Journal of Accountancy

Volume 20 | Issue 3 Article 5

9-1915

Annuities and Bond Discount

Otto A. Spies

Annuities and Bond Discount

By Otto A. Spies.

A perusal of Mr. Bennett's article in the June and July Journal of Accountancy is highly interesting and instructive. However, the solution methods used require the application of bond and compound interest tables, and it appears that the stated problems can be made clearer to the mind of the student and those not having access to elaborate bond tables, by the formation of equations, the solution of which should be an easy matter to those familiar with algebra and logarithms.

The more complicated problems being 11, 12 and 13 of the June number, and 15 of the July number, have been solved by the suggested process, and are here submitted to the readers of The Journal of Accountancy, with the hope that the method applied will not only prove of interest but of practical value for future reference and use.

PROBLEM STATED ON PAGE 419, JUNE, 1915 THE JOURNAL OF ACCOUNTANCY

A municipality borrowed $40,000.00 at 5% interest, for local improvements, to be repaid in 15 years, by equal annual instalments including principal and interest. What is the amount of the annual payment? Show the respective amounts paid for principal and interest for the first three years.

SOLUTION

The accumulated amount of the debt of $40,000.00 borrowed at 5% annual interest, compounded for a period of 15 years, must have, according to the provisions of the problem, the same value as the accumulated amounts of a series of 15 equal annual payments, representing principal and interest.

From this condition results the fundamental equation, which may be used for the solution of the problem.

It is evident that the accumulated amounts of the series of 15 equal annual payments must all have the same geometrical ratio of increase, so that, if the annual equal payment be designated by X, and the interest be computed at 5% per annum, the sum of the accumulated amounts of the equal payments will be stated in equation form as follows:

1. $\text{Sum} = X\,(1 + 1.05 + 1.05^2 + + + + 1.05^{13} + 1.05^{14})$

The amounts within parentheses can be greatly contracted as follows:

Multiply on both sides with the factor, which results by division of one member of the geometrical progression by the next preceding one. This factor would be 1.05. Designating the sum by S, the following or second equation is produced.

2. $S\,1.05 = X\,(1.05 + 1.05^2 + 1.05^3 + + + 1.05^{14} + 1.05^{15})$

Deducting equation 1 from equation 2, and again expressing the first equation as follows:

1. $S = X\,(1 + 1.05 + 1.05^2 + + + + 1.05^{13} + 1.05^{14})$

the result is

$$S\,(1.05 - 1) = X\,(1.05^{15} - 1) \text{ or}$$

3. $\text{Sum} = X\,\dfrac{(1.05^{15} - 1)}{0.05}$

The fundamental equation for the solution of the given problem is now expressed:

Accumulated amount of $40,000.00 at 5% annual interest compounded for 15 years, equals: sum of accumulated amounts of annual equal payments, compounded at 5% per annum, or

4. $40{,}000 \times 1.05^{15} = X\,\dfrac{(1.05^{15} - 1)}{0.05}$ or

$$X = \frac{40{,}000 \times 1.05^{15}}{\left(\dfrac{1.05^{15} - 1}{0.05}\right)} = \frac{40{,}000 \times 1.05^{15} \times 0.05}{1.05^{15} - 1}$$

or annual equal payment:

5. $X = \dfrac{2{,}000 \times 1.05^{15}}{1.05^{15} - 1}$

Applying logarithms for the computations as follows, the answer is readily reached.

$$
\begin{aligned}
\log 1.05 &= 0.0211893 \\
15 \log 1.05 &= 0.3178395 \\
1.05^{15} &= 2.0789282 \\
1.05^{15} - 1 &= 1.0789282
\end{aligned}
$$

$$\log X = \log 2000 + \log 1.05^{15} - \log (1.05^{15} - 1)$$

$$
\begin{aligned}
\log 2000 &= 3.3010300 \\
+ \log 1.05^{15} &= 0.3178395 \\
\hline
& 3.6188695 \\
- \log (1.05^{15} - 1) &= 0.0329882 \\
\hline
\log X & 3.5858813 \\
X &= \$3{,}853.70 \quad \text{answer}
\end{aligned}
$$

The respective amounts paid for principal and interest for the first three years, are to be:

	Balance of Principal	Principal payment	Interest payment	Total annual payment
	$40,000.00			
End of 1st year	38,146.30	$1,853.70	$2,000.00	$3,853.70
" " 2d "	36,199.92	1,946.38	1,907.32	3,853.70
" " 3d "	34,156.22	2,043.70	1,810.00	3,853.70
" " 4th "	32,010.33	2,145.89	1,707.81	3,853.70

PROBLEM STATED ON PAGE 420 JUNE, 1915 THE JOURNAL OF ACCOUNTANCY.

(Problem of the Pennsylvania examination of 1912)

A manufacturer owes $100,000.00 on his plant at 5% per annum, due at the end of five years from date. He secures an agreement, however, to pay the debt in equal instalments, which will include principal and interest. What amount is he required to pay each year?

1st SOLUTION:

The conditions of the problem demand that the sum of all principal payments, which are to be made at the end of the 1st, 2d, 3d, 4th and 5th years, must equal the sum of $100,000.00.

If the first principal payment to be made at the end of the first year is designated by X_1 and the succeeding principal payments by X_2; X_3; X_4; and X_5, the first fundamental equation is expressed as follows:

$$1.\quad X_1 + X_2 + X_3 + X_4 + X_5 = \$100{,}000.00$$

It is further known that 5% interest is to be paid every year on the unpaid balance of principal, and as the total payments of principal and interest must every year be the same, it follows that the interest payments must decrease, and the principal payments must increase.

It now becomes the question as to how much this increase of principal payment is, in order to satisfy the requirements of the problem.

If the first principal payment is X_1 and the annual interest is 5%, it follows that the:

2d principal payment $= X_2 = 1.05 \times X_1$, and the
3d " " $= X_3 = 1.05^2 \times X_1$, and the
4th " " $= X_4 = 1.05^3 \times X_1$, and the
5th " " $= X_5 = 1.05^4 \times X_1$.

As the sum of all five principal payments must of course equal $100,000.00 the fundamental equation is therefore restated as follows:

$$X_1 + 1.05\ X_1 + 1.05^2\ X_1 + 1.05^3\ X_1 + 1.05^4\ X_1 = \$100{,}000.00$$

or

$$2.\quad X_1\ (1 + 1.05 + 1.05^2 + 1.05^3 + 1.05^4) = \$100{,}000.00$$

The geometrical progression equals according to well known rules,

$$1 + 1.05 + 1.05^2 + 1.05^3 + 1.05^4 = \frac{1.05^5 - 1}{0.05}$$

This value inserted in equation 2,

$$X_1 = \frac{\$100{,}000.00}{\left(\frac{1.05^5 - 1}{0.05}\right)} = \frac{100{,}000.00 \times 0.05}{1.05^5 - 1}$$

or

$$3.\quad X_1 = \frac{5000}{1.05^5 - 1}$$

The computation is easiest with logarithms, as follows:

$$
\begin{aligned}
\log 1.05 &= 0.0211893 \\
5 \log 1.05 &= 0.1059465 \\
1.05^5 &= 1.27628153 \\
1.05^5 - 1 &= 0.27628153 \\
\log 5000 &= 3.6989700 \\
-\log (1.05^5 - 1) &= 0.4413521 - 1 \\
\hline
\log X_1 &= 4.2576179
\end{aligned}
$$

First principal payment $= X_1 =$ \$18,097.48.

The interest requirement at the end of the first year would be

$$\frac{5}{100} \times \$100{,}000.00 = \$5{,}000.00$$

and the total payment to be made at the end of the first year,

Principal	\$18,097.48
Interest	5,000.00
Total payment =	\$23,097.48 *Answer*

The same total payment is to be made, of course, at the end of the 2d, 3d, 4th, and 5th years, and the schedule of payments would be expressed in a graphic manner as follows:

	Principal payments	Interest payments	Total annual payments
End of 1st year	\$18,097.48	\$5,000.00	\$23,097.48
" " 2d "	19,002.35	4,095.13	23,097.48
" " 3d "	19,952.47	3,145.01	23,097.48
" " 4th "	20,950.10	2,147.38	23,097.48
" " 5th "	21,997.60	1,099.88	23,097.48
	\$100,000.00	\$15,487.40	\$115,487.40

2D SOLUTION

If it is considered that the debt of \$100,000.00 of the manufacturer, borrowed at the rate of 5% annual interest, is to be liquidated in five equal annual instalments, there would be an accumulated amount at the end of the fifth year, expressed as follows:

Accumulated amount $= 100{,}000 \times 1.05^5$

This amount must equal the accumulated amounts of a series of five annual equal payments, representing principal and interest, compounded at the rate of five per cent per annum. If the annual equal payment is designated by X there would be:

Accumulated amounts of annual payments $= X\ (1.05^4 + 1.05^3 + 1.05^2 + 1.05 + 1)$.

This value, according to the conditions of the problem, must equal the accumulated amount of the debt at the end of the fifth year, or expressed as an equation:

1. $100{,}000 \times 1.05^5 = X\ (1.05^4 + 1.05^3 + 1.05^2 + 1.05 + 1)$

$$\text{or } X = \frac{100{,}000 \times 1.05^5}{1.05^4 + 1.05^3 + 1.05^2 + 1.05 + 1}$$

or the value of $1.05^4 + 1.05^3 + 1.05^2 + 1.05 + 1 = \dfrac{1.05^5 - 1}{0.05}$

inserted, we have for X the annual equal payment,

$$X = \frac{100{,}000 \times 1.05^5 \times 0.05}{1.05^5 - 1} \quad \text{or}$$

2. $X = \dfrac{5{,}000 \times 1.05^5}{1.05^5 - 1}$

By comparing this value with the value of the first principal payment X_1 as computed in the first solution, it will be observed that the first principal payment X_1 is to the combined interest and principal payment X, as $\dfrac{X_1}{X} = \dfrac{1}{1.05^5} = \dfrac{1}{1.27628156}$

$$\text{or } X = 1.27628153 \times \$18{,}097.48$$

$$\text{or } X = \$23{,}097.48 \quad \textit{Answer}$$

PROBLEM STATED ON PAGE 422, JUNE, 1915, THE JOURNAL OF ACCOUNTANCY

A company on January 1, 1911, contemplates the purchase of the equity in a contract which yields a net income of $100,000.00 semi-annually over a period of ten years.

From the income received, the company desires to pay 3 per cent semi-annually on its investment, and to set aside the balance in a sinking fund which, invested at 3% per annum, compounded semi-annually, will produce the amount of the original investment.

What are the amounts of the semi-annual sinking fund, and of the original investment?

SOLUTION

In this problem are two unknown quantities, namely,

The original investment or purchase price, designated by X,

The amount of the sinking fund, which invested at 3% per annum, compounded, semi-annually, will produce the amount of the original investment, designated by Y.

According to the conditions of the problem, the accumulations of the sinking fund contributions, compounded semi-annually at 3% per annum, must equal the original purchase price X, or expressed by an equation:

$$1.\quad X = Y\ (1.015^{19} + 1.015^{18} + \text{———————} + 1.015 + 1)$$

or contracted, according to well known rules,

$$2.\quad X = Y\ \frac{(1.015^{20} - 1)}{0.015}$$

A certain part of the semi-annual income of $100,000.00 has to be set aside as sinking fund contribution = Y, and the balance of the $100,000.00 represents and must equal 3% semi-annual interest on the purchase price X.

From this condition follows a second fundamental equation:

$$100{,}000 = Y + \frac{3\,X}{100} \quad \text{or}$$

$$3.\quad Y = 1{,}000 - \frac{3\,X}{100}$$

If this value of Y is inserted in the equation 2, there is obtained a very simple equation of the first degree for X, from which the unknown purchase price can be easily computed, namely:

4. $X = \left(100{,}000 - \frac{3\,X}{100}\right) \qquad \frac{(1.015^{20} - 1)}{0.015}$

or dissolved

$$X + \frac{3\,X}{100}\;\frac{(1.015^{20} - 1)}{0.015} = 100{,}000\,\frac{(1.015^{20} - 1)}{0.015} \quad \text{or}$$

5. Purchase price $= X = \dfrac{100{,}000\left(\dfrac{1.015^{20} - 1}{0.015}\right)}{1 + \dfrac{3}{100}\left(\dfrac{1.015^{20} - 1}{0.015}\right)}$

The value $\dfrac{(1.015^{20} - 1)}{0.015}$ is to be best computed with logarithms, as follows:

$$\begin{aligned} \log 1.015 &= 0.00646604 \\ 20 \log 1.015 &= 0.1293208 \\ 1.015^{20} &= 1.34685501 \\ 1.015^{20} - 1 &= 0.34685501 \\ \frac{1.015^{20} - 1}{0.015} &= 23.123667333 \end{aligned}$$

This value inserted in equation 5, results in

$$X = \frac{100{,}000 \times 23.123667333}{1 + \frac{3}{100}\,(23.123667333)} \quad \text{or}$$

6. $X = \dfrac{2312366.7333}{1.69371002} = \$1{,}365{,}267.19$ *Answer*

For computation of the semi-annual sinking fund contribution, we use best equation 3, namely,

3. Sinking fund contribution $= Y = 100{,}000 - \dfrac{3\,X}{100}$

or inserting the value of X,

$$Y = 100{,}000 - \frac{3 \times 1{,}365{,}267.19}{100} \quad \text{or}$$

$$Y = 100{,}000 - 40{,}958.0157$$

7. $Y = \$59{,}041.98$ *Answer*

With these values for X and Y, the conditions of the problem are satisfied in every respect, namely:

2. $X = Y \frac{(1.015^{20} - 1)}{0.015}$ or values inserted:

$1{,}365{,}267.19 = 59{,}041.98 \times 23.123667333$ and also

3. $Y = 100{,}000 - \frac{3\ X}{100}$ or values inserted:

$$59{,}041.98 = 100{,}000 - \frac{4{,}095{,}801.57}{100}$$

EXAMPLE 15, PAGE 10, JULY, 1915, THE JOURNAL OF ACCOUNTANCY

You purchased January 1, 1915, a \$10,000.00 bond bearing semi-annual interest at the rate of 6% per annum, for \$10,275.00.

What per cent did you make on your investment? Bond period 3 years.

Show the entries involved, and the records for bond and amortization for the entire time.

SOLUTION:

The semi-annual amount of interest of the bond is \$300.00.

The accumulation of all interest payments at a certain unknown rate of interest for six semi-annual periods is:

$$300\ (1.0X^5 + 1.0X^4 + \text{-------} + 1.0X + 1) \text{ or}$$

$$300 \left(\frac{1.0X^6 - 1}{0.0X}\right)$$

The bond of \$10,000 is payable at the end of the sixth period, and the amount of the bond plus the accumulation of all interest payments must equal the purchase price of the bond, invested for a period of 3 years at an unknown rate of interest (X), to be compounded semi-annually.

The fundamental equation for this condition is therefore:

1. $10275 \times 1.0X^6 = \frac{300}{0.0X}(1.0X^6 - 1) + 10{,}000.$

or the purchase price:

2. $10275 = \frac{300}{0.0X}\left(\frac{1.0X^6 - 1}{1.0X^6}\right) + \frac{10000}{1.0X^6}$

This equation (2) has only one unknown quantity, X, the desired rate of interest, but cannot be solved directly as X appears at the 6th power. The practical way to proceed, in order to compute X, is to take approximate values for X, and insert these values in equation (2).

If such an inserted value for X satisfies the equation (2) perfectly, it must be the exact and mathematically correct amount of the desired rate of interest.

The nominal rate of interest being 3% semi-annually, and the bond premium being $275.00, it follows that the effective rate of interest must be less than 3%.

An approximate correct amount is ascertained from the approximate return of the bond.

In the given example, there is a nominal periodical return of $300.00.

If the approximate periodical amortization $\frac{275}{6} = \$45.83$ is deducted from the nominal return, there results the approximate return of the bond of $10,000.

$\$300 - \$45.83 = \$254.17$, or for a $100.00 bond $= \$2,5417$.

From this result it may be concluded that the true effective semi-annual interest is close to 2.5%.

If for the purpose of verification, this value for $X = 2.5$ is inserted in equation (2), there would result:

$$3.\quad 10275 = \frac{300}{0.025}\left(\frac{1.025^6 - 1}{1.025^6}\right) + \frac{10000}{1.025^6} \text{ or}$$

$$10275 = 12000\left(\frac{0.1596933}{1.1596933}\right) + \frac{10000}{1.1596933}$$

4. or $10275 = 1652.44 + 8622.96 = 10275.40$.
or indicating that the semi-annual rate of 2½% is within 40 cents mathematically correct and suffices for practical purposes.

The mathematically correct amount for X may be ascertained in the following way:

If in equation (2) $X = 2.51$ is inserted, the result would be:

5. $10275 = 1651.88 + 8617.93 = 10269.81$ whereas if $X = 2.50$

4. $10275 = 1652.44 + 8622.96 = 10275.40$.

From this may be concluded that if X increases by 0.01, the result of the equation decreases by \$5.59. The correct increase of X, in order to satisfy equation (2) perfectly, must be:

$$\text{Increase of X} = \frac{40 \times 0.01}{559} = \frac{40}{55900} = 0.0007$$

The true value of X is therefore $X = 2.5 + 0.0007 = 2.5007$
If this value is inserted in equation (2) the result is:

$$6. \quad 10275 = \frac{300}{0.025007}\left(\frac{1.025007^6 - 1}{1.025007^6}\right) + \frac{10000}{1.025007^6}$$

or computed with logarithms:

$$\begin{aligned} \log 1.025007 &= 0.0107269 \\ 6 \log 1.025007 &= 0.0643614 \\ 1.025007^6 &= 1.159742 \\ 1.025007^6 - 1 &= 0.159742 \end{aligned}$$

$$10275 = \frac{300}{0.025007}\left(\frac{0.159742}{1.159742}\right) + \frac{10000}{1.159742} = \frac{47.9226}{0.0290016} + \frac{10000}{1159742}$$

$10275 = \$1652.40 + 8622.60 = 10275.$

This proves that the mathematically correct amount of the semi-annual effective rate of interest is $= 2.5007\%$.

PERIODICAL AMORTIZATION BASED ON TRUE OR EFFECTIVE RATE OF INTEREST

Take the correct effective rate of semi-annual interest or the rate of yield on the purchase price of the bond at 2.5007%, and the nominal income, \$300.00.

There is for the first semi-annual period, a true income of 10275×0.025007. True income $=$ \$256.95. The difference between the nominal income and the true income is \$43.05.

This amount of \$43.05 is the first periodical amortization and should be charged at the end of the first period to income and credited to bond investment, if the bond investment was charged with the purchase price.

Dr. Income................	\$43.05
Cr. Bond investment.......	43.05

Or, if a separate bond premium account is kept:

Dr. Income
Cr. Bond premium account.

For the second period, the bond investment has decreased by \$43.05 and is now 10275 — 43.05 = 10231.95, on which the true income is 10231.95 × 0.025007 = 255.87, and the amortization \$44.13.

These operations are to be continued semi-annually six times, until at the end of the third year, the entire bond premium is entirely charged off.

The periodical amortization, true income and bond balances, computed on the basis of true effective interest rate for the full period, is as follows:

Period	Nominal income	True income	Bond amortization	Bond purch. price	Bond balance
1	\$300.00	\$256.95	\$43.05	\$10,275.00	\$10,231.95
2	300.00	255.87	44.13		10,187.82
3	300.00	254.77	45.23		10.142.59
4	300.00	253.64	46.36		10,096.23
5	300.00	252.48	47.52		10,048.71
6	300.00	251.29	48.71		10,000.00
	\$1,800.00	\$1,525.00	\$275.00	\$10,275.00	

Another, more scientific way for computing the exact periodical amortization, after the true effective rate of interest has been obtained, is as follows:

If the first periodical amortization is designated by X, the aggregate accumulation for 6 periods at 2.5007% true interest compounded semi-annually must equal the bond premium. Or expressed by equation

7. $X\ (1 + 1.025007 + 1.025007^2 + + + 1.025007^5) = \$275.00$

$$\text{or } X\left(\frac{1.025007^6 - 1}{0.025007}\right) = 275$$

$$\text{or } X = \frac{275 \times 0.025007}{1.025007^6 - 1}$$

$$X = \frac{6.876925}{1.025007^6 - 1}$$

Computed with logarithms:

$$\begin{aligned} \log 1.025007 &= 0.0107269 \\ 6 \log 1.025007 &= 0.0643614 \\ 1.025007^6 &= 1.159742 \\ 1.025007^6 - 1 &= 0.159742 \end{aligned}$$

$$\text{or } X = \frac{6.876925}{0.159742} = \$43.05$$

Equation (7) determines the following amortizations as:

1st	amortization	X =	$43.05
2d	"	43.05×1.025007 =	44.13
3d	"	43.05×1.025007^2 =	45.23
4th	"	43.05×1.025007^3 =	46.36
5th	"	43.05×1.025007^4 =	47.52
6th	"	43.05×1.025007^5 =	48.71
			$275.00

For the purpose of comparing results, there is given a tabulation of the mathematically correct figures and those resulting from approximate methods as given on page 18 of the July issue of The Journal of Accountancy.

Period	True income	Exact amortization	Approximate amortization	True bond balance	Approximate bond balance
1st	$256.95	$43.05	$43.12	$10,231.95	$10,231.88
2d	255.87	44.13	44.20	10,187.82	10,187.68
3d	254.77	45.23	45.31	10,142.59	10,142.37
4th	253.64	46.36	46.44	10,096.23	10,095.23
5th	252.48	47.52	47.60	10,048.71	10,048.33
6th	251.29	48.71	48.33*	10,000.00	10,000.00
	$1,525.00	$275.00	$275.00		

*Adjusted by 46 cents from $48.79 to $48.33.