$. □

Lemma 4.2.4. *If $k \geq 3$ and $c_1 = 1/(\sqrt[3]{256} + 0.02481)^{3/5} \approx 0.3291$, then $k - 2 \geq c_1 k^{3/5}$.*

Proof. Let $f(k) = k - 2 - c_1 k^{\frac{3}{5}}$. Then $f'(k) = 1 - \frac{3c_1}{5}k^{-\frac{2}{5}} > 0$ for all $k \geq 1$. Thus $f(k)$ is increasing for all $k \geq 1$. In particular, if $k \geq 3$, $f(k) \geq f(3) = 3 - 2 - c_1 \cdot 3^{\frac{3}{5}} > 0$. □

4.3 Proof of the Main Results

We will prove our main theorems in this section. The idea in the proofs for Theorems 4.1.1 and 4.1.2 is to first let $|V(C) \cap V(D)| = m$ for m less than our bound for contradiction. We show that if there is no Hamiltonian cycle, then by Theorem 1.2.4, we obtain a cycle of length at least $2k$. Next, we assume $k \geq 11$ in Theorem 4.1.1 and explore small k in Theorem 4.1.2, and use this to guarantee at least $k - m$ vertices in the segments of $C - T$ and $D - T$. Then, as G is k-connected and $|T| = m$, we can show $G - T$ is $(k-m)$-connected, and obtain $k - m$ independent internally disjoint paths between the segments of $C - T$ and $D - T$. Constructing an auxilary subgraph H as described later, H has at least $k - m$ edges. Using Claims 4.3.1, 4.3.2, and 4.3.3 below, we obtain a minimum number of edges for $si(H)$. Then we show that $|E(si(H))| \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$. Now, by Corollary 1.3.4, $si(H)$ has a $K_{3,257}$-subgraph. From here, we are able to show that our graph contains two new cycles, whose combined length leads to a contradiction, concluding our proofs.

Proof. (Proof of Theorem 4.1.1) Let G be a k-connected graph, $k \geq 11$, and let C and D be two cycles such that $|C| + |D| \geq 2c(G) - 7$ and $|V(C) \cap V(D)| = m$, for $m < ck^{3/5}$, and $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$.

By Lemma 1.2.4, either G has a Hamiltonian cycle, or G has a cycle of length $\geq 2k$. If G has a Hamiltonian cycle, then $|V(C) \cap V(D)| = |C| + |D| - |V(C) \cup V(D)| \geq 2c(G) - 7 - n = n - 7 \geq k - 6 > ck^{3/5}$ by Lemma 4.2.3. So suppose G does not have a Hamiltonian cycle. Then, $c(G) \geq 2k$. Note then that as $|C| + |D| \geq 2c(G) - 7 = c(G) + (c(G) - 7)$, we have that $|V(C)| \geq c(G) - 7$. Now, as $c(G) \geq 2k$, we have that $|V(C)| \geq 2k - 7$, and similarly, $|V(D)| \geq 2k - 7$. Now, $|\bigcup_{i=1}^{m} P_i| \geq c(G) - 7 - m \geq 2k - 7 - m = k - m + k - 7 > k - m$, as $k \geq 11$. Similarly, $|\bigcup_{j=1}^{m} Q_j| > k - m$.

As G is k-connected, note that $G - T$ is $(k - m)$-connected and $G - T$ has at least $k - m$ vertex-disjoint paths (denote this set of paths by $\mathcal{P}$ from $\bigcup_{i=1}^{m} P_i$ to $\bigcup_{j=1}^{m} Q_j$. Now, construct an auxiliary bipartite graph H as follows, and note that $|E(H)| \geq k - m$. H is an auxiliary graph of G such that for the segments $X_1, X_2, ..., X_m$ of $C - T$ and $Y_1, Y_2, ..., Y_m$ of $D - T$, $V(H) = \{X_1, X_2, ..., X_m, Y_1, Y_2, ..., Y_m\}$, and for every path $R_{i,j}$ in $\mathcal{P}$ from X_i to Y_j in G, noting that different $R_{i,j}$ are vertex disjoint paths, we say there exists an edge joining X_i and Y_j in H. Following [1], we orient the cycles C and D as follows. Beginning with vertex v in C, $\forall x_1, x_2 \in V(C), x_1 \prec x_2$ if $x_1 \in P[v, x_2]$. Additionally, if S_1 and S_2 are disjoint segments of $C - \{v\}$, then we define $S_1 \prec S_2$ if S_1 lies between v and S_2 along C. For D, one can make a similar definition.

The auxiliary graph H may not be simple. In the next three claims, we study H and obtain a lower bound on the number of edges in the simplified graph of H, denoted by $si(H)$.

Claim 4.3.1. *Each edge of H has multiplicity of size at most* 9.

Proof. Suppose for contradiction without loss of generality that the edge P_1Q_1 has a multiplicity of size at least 10. Then in G, there exists at least 10 paths, $R_i[x_i, y_{n_i}]$ for $i \geq 10$, with endpoints x_i and y_{n_i} in the segments P_1 and Q_1 respectively. Relabel the endpoints of these paths in P_1 if necessary, and assume that $x_1 \prec x_2 \prec ... \prec x_{10}$.

By Corollary 1.3.1, As $10 = 3^2 + 1$, the sequence $\{n_i\}(1 \leq i \leq 10)$ contain a monotone subsequence of length at least $3 + 1 = 4$ with respect to their index along the orientation of

Q_1. Call these endpoints $y_{a_1}, y_{b_1}, y_{c_1}, y_{d_1}$ and relabel their corresponding endpoints in P_1 as x_a, x_b, x_c, x_d. Relabel the paths between them R_a, R_b, R_c, R_d such that they have endpoints x_a and y_{a_1}, x_b and y_{b_1}, x_c and y_{c_1}, and x_d and y_{d_1} respectively. Reorienting D if necessary, we may assume that $y_{a_1} \prec y_{b_1} \prec y_{c_1} \prec y_{d_1}$. Then we have a new cycle C' in G, by replacing the $P_1[x_a, x_d]$-segment in P_1 by a new segment $R_a[x_a, y_{a_1}]Q_1[y_{a_1}, y_{b_1}]\overleftarrow{R}_b[y_{b_1}, x_b]P_1[x_b, x_c]R_c[x_c, y_{c_1}]$ $Q_1[y_{c_1}, y_{d_1}]\overleftarrow{R}_d[y_{d_1}, x_d]$. Similarly, we obtain a new cycle D' by replacing the $Q_1[y_{a_1}, y_{d_1}]$-segment in Q_1 by $\overleftarrow{R}_a[y_{a_1}, x_a]P_1[x_a, x_b]R_b[x_b, y_{b_1}]Q_1[y_{b_1}, y_{c_1}]\overleftarrow{R}_c[y_{c_1}, x_c]P_1[x_c, x_d]R_d[x_d, y_{d_1}]$.

However, $|C'|+|D'| \geq |C|+|D|+2(|R_a[x_a, y_{a_1}]|+|R_b[x_b, y_{b_1}]|+|R_c[x_c, y_{c_1}]|+|R_d[x_d, y_{d_1}]|) \geq |C| + |D| + 8 \geq 2c_F(G) - 7 + 8 > 2c_F(G)$, a contradiction.

□

Note that the above proof can be easily modified to show that if $m = 0$, then we can find two new cycles C' and D' such that $|C'| + |D'| \geq |C| + |D| + 8 \geq 2c_F(G) - 7 + 8 > 2c_F(G)$; a contradiction. This $m \geq 1$.

Claim 4.3.2. *If two edges of H, P_iQ_j and P_sQ_t, both have multiplicity more than one, then P_iQ_j and P_sQ_t must be adjacent edges.*

Proof. Suppose $R_1[x_1, y_1]$ and $R_2[x_2, y_2]$ are two vertex-disjoint paths between P_i and Q_j where $\{x_1, x_2\} \subseteq V(P_i)$ and $\{y_1, y_2\} \subseteq V(Q_j)$, and $S_1[x_3, y_3]$ and $S_2[x_4, y_4]$ are two vertex-disjoint paths between P_s and Q_t where $\{x_3, x_4\} \subseteq V(P_s)$, and $\{y_3, y_4\} \subseteq V(Q_t)$. Moreover, R_1, R_2 and S_1, S_2 are vertex-disjoint. We construct two new cycles C' and D' as follows: C' is obtained from C by replacing two segments $C[x_1, x_2]$ by $R_1[x_1, y_1]D[y_1, y_2]\overleftarrow{R}_2[y_2, x_2]$, and $C[x_3, x_4]$ by $S_1[x_3, y_3]D[y_3, y_4]\overleftarrow{S}_2[y_4, x_4]$. Similarly, we construct D' from D by replacing two segments $D[y_1, y_2]$ by $\overleftarrow{R}_1[y_1, x_1]C[x_1, x_2]R_2[x_2, y_2]$, and $D[y_3, y_4]$ by $\overleftarrow{S}_1[y_3, x_3]C[x_3, x_4]S_2[x_4, y_4]$. Then C' and D' are such that

$$|C'| + |D'| = |C| + |D| + 2(|R_1| + |R_2| + |S_1| + |S_2|) \geq |C| + |D| + 8 \geq 2c(G) + 1;$$

a contradiction. □

Claim 4.3.3. *The simplified graph of H, $si(H)$, has at least $k - 9m$ edges.*

Proof. Let $|E(H)| \geq k - m$. To find $|E(si(H))|$, we must subtract the number of possible multiple edges in H from $k - m$. By Claim 4.3.2, any edge of H with multiplicity more than one must be covered by a single vertex as any two such edges are adjacent. Now, the vertex covering all edges that could be multiple can be incident to at most m edges as $H \subseteq K_{m,m}$. By Claim 4.3.1, each of these m edges has multiplicity of at most nine. So, to subtract the number of multiple edges in H, we must subtract at most eight edges for each of the m possible edges. Thus, $|E(si(H))| \geq (k - m) - 8m = k - 9m$. □

If $m = 1$, then $si(H)$ has at most one edge. However, from the last claim, we deduce that $|E(si(H))| \geq k - 9m \geq 11 - 9 = 2$, a contradiction. If $m = 2$, then $si(H))$ has at most four edges as it is a subgraph is $K_{2,2}$. However, as $m = 2 \leq ck^{3/5}$, we deduce that $k \geq 24$. From the last claim, we deduce that $|E(si(H))| \geq k - 9m \geq 24 - 18 = 6$, a contradiction again. Therefore, $m \geq 3$.

By Lemma 4.2.1, if $m < ck^{3/5}$ where $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$, then $k - 9m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$. Now, as $si(H) \subseteq K_{m,m}$, we have from Corollary 1.3.4 that $si(H)$ contains a $K_{3,257}$-subgraph.

When C and D are longest cycles of G, Chen, Faudree, and Gould [1, page 14, line -4] showed that if H has a $K_{3,257}$-subgraph, then there are two new cycles C' and D' such that either C' or D' is longer than $|C| = |D|$; thus reaching a contradiction. We observe that, in the following claim, our cycles C and D are not necessarily longest cycles of G, and we can still get the following result which follows essentially by the proof in [1, page 14, line -4].

Claim 4.3.4. *[1, page 14, line -4] If H contains a $K_{3,257}$-subgraph with vertex set $\{X_1, X_2, X_3, Y_1, Y_2, ..., Y_{257}\}$, then G contains two new cycles C' and D' such that $|C'| + |D'| \geq |C| + |D| + 8$.*

We outline the proof of [1] for this claim here for completion. Relabelling as necessary, assume that $X_1, X_2, X_3, Y_1, ..., Y_{257}$ induce a $K_{3,257}$-subgraph and let $R_{i,j}[u_{i,j}, v_{i,j}]$ denote the path from X_i to Y_j for $1 \leq i \leq 3$ and $1 \leq j \leq 257$. Recall that we oriented the cycles C and D.

In [1], the authors prove that, Using Lemma 1.3.2, reversing the orientation and relabeling X_i and Y_j if necessary, we can assume that $u_{1,1} \prec u_{1,2} \prec u_{1,3}$ and $u_{2,1} \prec u_{2,2} \prec u_{2,3}$, and either $u_{3,1} \prec u_{3,2} \prec u_{3,3}$ or $u_{3,3} \prec u_{3,2} \prec u_{3,1}$.

Then the authors show that there are i, j, where $1 \leq i < j \leq 3$, such that either

- (i) $v_{1,i} \prec v_{2,i}$ and $v_{1,j} \prec v_{2,j}$, or

- (ii) $v_{2,i} \prec v_{1,i}$ and $v_{2,j} \prec v_{1,j}$.

Indeed, assume that $v_{1,1} \prec v_{2,1}$. If $v_{1,2} \prec v_{2,2}$, then choose $(i, j) = (1, 2)$, and if $v_{1,3} \prec v_{2,3}$, then choose $(i, j) = (1, 3)$. Otherwise, If $v_{2,2} \prec v_{1,2}$, and $v_{2,3} \prec v_{1,3}$, then choose $(i, j) = (2, 3)$. The case for $v_{2,1} \prec v_{1,1}$ is similar.

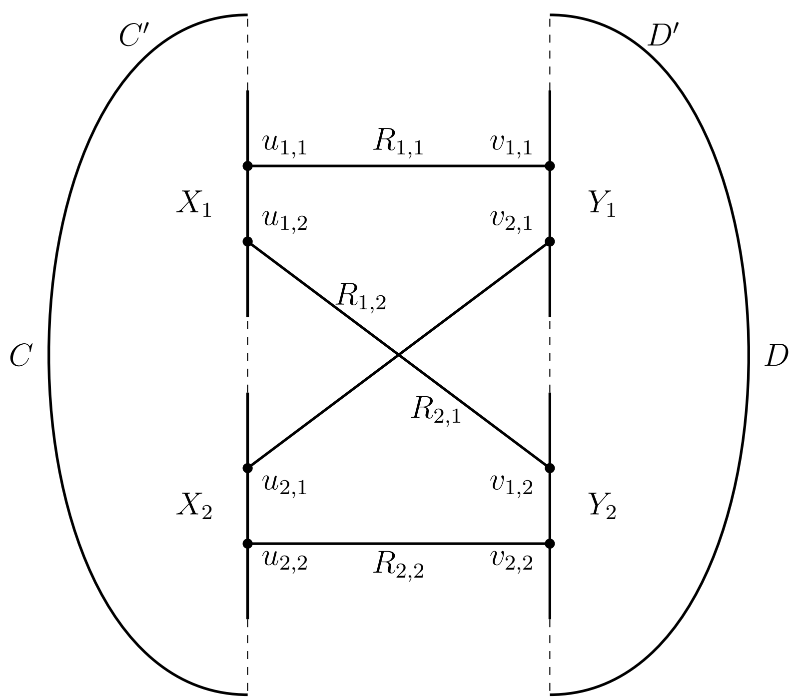


Figure 4.1: The new cycles C' and D' in G

Reorient D if necessary one can assume that $v_{1,1} \prec v_{2,1}$. Now in each case one can reroute and obtain two new cycles C' and D' such that $|C'| + |D'| \geq |C| + |D| + 8$. All cases are similar and without loss of generality, we will only describe the case when $i = 1$ and $j = 2$ and $v_{1,1} \prec v_{2,1}$ and $v_{1,2} \prec v_{2,2}$. Then one can get two new cycles as follows:

$$C' = u_{1,1}R_{1,1}v_{1,1}Dv_{2,1}\overleftarrow{R}_{2,1}u_{2,1}\overleftarrow{C}u_{1,2}R_{1,2}v_{1,2}Dv_{2,2}\overleftarrow{R}_{2,2}u_{2,2}Cu_{1,1}$$
$$D' = v_{1,1}\overleftarrow{R}_{1,1}u_{1,1}Cu_{1,2}R_{1,2}v_{1,2}\overleftarrow{D}v_{2,1}\overleftarrow{R}_{2,1}u_{2,1}Cu_{2,2}R_{2,2}v_{2,2}Dv_{1,1}$$

Now $|C'| + |D'| = |C| + |D| + 2(|R_{1,1}| + |R_{1,2}| + |R_{2,1}| + |R_{2,2}|) \geq |C| + |D| + 8$.

Thus, by Claim 4.3.4, since $si(H)$ has a $K_{3,257}$-subgraph, there exists two cycles C' and D' such that $|C'| + |D'| \geq |C| + |D| + 8 \geq 2c(G) - 7 + 8 > 2c(G)$, a contradiction. Thus $|V(C) \cap V(D)| \geq ck^{3/5}$. □

Proof. (Theorem 4.1.2) Let G be a k-connected graph, $k \geq 2$, and let C and D be two cycles such that $|C| + |D| \geq 2c(G) - 3$ and $|V(C) \cap V(D)| = m$, for $m < c_1k^{3/5}$, and $c_1 = 1/(\sqrt[3]{256} + 0.02481)^{3/5} \approx 0.3291$. Again we orient the cycles C and D. Beginning with vertex v, $\forall x_1, x_2 \in V(C), x_1 \prec x_2$ if $x_1 \in P[v, x_2]$.

We now show that the theorem holds for $k = 2$. Suppose $|V(C) \cap V(D)| = 0$. As G is 2-connected, there exist two internally disjoint paths connecting C and D, say $R_1[x_1, y_1]$ and $R_2[x_2, y_2]$, where $x_1, x_2 \in C$ and $y_1, y_2 \in D$. Reorient C and D if necessary, we assume that $x_1 \prec x_2$ and $y_1 \prec y_2$. Then, construct two new cycles

$$C' = R_1[x_1, y_1]D[y_1, y_2]\overleftarrow{R}_2[y_2, x_2]C[x_2, x_1]$$
$$D' = \overleftarrow{R}_1[y_1, x_1]C[x_1, x_2]R_2[x_2, y_2]D[y_2, y_1].$$

Now note that $|C'|+|D'| = |C|+|D|+2|R_1|+2|R_2| \geq |C|+|D|+4 \geq 2c(G)-3+4 > 2c(G)$, a contradiction. Thus the intersection of C and D must be nonempty, and we are done, as we assume $m < c_1k^{3/5}$, which for $k = 2$ gives $c_1k^{3/5} = c_1(2)^{3/5} \approx 0.4988 < 1$. For the rest of the proof we may now assume that $k \geq 3$. If G has a Hamiltonian cycle, then $|V(C) \cap V(D)| = |C| + |D| - |V(C) \cup V(D)| \geq 2c(G) - 3 - n = n - 3 \geq k - 2 \geq ck^{3/5}$ by

Lemma 4.2.4. So suppose not. Then, by Lemma 1.2.4, G has a cycle of length at least $2k$ which implies $c(G) \geq 2k$. Note then that as $|C| + |D| \geq 2c(G) - 3 = c(G) + (c(G) - 3)$, we have that $|V(C)| \geq c(G) - 3$. Now, as $c(G) \geq 2k$, we have that $|V(C)| \geq 2k - 3$, and $|V(D)| \geq 2k - 3$. Thus, $|\bigcup_{i=1}^{m} P_i| \geq c(G) - 3 - m \geq 2k - 3 - m = k - m + k - 3 \geq k - m$, as $k \geq 3$. Similarly, $|\bigcup_{j=1}^{m} Q_j| \geq k - m$.

As G is k-connected, note that $G - T$ is $(k - m)$-connected and $G - T$ has at least $k - m$ independent paths from $\bigcup_{i=1}^{m} P_i$ to $\bigcup_{j=1}^{m} Q_j$. Now, construct an auxiliary bipartite graph H as described above, and note that $|E(H)| \geq k - m$. Suppose that H could have multiple edges, say $P_1 = P_1[u_1, v_1]$ and $P_2 = P_2[u_2, v_2]$, such that u_1 and u_2 are in X_1, and v_1 and v_2 are in Y_1. Reorient C and D if necessary, assume that $u_1 \prec u_2$ and $v_1 \prec v_2$.

Furthermore, suppose $C[u_1, u_2] \subseteq X_1$ and $D[v_1, v_2] \subseteq Y_1$. Then G would contain two cycles

$$C^* = C[u_2, u_1]P_1[u_1, v_1]D[v_1, v_2]\overleftarrow{P}_2[v_2, u_2]$$
$$D^* = D[v_2, v_1]\overleftarrow{P}_1[v_1, u_1]C[u_1, u_2]P_2[u_2, v_2],$$

however $|C^*| + |D^*| = |C| + |D| + 2|P_1| + 2|P_2| \geq 2c(G) - 3 + 4 > 2c(G)$, a contradiction. Thus, H has no multiple edges. Thus we have that H is a simple bipartite graph such that $|E(H)| \geq k - m$.

Modifying the proof in the last paragraph, one can show that $m > 0$. If $m = 1$, then H has at most one edge, but as $m < c_1 k^{3/5}$, we deduce that $k > 6$ and thus H has at least $k - m > 5$ edges; a contradiction. If $m = 2$, then from $m < c_1 k^{3/5}$, we deduce that $k > 20$, thus H should have at least $k - m \geq 18$ edges; a contradiction as H is a subgraph of $K_{2,2}$. Thus $m \geq 3$.

With our chosen bound of $c_1 k^{3/5}$, Lemma 4.2.2 gives us that $|E(H)| \geq k - m \geq \sqrt[3]{256}(m - 2)m^{2/3} + 2m$. Now, as $H \subseteq K_{m,m}$, we have from Corollary 1.3.4 that H contains a $K_{3,257}$-subgraph.

Then by Claim 4.3.4, since H has a $K_{3,257}$-subgraph, there exists two cycles C' and D' such that $|C'| + |D'| \geq |C| + |D| + 8 \geq 2c(G) - 3 + 8 > 2c(G)$, a contradiction. $\square$

5 INTERSECTION OF LONG CYCLES CONTAINING A LINEAR FOREST SUBGRAPH IN k-CONNECTED GRAPHS

5.1 Introduction and Main Results

In Chapter 2, we raised the following general conjecture for k-connected graphs ($k \geq 2$): Let C and D be two longest cycles in a k-connected graph containing a linear forest subgraph F with at most $k-1$ vertices, then C and D must meet in at least k common vertices. In Chapter 3, we prove this conjecture for $2 \leq k \leq 6$ (in fact, we prove a slightly stronger result by allowing $|C| + |D| \geq 2c_F(G) - 1$). In Chapter 4, we extend the result of Chen, Faudree, and Gould [1] who gave a lower bound on the intersection of two longest cycles for k-connected graphs.

In this chapter, we give a lower bound on the number of common vertices between two long cycles containing a linear forest. This linear forest could be empty. Our main results give a generalization of Chen et al.'s result [1]. Our main results also give generalizations of the main results in the last chapter.

Recall that a graph F is called a linear forest if $V(F) = E(F) = \emptyset$ or every component of F is a path [13]. Again, we use $c_F(G)$ as the length of a longest cycle of G containing F. The following are the main results of this chapter.

Theorem 5.1.1. *Let C and D be two cycles containing a specified linear forest subgraph F with at most $k-1$ vertices of a k-connected graph G, such that $|C| + |D| \geq 2c_F(G) - 7$. In addition, we assume $k \geq 11$ if $|V(F)| \leq 1$. Then $|V(C) \cap V(D)| \geq ck^{3/5}$, where $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$.*

Theorem 5.1.2. *Let C and D be two cycles containing a specified linear forest subgraph F with at most $k-1$ vertices of a k-connected graph G ($k \geq 2$) such that $|C|+|D| \geq 2c_F(G)-3$. Then $|V(C) \cap V(D)| \geq c_1 k^{3/5}$, where $c_1 = 1/(\sqrt[3]{256} + 0.02481)^{3/5} \approx 0.3291$.*

Note that if F is the empty set, then the above results imply Theorems 4.1.1 and 4.1.2, respectively. Additionally, if F is a single vertex, then the above results imply the following, considering two cycles containing a specified vertex, where $c_v(G)$ is the length of a longest cycle containing the vertex v.

Corollary 5.1.3. *Let C and D be cycles containing a specified vertex v of a k-connected graph G, $k \geq 11$, such that $|C| + |D| \geq 2c_v(G) - 7$. Then, $|V(C) \cap V(D)| \geq ck^{3/5}$, where $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$.*

Corollary 5.1.4. *Let C and D be cycles containing a specified vertex v of a k-connected graph G, such that $|C| + |D| \geq 2c_v(G) - 3$. Then, $|V(C) \cap V(D)| \geq c_1 k^{3/5}$, where $c_1 = 1/(\sqrt[3]{256} + 0.02481)^{3/5} \approx 0.3291$.*

All the proofs of the above results will be given in the next section.

5.2 Proof of the Main Results

Proof. (Proof of Theorem 5.1.1) Let G be a k-connected graph and let C and D be two cycles containing a linear forest subgraph F with at most $k-1$ vertices such that $|C|+|D| \geq 2c_F(G) - 7$. If $|(F)| \leq 1$, we also suppose $k \geq 11$. Let $|V(C) \cap V(D)| = m$ for $m < ck^{3/5}$ and $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$.

If $n = |V(G)| \leq 2\delta - l$, by Corollary 1.2.8 and Dirac's Theorem, G has a Hamiltonian cycle containing F. Then $|V(C) \cap V(D)| = |V(C)| + |V(D)| - |V(C) \cup V(D)| \geq 2c_F(G) - 7 - n = 2n - 7 - n = n - 7 \geq k + 1 - 7 = k - 6$. By Lemma 4.2.3, $k - 6 \geq ck^{3/5}$, and we are done. Thus we may assume $n > 2\delta - l$, and assume that G does not have a Hamiltonian cycle containing F. Then by Corollary 1.2.8 and Theorem 1.2.9, $c_F(G) \geq 2\delta - l$. Now, $|C| + |D| \geq 2c_F(G) - 7 = c_F(G) + (c_F(G) - 7)$. Thus $|C|, |D| \geq c_F(G) - 7$.

Now, $|\bigcup_{i=1}^{m} P_i| \geq c_F(G) - 7 - m \geq (2k-l) - 7 - m = (k-m) + (k-l-7)$. If $k-l-7 \leq 0$, then $l \geq k-7$ and thus $|V(C) \cap V(D)| \geq k-6 \geq ck^{3/5}$. So assume $k-l-7 > 0$. Then $|\bigcup_{i=1}^{m} P_i| > k-m$, and similarly, $|\bigcup_{j=1}^{m} Q_j| > k-m$.

Following [1], orient the cycles C and D. Beginning with a vertex v of C, $\forall x_1, x_2 \in V(C), x_1 \prec x_2$ if $x_1 \in P[v, x_2]$. For D, one can make a similar definition.

As G is k-connected, note that $G-T$ is $(k-m)$-connected and $G-T$ has at least $k-m$ vertex-disjoint paths (denote this set of paths by $\mathcal{P}$) from $\bigcup_{i=1}^{m} P_i$ to $\bigcup_{j=1}^{m} Q_j$. Now, construct an auxiliary bipartite graph H as in [1] and in Chapter 4, and note that $|E(H)| \geq k-m$. Here, H is an auxiliary graph of G such that for the segments $X_1, X_2, ..., X_m$ of $C-T$ and $Y_1, Y_2, ..., Y_m$ of $D-T$, $V(H) = \{X_1, X_2, ..., X_m, Y_1, Y_2, ..., Y_m\}$, and for every path $R_{i,j}$ in $\mathcal{P}$ from X_i to Y_j in G, noting that different $R_{i,j}$ are vertex disjoint paths, we say there exists an edge joining X_i and Y_j in H. Next we study the number of edges in $si(H)$.

Claim 5.2.1. *Each edge of H has a multiplicity of size at most nine.*

Proof. Similar to the proof of Claim 4.3.1, suppose for contradiction without loss of generality that the edge P_1Q_1 has a multiplicity of size at least 10. Then in G, there exists at least 10 paths, $R_i[x_i, y_{n_i}]$ for $i \geq 10$, with endpoints x_i and y_{n_i} in the segments P_1 and Q_1 respectively. Relabel the endpoints of these paths in P_1 if necessary, and assume that $x_1 \prec x_2 \prec ... \prec x_{10}$.

By Corollary 1.3.1, As $10 = 3^2 + 1$, the sequence $\{n_i\}(1 \leq i \leq 10)$ contain a monotone subsequence of length at least $3+1=4$ with respect to their index along the orientation of Q_1. Call these endpoints $y_{a_1}, y_{b_1}, y_{c_1}, y_{d_1}$ and relabel their corresponding endpoints in P_1 as x_a, x_b, x_c, x_d. Relabel the paths between them R_a, R_b, R_c, R_d such that they have endpoints x_a and y_{a_1}, x_b and y_{b_1}, x_c and y_{c_1}, and x_d and y_{d_1} respectively. Reorienting D if necessary, we may assume that $y_{a_1} \prec y_{b_1} \prec y_{c_1} \prec y_{d_1}$. Then we have a new cycle C' in G, by replacing the $P_1[x_a, x_d]$-segment in P_1 of C by $R_a[x_a, y_{a_1}]Q_1[y_{a_1}, y_{b_1}]\overleftarrow{R}_b[y_{b_1}, x_b]P_1[x_b, x_c]R_c[x_c, y_{c_1}]$
$Q_1[y_{c_1}, y_{d_1}]\overleftarrow{R}_d[y_{d_1}, x_d]$.

Similarly, we obtain a new cycle D' by replacing the $Q_1[y_{a_1}, y_{d_1}]$-segment in Q_1 of D by $\overleftarrow{R}_a[y_{a_1}, x_a]P_1[x_a, x_b]R_b[x_b, y_{b_1}]Q_1[y_{b_1}, y_{c_1}]\overleftarrow{R}_c[y_{c_1}, x_c]P_1[x_c, x_d]R_d[x_d, y_{d_1}]$.

Note that neither P_1 nor Q_1 contains any vertices or edges of F by the definition of H. When we construct the new cycle C', we only replace two segments of P_1, therefore the new cycle C' still contains the linear forest F. Similarly, D' also contains F. However, $|C'| + |D'| \geq |C| + |D| + 2(|R_a[x_a, y_{a_1}]| + |R_b[x_b, y_{b_1}]| + |R_c[x_c, y_{c_1}]| + |R_d[x_d, y_{d_1}]|) \geq |C| + |D| + 8 \geq 2c_F(G) - 7 + 8 > 2c_F(G)$, a contradiction. □

Note that the proof of the last claim can be easily modified to show that m is not zero. Otherwise, one gets two new cycles C' and D' containing F so that $|C'| + |D'| > 2c_F(G)$, a contradiction. Thus $m \geq 1$.

Claim 5.2.2. *If two edges of H, P_iQ_j and P_sQ_t, both have a multiplicity of more than one, then P_iQ_j and P_sQ_t must be adjacent edges in H.*

Proof. Suppose $R_1[x_1, y_1]$ and $R_2[x_2, y_2]$ are two vertex-disjoint paths between P_i and Q_j where $\{x_1, x_2\} \subseteq V(P_i)$ and $\{y_1, y_2\} \subseteq V(Q_j)$, and $S_1[x_3, y_3]$ and $S_2[x_4, y_4]$ are two vertex-disjoint paths between P_s and Q_t where $\{x_3, x_4\} \subseteq V(P_s)$ and $\{y_3, y_4\} \subseteq V(Q_t)$. Moreover, R_1, R_2 and S_1, S_2 are vertex-disjoint.

We construct two new cycles C' and D' as follows: C' is obtained from C by replacing the segment $C[x_1, y_1]$ by $R_1[x_1, y_1]D[y_1, y_2]\overleftarrow{R}_2[y_2, x_2]$, and $C[x_3, x_4]$ by $S_1[x_3, y_3]D[y_3, y_4]\overleftarrow{S}_2[y_4, x_4]$.

Similarly, we construct D' from D by replacing the segment $D[y_1, y_2]$ by $\overleftarrow{R}_1[y_1, x_1]C[x_1, x_2]$ $R_2[x_2, y_2]$, and $D[y_3, y_4]$ by $\overleftarrow{S}_1[y_3, x_3]C[x_3, x_4]S_2[x_4, y_4]$.

By the definition of H, none of P_i, P_s, Q_j, Q_t contain any vertices or edges of F. Note that during the process of constructing C' and D', we just replaced part of P_i, $C[x_1, x_2]$, by a new path from x_1 to x_2, and we replaced part of P_s, $C[x_3, x_4]$, by a new path from x_3 to

x_4. Therefore, C' still contains F. Similarly, D' contains F as well. Moreover,

$$\begin{aligned}|C'| + |D'| &= |C| + |D| + 2(|R_1| + |R_2| + |S_1| + |S_2|)\\ &\geq |C| + |D| + 8 \geq 2c_F(G) + 1;\end{aligned}$$

a contradiction.

□

Claim 5.2.3. *The simplified graph of H, $si(H)$, has at least $k - 9m$ edges.*

Proof. Let $|E(H)| \geq k - m$. To find $|E(si(H))|$, we must subtract the number of possible multiple edges in H from $k - m$. By Claim 5.2.2, any edge of H with multiplicity more than one must be covered by a single vertex as any two such edges are adjacent. Now, the vertex covering all edges that could be multiple can be incident to at most m possible edges as $H \subseteq K_{m,m}$. By Claim 5.2.1, each of these m edges has a multiplicity of at most nine. So, to subtract the number of multiple edges in H, we must subtract at most eight edges for each of the m possible edges. Thus, $|E(si(H))| \geq (k - m) - 8m = k - 9m$. □

If $m = 1$, then $si(H))$ has at most one edge. However, from the last claim, we deduce that $|E(si(H))| \geq k - 9m \geq 11 - 9 = 2$, a contradiction. If $m = 2$, then $si(H))$ has at most four edges as it is a subgraph of $K_{2,2}$. However, as $m = 2 \leq ck^{3/5}$, we deduce that $k \geq 24$. From the last claim, we deduce that $|E(si(H))| \geq k - 9m \geq 24 - 18 = 6$, a contradiction again. Therefore, $m \geq 3$.

By Lemma 4.2.1, if $m < ck^{3/5}$ where $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$, then $|E(si(H))| \geq k - 9m \geq \sqrt[3]{256}(m - 2)m^{2/3} + 2m$. Now, as $si(H) \subseteq K_{m,m}$, we have from Corollary 1.3.4 that $si(H)$ contains a $K_{3,257}$-subgraph.

Claim 5.2.4. *G must contain two new cycles C' and D' both containing the linear forest subgraph F, such that $|C'| + |D'| \geq |C| + |D| + 8$.*

This claim differs from the result of [1, page 14, line -4] by allowing C and D to be long cycles containing a linear forest, and the obtained two new cycles also contain the linear forest. The proof we provide next, again, essentially follow along the line from [1] (see Claim 4.3.4). For completion, we outline their original proof while showing both new cycles constructed contain the linear forest F.

Proof. We first outline the proof of the corresponding result of [1, pages 13-14], then we prove that both new cycles constructed also contain the linear forest F.

Relabelling as necessary, assume that $X_1, X_2, X_3, Y_1, ..., Y_{257}$ induce a $K_{3,257}$-subgraph and let $R_{i,j}[u_{i,j}, v_{i,j}]$ denote the path from X_i to Y_j for $1 \leq i \leq 3$ and $1 \leq j \leq 257$. Recall that we oriented the cycles C and D as follows: beginning with vertex v, $\forall x_1, x_2 \in V(C), x_1 \prec x_2$ if $x_1 \in P[v, x_2]$.

In [1], the authors prove that, Using Lemma 1.3.2, reversing the orientation and relabeling X_i and Y_j if necessary, we can assume that $u_{1,1} \prec u_{1,2} \prec u_{1,3}$ and $u_{2,1} \prec u_{2,2} \prec u_{2,3}$, and either $u_{3,1} \prec u_{3,2} \prec u_{3,3}$ or $u_{3,3} \prec u_{3,2} \prec u_{3,1}$.

Then the authors show that there are i, j, where $1 \leq i < j \leq 3$, such that either

- (i) $v_{1,i} \prec v_{2,i}$ and $v_{1,j} \prec v_{2,j}$, or
- (ii) $v_{2,i} \prec v_{1,i}$ and $v_{2,j} \prec v_{1,j}$.

Indeed, assume that $v_{1,1} \prec v_{2,1}$. If $v_{1,2} \prec v_{2,2}$, then choose $(i, j) = (1, 2)$ in (i), and if $v_{1,3} \prec v_{2,3}$, then choose $(i, j) = (1, 3)$ in (i). Otherwise, If $v_{2,2} \prec v_{1,2}$, and $v_{2,3} \prec v_{1,3}$, then choose $(i, j) = (2, 3)$ in (ii). The case for $v_{2,1} \prec v_{1,1}$ is similar.

Reorient D if necessary one can assume that $v_{1,1} \prec v_{2,1}$. Now in each case we we can reroute and obtain two new cycles C' and D', both containg F such that $|C'| + |D'| \geq |C| + |D| + 8$.

Without loss of generality, next we will only describe the following cases:

(a) $v_{1,1} \prec v_{2,1}$ and $v_{1,2} \prec v_{2,2}$.

(b) $v_{2,2} \prec v_{1,2}$ and $v_{2,3} \prec v_{1,3}$.

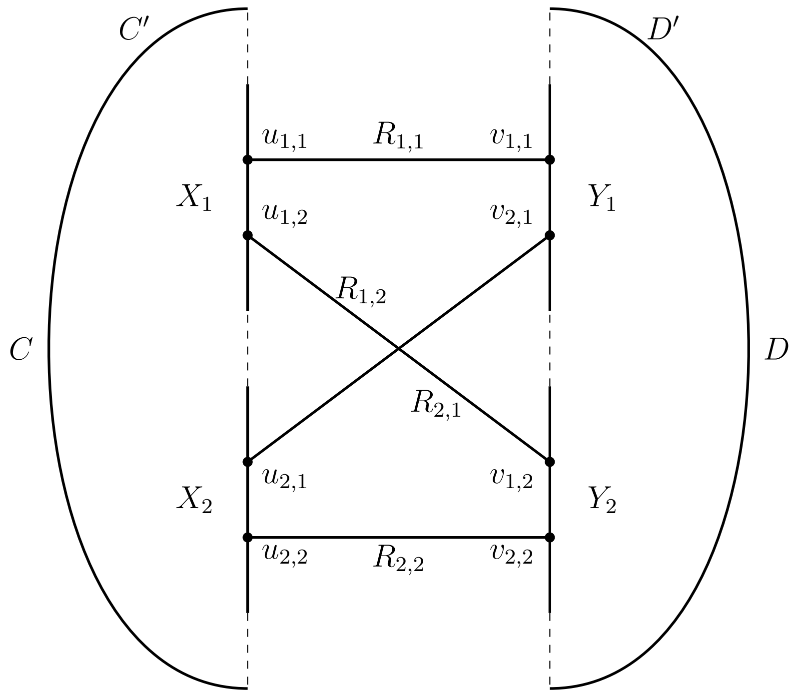


Figure 5.1: The new cycles C' and D' in G with $v_{1,1} \prec v_{2,1}$ and $v_{1,2} \prec v_{2,2}$, each containing every vertex and edge in the intersection of C and D

In case (a), we construct two new cycles as follows (see Fig 5.1):

$$C' = u_{1,1}R_{1,1}v_{1,1}Dv_{2,1}\overleftarrow{R}_{2,1}u_{2,1}\overleftarrow{C}u_{1,2}R_{1,2}v_{1,2}Dv_{2,2}\overleftarrow{R}_{2,2}u_{2,2}Cu_{1,1}$$
$$D' = v_{1,1}\overleftarrow{R}_{1,1}u_{1,1}Cu_{1,2}R_{1,2}v_{1,2}\overleftarrow{D}v_{2,1}\overleftarrow{R}_{2,1}u_{2,1}Cu_{2,2}R_{2,2}v_{2,2}Dv_{1,1}.$$

By the definition of H, none of X_i and Y_i $(1 \leq i \leq 3)$ contain any vertices or edges of F, as they are segments of $C - T$. Note that C' misses only internal vertices of X_1 and X_2 from C, and therefore these internal vertices of X_1 and X_2 contain no vertices or edges of F. Therefore, C' contains F. Similarly, D' also contains F.

In case (b), we construct two new cycles as follows (see Fig 5.2):

$$C' = u_{1,1}R_{1,2}v_{1,2}\overleftarrow{D}v_{2,2}\overleftarrow{R}_{2,2}u_{2,2}\overleftarrow{C}u_{1,3}R_{1,3}v_{1,3}\overleftarrow{D}v_{2,3}\overleftarrow{R}_{2,3}u_{2,3}Cu_{1,1},$$
$$D' = v_{2,2}\overleftarrow{R}_{2,2}u_{2,2}Cu_{2,3}R_{2,3}v_{2,3}\overleftarrow{D}v_{1,2}\overleftarrow{R}_{1,2}u_{1,1}Cu_{1,3}R_{1,3}v_{1,3}Dv_{2,2}.$$

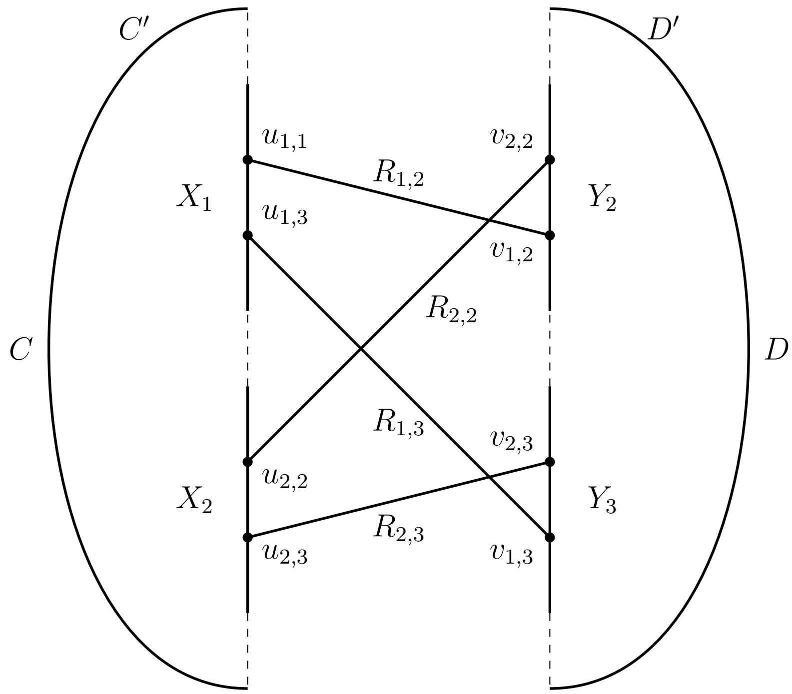


Figure 5.2: The new cycles C' and D' in G with $v_{2,2} \prec v_{1,2}$ and $v_{2,3} \prec v_{1,3}$, each containing every vertex and edge in the intersection of C and D

Again none of X_i and Y_i $(1 \leq i \leq 3)$ contain any vertices or edges of F, as they are segments of $C - T$. Note that D' misses only the internal vertices of Y_2 and Y_3 from D, and therefore these internal vertices of Y_2 and Y_3 contain no vertices or edges of F. Therefore, D' contains F. Similarly, C' also contains F.

Moreover, in case (a), $|C'|+|D'| = |C|+|D|+2(|R_{1,1}|+|R_{1,2}|+|R_{2,1}|+|R_{2,2}|) \geq |C|+|D|+8$. In case (b), $|C'| + |D'| = |C| + |D| + 2(|R_{1,2}| + |R_{2,2}| + |R_{1,3}| + |R_{2,3}|) \geq |C| + |D| + 8$. This completes the proof of the claim. □

Now, by Claim 5.2.4, G contains two new cycles, C' and D', both containing F, such that $|C'| + |D'| \geq |C| + |D| + 8 \geq 2c_F(G) - 7 + 8 > 2c_F(G)$, a contradiction. This completes the proof of the theorem. □

Proof. (Proof of Theorem 5.1.2) Let G be a k-connected graph, $k \geq 2$, and let C and D be two cycles containing a linear forest subgraph F such that $|C| + |D| \geq 2c_F(G) - 3$ and $|V(C) \cap V(D)| = m$, for $m < c_1 k^{3/5}$, and $c_1 = 1/(\sqrt[3]{256} + 0.02481)^{3/5} \approx 0.3291$. If $n \leq 2\delta - l$, by Corollary 1.2.8 and Dirac's Theorem, G has a Hamiltonian cycle containing F. Then $|V(C) \cap V(D)| = |C| + |D| - |V(C) \cup V(D)| \geq 2c_F(G) - 3 - n = n - 3 \geq k - 2 \geq c_1 k^{3/5}$ by Lemma 4.2.4. Thus we may assume $n > 2\delta - l$. By Corollary 1.2.8 and Theorem 1.2.9, $c_F(G) \geq 2\delta - l \geq 2k - l$. Now, $|C| + |D| \geq 2c_F(G) - 3 = c_F(G) + (c_F(G) - 3)$. Thus $|C|, |D| \geq c_F(G) - 3$.

Now, $|\bigcup_{i=1}^{m} P_i| \geq c_F(G) - 3 - m \geq (2k - l) - 3 - m = (k - m) + (k - l - 3)$. If $k - l - 3 \leq 0$, then $l \geq k - 3$ and thus $|V(C) \cap V(D)| \geq k - 2 \geq c_1 k^{3/5}$. So assume $k - l - 3 \geq 0$. Then $|\bigcup_{i=1}^{m} P_i| \geq k - m$, and similarly, $|\bigcup_{j=1}^{m} Q_j| \geq k - m$.

As G is k-connected, note that $G - T$ is $(k - m)$-connected and $G - T$ has at least $k - m$ independent paths from $\bigcup_{i=1}^{m} P_i$ to $\bigcup_{j=1}^{m} Q_j$. Now, construct an auxiliary bipartite graph H as described above, and note that $|E(H)| \geq k - m$. Suppose that H could have multiple edges, say $P_1 = P_1[u_1, v_1]$ and $P_2 = P_2[u_2, v_2]$ (vertex disjoint paths) such that u_1 and u_2 are in X_1 and v_1, and v_2 are in Y_1. Furthermore, suppose $C[u_1, u_2] \subseteq X_1$ and $D[v_1, v_2] \subseteq Y_1$. Then G would contain two cycles $C^* = u_1 P_1 v_1 D v_2 \overleftarrow{P}_2 u_2 C u_1$ and $D^* = v_1 \overleftarrow{P}_1 u_1 C u_2 P_2 v_2 D v_1$ such that $|C^*| + |D^*| = |C| + |D| + 2|P_1| + 2|P_2| \geq 2c_F(G) - 3 + 4 > 2c_F(G)$. Note that C^* only misses some internal vertices of X_1 from C, and as any vertex of F is not an internal vertex of X_1, C^* contains all vertices of F. Similarly, D^* contains every vertex of F. If X_1 contained an edge of F, then X_1 would be the empty set. Thus, all edges of F are contained in both C^* and D^* such that $|C^*| + |D^*| > 2c_F(G)$, a contradiction. Thus, H has no multiple edges and we have that H is a simple bipartite graph such that $|E(H)| \geq k - m$.

Modifying the proof in the last paragraph, one can show that $m > 0$. If $m = 1$, then H has at most one edge, but as $m < c_1 k^{3/5}$, we deduce that $k > 6$ and thus H has at least $k - m > 5$ edges; a contradiction. If $m = 2$, then from $m < c_1 k^{3/5}$, we deduce that $k > 20$, thus H should have at least $k - m \geq 18$ edges; a contradiction as H is a subgraph of $K_{2,2}$. Thus $m \geq 3$.

With our chosen bound of $c_1k^{3/5}$, Lemma 4.2.2 gives us that $|E(H)| \geq k - m \geq \sqrt[3]{256}(m-2)m^{2/3} + 2m$. Now, as $H \subseteq K_{m,m}$, we have from Corollary 1.3.4 that H contains a $K_{3,257}$-subgraph. Then by Claim 5.2.4, since H has a $K_{3,257}$-subgraph, there exists two cycles C' and D', both containing F, such that $|C'| + |D'| \geq |C| + |D| + 8 \geq 2c_F(G) - 3 + 8 > 2c_F(G)$, a contradiction. This completes the proof of the theorem. □

6 CONSEQUENCES ON THE INTERSECTION OF LONG PATHS CONTAINING A LINEAR FOREST SUBGRAPH IN k-CONNECTED GRAPHS

6.1 Introduction and Main Results

In this chapter, we present our results related to the intersection of long paths containing a linear forest subgraph. We obtain a result that partially proves a conjecture related to Conjecture 2.1.7, and also give a lower bound on the intersection of two long paths containing a linear forest subgraph. All results in this chapter are obtained as consequences of theorems proved in earlier chapters for cycles.

The first main result for paths is similar to Theorem 5.1.1 for cycles. We note that the small difference between this result and Theorem 5.1.1 arises from the construction of the new cycles and application of the cycle result. Let F be a linear forest subgraph of G. We use $p(G)$ to denote the longest path length of G and $p_F(G)$ to denote the length of a longest path containing F. For a path P, we use $l(P)$ to denote the length of the path.

Theorem 6.1.1. *Let P and Q be two paths containing a specified linear forest subgraph F with at most $k-1$ vertices of a k-connected graph G, such that $l(P) + l(Q) \geq 2p_F(G) - 7$. In addition, we assume $k \geq 10$ if $|V(F)| = \emptyset$. Then $|V(P) \cap V(Q)| \geq c(k+1)^{3/5} - 1$, where $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$.*

Corollary 6.1.2. *Let P and Q be two paths of a k-connected graph G for $k \geq 10$, such that $l(P) + l(Q) \geq 2p(G) - 7$. Then $|V(P) \cap V(Q)| \geq c(k+1)^{3/5} - 1$, where $c = 1/(\sqrt[3]{256} + 1.22271)^{3/5} \approx 0.2968$.*

The second result is similar to the above and uses an application of Theorem 5.1.2. We omit the proof as it follows similarly to the proof of Theorem 6.1.1.

Theorem 6.1.3. *Let P and Q be two paths containing a specified linear forest subgraph F with at most $k-1$ vertices of a k-connected graph G, such that $l(P)+l(Q) \geq 2p_F(G)-3$. Then $|V(P) \cap V(Q)| \geq c_1(k+1)^{3/5}-1$, where $c_1 = 1/(\sqrt[3]{256}+0.02481)^{3/5} \approx 0.3291$.*

Corollary 6.1.4. *Let P and Q be two paths of a k-connected graph G, such that $l(P)+l(Q) \geq 2p(G)-3$. Then $|V(P) \cap V(Q)| \geq c_1(k+1)^{3/5}-1$, where $c_1 = 1/(\sqrt[3]{256}+0.02481)^{3/5} \approx 0.3291$.*

The final main result is similar to Theorem 3.2.4, and uses an application of the theorem in its proof. It confirms our previous Conjecture 2.1.7 for $k \leq 5$. From the proof, we will see that the path version conjecture is weaker than the cycle version conjecture.

Theorem 6.1.5. *Let G be a k-connected graph where $1 \leq k \leq 5$, and F be a specified linear forest subgraph of G with at most $k-1$ vertices. If P and Q are paths containing F such that $l(P)+l(Q) \geq 2p_F(G)-1$, then P and Q must meet in at least k common vertices.*

6.2 Proof of the Main Results

Proof. (Proof of Theorem 6.1.1) Let G be a k-connected graph and P and Q be two paths of G containing a specified linear forest subgraph F with at most $k-1$ vertices. Suppose $l(P)+l(Q) \geq 2p_F(G)-7$ and assume $k \geq 10$ if $V(F)=\emptyset$.

Construct a new graph $G+v$ by adding a vertex v, and adding an edge from v to every vertex of G. As G is k-connected, $G+v$ is then $(k+1)$-connected. Also note that P and the two edges between v and the endpoints of P form a cycle, denoted C. Similarly, Q and the two edges between v and the endpoints of Q form a cycle, denoted D. As C and D contain every vertex and edge of P and Q respectively, C and D both contain the linear forest subgraph F.

Now, suppose R is a longest path containing F of G with endpoints x_1 and x_2. Then tx_1Rx_2t is a cycle of $G+v$ containing R and thus containing F with length $l(R)+2=$

$p_F(G)+2$. Suppose for contradiction there exists a longer cycle S in $G+v$ containing F, with length at least $p_F(G)+3$. If v is not contained in S, then S contains a path containing F with length at least $p_F(G)+2$ in G, a contradiction. If S contains v, then $S\backslash v$ is a path in G containing F and has length at least $p_F(G)+3-2=p_F(G)+1$, again, a contradiction. Thus $c_F(G)=p_F(G)+2$. Now, $|C|+|D|=l(P)+l(Q)+4\geq 2p_F(G)+4-7=2c_F(G+v)-7$. By Theorem 5.1.1, as $G+v$ is $(k+1)$-connected, $|V(C)\cap V(D)|\geq c(k+1)^{3/5}$. Thus $1+|V(P)\cap V(Q)|\geq c(k+1)^{3/5}$, and thus P and Q must meet in at least $c(k+1)^{3/5}-1$ vertices in G. □

Proof. (Proof of Theorem 6.1.5) Suppose G is a k-connected graph, $1\leq k\leq 5$, and F is a specified linear forest subgraph of G with at most $k-1$ vertices. Let P and Q be paths containing F such that $l(P)+l(Q)\geq 2p_F(G)-1$.

Now, similar to the above proof, construct a new graph $G+v$ by adding a vertex v, and adding an edge from v to every vertex in G. As G is k-connected, $G+v$ is then $(k+1)$-connected, for $2\leq k+1\leq 6$. Also note that P and the two edges between v and the endpoints of P form a cycle, denoted C. Similarly, Q and the two edges between v and the endpoints of Q form a cycle, denoted D. As C and D contain every vertex and edge of P and Q respectively, C and D both contain the linear forest subgraph F. Moreover, $c_F(G+v)=p_F(G)+2$.

Now, $|C|+|D|=l(P)+l(Q)+4\geq 2p_F(G)-1+4=2c_F(G+v)-1$. As $k+1\in\{2,3,4,5,6\}$, and $G+v$ is $(k+1)$-connected, by Theorem 3.2.4, C and D must meet in at least $k+1$ common vertices. Therefore, P and Q must meet in at least k vertices in G. □

Smith conjectures that any two longest cycles in a k-connected graph ($k\geq 2$) meet in at least k common vertices. Hippchen [12] conjectures that any two longest paths in a k-connected graph must have at least k common vertices. This has been verified for $k=4$ in [11], $k=5$ in [3], and for $k\leq 6$ in [15]. Modifying our above proof in Theorem 6.1.5, we can see that if Simth's conjecture is true for $2\leq k\leq t$, then Hippchen's path version conjecture is true for $1 \leq k \leq t-1$. As Smith's conjecture is true for $2 \leq k \leq 8$, we deduce that Hippchen's above conjecture [12] is true for $1 \leq k \leq 7$.

BIBLIOGRAPHY

Bibliography

[1] G. Chen, R.J. Faudree, and R.J. Gould, *Intersections of longest cycles in k-connected graphs*, J. Combin. Theory Ser. B **72** (1998), no. 1, 143–149. MR 1604717

[2] J. Chen, L Chen, and D. Liu, *Intersections of cycles in k-connected graphs*, Graphs Combin. **31** (2015), no. 4, 897–914. MR 3357663

[3] E. Cho, I. Choi, and B. Park, *Improvements on Hippchen's conjecture*, Discrete Math. **345** (2022), no. 11, Paper No. 113029, 8. MR 4438135

[4] G. A. Dirac, *Some theorems on abstract graphs*, Proc. London Math. Soc. (3) **2** (1952), 69–81. MR 47308

[5] Y. Egawa, R. Glas, and S. C. Locke, *Cycles and paths through specified vertices in k-connected graphs*, J. Combin. Theory Ser. B **52** (1991), no. 1, 20–29. MR 1109415

[6] P. Erdös and G. Szekeres, *A combinatorial problem in geometry*, Compositio Math. **2** (1935), 463–470. MR 1556929

[7] M. Grötschel, *Graphs with cycles containing given paths*, Studies in integer programming (Proc. Workshop, Bonn, 1975), 1977, pp. 233–245. Ann. of Discrete Math., Vol. 1. MR 0491332

[8] ______, *On intersections of longest cycles*, Graph theory and combinatorics (Cambridge, 1983), Academic Press, London, 1984, pp. 171–189. MR 777174

[9] M. Grötschel and G.L. Nemhauser, *A polynomial algorithm for the max-cut problem on graphs without long odd cycles*, Math. Programming **29** (1984), no. 1, 28–40. MR 740503

[10] J. Gutiérrez, *On the intersection of two longest paths in k-connected graphs*, Pro Math. **31** (2021), no. 62, 11–23. MR 4568317

[11] ______, *Transversals of longest cycles in partial k-trees and chordal graphs*, J. Graph Theory **98** (2021), no. 4, 589–603. MR 4371466

[12] T. Hippchen, *Intersections of longest paths and cycles*, Master's thesis, Georgia State University, 2008.

[13] Z. Hu, F. Tian, and B. Wei, *Long cycles through a linear forest*, J. Combin. Theory Ser. B **82** (2001), no. 1, 67–80. MR 1828436

[14] C. Hyltén-Cavallius, *On a combinatorical problem*, Colloq. Math. **6** (1958), 59–65. MR 103158

[15] J. Gutiérrez and C. Valqui, *Bi-traceable graphs, the intersection of three longest paths and hippchen's conjecture*, https://doi.org/10.48550/arXiv.2101.07859, 2021.

[16] M. Lemos, T.J. Reid, B. Williams, and H. Wu, *Pairs of largest circuits in 3-connected matroids*, Linear Algebra Appl. **427** (2007), no. 2-3, 313–316. MR 2351362

[17] S. McGuinness, *Bonds intersecting cycles in a graph*, Combinatorica **25** (2005), no. 4, 439–450. MR 2143249

[18] N. McMurray, T.J. Reid, L. Sheppardson, B. Wei, and H. Wu, *On deletions of largest bonds in graphs*, Discrete Math. **308** (2008), no. 7, 1137–1141. MR 2382352

[19] N. McMurray, T.J. Reid, B. Wei, and H. Wu, *Largest circuits in matroids*, Adv. in Appl. Math. **34** (2005), no. 1, 213–216. MR 2102283

[20] J. Oxley, *Matroid theory*, second ed., Oxford Graduate Texts in Mathematics, vol. 21, Oxford University Press, Oxford, 2011. MR 2849819

[21] A. Shabbir, C.T. Zamfirescu, and T.I. Zamfirescu, *Intersecting longest paths and longest cycles: a survey*, Electron. J. Graph Theory Appl. (EJGTA) **1** (2013), no. 1, 56–76. MR 3093252

[22] L. Sheppardson, *Intersections of largest bonds in k-connected graphs*, Discrete Math. **309** (2009), no. 12, 4116–4122. MR 2537404

[23] I.A. Stewart and B. Thompson, *On the intersections of longest cycles in a graph*, Experiment. Math. **4** (1995), no. 1, 41–48. MR 1359416

[24] D.B. West, *Introduction to graph theory*, 2 ed., Prentice Hall, Inc., Upper Saddle River, NJ, 2001.

[25] H. Wu, *Intersection of cycles and paths in k-connected graphs*, submitted.

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